Existence of strong solutions in a larger space for the shallow-water system

Frédéric Charve, Boris Haspot

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Abstract

This paper is dedicated to the study of both viscous compressible barotropic fluids and Navier-Stokes equation with dependant density, when the viscosity coefficients are variable, in dimension $d \geq 2$. We aim at proving the local and global well-posedness for respectively *large* and *small* initial data having critical Besov regularity and more precisely we are interested in extending the class of initial data velocity when we consider the shallow water system, improving the results in [17] and [15]. Our result relies on the fact that the velocity u can be written as the sum of the solution u_L of the associated linear system and a remainder velocity term $\bar{u}$; then in the specific case of the shallow-water system the remainder term $\bar{u}$ is more regular than u_L by taking into account the regularizing effects induced on the bilinear convection term. In particular we are able to deal with initial velocity in $\dot{H}^{\frac{d}{2}-1}$ as Fujita and Kato for the incompressible Navier-Stokes equations (see [12]) with an additional condition of type $u_0 \in B^{-1}_{\infty,1}$. We would like to point out that this type of result is of particular interest when we want to deal with the problem of the convergence of the solution of compressible system to the incompressible system when the Mach number goes to 0.

1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) + \nabla P(\rho) = \rho f, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^d$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density. As usual, $D(u)$ stands for the strain tensor, with $D(u) = \frac{1}{2}\left({}^t\nabla u + \nabla u\right)$.

The pressure P is a suitable smooth function depending on the density ρ. We denote by λ and μ the two viscosity coefficients of the fluid, which are also assumed to depend on the density and which verify some parabolic conditions for the momentum equation $\mu > 0$ and $\lambda + 2\mu > 0$ (in the physical cases the viscosity coefficients verify $\lambda + \frac{2\mu}{d} > 0$

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which is a particular case of the previous assumption). We supplement the problem with initial condition (ρ_0, u_0) and an external force f. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^d$ or to the periodic box $\mathbb{T}^d_a$ with period a_i, in the i-th direction. We restrict ourselves to the case $d \geq 2$. Let us recall that in the case of constant viscosity coefficients, existence and uniqueness for (1.1) in the case of smooth data with no vacuum has been stated in the pioneering works by Nash (see [23]) and Matsumura and Nishida (see [19, 20]).
In this article we obtain the existence of strong solution (in finite time with large initial data or in global time for small initial data) for a family of initial data which is close from being optimal in terms of regularity (indeed the third index of the Besov space is not necessary 1, it should be considered like an extension to the compressible case of the result of Fujita and Kato, [12]). To do this, we combine two different ingredients, first the notion of scaling and second taking advantage of suitable choices on the viscosity coefficients which may confer specific structures in terms of regularity. We will detail more this last point later, but as a first example of the importance of the viscosity coefficients we recall that in [21] Mellet and Vasseur have obtained the stability of the global weak solutions for the Saint-Venant system by using new entropy giving additional regularity on the gradient of the density and on the integrability of the velocity.
Let us recall the fundamental notion of scaling for system (1.1). Indeed guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see in particular the pioneering works of Fujita and Kato concerning the existence of strong solutions for the incompressible Navier-Stokes equations in [12]), we aim at solving (1.1) in the case where the data (ρ_0, u_0) have *critical* regularity. By critical, we mean that we want to solve the system (1.1) in functional spaces with invariant norm by the natural changes of scales which leave (1.1) invariant. More precisely in the case of barotropic fluids, the following transformations:

$$(\rho(t,x), u(t,x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)), \quad l \in \mathbb{R}, \tag{1.2}$$

