Existence of global strong solutions in critical spaces for barotropic viscous fluids

Boris Haspot *

Abstract

This paper is dedicated to the study of viscous compressible barotropic fluids in dimension $N \geq 2$. We address the question of the global existence of strong solutions for initial data close from a constant state having critical Besov regularity. In a first time, this article show the recent results of [6] and [10] with a new proof. Our result relies on a new a priori estimate for the velocity, where we introduce a new structure to *kill* the coupling between the density and the velocity as in [19]. We study so a new variable that we call effective velocity. In a second time we improve the results of [6] and [10] by adding some regularity on the initial data in particular ρ_0 is in H^1. In this case we obtain global strong solutions for a class of large initial data on the density and the velocity which in particular improve the results of D. Hoff in [22]. We conclude by generalizing these results for general viscosity coefficients.

1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu(\rho) D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) + \nabla P(\rho) = \rho f, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density. The pressure P is a suitable smooth function of ρ. We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$ (in the sequel to simplify the calculus we will assume the viscosity coefficients as constants). Such a conditions ensures ellipticity for the momentum equation and is satisfied in the physical cases where $\lambda + \frac{2\mu}{N} > 0$. We supplement the problem with initial condition (ρ_0, u_0) and an outer force f. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$ or to the periodic box $\mathcal{T}_a^N$ with period a_i, in the i-th direction. We restrict ourselves the case $N \geq 2$.
The problem of existence of global solution in time for Navier-Stokes equations was addressed in one dimension for smooth enough data by Kazhikov and Shelukin in [27], and for discontinuous ones, but still with densities away from zero, by Serre in [33] and Hoff in

*Karls Ruprecht Universität Heidelberg, Institut for Applied Mathematics, Im Neuenheimer Feld 294, D-69120 Heildelberg, Germany. Tel. 49(0)6221-54-6112

[20]. Those results have been generalized to higher dimension by Matsumura and Nishida in [29] for smooth data close to equilibrium and by Hoff in the case of discontinuous data in [23, 24]. All those results do not require to be far from the vacuum. The existence and uniqueness of local classical solutions for (1.1) with smooth initial data such that the density ρ_0 is bounded and bounded away from zero (i.e., $0 < \underline{\rho} \leq \rho_0 \leq M$) has been stated by Nash in [31]. Let us emphasize that no stability condition was required there. On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [29]. Many works on the case of the one dimension have been devoted to the qualitative behavior of solutions for large time (see for example [20, 27]). Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [34], [35], [22], [26]. Let us recall that (local) existence and uniqueness for (1.1) in the case of smooth data with no vacuum has been stated for long in the pioneering works by J. Nash [31], and A. Matsumura, T. Nishida [29].
Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see e.g [30]):

$$(NS) \qquad \begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \text{div} v = 0, \end{cases}$$

we aim at solving (1.1) in the case where the data (ρ_0, u_0, f) have *critical* regularity.
By critical, we mean that we want to solve the system functional spaces with norm in invariant by the changes of scales which leaves (1.1) invariant. In the case of barotropic fluids, it is easy to see that the transformations:

$$(\rho(t,x), u(t,x)) \longrightarrow (\rho(l^2 t, lx), l u(l^2 t, lx)), \quad l \in \mathbb{R}, \tag{1.2}$$

have that property, provided that the pressure term has been changed accordingly.

Definition 1.1 *Let $\bar{\rho} > 0$. In the sequel we will note: $q = \frac{\rho - \bar{\rho}}{\bar{\rho}}$.*

The use of critical functional frameworks led to several new weel-posedness results for compressible fluids (see [12, 16, 17, 19]). In addition to have a norm invariant by (1.2), appropriate functional space for solving (1.1) must provide a control on the L^∞ norm of the density (in order to avoid vacuum and loss of ellipticity). For that reason, we restricted our study to the case where the initial data (ρ_0, u_0) and external force f are such that, for some positive constant $\bar{\rho}$:

$$q_0 \in B^{\frac{N}{p}}_{p,1}, \; u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1} \;\text{ and }\; f \in L^1_{loc}(\mathbb{R}^+, \in B^{\frac{N}{p_1}-1}_{p_1,1})$$

with $(p, p_1) \in [1, +\infty[$ good chosen.
Concerning the global existence of strong solutions for initial data with high regularity order and close to a stable equilibrium has been proved by Matsumura and Nishida in [29] for three-dimensional polytropic ideal fluids and no outer force. More precisely for $\bar{\rho} > 0$, the initial data are choosen small in the following spaces $(\rho_0 - \bar{\rho}, u_0) \in H^3 \times H^3$. More recently D. Hoff in [22, 21] stated the existence of global weak solutions with small

initial data including discontinuous initial data (namely q_0 is small in $L^2 \cap L^\infty$ and u_0 is small in L^4 if $N = 2$ and small in L^8 if $N = 3$). One of the major interest of the results of Hoff is to get some smoothing effects on the incompressible part of the velocity u and on the effective viscous flux $F = (2\mu + \lambda)\text{div}u - P(\rho) + P(\bar{\rho})$ are also pointed out. D. Hoff is the first author to have introduced the notion of effective flux which play a crucial role in the proof of P-L Lions for the existence of global weak solution. However if the results of Hoff are critical in the sense of the scaling for the density, it is not the case for the initial velocity. In [23], D. Hoff show a very interesting theorem of weak-strong uniqueness when $P(\rho) = K\rho$ with $K > 0$. To speak roughly under the conditions that two solutions (ρ, u), (ρ_1, u_1) check a control L^∞ on the density and a control Lipschitz on the velocity, with additional property of regularity on the strong solution (ρ_1, u_1) then we obtain $(\rho, u) = (\rho_1, u_1)$. D. Hoff use this result to show that the solutions of [23] are unique. We will use this theorem in the sequel, by showing that our solutions verify the hypothesis of D. Hoff in [23]. Finally R. Danchin in [13] show for the first time a result of existence of global strong solution close from a stable equilibrium in critical space for the scaling of the system. More precisely the initial data are choose as follows $(q_0, u_0) \in (B^{\frac{N}{2}}_{2,1} \cap B^{\frac{N}{2}-1}_{2,1}) \times B^{\frac{N}{2}-1}_{2,1}$. The main difficulty is to get estimates on the linearized system where the velocity and the density are coupled via the pressure, and what is crucial in this work is the smoothing effect on the velocity and a L^1 decay on $\rho - \bar{\rho}$ (this play a necessary role to control the pressure term). In this work, R. Danchin use some astucious inequality of energy on the system in variable Fourier where he has decomposed the space in dyadic shell. This explain in particular why the result is obtained in Besov space with a Lebesgue index $p = 2$. In the same time of the redaction of this paper, Q. Chen et al in [10] and F. Charve and R. Danchin in [6] improve the previous result by working in more general Besov space by studying the linear part of the system.

The goal of this article is to make a connection between the article of D. Hoff [22, 21] and those of Q. Chen et al and F. Charve and R. Danchin in [6] and [10]. In fact we extend the results [6] and [10] to the case where the Lebesgue index of Besov space are more general, it means $q_0 \in B^{\frac{N}{p}}_{p,1}$ and $u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1}$ with p and p_1 good choosen. In [6] and [10], the authors obtain global weak solutions when $p = p_1 < 2N$ and strong solution when $p = p_1 \leq N$, the restriction on the choice of p come from a very strong coupling between the pressure and the velocity. Indeed in this case the coupling is very strong in high frequencies because of the term of pressure, that's why we need to integrate completly the pressure term in the linear part. In the case of lows frequencies, according the point of view of the Fourier frequencies, the term of pressure is very regular so it does not make problem to consider in the rest. The study of the linear part in this case is crucial to get a gain L^1 of integrability on the density, and for low frequencies we follow the method of R. Danchin in [13]. In [22, 21], D. Hoff get global weak solution with a critical regularity on the density in the sense that $\rho_0 - \bar{\rho}$ is small in L^∞. It means that he does not ask any regularity on the initial density which is of this point of view besser than [13], [6] and [10]. However the velocity in his case in only L^1 *log* Lipschitz, that is why he can not obtain the uniqueness and have only weak solution. In this paper we improve the results of [22, 21] by the fact that we get strong solutions, and we will show that it is just enough to ask a arbitrary small ε regularity on the initial density to get uniqueness.

In [19], we improve the results of R. Danchin in [12, 16], in the sense that the initial density belongs to larger spaces $B^{\frac{N}{p}}_{p,1}$ with $p \in [1,+\infty[$. In the present paper, we address the question of global existence of strong solution in the critical functional framework under the assumption that the initial density belongs to critical Besov space with a index of integrability different of this of the velocity. To do that, as in [19] we introduce a new variable in high frequencies than the velocity that we call effective velocity in the goal to *kill* the relation of coupling between the velocity and the pressure. We observe that this new notion of effective velocity allow us easily to get as R. Danchin in [13] a L^1 decay on q. However this new variable is interesting only in high frequencies, indeed in low frequencies the term $\nabla P(\rho)$ is small in Fourier analysis. Moreover in the low frequency regime, the first order terms predominate and the viscous term Δu may be neglected in Fourier analysis, so that (1.1) has to be treat by means of hyperbolic energy methods (more particularly the velocity verifies in some way a wave equation). This implies that we can treat the low regime only in space construct on L^2, it is classical that the hyperbolic system are ill-posed in general L^p spaces. So as in [10] and [6], the system has to be handled differently in low and high frequencies. In short, we will use the analysis of R. Danchin in [13] in low frequencies and the introduction of this new variable the effctive velocity introduced in [19] in high frequencies. To simplify the notation, we assume from now on that $\bar{\rho}=1$. Hence as long as ρ does not vanish, the equations for $(q=\rho-1,u)$ read:

$$\begin{cases} \partial_t q + u\cdot\nabla q = -(1+q)\mathrm{div}u,\\ \partial_t u + u\cdot\nabla u - \dfrac{1}{1+q}\mathcal{A}u + \nabla P(1+q) = f, \end{cases} \tag{1.3}$$

In the sequel we will note $\mathcal{A}=\mu\Delta+(\lambda+\mu)\nabla\mathrm{div}$ and where g is a smooth function which may be computed from the pressure function P. One can now state our main result.

Theorem 1.1 *Let P a suitably smooth function of the density such that $P'(1)>0$, $f \in \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})$ and $1\leq p_1\leq p<+\infty$ such that $\frac{1}{p_1}\leq\frac{1}{N}+\frac{1}{p}$. Assume that $u_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}$, $f\in L^1_{loc}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})$ and $q_0\in\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}$.Then there exists a constant ε_0 such that if:*

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}} + \|f\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})} \leq \varepsilon_0,$$

then if $\frac{1}{p}+\frac{1}{p_1}>\frac{1}{N}$, $p<\max(4,N)$ and $\frac{1}{2}\leq\frac{1}{p}+\frac{1}{p_1}$ there exists a global solution (q,u) for system (1.1) with $1+q$ bounded away from zero and,

$$q\in\widetilde{C}(\mathbb{R},\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^1(\mathbb{R},\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}) \quad \text{and}$$

$$u\in\widetilde{C}(\mathbb{R};\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^1(\mathbb{R},\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}).$$

Moreover this solution is unique if $\frac{2}{N}\leq\frac{1}{p}+\frac{1}{p_1}$.