verify this property, provided that the pressure term has been changed accordingly. This notion of critical functional frameworks has been extensively used in order to obtain optimal class of initial data for the existence of global strong solution (see [5, 6, 10, 15, 14, 17] in the case of constant viscosity coefficients). In particular the first result on the existence of strong solutions in spaces invariant for the scaling of the equations (when the viscosity coefficients are constant) is due to Danchin in [6] when the initial data $(q_0 = \rho_0 - \bar{\rho}, u_0)$ (with $\bar{\rho} > 0$) are in $\dot{B}^{\frac{d}{2}}_{2,1} \times (\dot{B}^{\frac{d}{2}-1}_{2,1})^d$. In [10], Danchin generalizes the previous results by working with more general Besov space of the type $\dot{B}^{\frac{d}{p}}_{p,1} \times (\dot{B}^{\frac{d}{p}-1}_{p,1})^d$ with some restrictions on the choice of p ($p \leq d$) due to some limitation in the application of the paraproduct law. The fact that Danchin is working with the same Lebesgue index p both for the density and the velocity is a consequence of the strong coupling between the density and the velocity equations, indeed the pressure term is considered as a remainder for the parabolic operator in the momentum equation of (1.2). In [17], the second author generalizes the results of [10] as we have no restriction on the size of p for the initial density. To do this he is working with a new unknown, the effective velocity which allows to cancel out the coupling between the pressure and the velocity.
In the case of global strong solution in critical space for small initial data, we would

like to recall the works of Danchin in [7] who shows for the first time a result of global existence of strong solution close to a stable equilibrium when the initial data verify $(q_0,u_0)\in(\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}-1}_{2,1})\times\dot{B}^{\frac{d}{2}-1}_{2,1}$. The main difficulty is to get estimates on the linearized system given that the velocity and the density are coupled via the pressure. What is crucial in this work is the smoothing effect on the velocity and a L^1 decay on $\rho-\bar{\rho}$ (this plays a key role to control the pressure term). This work was generalized in the framework of Besov space with a Lebesgue index different of $p=2$ by Charve and Danchin in [5] and Haspot in [15].

However very few articles really take into account the structure of the viscosity coefficients, indeed most of them deal with constant viscosity coefficients or consider the system (1.1) as a perturbation of the previous case (indeed generally one writes the diffusive tensor like a sum of constant Laplacians with a remainder term which is considered like a small perturbation because $\rho-\bar{\rho}$ is small in suitable norms). In addition to have a norm invariant by (1.2), appropriate functional spaces for solving (1.1) must provide a control on the L^∞ norm of the density for at least two reasons; the first consists in avoiding the vacuum and to ensure the parabolicity of system (1.1), the second is related to some property of multiplier on the density in order to deal with the term $\frac{1}{\rho}\Delta u$. It explains why the authors restrict their study to the case where the initial density is assumed in $\dot{B}^{\frac{d}{p}}_{p,1}$ with $p\in[1,+\infty[$ suitably chosen, indeed $\dot{B}^{\frac{d}{p}}_{p,1}$ is embedded in L^∞. Furthermore in order to propagate this regularity on the density via the mass equation, it appears necessary to control the velocity in Lipschitz norm, it means $\nabla u\in L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})\hookrightarrow L^1_T(L^\infty)$ with $p\in[1,+\infty[$ and $T>0$. We would like to point out that this necessary Lipschitz control on the velocity seems to prevent any hope of existence of strong solutions when u_0 is only assumed to belong to $\dot{H}^{\frac{d}{2}-1}$.

Indeed in [5, 6, 10, 14, 17] the idea is to propagate a $\widetilde{L}^\infty(\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(\dot{B}^{\frac{d}{2}+1}_{2,1})$ regularity on the velocity u via the regularizing effects induced by the momentum equation written in its eulerian form:

$$\partial_t u-\frac{\mu}{\rho}\Delta u-\frac{\mu+\lambda}{\rho}\nabla\mathrm{div}u=R,\tag{1.3}$$