Remark 1 *This theorem is the same than Chen et al obtained in [10]. In particular we have strong restrictions on p, it means $p<\max(4,2N)$. This fact is due to the* *interactions between low and high frequencies in the paraproduct laws. However we obtain this result with a new method which seems more flexible and we will explain why in thecorrolary 1.*

Remark 2 *It seems possible to improve the theorem 1.1 by choosing initial data q_0 in $B^{\frac{N}{2}-1,\frac{N}{p}}_{(2,1),(p,\infty)} \cap B^{\frac{N}{2}-1,0}_{(2,1),(\infty,1)}$, however some supplementary conditions appear on p_1 in this case. Here $B^{s_1,s_2}_{(p_1,r_1),(p_2,r_2)}$ is a Besov space where the behavior is $B^{s_1}_{p_1,r_1}$ and $B^{s_2}_{p_2,r_2}$ in high frequencies.*

The key to theorem 1.1 is to introduce a new variable v_1 to control the velocity where to avoid the coupling between the density and the velocity, we analyze by a new way the pressure term. More precisely we write the gradient of the pressure as a Laplacian of the variable v_1, and we introduce this term in the linear part of the momentum equation. We have then a control on v_1 which can write roughly as $u-\mathcal{G}P(\rho)$ where $\mathcal{G}$ is a pseudodifferential operator of order -1. We will call $u-\mathcal{G}P(\rho)$ the effective velocity. By this way, we have canceled the coupling between v_1 and the density, we next verify easily that we have a control Lipschitz of the gradient of u (it is crucial to estimate the density by the mass equation). In the previous theorem 1.1, we have as in [10] very big restrictions on p ($p<\max(4,2N)$) because the behavior in low frequencies. At the difference with the results of strong solutions in finite time (see [19]), we can not choose p arbitrarly big. To overcome this difficulty, we need to add some additional conditions on (q_0,u_0) in low frequencies as in [6] to avoid these restrictions in the use of the paraproduct laws. We obtain then the following corollary:

Corollary 1 *Let P a suitably smooth function of the density with $P^{'}(1)>0$, $f\in \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r})$ and $1\leq p_1\leq p<+\infty$ such that $\frac{1}{p_1}\leq\frac{1}{N}+\frac{1}{p}$. Assume that $u_0\in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r}$, $f\in L^1_{loc}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})$ and $q_0\in\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\cap B^{0,1}_{2,r}$ with $r=+\infty$ if $N\geq 3$ and $r=1$ if $N=2$.Then there exists a constant ε_0 such that if:*

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\cap B^{0,1}_{2,r}}+\|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r}}+\|f\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r})}\leq\varepsilon_0,$$

then if $\frac{1}{p}+\frac{1}{p_1}>\frac{1}{N}$, there exists a global solution (q,u) for system (1.1) with $1+q$ bounded away from zero and,

$$q\in\widetilde{C}(\mathbb{R},\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\cap B^{0,1}_{2,r})\cap\widetilde{L}^1(\mathbb{R},\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}\cap B^{2,1}_{2,r})\quad and$$
$$u\in\widetilde{C}\big(\mathbb{R};(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap B^0_{2,r}\big)\cap\widetilde{L}^1(\mathbb{R},\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}\cap B^2_{2,r}).$$

Moreover this solution is unique if $\frac{2}{N}\leq\frac{1}{p}+\frac{1}{p_1}$.

Remark 3 *We can observe that when p tends to infinity, we are close to get weak solution with the following initial data (q_0,u_0) in $B^{\frac{N}{2}-1,0}_{2,\infty,1}\times B^{\frac{N}{2}-1,1}_{2,N,1}$. It means that this theorem rely the result of D. Hoff where the initial density is assumed L^∞ but where the initial velocity is more regular (it means not critical) and the results of R. Danchin in [16].Moreover it is right for general pressure when in the works of D. Hoff the pressure verify $P(\rho)=K\rho$ with $K>0$.*

Remark 4 *If $r = +\infty$, then we replace above the strong continuity in $B^s_{2,r}$ by the weak continuity.*

Remark 5 *In some some sense, we could consider that the case $p > N$ is not so important because we obtain only the existence of global weak solution as in the works of D. Hoff in [22], [23]. However it stays very interesting, indeed as in the work of F. Charve and R. Danchin in [6], we could add some additional condition on the data such $u_0 \in B^0_{N,1}$. In this fact, it would be easy to show some results of persistency as for Navier-Stokes without condition of smallness on $\|u_0\|_{B^0_{N,1}}$, in particular the fact that $u \in \widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,0}_{2,N,1} + \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}) \cap \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})$. And in this case, we will case the existence of global strong solutions but with only a condition of smallness for the density on $\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,p,1} \cap \widetilde{B}^{0,1}_{2,r}}$. In particular it improves widely the results of [10] and [6], this result is close from the results of D. Hoff in [22] and [23] except that we are in critical space for the scaling (except concerning the fact that $q_0 \in \widetilde{B}^{0,1}_{2,r}$).*
In particular as in [6], we can take $u_0(x) = \phi(x) sin(\varepsilon^{-1} x \cdot \omega) n$ where ω and n stand for any unit vectors of $\mathbb{R}^N$ and ϕ for any smooth compactly supported function then we have if $p_1 > N$:

$$\|u_0\|_{B^{\frac{N}{p_1}-1}_{p_1,1}} \leq C\varepsilon^{1-\frac{N}{p_1}},$$

so that the smallness condition is satisfied by u_0 if ε small enough. However we remark that u_0 is arbitrarly big in L^3. On the other hand, u_0 belongs to Scwartz class $\mathcal{S}$ hence also to $B^0_{N,1}$ so that uniqueness holds true by persistency results.

Remark 6 *We can observe that $u_0 \in B^0_{2,r}$ corresponds exactly to the energy space when $r = 2$, in this sense this additional regularity on the velocity seems very natural. We will explain in the corollary 2 why it is perfectly adapted in the case of specific viscosity coefficients. Indeed in the general case $q_0 \in B^1_{2,r}$ is not in the energy space.*

We want treat now the special case of the *BD viscosity coefficients*. Indeed in [5] Bresch and Desjardins have discovered a new entropy inequality when in (1.1), we have:

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho).$$

In this case they show that we can control $\sqrt{\rho}\nabla\varphi(\rho)$ in $L^\infty(L^2)$ where $\varphi'(\rho) = \frac{\mu'(\rho)}{\rho}$. Roughly it means that we controll the density ρ in $L^\infty(H^1)$. It is the additional condition that we ask in the corollary 1. In the following result, we prove that we can extend the corollary 1 to the case of general viscosity.

Corollary 2 *Let P a suitably smooth function of the density with $P'(1) > 0$, $f \in \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1} \cap B^0_{2,r})$ and μ, λ are general regular functions such that $\mu(1) > 0$ and $\mu(1) + \lambda(1) > 0$ and $1 \leq p_1 \leq p < +\infty$ such that $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p}$. Assume that $u_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1} \cap B^0_{2,r}$, $f \in L^1_{loc}(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})$ and $q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1} \cap B^{0,1}_{2,r}$ with $r = +\infty$ if $N \geq 3$ and $r = 1$ if $N = 2$. Then there exists a constant ε_0 such that if:*

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1} \cap B^{0,1}_{2,r}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1} \cap B^0_{2,r}} + \|f\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1} \cap B^0_{2,r})} \leq \varepsilon_0,$$

then if $\frac{1}{p}+\frac{1}{p_1}>\frac{1}{N}$*, there exists a global solution* (q,u) *for system (1.1) with* $1+q$ *bounded away from zero and,*

$$q\in\widetilde{C}(\mathbb{R},\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}\cap B^{0,1}_{2,r})\cap\widetilde{L}^1(\mathbb{R},\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}\cap B^{2,1}_{2,r})\quad and$$
$$u\in\widetilde{C}\big(\mathbb{R};(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap B^{0}_{2,r}\big)\cap\widetilde{L}^1(\mathbb{R},\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}\cap B^{2}_{2,r}).$$

Moreover this solution is unique if $\frac{2}{N}\leq\frac{1}{p}+\frac{1}{p_1}$.

Remark 7 *This result is very interesting in the case of the BD viscosity coefficients. In this case our result is very close of the energy initial data with the optimal condition for the scaling* $(q_0,u_0)\in B^0_{\infty,1}\times B^0_{N,1})$*. In particular it concerns the shallow-water system.*

Remark 8 *Moreover our method is more flexible than the proofs of D. Hoff in [23], [24], [22] as these works are based crucially on the notion of effective pressure and on a gain of integrability on the velocity which are right only in the case of constant viscosity coefficients.*

Our paper is structured as follows. In section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section 3, we prove estimate on the transport equation. In section 4, we prove the theorem 1.1.In section 5 we prove the corollaries 1 and 2. Two inescapable technical commutator estimates and the proof of paraproduct in hybrid Besov spacesare postponed in appendix.

2 Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A\lesssim B$ means that $A\leq CB$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p\in[1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t\to\|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any $\varphi\in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C}=\{\xi\in\mathbb{R}^N/\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ such that:

$$\sum_{l\in\mathbb{Z}}\varphi(2^{-l}\xi)=1\ \ \text{if}\ \ \xi\neq0.$$

Denoting $h=\mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\Delta_l u=\varphi(2^{-l}D)u=2^{lN}\int_{\mathbb{R}^N}h(2^l y)u(x-y)dy\ \ \text{and}\ \ S_l u=\sum_{k\leq l-1}\Delta_k u\,.$$

Formally, one can write that:

$$u=\sum_{k\in\mathbb{Z}}\Delta_k u\,.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^N)$ for two reasons:

1. The right hand-side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^N)$.
2. Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^N)$ (consider the case of the polynomials).

2.1 Homogeneous Besov spaces and first properties

Definition 2.2 *For $s \in \mathbb{R}$, $p \in [1,+\infty]$, $q \in [1,+\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (\sum_{l\in\mathbb{Z}}(2^{ls}\|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

The Besov space $B^s_{p,q}$ is the set of temperate distribution u such that $\|u\|_{B^s_{p,q}} < +\infty$.

Remark 9 *The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the nonhomogeneous Hölder space C^s and that $B^s_{2,2}$ is the nonhomogeneous space H^s.*

Proposition 2.1 *The following properties holds:*

1. *there exists a constant universal C such that:*
   $C^{-1}\|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}$.
2. *If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*
3. *$B^{s'}_{p,r_1} \hookrightarrow B^s_{p,r}$ if $s' > s$ or if $s = s'$ and $r_1 \leq r$.*

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [4]) and rewrite on a generalized form in [1] by H. Abidi and M. Paicu (in this article the results are written in the case of homogeneous sapces but it can easily generalize for the nonhomogeneous Besov spaces).

Proposition 2.2 *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*
  $$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}})\,. \tag{2.4}$$
- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,+\infty]^2$ such that:$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
  if $s_1 + s_2 + N\inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:
  $$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.5}$$
  when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take

$r = 1$.
If $s_1 + s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$ *and* $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ *then:*

$$\|uv\|_{B_{p,\infty}^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B^{s_1}_{p_1,1}} \|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.6}$$

If $|s| < \frac{N}{p}$ *for* $p \ge 2$ *and* $-\frac{N}{p'} < s < \frac{N}{p}$ *else, we have:*

$$\|uv\|_{B^s_{p,r}} \le C\|u\|_{B^s_{p,r}} \|v\|_{B^{\frac{N}{p}}_{p,\infty} \cap L^\infty}. \tag{2.7}$$

Remark 10 *In the sequel* p *will be either* p_1 *or* p_2 *and in this case* $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ *if* $p_1 \le p_2$*, resp* $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ *if* $p_2 \le p_1$*.*

Corollary 3 *Let* $r \in [1, +\infty]$, $1 \le p \le p_1 \le +\infty$ *and* s *such that:*

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} \le 1$,
- $s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} > 1$,

then we have if $u \in B^s_{p,r}$ *and* $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$*:*

$$\|uv\|_{B^s_{p,r}} \le C\|u\|_{B^s_{p,r}} \|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty}.$$

The study of non stationary PDE's requires space of type $L^\rho(0, T, X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0, T, X)$ (except if $r = p$). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.3 *Let* $\rho \in [1, +\infty]$, $T \in [1, +\infty]$ *and* $s_1 \in \mathbb{R}$*. We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} = \big(\sum_{l\in\mathbb{Z}} 2^{lrs_1} \|\Delta_l u(t)\|^r_{L^\rho(L^p)}\big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *as the set of temperate distribution* u *over* $(0, T) \times \mathbb{R}^N$ *such that* $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} < +\infty$*.*

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r}) = \widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r}) \cap \mathcal{C}([0, T], B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \le \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \ge \rho, \quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \ge \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \le \rho.$$

Remark 11 *It is easy to generalize proposition 2.2, to* $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *spaces. The indices* s_1*,* p*,* r *behave just as in the stationary case whereas the time exponent* ρ *behaves according to Hölder inequality.*

In the sequel we will need of composition lemma in $\widetilde{L}^\rho_T(B^s_{p,r})$ spaces.

Lemma 1 *Let $s>0$, $(p,r)\in[1,+\infty]$ and $u\in\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})\cap L^{\infty}_{T}(L^{\infty})$.*

1. *Let $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)\in\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})}\leq C(\|u\|_{L^{\infty}_{T}(L^{\infty})}\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})}.$$

2. *Let $F\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)-F^{'}(0)u\in\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)-F^{'}(0)u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})}\leq C(\|u\|_{L^{\infty}_{T}(L^{\infty})}\|u\|^{2}_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,r})}.$$

Here we recall a result of interpolation which explains the link of the space $B^{s}_{p,1}$ with the space $B^{s}_{p,\infty}$, see [11].

Proposition 2.3 *There exists a constant C such that for all $s\in\mathbb{R}$, $\varepsilon>0$ and $1\leq p<+\infty$,*

$$\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,1})}\leq C\frac{1+\varepsilon}{\varepsilon}\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,\infty})}\Big(1+\log\frac{\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s+\varepsilon}_{p,\infty})}}{\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p,\infty})}}\Big).$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [11]).

Definition 2.4 *Let $m\in\mathbb{R}$. A smooth function function $f:\mathbb{R}^N\rightarrow\mathbb{R}$ is said to be a $\mathcal{S}^m$ multiplier if for all muti-index α, there exists a constant C_α such that:*

$$\forall\xi\in\mathbb{R}^N,\ \ |\partial^{\alpha}f(\xi)|\leq C_{\alpha}(1+|\xi|)^{m-|\alpha|}.$$

Proposition 2.4 *Let $m\in\mathbb{R}$ and f be a $\mathcal{S}^m$ multiplier. Then for all $s\in\mathbb{R}$ and $1\leq p,r\leq+\infty$ the operator $f(D)$ is continuous from $B^{s}_{p,r}$ to $B^{s-m}_{p,r}$.*

Let us now give some estimates for the heat equation:

Proposition 2.5 *Let $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$ and $1\leq\rho_2\leq\rho_1\leq+\infty$. Assume that $u_0\in B^{s}_{p,r}$ and $f\in\widetilde{L}^{\rho_2}_{T}(\widetilde{B}^{s-2+2/\rho_2}_{p,r})$. Let u be a solution of:*

$$\begin{cases}\partial_t u-\mu\Delta u=f\\ u_{t=0}=u_0\,.\end{cases}$$

Then there exists $C>0$ depending only on N,μ,ρ_1 and ρ_2 such that:

$$\|u\|_{\widetilde{L}^{\rho_1}_{T}(\widetilde{B}^{s+2/\rho_1}_{p,r})}\leq C\big(\|u_0\|_{B^{s}_{p,r}}+\mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}^{\rho_2}_{T}(\widetilde{B}^{s-2+2/\rho_2}_{p,r})}\big)\,.$$

If in addition r is finite then u belongs to $C([0,T],B^{s}_{p,r})$.

2.2 Hybrid Besov spaces

The homogeneous Besov spaces fail to have nice inclusion properties: owing to the low frequencies, the embedding $B^s_{p,1} \hookrightarrow B^t_{p,1}$ does not hold for $s > t$. Still, the functions of $B^s_{p,1}$ are locally more regular than those of $B^t_{p,1}$: for any $\phi \in C_0^\infty$ and $u \in B^s_{p,1}$, the function $\phi u \in B^t_{p,1}$. This motivates the definition of Hybrid Besov spaces introduced by R. Danchin in [13] where the growth conditions satisfied by the dyadic blocks and the coefficient of integrability are not the same for low and high frequencies. Hybrid Besov spaces have been used in [14] to prove global well-posedness for compressible gases in critical spaces. We generalize here a little bit the definition by distinguishing the coefficients of integrability.

Definition 2.5 *Let $s, t \in \mathbb{R}$ and $(p, q) \in [1, +\infty]$. We set:*

$$\|u\|_{\widetilde{B}^{s,t}_{p,q,1}} = \sum_{l\leq 0} 2^{ls}\|\Delta_l u\|_{L^p} + \sum_{l\leq 0} 2^{lt}\|\Delta_l u\|_{L^q}.$$

Notation 1 *We will often use the following notation:*

$$u_{BF} = \sum_{l\leq 0} \Delta_l u \quad \text{and} \quad u_{HF} = \sum_{l>0} \Delta_l u.$$

Remark 12 *We have the following properties:*

- *We have $\widetilde{B}^{s,s}_{p,p,1} = B^s_{p,1}$.*
- *If $s_1 \geq s_3$ and $s_2 \geq s_4$ then $\widetilde{B}^{s_3,s_2}_{p,q,1} \hookrightarrow \widetilde{B}^{s_1,s_4}_{p,q,1}$.*

We shall also make use of hybrid Besov-spaces. For them, one can prove results analoguous to proposition 2.2, we refer to proposition 6.9 in the appendix.

3 The mass conservation equation

We begin this section by recalling some estimates in Besov spaces for transport and heat equations. For more details, the reader is referred to [3].

Proposition 3.6 *Let $1 \leq p_1 \leq p \leq +\infty$, $r \in [1, +\infty]$ and $s \in \mathbb{R}$ be such that:*

$$-N \min(\frac{1}{p_1}, \frac{1}{p'}) < s < 1 + \frac{N}{p_1}.$$

Suppose that $q_0 \in B^s_{p,r}$, $F \in L^1(0, T, B^s_{p,r})$ and that $q \in L^\infty_T(B^s_{p,r}) \cap C([0, T]; \mathcal{S}')$ solves the following transport equation:

$$\begin{cases} \partial_t q + u \cdot \nabla q = F, \\ q_{\ t=0} = q_0. \end{cases}$$

There exists a constant C depending only on N, p, p_1, r and s such that , we have for a.e $t \in [0, T]$:

$$\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq e^{CU(t)}\big(\|q_0\|_{B^s_{p,r}} + \int_0^t e^{-CU(\tau)}\|F(\tau)\|_{B^s_{p,r}} d\tau\big), \tag{3.8}$$

with: $U(t) = \int_0^t \|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty} d\tau$.

We want study now the following problem:

$$(\mathcal{H}) \qquad \begin{cases} \partial_t q + u \cdot \nabla q + \alpha q = F, \\ q_{\ t=0} = q_0. \end{cases}$$

Above a is the unknown function. We assume that $F \in L^r(0,T;B^s_{p,r})$, that v is time dependent vector-fields with coefficients in $L^1(0,T;B^{\frac{N}{p_1}+1}_{p_1,1})$ and $\alpha > 0$.
Indeed we recall that we can rewrite the transport equation on the following form:

$$\partial_t q + u \cdot \nabla q + (q+1)(P(1+q) - P(1)) = -(1+q)\text{div}v_1,$$

where we refer to the section 4.1 for the definition of v_1.

Proposition 3.7 *Let $1 \leq p_1 \leq p \leq +\infty$, $r \in [1,+\infty]$ and $s \in \mathbb{R}$ be such that:*

$$-N\min(\frac{1}{p_1},\frac{1}{p'}) < s < 1 + \frac{N}{p_1}.$$

There exists a constant C depending only on N, p, p_1, r and s such that for all $a \in L^\infty([0,T],B^\sigma_{p,r})$ of $(\mathcal{H})$ with initial data a_0 in $B^s_{p,r}$ and $g \in L^1([0,T],B^s_{p,r})$, we have for a.e $t \in [0,T]$:

$$\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \|q\|_{\widetilde{L}^1_t(B^s_{p,r})} \leq e^{CU(t)}\big(\|q_0\|_{B^s_{p,r}} + \int_0^t e^{-CU(\tau)}\|F(\tau)\|_{B^s_{p,r}}d\tau\big), \tag{3.9}$$

with: $U(t) = \int_0^t \|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}d\tau$.

Proof: Applying Δ_l to $(\mathcal{H})$ yields:

$$\partial_t\Delta_l q + u \cdot \nabla\Delta_l q + \alpha\Delta_l q = R_l + \Delta_l F,$$

with $R_l = [u\cdot\nabla,\Delta_l]q$. Multiplying by $\Delta_l a|\Delta_l a|^{p-2}$ then performing a time integration, we easily get:

$$\|\Delta_l q(t)\|_{L^p} + \alpha\int_0^t \|\Delta_l q(s)\|_{L^p}ds \leq \|\Delta_l q_0\|_{L^p} + \int_0^t \big(\|R_l\|_{L^p} + \frac{1}{p}\|\text{div}u\|_{L^\infty}\|\Delta_l q\|_{L^p} + \|\Delta_l F\|_{L^p}\big)d\tau.$$

Next the term $\|R_l\|_{L^p}$may be bounded according to lemma 2 in appendix. We get then:

$$\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \alpha\|q\|_{\widetilde{L}^1_t(B^s_{p,r})}ds \leq \|\Delta_l q_0\|_{B^s_{p,r}} + \int_0^t \big(\|F(\tau)\|_{B^s_{p,r}} + CU'(\tau)\|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})}\big)d\tau.$$

We end up with Gronwall lemma by letting $X(t) = \|q\|_{\widetilde{L}^\infty_t(B^s_{p,r})} + \alpha\|q\|_{\widetilde{L}^1_t(B^s_{p,r})}$.

4 The proof of theorem 1.1

4.1 Strategy of the proof

To improve the results of Danchin in [13], Charve and Danchin in [6] and Chen et al in [10], it is crucial to kill the coupling between the velocity and the pressure which exists in these works . In this goal, we need to integrate the pressure term in the study of the linearized equation of the momentum equation as in [19]. For making, we will try to express the gradient of the pressure as a Laplacian term, so we set for $\bar{\rho} > 0$ a constant state:

$$\mathrm{div} v = P(\rho) - P(\bar{\rho}).$$

Let $\mathcal{E}$ the fundamental solution of the Laplace operator.

We will set in the sequel: $v = \nabla\mathcal{E} * \big(P(\rho) - P(\bar{\rho})\big) = \nabla\big(\mathcal{E} * [P(\rho) - P(\bar{\rho})]\big)$ ($*$ here means the operator of convolution). We verify next that:

$$\nabla\mathrm{div} v = \nabla\Delta\big(\mathcal{E} * [P(\rho) - P(\bar{\rho})]\big) = \Delta\nabla\big(\mathcal{E} * [P(\rho) - P(\bar{\rho})]\big) = \Delta v = \nabla P(\rho).$$

By this way we can now rewrite the momentum equation of (1.3). We obtain the following equation where we have set $\nu = 2\mu + \lambda$:

$$\partial_t u + u \cdot \nabla u - \frac{\mu}{\rho}\Delta\big(u - \frac{1}{\nu}v\big) - \frac{\lambda + \mu}{\rho}\nabla\mathrm{div}\big(u - \frac{1}{\nu}v\big) = f.$$

We want now calculate $\partial_t v$, by the transport equation we get:

$$\partial_t v = \nabla\mathcal{E} * \partial_t P(\rho) = -\nabla\mathcal{E} * \big(P'(\rho)\mathrm{div}(\rho u)\big).$$

We have finally:

$$\Delta(\partial_t F) = -P'(\rho)\mathrm{div}(\rho u).$$

Notation 2 *To simplify the notation, we will note in the sequel*

$$\nabla\mathcal{E} * \big(P'(\rho)\mathrm{div}(\rho u)\big) = \nabla(\Delta)^{-1}\big(P'(\rho)\mathrm{div}(\rho u)\big).$$

Finally we can now rewrite the system (1.3) as follows:

$$\begin{cases} \partial_t q + (v_1 + \frac{1}{\nu}v) \cdot \nabla q + \frac{1}{\nu}(1+q)(P(\rho) - P(1)) = -(1+q)\mathrm{div} v_1, \\ \partial_t v_1 - \frac{1}{1+q}\mathcal{A}v_1 = f - u \cdot \nabla u + \frac{1}{\nu}\nabla(\Delta)^{-1}\big(P'(\rho)\mathrm{div}(\rho u)\big), \\ q_{/t=0} = a_0, \ (v_1)_{/t=0} = (v_1)_0. \end{cases} \tag{4.10}$$

where $v_1 = u - \frac{1}{\nu}v$ is called the effective velocity. In the sequel we will study this system by exctracting some uniform bounds in Besov spaces on (q, v_1). The advantage of the system (4.10) is that we have *kill* the coupling between v_1 and a term of pressure. Indeed in the works [6] and [10], the pressure was included in the study of the linear system, it means mandatory a coupling between the density and the velocity. In particular it was

impossible to distinguish the index of integration for the Besov spaces.
However we can remark that this change of variable v_1 is interesting only in the case of low frequencies, indeed heuristically in low frequencies $\nabla P(\rho)$ is small in Fourier variable so it is not a matter.
It is natural in this case to study the variable u in low frequencies. Moreover as explained in the introduction, the system (1.1) has a hyperbolic behavior, which means that we can work only with spaces builtet on L^2 (indeed classicaly the hyperbolic system are ill-posed in space constructed on general L^p). In the following ssection, we will explain how to treat the case in low frequencies and how we will use the fact that q behaves in low frequencies as an heat equation.