with: $R=-u\cdot\nabla u-\frac{\nabla P(\rho)}{\rho}+f$. However if we only were interested in obtaining a regularity in $\widetilde{L}^\infty(\dot{H}^{\frac{d}{2}-1})\cap\widetilde{L}^1(\dot{H}^{\frac{d}{2}+1})$ (we refer to the next section for the definition of such spaces), it would be necessary to deal with the term $\frac{1}{\rho}\Delta u$ and getting enough regularity in order that $\frac{1}{\rho}$ remains in a multiplier space $\mathcal{M}(\dot{H}^{\frac{d}{2}-1})$ of $\dot{H}^{\frac{d}{2}-1}$. Typically $\dot{H}^{\frac{d}{2}}\cap L^\infty$ is embedded in $\mathcal{M}(\dot{H}^{\frac{d}{2}-1})$. However in this case how to propagate the regularity $\dot{H}^{\frac{d}{2}}\cap L^\infty$ on the density as our velocity is only assumed in $\widetilde{L}^1(\dot{H}^{\frac{d}{2}+1})$ (it means not necessary Lipshitz)? We want to partially solve this question in the case of specific viscosity coefficients. As we explained above, one of the main difficulty is linked to the treatment of heat equation with variable coefficients. We would like to work with the shallow water system (i.e $\mu(\rho)=\mu\rho$ and $\lambda(\rho)=0$, and to simplify we will take $\mu=1$) that we can rewrite in the following form:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u\cdot\nabla u - \mathcal{A}u - 2D(u).\nabla(\ln\rho) + \nabla(G(\rho)) = 0, \\ (\rho,u)_{/t=0} = (\rho_0,u_0), \end{cases} \tag{1.4}$$

where the operator $\mathcal{A}$ is defined by: $\mathcal{A}u = \Delta u + \nabla\operatorname{div} u$ and $G^{'}(\rho) = \frac{P^{'}(\rho)}{\rho}$. Roughly speaking, we are led to consider a basic heat equation with remainder terms, and in particular it seems possible to propagate in this case the regularity $\widetilde{L}^\infty(\dot{H}^{\frac{d}{2}-1})\cap\widetilde{L}^1(\dot{H}^{\frac{d}{2}+1})$. However due to the strict coupling between the pressure and the velocity, it will be necessary to control the density in $L^1_T(\dot{B}^{\frac{d}{2}}_{2,1})$ (the third index 1 is due to multiplier space reason). More precisely we will split the solution u into the following sum $u = u_L + \bar{u}$ with:

$$\begin{cases} \partial_t u_L - \mathcal{A}u_L = 0 \\ u_{/t=0} = u_0 \end{cases} \tag{1.5}$$

In the present article, if we assume that u_0 belongs to $\dot{H}^{\frac{d}{2}-1}\cap\dot{B}^{-1}_{\infty,1}$ then, with usual methods we show that $\bar{u}$ is more regular than u_L ($\bar{u}$ will be in $L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})$) which will help us propagating the regularity on u.
To simplify the notation, we assume from now on that $\bar{\rho}=1$. Hence as long as ρ does not vanish, the equations for ($q=\rho-1$,u) read:

$$\begin{cases} \partial_t q + u\cdot\nabla q + (1+q)\operatorname{div}u = 0, \\ \partial_t u + u\cdot\nabla u - \mathcal{A}u - 2D(u).\nabla(\ln(1+q)) + \nabla(G(1+q)) = 0, \end{cases} \tag{1.6}$$

We can now state our main result:

Theorem 1 *Let P be a suitably smooth function of the density. Assume that $u_0 \in \dot{B}^{\frac{d}{2}-1}_{2,2}\cap\dot{B}^{-1}_{\infty,1}$, $\operatorname{div} u_0 \in \dot{B}^{\frac{d}{2}-2}_{2,1}$, $q_0\in\dot{B}^{\frac{d}{2}}_{2,1}$ and that there exists $c>0$ such that $\rho_0\geq c$. Then there exists a time T such that there exists a unique solution (q,u) for system (1.6) on $[0,T]$ with $1+q$ bounded away from zero and,*

$$q\in\widetilde{C}_T(\dot{B}^{\frac{d}{2}}_{2,1}) \quad\text{and}\quad u\in\widetilde{C}_T(\dot{B}^{\frac{d}{2}-1}_{2,2})\cap\widetilde{L}^1_T(\dot{B}^{\frac{d}{2}+1}_{2,2})\cap\widetilde{C}_T(\dot{B}^{-1}_{\infty,1})\cap L^1_T(\dot{B}^{1}_{\infty,1}).$$

In addition $\operatorname{div}u$ belongs to $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-2}_{2,1})\cap L^1_T(\dot{B}^{\frac{d}{2}}_{2,1})$.