4.2 A linear model with convection

In this section, we will explain how we treat the low frequency regime by following the approch of Charve and Danchin in [6]. In low frequencies, the first order terms predominate and the viscous term Δu may be neglected so that (1.1) has to be treated by means of hyperbolic energy methods. It means that we can only work in spaces constructed on L^2. Moreover in the case of low frequencies the effective velocity is not a adapted variable in the sense that it is less regular than u as $(\Delta)^{-1}\nabla P(\rho)$ is not very regular. It is better in this case to work with u.The first idea would be to study the linear system associated to (1.1), it means:

$$(PH)\qquad \begin{cases} \partial_t q + \mathrm{div} u = F^{'}, \\ \partial_t u - \mu\Delta u - \lambda\nabla\mathrm{div} u + \nabla q = G^{'}. \end{cases}$$

This system has been studied by D. Hoff and K. Zumbrum in [25]. There, they investigate the decay estimates, and exhibit the parabolic smoothing effect on u and on the low frequencies of q, and a damping effect on the high frequencies of q.
The problem is that if we focus on this linear system, it appears impossible to control the term of convection $u\cdot\nabla q$ which is one derivative less regular than q. However in low frequencies the Green matrix of the linearized systems behaves as the heat kernel (see [10]), the terms $v\cdot\nabla q$ and $v\cdot\nabla u$ can be handled as the perturbation terms. We study then the following system:

$$(LH)^{'}\qquad \begin{cases} \partial_t q + \mathrm{div} u = - + v\cdot\nabla q + F, \\ \partial_t d - \mu\Delta u - \lambda\nabla\mathrm{div} u + \nabla q = -v\cdot\nabla u + G, \end{cases}$$

We obtain then the following proposition:

Proposition 4.8 *Let (q,u) a solution of $(LH)^{'}$, let $s\in\mathbb{R}$. The following estimate holds:*

$$\|(q,u)_{BF}\|_{\widetilde{L}^\infty(\widetilde{B}^s_{2,1})} + \|(q,u)_{BF}\|_{\widetilde{L}^1(\widetilde{B}^{s+2}_{2,1})} \leq \|(q_0,u_0)_{BF}\|_{\widetilde{B}^s_{2,1}} + \|(F,G)_{BF}\|_{\widetilde{L}^1(\widetilde{B}^s_{2,1})} + \|(v\cdot\nabla q, v\cdot\nabla u)BF\|_{\widetilde{L}^1(\widetilde{B}^s_{2,1})}.$$

Proof:

In this case for $j \leq 0$, in terms of Green matrix (see [10]), the solution of $(LH)'$ can be expressed as:

$$\begin{pmatrix} \Delta_j q(t) \\ \Delta_j u(t) \end{pmatrix} = W(t) \begin{pmatrix} \Delta_j q_0 \\ \Delta_j u_0 \end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix} \Delta_j F(s) - \Delta_j(v \cdot \nabla q) \\ \Delta_j G(s) - \Delta_j(v \cdot \nabla u) \end{pmatrix} ds \, .$$

with W the Green matrix. From proposition 4.4 in [10] and Young's inequality we obtain the result. □

4.3 Proof of the existence

Construction of approximate solutions

We use a standard scheme:

1. We smooth out the data and get a sequence of global smooth solutions $(q^n, u^n)_{n\in\mathbb{N}}$ to (1.1) on $\mathbb{R}$ by using the results of [6] and [10].
2. We prove uniform estimates on (q^n, v_1^n) in high frequencies and on (q^n, u^n) in low frequencies.
3. We use compactness to prove that the sequence (q^n, u^n) converges, up to extraction, to a solution of (1.1).

First step

We smooth out the data as follows:

$$q_0^n = S_n q_0, \;\; u_0^n = S_n u_0 \;\; \text{and} \;\; f^n = S_n f.$$

Note that we have:

$$\forall l \in \mathbb{Z}, \;\; \|\Delta_l q_0^n\|_{L^p} \leq \|\Delta_l q_0\|_{L^p} \;\; \text{and} \;\; \|q_0^n\|_{B^{\frac{N}{p}}_{p,1} \cap B^{\frac{N}{p}-1}_{p,1}} \leq \|q_0\|_{B^{\frac{N}{p}}_{p,1} \cap B^{\frac{N}{p}-1}_{p,1}},$$

and similar properties for u_0^n and f^n, a fact which will be used repeatedly during the next steps. Now, according [13], one can solve (1.1) with the smooth data (q_0^n, u_0^n, f^n). We get a solution (q^n, u^n) such that:

$$q^n \in \widetilde{C}(\mathbb{R}, B^N_{2,1} \cap B^{\frac{N}{2}-1}_{2,1}) \;\; \text{and} \;\; u^n \in \widetilde{C}(\mathbb{R}, B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1(\mathbb{R}, B^{\frac{N}{2}+1}_{2,1}). \tag{4.11}$$

Uniform bounds

We set now

$$v_n = \nabla\big(\mathcal{E} * [P(\rho^n) - P(1)]\big) \;\; \text{with} \;\; \text{div} v^n = P(\rho^n) - P(1) \;\; \text{and} \;\; v_1^n = u^n - \frac{1}{\nu} v^n,$$

with $\mathcal{E}$ the fundamental solution of the Laplace operator and $\nu = \lambda + \mu$. In the sequel we will note $g(q^n) = P(\rho^n) - P(1)$ where g is a regular function. In this part, we aim at getting uniform estimates on $(q^n_{HF},(v_1)^n_{HF})$ in high frequencies and on (q^n_{BF},u^n_{BF}) in low frequencies in the following space $E^{'}$ and $F^{'}$:

$$\begin{aligned}
E^{'}&=\big(\widetilde{L}^{\infty}(B^{\frac{N}{p}}_{p,1})\cap\widetilde{L}^{1}(B^{\frac{N}{p}}_{p,1})\big)\times\big(\widetilde{L}^{\infty}(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})+\widetilde{L}^{1}(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})\big).\\
F^{'}&=\big(\widetilde{L}^{\infty}(B^{\frac{N}{2}-1}_{2,1})\cap\widetilde{L}^{1}(B^{\frac{N}{2}+1}_{2,1})\big)\times\big(\widetilde{L}^{\infty}(B^{\frac{N}{2}-1}_{2,1})+\widetilde{L}^{1}(B^{\frac{N}{2}+1}_{2,1})\big).
\end{aligned}$$

More precisely we will obtain uniform estimates on (q^n,u^n) in E and on (q^n,v_1^n) in F whith:

$$\begin{aligned}
E=&\big(\widetilde{L}^{\infty}(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^{1}(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})\big)\times\big(\widetilde{L}^{\infty}(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})\\
&\qquad\cap\widetilde{L}^{1}(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})\big).
\end{aligned}$$

$$\begin{aligned}
F=&\widetilde{L}^{\infty}(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^{1}(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})\times\big(\widetilde{L}^{\infty}(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1}+\widetilde{B}^{\frac{N}{2},\frac{N}{p_1}-1}_{2,p_1,1})\\
&\qquad\cap\widetilde{L}^{1}(\widetilde{B}^{\frac{N}{2}+2,\frac{N}{p}+2}_{2,p,1}+\widetilde{B}^{\frac{N}{2}+2,\frac{N}{p_1}+1}_{2,p_1,1})\big).
\end{aligned}$$

We will work finally in the space H with:

$$(q,u)\in H\Leftrightarrow(q,u)_{BF}\in E^{'}\ \text{ and }\ (q,v_1)_{HF}\in F^{'}.$$

We have then: $\|(q,u)\|_H=\|(q,u)_{BF}\|_{E'}+\|(q,v_1)_{HF}\|_{F'}$. We can now check that (q^n,v_1^n) satisfy the following system:

$$\begin{cases}\partial_t q^n+u^n\cdot\nabla q^n+\dfrac{P^{'}(1)}{\nu}q^n=F_1^n,\\ \partial_t v_1^n-\mathcal{A}v_1^n=F_2^n+f,\\ q_0^n=q_0,\ (v_1^n)_{/t=0}=u_0^n-\dfrac{1}{\nu}v_0^n.\end{cases}\tag{4.12}$$

which is a transport equation and a heat equation.

$$\begin{aligned}
F_1^n&=-(1+q^n)\mathrm{div}v_1^n-\frac{1}{\nu}(P(1+q^n)-P(1)-P^{'}(1)q^n)-\frac{1}{\nu}q^n(P(1+q^n)-P(1)),\\
G_1^n&=(\frac{1}{1+q^n}-1)\mathcal{A}v_1^n-u^n\cdot\nabla u^n+\frac{1}{\nu}\nabla(\Delta)^{-1}(P^{'}(\rho^n)\mathrm{div}(\rho^n u^n)).
\end{aligned}$$

Moreover $(q^n,u^n)_{n\in\mathbb{N}}$ is the solution of the following system:

$$\begin{cases}\partial_t q^n+u^n\cdot\nabla q^n+\mathrm{div}u^n=F^n\\ \partial_t u^n+u^n\cdot\nabla u^n-\mathcal{A}u^n+P^{'}(1)\nabla q^n=G^n+f^n\\ (q^n,u^n)_{/t=0}=(q_0^n,u_0^n),\end{cases}\tag{4.13}$$

which has been studied for low frequencies in proposition 4.8 with:

$$\begin{aligned}
F^n&=-q^n\mathrm{div}u^n,\\
G^n&=-\frac{q^n}{1+q^n}\mathcal{A}u^n+(P^{'}(1)-P^{'}(1+q^n))\nabla q^n.
\end{aligned}$$

Let us set:

$$E(q,u)=\|q\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|u\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}+\|q\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})}$$
$$+\|u\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})},$$
$$E_1(q,u)=\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,1})}+\|u\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,1})}+\|q\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,1})}+\|u\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,1})}.$$
$$E_2(q,u)=\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{p}}_{p,1})}+\|u\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}+\|q\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}$$
$$+\|u\|_{\widetilde{L}^1(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})}.$$

One can now apply the propositions 2.5 at our system to obtain uniform bounds, so we have in high frequencies to control (v_1^n,q^n) and in low frequencies (q^n,u^n):

$$E_2((q^n,v_1^n)_{HF})\le C\big(\|(q_0)_{HF}\|_{B^{\frac{N}{p}-1}_{p,1}+B^{\frac{N}{p}}_{p,1}}+\|(u_0)_{HF}\|_{B^{\frac{N}{p_1}-1}_{p_1,1}}$$
$$+\|(F_1^n)_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}+\|G_1^n\|_{\widetilde{L}^1(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}\big),$$

and

$$E_1((q^n,u^n)_{BF})\le C\big(\|(q_0)_{BF}\|_{B^{\frac{N}{2}-1}_{2,1}}+\|(u_0)_{BF}\|_{B^{\frac{N}{2}-1}_{2,1}}$$
$$+\|(F^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}+\|G^n\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\big),$$