Moreover if $P'(1)>0$ and $q_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$, there exists a constant ε_0 which depends in particular on $\|u_0\|_{B^{-1}_{\infty,1}}$ such that if:

$$\|q_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2}} + \|\operatorname{div}u_0\|_{\dot{B}^{\frac{d}{2}-2}_{2,1}} \leq \varepsilon_0,$$

then the solution is global.

Remark 1 *We would like to mention that this theorem could easily be extended to the case of Besov space constructed on Lebesgue spaces with index $p\neq 2$. More precisely* *as in [6], we could obtain the existence of solution under the condition than $1 \leq p < 2d$ and the uniqueness for $1 \leq p \leq d$ when (q_0, u_0) belong to $\dot{B}^{\frac{d}{p}}_{p,1} \times (\dot{B}^{\frac{d}{p}-1}_{p,2} \cap \dot{B}^{-1}_{\infty,1})$. These restrictions on the size of p are essentially due to some limitations on the use of the paraproduct when we are dealing with the convection term $u \cdot \nabla u$.*

Remark 2 *In [21] the authors prove the stability of the global weak solution for the shallow-water system. In particular in a crucial way they take profit of the entropy inequalities which enable us to control $\sqrt{\rho}$ in $L^\infty(L^2)$. Roughly speaking it means that we control the density ρ in $L^\infty(\dot{H}^1)$. It means in particular that our initial data are very close to the energy data in dimension $d = 2$. Indeed we essentially assume only additional condition in terms of vacuum ($\frac{1}{\rho_0} \in L^\infty$) and a control on the L^∞ norm of ρ_0. These conditions are natural in order to deal with the non linear terms but also for preserving the parabolicity of the momentum equation. Moreover we suppose that u_0 is in $\dot{B}^{-1}_{\infty,1}$. We would like to emphasize that this last condition is not so far from being quite optimal for incompressible Navier-Stokes system. Indeed Koch and Tataru in [18] proved that the Navier-Stokes equations are well posed in BMO^{-1}. We would like to point out that $\dot{B}^{-1}_{\infty,2}$ is embedded in BMO^{-1} and recently Bourgain and Pavlovič proved in [4] that the Navier-Stokes system is ill-posed in the sense of the explosion of the norm when u_0 is in $\dot{B}^{-1}_{\infty,\infty}$ (this problem was open during a long time and very relevant in the sense that $\dot{B}^{-1}_{\infty,\infty}$ is the largest invariant space by the scaling of the equations). This last result was extended by Yoneda in [24] to the case where u_0 is in $\dot{B}^{-1}_{\infty,r}$ with $r > 2$. It shows in particular that BMO^{-1} is optimal for the well-posedness of the Navier-Stokes equations. Let us mention that the compressible Navier-Stokes equations are probably ill-posed in $\dot{B}^{\frac{d}{2}-1}_{2,r}$ with $r > 1$ and without any additionnal assumption on the initial velocity. The reason why is that there is no hope to obtain a Lipschitz velocity, which prevents in particular any control on the density.*

Remark 3 *This result is also very interesting in the case of the convergence of the solution of compressible system to the incompressible system when the Mach number goes to 0. Indeed in a forthcoming paper we would like to deal with initial data of Fujita-Kato type, it means here $u_0 \in \dot{H}^{\frac{d}{2}-1} \cap \dot{B}^{-1}_{\infty,1}$ which should to improve the result of R.Danchin (see [8]) who needs to assume $u_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}$.*

Remark 4 *The additionnal assumption $\operatorname{div} u_0 \in \dot{B}^{\frac{d}{2}-2}_{2,1}$ is quite natural as it expresses the compressibility of the fluid. Indeed it is necessary, in order to control the density in $L^\infty \dot{B}^{\frac{d}{2}}_{2,1}$, that $\operatorname{div} u$ be in $L^1 \dot{B}^{\frac{d}{2}}_{2,1}$ (obviously it is induced by the fact that $\operatorname{div} u_0$ belongs to $\dot{B}^{\frac{d}{2}-2}_{2,1}$).*

The previous theorem can be easily adapted to the incompressible density dependent Navier-Stokes equations. We recall here the equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\rho D u) + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.7}$$

Following the same ideas as in theorem 1, we obtain the following result:

Theorem 2 *Let $d \geq 3$. Assume that $u_0 \in \dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}$ with $\operatorname{div} u_0 = 0$, $q_0 \in \dot{B}^{\frac{d}{2}}_{2,1}$ and that there exists $c > 0$ such that $\rho_0 \geq c$. Then there exists a time T such that there exists a unique solution (q, u) for system (1.1) on $[0, T]$ with $1 + q$ bounded away from zero and,*

$$q \in \widetilde{C}_T(\dot{B}^{\frac{d}{2}}_{2,1}) \quad \text{and} \quad u \in \widetilde{C}_T(\dot{B}^{\frac{d}{2}-1}_{2,2}) \cap L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,2}) \cap \widetilde{C}_T(\dot{B}^{-1}_{\infty,1}) \cap L^1_T(\dot{B}^{1}_{\infty,1})$$
$$\text{and } \nabla\Pi \in \widetilde{L}^1_T(\dot{B}^{\frac{d}{2}-1}_{2,2}) \cap \widetilde{L}^1_T(\dot{B}^{-1}_{\infty,1}).$$

Moreover there exists a constant ε_0 which depends in particular on $\|u_0\|_{B^{-1}_{\infty,1}}$ such that if:

$$\|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2}} \leq \varepsilon_0,$$

then the solution is global.

Remark 5 *This result may be considered as an extension of the Fujita-Kato theorem (see [12]) to the case of incompressible density dependent Navier-Stokes equations. In particular it improves the analysis of [1], [11] and [16] because we deal with a velocity in a Besov space such that the third index is different of 1 and with a critical initial density in terms of scaling.*

Our paper is structured as follows. In section 2, we give a few notations and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section 3 and section 4, we prove theorem 1 and more particular the existence of such solution in section 3 and the uniqueness in section 4. In section 5 we are proving the global well-posedness of theorem 1. In section 6, we are dealing with theorem 2.

2 Littlewood-Paley theory and Besov spaces

As usual, the Fourier transform of u with respect to the space variable will be denoted by $\mathcal{F}(u)$ or $\widehat{u}$. In this section we will state classical definitions and properties concerning the homogeneous dyadic decomposition with respect to the Fourier variable. We will recall some classical results and we refer to [2] (Chapter 2) for proofs (and more general properties).

To build the Littlewood-Paley decomposition, we need to fix a smooth radial function χ supported in (for example) the ball $B(0, \frac{4}{3})$, equal to 1 in a neighborhood of $B(0, \frac{3}{4})$ and such that $r \mapsto \chi(r.e_r)$ is nonincreasing over $\mathbb{R}_+$. So that if we define $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$, then φ is compactly supported in the annulus $\{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and we have that,