Therefore, it is only a matter of proving appropriate estimates for F_1^n, G_1^n, F^n and G^n by using properties of continuity on the paraproduct and proposition **??**, 2.5 and 4.8. We begin by estimating $\|(F_1^n)_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}$ and $\|(G_1^n)_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}$, we have to use proposition 2.2 and proposition 6.9 and the fact that by interpolation ${\rm div}v_1^n$ is in $\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})$ because $\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}+\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p_1}}_{2,p_1,1})\hookrightarrow\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})$ as $p_1\le p$:

$$\|\big((1+q^n){\rm div}v_1^n\big)_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}\le\|{\rm div}v_1^n\|_{\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})}+\|q\|_{L^\infty(L^\infty)}\|{\rm div}v_1^n\|_{\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1})}$$
$$+\|{\rm div}v_1^n\|_{L^1(L^\infty)}\|q^n\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}.$$

$$\|\big[(P(1+q^n)-P(1)-P'(1)q^n)\big]_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}\le C\|q^n\|^2_{\widetilde{L}^2(B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})},$$

$$\|\big[q^n(P(1+q^n)-P(1))\big]_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}\le C\|q^n\|^2_{\widetilde{L}^2(B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})},$$

Next we have to treat the term $[\frac{q^n}{1+q^n}\mathcal{A}v_1^n]_{HF}$ in $\widetilde{L}^1(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})$, where we can split $\mathcal{A}v_1^n$ on the form:

$$v_1^n=h^n+g^n,$$

with: $h^n\in\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})\cap\widetilde{L}^1(B^{\frac{N}{2}+2,\frac{N}{p_1}+1}_{2,p_1,1})$ and $g^n\in\widetilde{L}^\infty(B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^1(B^{\frac{N}{2}+2,\frac{N}{p}+2}_{2,p,1})$. We obtain then by proposition 6.9:

$$\begin{aligned}\|\big[\frac{q^n}{1+q^n}\mathcal{A}g^n\big]_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}&\leq\|T_{\frac{q^n}{1+q^n}}\mathcal{A}g^n\|_{\widetilde{L}^1(B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}+\|T_{\mathcal{A}g^n}\frac{q^n}{1+q^n}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\\&\quad+\|R(\mathcal{A}g^n,\frac{q^n}{1+q^n})\|_{\widetilde{L}^1(B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})},\\&\leq C\|q^n\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\|\mathcal{A}g^n\|_{\widetilde{L}^1(B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})},\end{aligned}$$

Next we have to use proposition 6.9 to treat the term $T_{\mathcal{A}h^n}(\frac{1}{1+q^n}-1)$ and $R(\mathcal{A}h^n,\frac{1}{1+q^n}-1)$ when $p_1>N$, we have then:

$$\|T_{\mathcal{A}h^n}\frac{q^n}{1+q^n}\|_{\widetilde{L}^1(B^{\frac{N}{2},\frac{N}{p_1}-1}_{2,p_1,1})}\leq\|\mathcal{A}h^n\|_{\widetilde{L}^1(B^{\frac{N}{2},\frac{N}{p_1}-1}_{2,p_1,1})}\|\frac{q^n}{1+q^n}\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})},$$

where following the proposition 6.9, we have chose $p=2$, $q=p_1$, and as $p\geq p_1$ we have $\frac{1}{\lambda'}=\frac{1}{p_1}-\frac{1}{p}$ and $\lambda=+\infty$. It means that: $\frac{N}{p_1}-1\leq\frac{N}{p}$ (what is assumed) and $2\leq\lambda'$ if $2\geq\frac{p_1p}{p-p_1}$. It means that we need of the following condition:

$$2\leq\frac{p_1p}{p-p_1}\quad\text{and}\quad\frac{N}{p_1}-1\leq\frac{N}{p}.\tag{4.14}$$

Next we have as $\frac{N}{p_1}+\frac{N}{p}-1>0$ by proposition 6.9 for the rest term on the high frequencies:

$$\|\big[R(\mathcal{A}h^n,(\frac{1}{1+q^n}-1))\big]_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p_1}-1}_{p_1,1})}\leq\|h^n\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})}\|\frac{1}{1+q^n}-1\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}.$$

We have seen that we need to treat this term of the condition:

$$\frac{N}{p_1}+\frac{N}{p}-1>0.\tag{4.15}$$

Easily we have by proposition 6.9 as $\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\hookrightarrow L^\infty$:

$$\|T_{\frac{q^n}{1+q^n}-1}\mathcal{A}h^n]_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p_1}-1}_{p_1,1})}\leq\|\mathcal{A}h^n\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})}\|\frac{q^n}{1+q^n}\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}.$$

We treat now the term $u^n\cdot\nabla u^n$ and we have as $u^n\in E$, it exists h_1^n and g_1^n such that $u^n=g_1^n+h_1^n$ with $h_1^n\in\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})$ and $g_1^n\in\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})\cap\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})$. We have then by proposition 6.9:

$$\begin{aligned}\|\big(h_1^n\cdot\nabla h_1^n\big)_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{p}}_{p,1})}&\leq\|T_{h_1^n}\nabla h_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}+\|T_{\nabla h_1^n}h_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\\&\quad+\|R(\nabla h_1^n,h_1^n)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})},\\&\leq\|h_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}\|h_1^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\end{aligned}$$

Next we have to treat the term $T_{g_1^n}\nabla g_1^n$ by using the proposition 6.9 with $\frac{1}{\lambda'}=\frac{1}{p_1}-\frac{1}{p}$, $2\leq\lambda'$ and $\frac{N}{p_1}-1\leq\frac{N}{p}$ then:

$$\|T_{g_1^n}\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}_{2,p_1,1}^{\frac{N}{2}-1,\frac{N}{p_1}-1})}\leq\|g_1^n\|_{\widetilde{L}^\infty(\widetilde{B}_{2,p_1,1}^{\frac{N}{2}-1,\frac{N}{p_1}-1})}\|\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}.$$

We have seen that we need of the conditions:

$$\frac{N}{p_1}-1\leq\frac{N}{p}\ \text{ and }\ 2\leq\frac{p_1p}{p-p_1}. \tag{4.16}$$

Easily we have as $\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}}\hookrightarrow L^\infty$ by proposition 6.9:

$$\|T_{\nabla g_1^n}g_1^n\|_{\widetilde{L}^1(\widetilde{B}_{2,p_1,1}^{\frac{N}{2}-1,\frac{N}{p_1}-1})}\leq C\|g_1^n\|_{\widetilde{L}^\infty(\widetilde{B}_{2,p_1,1}^{\frac{N}{2}-1,\frac{N}{p_1}-1})}\|\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}.$$

To finish with the term $g_1^n\nabla g_1^n$, we have to treat the term $\big(R(g_1^n,\nabla g_1^n)\big)_{HF}$. By proposition 6.9, as $\frac{N}{p}+\frac{N}{p_1}-1>0$ we have:

$$\|\big(R(g_1^n,\nabla g_1^n)\big)_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1})}\leq C\|g_1^n\|_{\widetilde{L}^\infty(\widetilde{B}_{2,p_1,1}^{\frac{N}{2}-1,\frac{N}{p_1}-1})}\|\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}.$$

We have seen that we need of the conditions:

$$\frac{N}{p_1}-1+\frac{N}{p}>0. \tag{4.17}$$

From the previous inequalities, we have obtained:

$$\|\big(g_1^n\cdot\nabla g_1^n\big)_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1})}\leq C\|g_1^n\|_{\widetilde{L}^\infty(\widetilde{B}_{2,p_1,1}^{\frac{N}{2}-1,\frac{N}{p_1}-1})}\|\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}.$$

We can treat similarly the terms $g_1^n\cdot\nabla h_1^n$ and $h_1^n\cdot\nabla g_1^n$. We have finally under the conditions (4.14), (4.15), (4.16) and (4.17):

$$\|\big(u^n\cdot\nabla u^n\big)_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1}+B_{p,1}^{\frac{N}{p}})}\leq C\|u^n\|_E^2.$$

We finish with the following term where f is a regular function such that $f(0)=0$:

$$\begin{aligned}
&\|\big[\nabla(\Delta)^{-1}(P'(\rho^n)\mathrm{div}(\rho^n u^n))\big]_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1}+B_{p,1}^{\frac{N}{p}})}\\
&\qquad\qquad\leq\|\big[\nabla(\Delta)^{-1}(f(q^n)\mathrm{div}(q^n u^n))\big]_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1}+B_{p,1}^{\frac{N}{p}})}\\
&+\|\big[\nabla(\Delta)^{-1}(\mathrm{div}(q^n u^n))\big]_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1}+B_{p,1}^{\frac{N}{p}})}+\|\big[\nabla(\Delta)^{-1}\mathrm{div}(u^n)\big]_{HF}\|_{\widetilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}-1}+B_{p,1}^{\frac{N}{p}})},\\
&\leq C\|\mathrm{div}(u^n)\|_{\widetilde{L}^1(B_{2,p,1}^{\frac{N}{2},\frac{N}{p}+1})}(1+\|q^n\|_{\widetilde{L}^\infty(B_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}+\|q^n\|^2_{\widetilde{L}^\infty(B_{2,p,1}^{\frac{N}{2},\frac{N}{p}})}).
\end{aligned}$$

We have now to treat the case of low frequencies and in particular estimating $\|(F^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}$ and $\|(G^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}$, we begin with $\|(F^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}$. We have then according proposition 6.9 if $p<\max(4,2N)$:

$$\begin{aligned}\|(q^n{\rm div}u^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq&\|T_{q^n}({\rm div}u^n)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}+\|T_{{\rm div}u^n}q^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\\&+\|\big(R(q^n,{\rm div}u^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})},\\ \leq C\big(&\|q^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}\|u^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}+\|q^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\|u^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1})}\big).\end{aligned}$$

Here the only difficulty was to treat the term $R(q^n,{\rm div}u^n)$ when $N=2$, we need in this case of the previous condition:

$$p<\max(4,2N).\tag{4.18}$$

and:

$$\|\big(R(q^n,{\rm div}u^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq C\|q^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}\|u^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

Similarly we have by using proposition 6.9 with condition (4.18):

$$\|(u^n\cdot\nabla q^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq C\|q^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}\|u^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

We now want to estimate $\|G^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}$, we begin with $\|(\frac{q^n}{1+q^n}\mathcal{A}u^n)_{HF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}$, the main difficulty corresponds to treat $T_{\mathcal{A}u^n}\frac{q^n}{1+q^n}$ and $R(\frac{q^n}{1+q^n},\mathcal{A}u^n)$. We have by using proposition 6.9 if: $\frac{1}{2}\leq\frac{2}{p}$, $N-1>0$, $2\frac{N}{p}-1>0$. We recall here that $\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})\hookrightarrow\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})$, we have then:

$$\|\big(R(\frac{q^n}{1+q^n},\mathcal{A}u^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq C\|\frac{q^n}{1+q^n}\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}\|\mathcal{A}u^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}.$$

We need then of the following conditions:

$$p<\max(4,2N).\tag{4.19}$$

Next we have according proposition 6.9 with $\lambda=\lambda'=+\infty$:

$$\begin{aligned}\|\big(T_{\mathcal{A}u^n}\frac{q^n}{1+q^n}\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}&\leq\|T_{\mathcal{A}u^n}\frac{q^n}{1+q^n}\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})},\\&\leq C\|\frac{q^n}{1+q^n}\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}\|\mathcal{A}u^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\end{aligned}$$

Finally we have:

$$\|(K(q^n)\nabla q^n)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq\|q^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

To finish, it stays in low frequencies the terms: $T_{\nabla u^n} u^n$, $T_{u^n}\nabla u^n$ and $R(u^n,\nabla u^n)$. We have then by proposition 6.9 if $p<\max(4,2N)$):

$$\|(R(h_1^n,\nabla h_1^n))_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq C\|h_1^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}\|\nabla h_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

We have seen that we need again of condition (4.18).
We want treat now $(R(g_1^n,\nabla g_1^n))_{BF}$, we have then by proposition 6.9 if $\frac{N}{p_1}+\frac{N}{p}-1>0$ and $\frac{1}{2}\leq\frac{1}{p}+\frac{1}{p_1}$.

$$\|(R(g_1^n,\nabla g_1^n))_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})}\leq C\|g_1^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p-1,1})}\|\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

We have seen that we need of the following conditions:

$$\frac{1}{2}\leq\frac{1}{p}+\frac{1}{p_1}\;\;\text{and}\;\;\frac{N}{p_1}+\frac{N}{p}-1>0. \tag{4.20}$$

Next we have by proposition 6.9 if:

$$\|T_{\nabla h_1^n}h_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\leq C\|h_1^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\|\nabla h_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

and

$$\|T_{\nabla g_1^n}g_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}\leq C\|g_1^n\|_{\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p-1,1})}\|\nabla g_1^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.$$

We proced similarly to control $T_{u^n}\nabla u^n$. Therefore the above inequalities with conditions (4.14), (4.15), (4.16), (4.17), (4.18), (4.19) and (4.20) imply that for all $t\in\mathbb{R}$ we have :

$$\|(q^n,u^n)\|_{H_t}\leq Ce^{C\|(q^n,u^n)\|_{H_t}}(\|q_0\|_{B^{\frac{N}{2}-1\frac{N}{p}}_{2,p,1}}+\|u_0\|_{B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}}+\|f\|_{\widetilde{L}^1(B^{\frac{N}{2}-1\frac{N}{p_1}-1}_{2,p_1,1})}+\|(q^n,u^n)\|^2_{H_t}).$$

From a standard bootstrap argument, it is now easy to conclude that there exists a positive constant c such that if the data has been chosen so small as to satisfy:

$$\|q_0\|_{B^{\frac{N}{2}-1\frac{N}{p}}_{2,p,1}}+\|u_0\|_{B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}}+\|f\|_{\widetilde{L}^1(B^{\frac{N}{2}-1\frac{N}{p_1}-1}_{2,p_1,1})}\leq c.$$

then it exists $C>0$ such that for all $t\in\mathbb{R}$:

$$\|(q^n,u^n)\|_{H_t}\leq C,\;\;\forall t\in\mathbb{R}.$$

Compactness arguments

Let us first focus on the convergence of $(q^n)_{n\in\mathbb{N}}$. We claim that, up to extraction, $(q^n)_{n\in\mathbb{N}}$ converges in the distributional sense to some function q such that:

$$q\in\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\cap\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{p}}_{2,p,1}). \tag{4.21}$$

The proof is based on Ascoli's theorem and compact embedding for Besov spaces. As similar arguments have been employed in [12] or [16], we only give the outlines of the proof. We may write that:

$$\partial_t q^n=-u^n\cdot\nabla q^n-(1+q^n)\mathrm{div}u^n.$$

Since $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^2_T(B^{\frac{N}{2},\frac{N}{p}+1}_{2,p,1}+B^{\frac{N}{2},\frac{N}{p_1}}_{2,p_1,1})$ and $q^n\in\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})$, we have $(1+q^n)\mathrm{div}u^n$ which is bounded in $\widetilde{L}^2_T(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}+B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})$ with the conditions between p and p_1 in theorem 1.1. Similarly $u^n\cdot\nabla q^n$ is bounded in $\widetilde{L}^2_T(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}+B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})$. Finally as $p\geq p_1$, we have proved that $\partial_t q^n$ is bounded in $\widetilde{L}^2_T(B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})$, it means that $(q^n)_{n\in\mathbb{N}}$ seen as a sequence of $B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}$ valued functions is equicontinuous in $\mathbb{R}$. In addition $(q^n)_{n\in\mathbb{N}}$ is bounded in $C(\mathbb{R},B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}\cap B^{\frac{N}{2},\frac{N}{p}}_{2,p,1})$. As the embedding $B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1}\cap B^{\frac{N}{2},\frac{N}{p}}_{2,p,1}$ is locally compact (see [3], Chap2), one can thus conclude by means of Ascoli's theorem and Cantor diagonal extraction process that there exists some distribution q such that up to an omitted extraction $(\psi q^n)_{n\in\mathbb{N}}$ converges to ψq in $\mathbb{C}(\mathbb{R},B^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})$ for all smooth ψ with compact support in $\mathbb{R}^=\times\mathbb{R}^N$. Then by using the so-called Fatou property for the Besov spaces, one can conclude that (4.21) is satisfied. (the reader may consult [3], Chap 10 too). By proceeding similarly, we can prove that up to extraction, $(u^n)_{n\in\mathbb{N}}$ converges in the distributional sense to some function u such that:

$$u\in\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}+B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p_1,1})\cap\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,1}). \tag{4.22}$$

In order to complete the proof of the existence part of theorem 1.1, it is only a matter of checking the continuity properties with respect to time, namely that:

$$q\in\widetilde{C}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})\ \text{ and }\ u\in\widetilde{C}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1}+\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}).$$

As regards q, it suffices to notice that, according to (4.21), (4.22) and to the product laws in the Besov spaces, we have:

$$\partial_t q+u\cdot q=-(1+q)\mathrm{div}u\in\widetilde{L}^1(B^{\frac{N}{2},NN}_{2,p,1}.$$

As $q_0\in B^{\frac{N}{2},\frac{N}{p}}_{2,p,1}$, classical results for the transport equation (see [3], Chap 3) ensure that $q\in\widetilde{C}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})$. And as previously, we have shown that $q\in\widetilde{C}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})$, it means clearly that $q\in\widetilde{C}(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})$.
For getting the continuity result for u, one may similarly use the properties of the heat equation on v_1 in high frequncies and on u in low frequencies.

The proof of the uniqueness

In the case $\frac{2}{N}\leq\frac{1}{p}+\frac{1}{p_1}$, the uniqueness has been established in [12, 19].

5 Proof of corollary 1 and 2

5.1 Proof of corollary 1

We want here to avoid the condition $p<\max(4,2N)$. For simplicity we will treat only the case $N=3$. This condition appears when we want treat the terms of rest in low

frequencies. For resolving this problem as in the paper of F. Charve and R. Danchin in [6], we need of additionnal condition in high frequencies on q_0 and u_0.
We want then to follow the same strategy as in the proof of theorem 1.1. It means that we use the same standard scheme which consists in the construction of approximate solutions, some uniform bounds and results of compactness. We will use the same notations as in proofof theorem 1.1. We just want treat the non linear term where appears the condition $p < \max(2N, 4)$ in an other way by using the additional hypothesis that we have on $(q_0, u_0) \in \widetilde{B}^{0,1}_{2,\infty} \times B^0_{2,\infty}$. The rest of the proof will be the same as in theorem 1.1. We will work with the same space as in the proof 1.1 except that we attend additional regularity on (q^n, u^n) in $E^{'}$ with:

$$E^{'}_1 = (\widetilde{L}^\infty(\widetilde{B}^{0,1}_{2,\infty} \cap \widetilde{L}^1(\widetilde{B}^{2,1}_{2,\infty}) \times (\widetilde{L}^\infty(\widetilde{B}^0_{2,\infty} \cap \widetilde{L}^1(\widetilde{B}^2_{2,\infty}).$$

Here $(q^n, u^n)_{n\in\mathbb{N}}$ is the solution of the following system:

$$\begin{cases} \partial_t q^n + u^n \cdot \nabla q^n + \mathrm{div} u^n = F^n \\ \partial_t u^n + u^n \cdot \nabla u^n - \mathcal{A} u^n + P^{'}(1)\nabla q^n = G^n + f^n \\ (q^n, u^n)_{/t=0} = (q^n_0, u^n_0), \end{cases} \tag{5.23}$$

which verifies proposition 4 in [6] with:

$$\begin{aligned} &F^n = -q^n \mathrm{div} u^n, \\ &G^n = -\frac{q^n}{1+q^n}\mathcal{A} u^n + (P^{'}(1) - P^{'}(1+q^n))\nabla q^n. \end{aligned}$$

We apply exactly the same proof than for theorem 1.1, however we have to complete the uniform bounds by showing that (q^n, u^n) is uniformly bounded in $H^{'} \cap E^{'}_1$, moreover we have to treat differently the term in low frequencies where appears the conditions $p < \max(4, 2N)$ and $\frac{1}{2} \leq \frac{1}{p} + \frac{1}{p_1}$ by using the fact that (q^n, u^n) in $E^{'}_1$.
We begin with treatinf the terms $\|F^n\|_{\widetilde{L}^1(B^{0,1}_{2,\infty})}$ and $\|G^n\|_{\widetilde{L}^1(B^0_{2,\infty})}$ by using properties of continuity on the paraproduct and proposition 4 of [?]. We have then:

$$\|T^{'}_{q^n} \mathrm{div} u^n\|_{\widetilde{L}^1(B^1_{2,\infty})} \leq \|q^n\|_{L^\infty(L^\infty)} \|\mathrm{div} u^n\|_{\widetilde{L}^1(B^1_{2,\infty})}.$$

Similarly:

$$\|T^{'}_{q^n} \mathrm{div} u^n\|_{\widetilde{L}^1(B^0_{2,\infty})} \leq \|q^n\|_{L^2(L^\infty)} \|\mathrm{div} u^n\|_{\widetilde{L}^2(B^0_{2,\infty})}.$$

Next we have:

$$\|K(q)\nabla q\|_{\widetilde{L}^1(B^0_{2,\infty})} \leq \|q\|_{L^2(L^\infty)} \|K(q)\|_{\widetilde{L}^2(B^1_{2,\infty})}.$$

For the term $(\frac{1}{\rho} - \frac{1}{\bar{\rho}})\Delta u = J(q)\Delta u$ with J regular and $J(0) = 0$, we have:

$$\begin{aligned} &\|T_{J(q)}\Delta u\|_{\widetilde{L}^1(B^0_{2,\infty})} \leq \|J(q)\|_{L^\infty(L^\infty)} \|\Delta u\|_{\widetilde{L}^1(B^0_{2,\infty})}, \\ &\|T_{\Delta u} J(q)\|_{\widetilde{L}^1(B^0_{2,\infty})} \leq \|J(q)\|_{\widetilde{L}^\infty(B^1_{2,\infty})} \|\Delta u\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,\infty})}, \end{aligned}$$

Concerning the remainder, we have if $p \geq 2$:

$$\|R(J(q), \Delta u)\|_{\widetilde{L}^1(B^0_{2,\infty})} \leq \|J(q)\|_{\widetilde{L}^\infty(B^1_{2,\infty})} \|\Delta u\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,\infty})}.$$

We have then obtained:

$$\|J(q)\Delta u\|_{\widetilde{L}^1(B^0_{2,\infty})}\leq\|q\|_{\widetilde{L}^\infty(B^1_{2,\infty})}\|u\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{p}+1}_{2,p,\infty})}+\|J(q)\|_{L^\infty(L^\infty)}\|\Delta u\|_{\widetilde{L}^1(B^0_{2,\infty})}.$$

It stays now to treat now the terms where appears the conditions $p<\max(4,2N)$ and $\frac{1}{2}\leq\frac{1}{p}+\frac{1}{p_1}$ in an other way. As $u^n\in\widetilde{L}^2(B^{\frac{N}{2},\frac{N}{p}+1}_{2,p,1}+B^{\frac{N}{2},\frac{N}{p_1}}_{2,p_1,1})$, we set $u^n=g^n+h^n$ with $g^n\in\widetilde{L}^2(B^{\frac{N}{2},\frac{N}{p}+1}_{2,p,1})$ and $h^n\in\widetilde{L}^2(B^{\frac{N}{2},\frac{N}{p_1}}_{2,p_1,1})$. According to proposition 6.9, we have:

$$\begin{aligned}\|\big(R(q^n,\text{div}g^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,}_{2,1})}&\leq C\|\big(R(q^n,\text{div}g^n)\big)_{BF}\|_{\widetilde{L}^1(B^0_{2,1})},\\&\leq C\|q^n\|_{\widetilde{L}^2(\widetilde{B}^1_{2,\infty})}\|\text{div}g^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})},\\&\leq C\|q^n\|_{\widetilde{L}^2(\widetilde{B}^1_{2,\infty})}\|\text{div}g^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})}.\end{aligned}$$

$$\begin{aligned}\|\big(R(q^n,\text{div}h^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,}_{2,1})}&\leq C\|\big(R(q^n,\text{div}h^n)\big)_{BF}\|_{\widetilde{L}^1(B^0_{2,1})},\\&\leq C\|q^n\|_{\widetilde{L}^2(\widetilde{B}^1_{2,\infty})}\|\text{div}h^n\|_{\widetilde{L}^2(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1})}.\end{aligned}$$