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1. \tag{2.8}$$

Then we can define the *dyadic blocks* $(\dot{\Delta}_l)_{l\in\mathbb{Z}}$ by $\dot{\Delta}_l := \varphi(2^{-l}D)$ (that is $\widehat{\dot{\Delta}_l u} = \varphi(2^{-l}\xi)\widehat{u}(\xi)$) so that, formally, we have

$$u = \sum_l \dot{\Delta}_l u \tag{2.9}$$

As (2.8) is satisfied for $\xi \neq 0$, the previous formal equality holds true for tempered distributions *modulo polynomials*. A way to avoid working modulo polynomials is to consider the set $\mathcal{S}'_h$ of tempered distributions u such that

$$\lim_{l\to-\infty} \|\dot{S}_l u\|_{L^\infty} = 0,$$

where $\dot{S}_l$ stands for the low frequency cut-off defined by $\dot{S}_l := \chi(2^{-l}D)$. If $u \in \mathcal{S}'_h$, (2.9) is true and we can write that $\dot{S}_l u = \sum_{q\leq l-1} \dot{\Delta}_q u$. We can now define the homogeneous Besov spaces used in this article:

Definition 1 *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we set*

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_l 2^{rls}\|\Delta_l u\|^r_{L^p}\right)^{\frac{1}{r}} \quad \textit{if} \;\; r<\infty \quad \textit{and} \quad \|u\|_{\dot{B}^s_{p,\infty}} := \sup_l 2^{ls}\|\Delta_l u\|_{L^p}.$$

We then define the space $\dot{B}^s_{p,r}$ as the subset of distributions $u \in \mathcal{S}'_h$ such that $\|u\|_{\dot{B}^s_{p,r}}$ is finite.

Once more, we refer to [2] (chapter 2) for properties of the inhomogeneous and homogeneous Besov spaces. Among these properties, let us mention:

- for any $p \in [1, \infty]$ we have the following chain of continuous embeddings:

$$\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty};$$

- if $p < \infty$ then $B^{\frac{d}{p}}_{p,1}$ is an algebra continuously embedded in the set of continuous functions decaying to 0 at infinity;
- for any smooth homogeneous of degree m function F on $\mathbb{R}^d\setminus\{0\}$ the operator $F(D)$ defined by $F(D)u = \mathcal{F}^{-1}\Big(F(\cdot)\mathcal{F}(u)(\cdot)\Big)$ maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-m}_{p,r}$. This implies that the gradient operator maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-1}_{p,r}$.

We refer to [2] (lemma 2.1) for the Bernstein lemma (describing how derivatives act on spectrally localized functions), that entails the following embedding result:

Proposition 1 *For all $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in the space $\dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

Then we have:

$$\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow L^\infty.$$

In this paper, we shall mainly work with functions or distributions depending on both the time variable t and the space variable x. We shall denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X. For $p \in [1, \infty]$, the notation $L^p(I; X)$ stands for the set of measurable functions on I with values in X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$.

In the case where $I=[0,T]$, the space $L^p([0,T];X)$ (resp. $\mathcal{C}([0,T];X)$) will also be denoted by $L^p_T X$ (resp. $\mathcal{C}_T X$). Finally, if $I=\mathbb{R}^+$ we shall alternately use the notation $L^p X$.

The Littlewood-Paley decomposition enables us to work with spectrally localized (hence smooth) functions rather than with rough objects. We naturally obtain bounds for each dyadic block in spaces of type $L^\rho_T L^p$. Going from those type of bounds to estimates in $L^\rho_T \dot{B}^s_{p,r}$ requires to perform a summation in $\ell^r(\mathbb{Z})$. When doing so however, we *do not* bound the $L^\rho_T \dot{B}^s_{p,r}$ norm for the time integration has been performed *before* the ℓ^r summation. This leads to the following notation:

Definition 2 *For $T>0$, $s\in\mathbb{R}$ and $1\le r,\sigma\le\infty$, we set*

$$\|u\|_{\widetilde{L}^\sigma_T \dot{B}^s_{p,r}} := \big\|2^{js}\|\dot{\Delta}_q u\|_{L^\sigma_T L^p}\big\|_{\ell^r(\mathbb{Z})}.$$