Next we have:

$$\begin{aligned}\|\big(R(u^n,\nabla u^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,}_{2,1})}&\leq C\|\big(R(u^n,\nabla u^n)\big)_{BF}\|_{\widetilde{L}^1(L^{B^0_{2,1}})},\\&\leq C\|u^n\|_{\widetilde{L}^\infty(B^0_{2,\infty})}\|\nabla u^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2},\frac{N}{p}}_{2,p,1})}.\end{aligned}$$

It stays to control the term $R(\frac{q^n}{1+q^n},\Delta u^n)$ in low frequencies:

$$\begin{aligned}\|\big(R(\frac{q^n}{1+q^n},\Delta u^n)\big)_{BF}\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,}_{2,1})}&\leq C\|\big(R(\frac{q^n}{1+q^n},\Delta u^n)\big)_{BF}\|_{\widetilde{L}^1(L^2)},\\&\leq C\|q^n\|_{\widetilde{L}^\infty(B^1_{2,\infty})}\|\Delta u^n\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{p}-1}_{2,p,1})}.\end{aligned}$$

Therefore the above nequalities imply that for all $t\in[0,T]$ we have :

$$\begin{aligned}\|(q^n,u^n)\|_{H_t\cap(E'_1)_t}\leq Ce^{C\|(q^n,u^n)\|_{H_t\cap(E'_1)_t}}(\|q_0\|_{B^{\frac{N}{2}-1\frac{N}{p}}_{2,p,1}\cap B^{0,1}_{2,r}}+\|u_0\|_{B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r}}\\+\|f\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r}}+\|(q^n,u^n)\|^2_{H_t\cap(E'_1)_t}).\end{aligned}$$

From a standard bootstrap argument, it is now easy to conclude that there exists a positive constant c such that if the data has been chosen so small as to satisfy:

$$\|q_0\|_{B^{\frac{N}{2}-1\frac{N}{p}}_{2,p,1}\cap B^{0,1}_{2,r}}+\|u_0\|_{B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r}}+\|f\|_{\widetilde{L}^1(B^{\frac{N}{2}-1,\frac{N}{p_1}-1}_{2,p_1,1}\cap B^0_{2,r}}\leq c.$$

then it exists $C>0$ such that for all $t\in\mathbb{R}$:

$$\|(q^n,u^n)\|_{H_t\cap E'_t}\leq C,\;\;\forall t\in\mathbb{R}.$$

To conclude we follow the previous proof of theorem 1.1. Compactness results go along the lines of the proof of theorem 1.1.

5.2 Proof of corollary 2

We follow here exactly the lines of the proof of theorem 1.1 except that we introduce a new effective velocity. Indeed in our case v verifies the following elliptic equation:

$$(\mu(1)Dv)+\nabla(\lambda(1)\mathrm{div}v)=\nabla P(\rho)+(f_1(q)Dv)+\nabla(f_2(q)\mathrm{div}v),$$

with $f_1(q)=\mu(1+q)-\mu(1)$ and $f_2(q)=\lambda(1)-\lambda(1+q)$. We can resolve this elliptic equation as $\mu(1)\geq c>0$ and $\mu(1)+\lambda(1)\geq c>0$, indeed in our case we work away from the vacuum. To do this we have to use the estimates on the Lamé operator of the appendix in [18]. More precisely we have as $q\in\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{p}}_{2,p,1})$ for $r\geq 1$, $p,q\geq 1$ and $|s_1|<\frac{N}{2}$, $|s_2|<\frac{N}{p}$¡:

$$\|v\|_{\widetilde{L}^r(B^{s_1,s_2}_{p,q,1})}\leq C\|q\|_{\widetilde{L}^r(B^{s_1-1,s_2-1}_{p,q,1})}.$$

Indeed as q is small, the terms of rest with $f_1(q)$ and $f_2(q)$ are easy to treat. It means as in the proof of theorem 1.1, v is one derivative more regular than q in high frequencies and that we can estimate v in function of q. Moreover we have $\partial_t v$ which verifies the following elliptic equation:

$$(\mu(\rho)D\partial_t v)+\nabla(\lambda(\rho)\mathrm{div}\partial_t v)=\nabla\partial_t P(\rho)-(\partial_t\mu(\rho)Dv)+\nabla(\partial_t\lambda(\rho)\mathrm{div}v).$$

We can in a similar way get estimates on $\partial_t v$ in function of q and u. The rest of the proof is exactly similar to the proof of theorem 1.1 and is nothing than tedious verifications. It is left to the reader.

6 Appendix

This section is devoted to the proof of proposition 6.9 and of commutators estimates which have been used in section 2 and 3. They are based on paradifferentiel calculus, a tool introduced by J.-M. Bony in [4]. The basic idea of paradifferential calculus is that any product of two distributions u and v can be formally decomposed into:

$$uv=T_u v+T_v u+R(u,v)=T_u v+T'_v u$$

where the paraproduct operator is defined by $T_u v=\sum_q S_{q-1}u\Delta_q v$, the remainder operator R, by $R(u,v)=\sum_q\Delta_q u(\Delta_{q-1}v+\Delta_q v+\Delta_{q+1}v)$ and $T'_v u=T_v u+R(u,v)$.

Proposition 6.9 *Let* $p_1,p_2,p_3,p_4\in[1,+\infty],(s_1,s_2,s_3,s_4)\in\mathbb{R}^4$ *and* $(p,q)\in[1,+\infty]^2$, *we have then the following inequalities:*

- *If* $\frac{1}{p}\leq\frac{1}{p_2}+\frac{1}{\lambda}\leq 1$, $\frac{1}{q}\leq\frac{1}{p_4}+\frac{1}{\lambda'}\leq 1$ *with* $(\lambda,\lambda')\in[1,+\infty]^2$ *and* $p_1\leq\lambda'$, $p_1\leq\lambda$, $p_3\leq\lambda'$ *then:*

$$\|T_u v\|_{\widetilde{B}^{s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2},s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4}}_{p,q,r}}\lesssim\|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}}\|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,r}},\qquad(6.24)$$

if $s_1+\frac{N}{\lambda'}\leq\frac{N}{p_1}$, $s_1+\frac{N}{\lambda}\leq\frac{N}{p_1}$ *and* $s_3+\frac{N}{\lambda'}\leq\frac{N}{p_3}$.

- *If* $\frac{1}{q}\leq\frac{1}{p_3}+\frac{1}{p_4}$ *and* $s_3+s_4+N\inf(0,1-\frac{1}{p_3}-\frac{1}{p_4})>0$ *then*

$$\sum_{l\geq 4}2^{l(s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4})}\|\Delta_l R(u,v)\|_{L^q}\lesssim\|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}}\|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,r}}. \tag{6.25}$$

- *If* $\frac{1}{p}\leq\frac{1}{p_3}+\frac{1}{p_4}\leq 1$, $\frac{1}{p}\leq\frac{1}{p_3}+\frac{1}{p_2}\leq 1$, $\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_4}\leq 1$, $\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_2}\leq 1$ *and* $s_3+s_4>0$, $s_3+s_2>0$, $s_4+s_1>0$, $s_1+s_2>0$ *then*

$$\sum_{l\leq 4}2^{l(s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2})}\|\Delta_l R(u,v)\|_{L^p}\lesssim\|u\|_{\widetilde{B}^{s_1,\frac{N}{p_3}-\frac{N}{p_1}+s_1}_{p_1,p_3,1}}\|v\|_{\widetilde{B}^{s_2,\frac{N}{p_4}-\frac{N}{p_2}+s_2}_{p_2,p_4,r}}. \tag{6.26}$$

with $s_3=\frac{N}{p_3}-\frac{N}{p_1}+s_1$ *and* $s_4=\frac{N}{p_4}-\frac{N}{p_2}+s_2$.

- *If* $u\in L^\infty$, *we also have:*

$$\|T_u v\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}}\lesssim\|u\|_{L^\infty}\|v\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}}, \tag{6.27}$$

and if $\min(s_1,s_2)>0$ *then:*

$$\|R(u,v)\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}}\lesssim\|u\|_{L^\infty}\|v\|_{\widetilde{B}^{s_1,s_2}_{p,q,r}}. \tag{6.28}$$

Proof: Let us prove (6.24). According to the decomposition of J.-M. Bony [4], we have:

$$uv=T_u v+T_v u+R(u,v),$$

so for all $l>0$:

$$\Delta_l T_u v=\sum_{|l-l'|\leq 3}\Delta_l(S_{l'-1}u\Delta_{l'}v),$$

For $\alpha,\beta\in\mathbb{R}$, let us define the following characteristic function on $\mathbb{Z}$

$$\varphi^{\alpha,\beta}=\alpha\ \text{ if }\ r\leq 0,$$
$$\varphi^{\alpha,\beta}=\beta\ \text{ if }\ r\geq 1.$$

if $\frac{1}{p}\leq\frac{1}{p_2}+\frac{1}{\lambda}\leq 1$ and $\frac{1}{q}\leq\frac{1}{p_4}+\frac{1}{\lambda'}\leq 1$ then

$$\|\Delta_l T_u v\|_{L^{\varphi^{p,q}(l)}}\lesssim 2^{lN\varphi^{\frac{1}{p_2}+\frac{1}{\lambda}-\frac{1}{p},\frac{1}{p_4}+\frac{1}{\lambda'}-\frac{1}{q}}(l)}\sum_{|l-l'|\leq 3}\|S_{l'-1}u\|_{L^{\varphi^{\lambda,\lambda'}(l')}}\|\Delta_{l'}v\|_{L^{\varphi^{p_2,p_4}(l')}}.$$

We have by Berstein inequalities and as $p_1\leq\lambda'$, $p_3\leq\lambda'$, $p_1\leq\lambda$ and $s_1+\frac{N}{\lambda}\leq\frac{N}{p_1}$, $s_1+\frac{N}{\lambda'}\leq\frac{N}{p_1}$, $s_3+\frac{N}{\lambda'}\leq\frac{N}{p_3}$:

$$\begin{aligned}\|S_{l'-1}u\|_{L^{\varphi^{\lambda,\lambda'}(l')}}&\lesssim\sum_{k\leq l'-2}2^{k(\varphi^{\frac{N}{p_1},\frac{N}{p_3}}(k)-\varphi^{\frac{N}{\lambda},\frac{N}{\lambda'}}(l'))}\|\Delta_k u\|_{L^{\varphi^{p_1,p_3}(k)}}\\&\lesssim\sum_{k\leq l'-2}2^{k(\varphi^{\frac{N}{p_1}-s_1,\frac{N}{p_3}-s_3}(k)-\varphi^{\frac{N}{\lambda},\frac{N}{\lambda'}}(l'))}2^{k\varphi^{s_1,s_3}(k)}\|\Delta_k u\|_{L^{\varphi^{p_1,p_3}(k)}}\\&\lesssim 2^{l'(\varphi^{\frac{N}{p_1}-s_1,\frac{N}{p_3}-s_3}(l')-\varphi^{\frac{N}{\lambda},\frac{N}{\lambda'}}(l'))}\|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}}.\end{aligned}$$

Since $\|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}(l')}} = c_{l'} 2^{-l'(\varphi^{s_2,s_4}(l'))} \|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,1}}$ with $\sum_{l'\in\mathbb{Z}} c_{l'} \leq 1$ we finally gather as $l > 0$:

$$\|\Delta_l T_u v\|_{L^q} \lesssim c_l 2^{l\varphi^{\frac{N}{p_1}+\frac{N}{p_2}-\frac{N}{p}-s_1-s_2,\frac{N}{p_2}+\frac{N}{p_4}-\frac{N}{q}-s_3-s_4}(l)} \|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}} \|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,1}}.$$

And we obtain (6.24).

Straightforward modification give (6.27). In this case as $\|S_{k-1}u\|_{L^\infty} \leq \|u\|_{L^\infty}$ we have:

$$\|\Delta_l T_u v\|_{L^{\varphi^{p,q}(l)}} \lesssim \sum_{|l-l'|\leq 3} \|u\|_{L^\infty} \|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}(l')}}.$$

Next we have:

$$2^{l\varphi^{p_2,p_4}(l)} \|\Delta_l T_u v\|_{L^{\varphi^{p,q}(l)}} \lesssim \|u\|_{L^\infty} \sum_{|l-l'|\leq 3} 2^{l\varphi^{p_2,p_4}(l) - l'\varphi^{p_2,p_4}(l'))} 2^{\varphi^{p_2,p_4}(l')} \|\Delta_{l'} v\|_{L^{\varphi^{p_2,p_4}(l')}}.$$

We conclude by convolution.

To prove (6.26), we write:

$$\Delta_l R(u,v) = \sum_{k\geq l-2} \Delta_l(\Delta_k u \widetilde{\Delta}_k v).$$

We consider now the case $l > 3$. By Bernstein and Hölder inequalities we obtain when $\frac{1}{q} \leq \frac{1}{p_3} + \frac{1}{p_4} \leq 1$:

$$\|\Delta_l R(u,v)\|_{L^q} \lesssim 2^{Nl(\frac{1}{p_3}+\frac{1}{p_4}-\frac{1}{q})} \sum_{k\geq l-2} \|\Delta_k u\|_{L^{p_3}} \|\widetilde{\Delta}_k v\|_{L^{p_4}}.$$

Next we have:

$$\begin{aligned} 2^{l(s_3+s_4+\frac{N}{q}-\frac{N}{p_3}-\frac{N}{p_4})} \|\Delta_l R(u,v)\|_{L^q} &\lesssim \sum_{k\geq l-2} 2^{(l-k)(s_3+s_4)} 2^{ks_3} \|\Delta_k u\|_{L^{p_3}} 2^{ks_4} \|\widetilde{\Delta}_k v\|_{L^{p_4}}, \\ &\lesssim (c_k) * (d_{k'}), \end{aligned}$$

with $c_k = 1_{[-\infty,2]}(k) 2^{k(s_3+s_4)}$ and $d_{k'} = 2^{k's_3} \|\Delta_k u\|_{L^{p_3}} 2^{k's_4} \|\widetilde{\Delta}_k v\|_{L^{p_4}}$. We conclude by Young inequality as $s_3 + s_4 > 0$.

We have to treat now the case when $l < 0$. We have then as $\frac{1}{p} \leq \frac{1}{p_3} + \frac{1}{p_4} \leq 1$ and $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$:

$$\begin{aligned} \|\Delta_l R(u,v)\|_{L^p} &\lesssim 2^{Nl(\frac{1}{p_3}+\frac{1}{p_4}-\frac{1}{p})} \sum_{k\geq 2} \|\Delta_k u\|_{L^{p_3}} \|\widetilde{\Delta}_k v\|_{L^{p_4}} \\ &\sum_{0\leq k\leq 1, |k-k'|\leq 1} \|\Delta_k u \Delta_{k'} v\|_{L^p} + 2^{Nl(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})} \sum_{l-2\leq k\leq -1} \|\Delta_k u\|_{L^{p_1}} \|\widetilde{\Delta}_k v\|_{L^{p_2}}. \end{aligned}$$

And by convolution on the middle frequencies:

$$\|\Delta_l R(u,v)\|_{L^p} \lesssim 2^{Nl(\frac{1}{p_3}+\frac{1}{p_4}-\frac{1}{p})}\sum_{k\geq 2}\|\Delta_k u\|_{L^{p_3}}\|\widetilde{\Delta}_k v\|_{L^{p_4}}$$
$$(2^{l(\frac{N}{p_3}+\frac{N}{p_2}-\frac{N}{p}-s_3-s_2)}+2^{l(\frac{N}{p_1}+\frac{N}{p_4}-\frac{N}{p}-s_1-s_4)})c_l+2^{Nl(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}\sum_{l-2\leq k\leq -1}\|\Delta_k u\|_{L^{p_1}}\|\widetilde{\Delta}_k v\|_{L^{p_2}},$$

with $c_l \in l^1(\mathbb{Z})$. Next by convolution we obtain:

$$\|\Delta_l R(u,v)\|_{L^p} \lesssim c_l(2^{l(\frac{N}{p_3}+\frac{N}{p_4}-\frac{N}{p}-s_3-s_4)}+2^{l(\frac{N}{p_3}+\frac{N}{p_2}-\frac{N}{p}-s_3-s_2)}+2^{l(\frac{N}{p_1}+\frac{N}{p_4}-\frac{N}{p}-s_1-s_4)}$$
$$+2^{l(\frac{N}{p_1}+\frac{N}{p_2}-\frac{N}{p}-s_1-s_2)})\|u\|_{\widetilde{B}^{s_1,s_3}_{p_1,p_3,1}}\|v\|_{\widetilde{B}^{s_2,s_4}_{p_2,p_4,r}}.$$

And we can conclude.
We want prove now the inequality (6.27). We have then:

$$2^{l\varphi^{s_1,s_2}(l)}\|\Delta_l R(u,v)\|_{L^p} \lesssim \sum_{k\geq l-2} 2^{(l-k)\varphi^{s_1,s_2}(l)}2^{k\varphi^{s_1,s_2}(l)}\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k v\|_{L^{\varphi^{p,q}(k)}},$$

And we conclude by Young inequality. □

Lemma 2 *Let $1\leq p_1\leq p\leq +\infty$ and $\sigma\in(-\min(\frac{N}{p},\frac{N}{p_1'}),\frac{N}{p}+1]$. There exists a sequence $c_q\in l^1(\mathbb{Z})$ such that $\|c_q\|_{l^1}=1$ and a constant C depending only on N and σ such that:*

$$\forall q\in\mathbb{Z},\;\;\|[v\cdot\nabla,\Delta_q]a\|_{L^{p_1}}\leq Cc_q 2^{-q\sigma}\|\nabla v\|_{B^{\frac{N}{p}}_{p,1}}\|a\|_{B^{\sigma}_{p_1,1}}. \tag{6.29}$$

In the limit case $\sigma=-\min(\frac{N}{p},\frac{N}{p_1'})$, we have:

$$\forall q\in\mathbb{Z},\;\;\|[v\cdot\nabla,\Delta_q]a\|_{L^{p_1}}\leq Cc_q 2^{q\frac{N}{p}}\|\nabla v\|_{B^{\frac{N}{p}}_{p,1}}\|a\|_{B^{-\frac{N}{p_1}}_{p,\infty}}. \tag{6.30}$$

Finally, for all $\sigma>0$ and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{1}{p}$, there exists a constant C depending only on N and on σ and a sequence $c_q\in l^1(\mathbb{Z})$ with norm 1 such that:

$$\forall q\in\mathbb{Z},\;\;\|[v\cdot\nabla,\Delta_q]v\|_{L^p}\leq Cc_q2^{-q\sigma}(\|\nabla v\|_{L^\infty}\|v\|_{B^{\sigma}_{p_1,1}}+\|\nabla v\|_{L^{p_2}}\|\nabla v\|_{B^{\sigma-1}_{p,1}}). \tag{6.31}$$

Proof: These results are proved in [3] chapter 2. □

References

[1] H. Abidi and M. Paicu. Équation de Navier-Stokes avec densité et viscosité variables dans l'espace critique. *Annales de l'institut Fourier*, 57 no. 3 (2007), p. 883-917.

[2] H. Bahouri and J.-Y. Chemin, Équations d'ondes quasilinéaires et estimation de Strichartz, *Amer. J. Mathematics.* 121 (1999) 1337-1377.

[3] H. Bahouri, J.-Y. Chemin and R. Danchin. Fourier analysis and nonlinear partial differential equations, *to appear in Springer.*

[4] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales Scientifiques de l'école Normale Supérieure.* 14 (1981) 209-246.

[5] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D Viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1-2): 211-223, 2003.

[6] F. Charve and R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, preprint and submitted.

[7] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel, *J.d'Analyse Math.* 77 (1999) 27-50.

[8] J.-Y. Chemin, About Navier-Stokes system, *Prépublication du Laboratoire d'Analyse Numérique de Paris 6*, R96023 (1996).

[9] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, *J.Differential Equations*, 121 (1992) 314-328.

[10] Q. Chen, C. Miao and Z. Zhang, Global well-posedness for the compressible Navier-Stokes equations with the highly oscillating initial velocity, *arXiv:0907.4540v2.*

[11] R. Danchin, Fourier analysis method for PDE's, *Preprint*, Novembre 2005.

[12] R. Danchin, Local Theory in critical Spaces for Compressible Viscous and Heat-Conductive Gases, *Communication in Partial Differential Equations*, 26 (78),1183-1233, (2001).

[13] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Inventiones Mathematicae*, 141, pages 579-614 (2000).

[14] R. Danchin, Global existence in critical spaces for compressible viscous and heat-conductive gases, *Archiv for Rational Mechanics and Analysis*, 160, pages 1-39 (2001).

[15] R. Danchin, On the uniqueness in critical spaces for compressible Navier-Stokes equations. *NoDEA Nonlinear Differentiel Equations Appl*, 12(1):111-128, 2005.

[16] R. Danchin, Well-Posedness in Critical Spaces for Barotropic Viscous Fluids with Truly Not Constant Density, *Communications in Partial Differential Equations*,32:9,1373-1397.

[17] B. Haspot,Cauchy problem for viscous shallow water equations with a term of capillarity , *accepted in HYP 2008.*

[18] B. Haspot, Local well-posedness results for density-dependent incompressible fluids, *Arxiv*, 0902.1982 (February 2009).

[19] B. Haspot, Well-posedness in critical spaces for barotropic viscous fluids, *Arxiv*, (March 2009).

[20] D. Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Amer. Math. Soc*, 303(1), 169-181, 1987.

[21] D. Hoff, Uniqueness of weak solutions of the NavierStokes equations of multidimensional, compressible flow, *SIAM J. Math. Anal.* 37 (6) (2006).

[22] D. Hoff. Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of the heat conducting fluids. *Arch. Rational Mech. Anal.*, 139, (1997), p. 303-354.

[23] D. Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations*, 120(1), 215-254, 1995.

[24] D. Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.*, 132(1), 1-14, 1995.

[25] D. Hoff and K. Zumbrum. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow, *Indiana University Mathematics Journal*, 1995, 44, 603-676.

[26] A. V. Kazhikov. The equation of potential flows of a compressible viscous fluid for small Reynolds numbers: existence, uniqueness and stabilization of solutions. *Sibirsk. Mat. Zh.*, 34 (1993), no. 3, p. 70-80.

[27] A. V. Kazhikov and V. V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one- dimensional equations of a viscous gas. *Prikl. Mat. Meh.*, 41(2): 282-291, 1977.

[28] Akitaka Matsumura and Takaaki Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1): 67-104, 1980.

[29] Akitaka Matsumura and Takaaki Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. *Proc. Japan Acad. Ser. A Math. Sci*, 55(9):337-342, 1979.

[30] Y. Meyer. Wavelets,paraproducts, and Navier-Stokes equation. *In Current developments in mathematics, 1996 (Cambridge, MA)*, page 105-212. Int. Press, Boston, MA, 1997.

[31] J. Nash. Le problème de Cauchy pour les équations différentielles d'un fluide général. *Bull. Soc. Math. France*, 90: 487-497, 1962.

[32] T. Runst and W. Sickel: Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. *de Gruyter Series in Nonlinear Analysis and Applications*, 3. Walter de Gruyter and Co., Berlin (1996)
na 21, no. 1 (2005), 1-24.

[33] D. Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible.*Comptes rendus de l'Académie des sciences.* Série 1, 303(13): 639-642, 1986.

[34] V. A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes systems. *Zap. Nauchn. Sem. LOMI*, 38, (1973), p.153-231; J. Soviet Math. 8, (1977), p. 467-529.

[35] V. Valli and W. Zajaczkowski. Navier-Stokes equations for compressible fluids: global existence and qualitative properties of the solutions in the general case. *Commun. Math. Phys.*, 103, (1986) no 2, p. 259-296.