One can then define the space $\widetilde{L}^\sigma_T \dot{B}^s_{p,r}$ as the set of tempered distributions u over $(0,T)\times\mathbb{R}^d$ such that $\lim_{q\to-\infty}\dot{S}_q u=0$ in $L^\sigma([0,T];L^\infty(\mathbb{R}^d))$ and $\|u\|_{\widetilde{L}^\sigma_T \dot{B}^s_{p,r}}<\infty$. The letter T is omitted for functions defined over $\mathbb{R}^+$. The spaces $\widetilde{L}^\sigma_T \dot{B}^s_{p,r}$ may be compared with the spaces $L^\sigma_T \dot{B}^s_{p,r}$ through the Minkowski inequality: we have

$$\|u\|_{\widetilde{L}^\sigma_T \dot{B}^s_{p,r}} \le \|u\|_{L^\sigma_T \dot{B}^s_{p,r}} \ \text{ if } \ r\ge\sigma \quad \text{and} \quad \|u\|_{\widetilde{L}^\sigma_T \dot{B}^s_{p,r}} \ge \|u\|_{L^\sigma_T \dot{B}^s_{p,r}} \ \text{ if } \ r\le\sigma.$$

All the properties of continuity for the product and composition which are true in Besov spaces remain true in the above spaces. The time exponent just behaves according to Hölder's inequality.

Let us now recall a few nonlinear estimates in Besov spaces. Formally, any product of two distributions u and v may be decomposed into

$$uv = T_u v + T_v u + R(u,v), \ \text{where} \tag{2.10}$$

$$T_u v := \sum_l \dot{S}_{l-1}u\dot{\Delta}_l v, \quad T_v u := \sum_l \dot{S}_{l-1}v\dot{\Delta}_l u \ \text{ and } \ R(u,v) := \sum_l \sum_{|l'-l|\le 1} \dot{\Delta}_l u\, \dot{\Delta}_{l'} v.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The decomposition (2.10) has been introduced by Bony in [3].

In this article we will frequently use the following estimates (we refer to [2] section 2.6, [7], for general statements, more properties of continuity for the paraproduct and remainder operators, sometimes adapted to $\widetilde{L}^\sigma_T \dot{B}^s_{p,r}$ spaces): under the same assumptions there exists a constant $C>0$ such that if $1/p_1+1/p_2=1/p$, and $1/r_1+1/r_2=1/r$:

$$\|\dot{T}_u v\|_{\dot{B}^s_{2,1}} \le C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{2,1}},$$

$$\|\dot{T}_u v\|_{\dot{B}^{s+t}_{p,r}} \le C\|u\|_{\dot{B}^t_{p_1,r_1}}\|v\|_{\dot{B}^s_{p_2,r_2}} \quad (t<0),$$

$$\|\dot{R}(u,v)\|_{\dot{B}^{s_1+s_2-\frac{d}{2}}_{p,r}} \le C\|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}} \quad (s_1+s_2>0). \tag{2.11}$$

Let us now turn to the composition estimates. We refer for example to [2] (Theorem 2.59, corollary 2.63)):

Proposition 2 *1. Let $s>0$, $u\in\dot{B}^s_{2,1}\cap L^\infty$ and $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^d)$ such that $F(0)=0$. Then $F(u)\in\dot{B}^s_{2,1}$ and there exists a function of one variable C_0 only depending on s, d and F such that*

$$\|F(u)\|_{\dot{B}^s_{2,1}}\leq C_0(\|u\|_{L^\infty})\|u\|_{\dot{B}^s_{2,1}}.$$

2. If u and $v\in\dot{B}^{\frac{d}{2}}_{2,1}$ and if $v-u\in\dot{B}^s_{2,1}$ for $s\in]-\frac{d}{2},\frac{d}{2}]$ and $G\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^d)$, then $G(v)-G(u)$ belongs to $\dot{B}^s_{2,1}$ and there exists a function of two variables C only depending on s, d and G such that

$$\|G(v)-G(u)\|_{\dot{B}^s_{2,1}}\leq C(\|u\|_{L^\infty},\|v\|_{L^\infty})\left(|G'(0)|+\|u\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|v\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)\|v-u\|_{\dot{B}^s_{2,1}}.$$

Let us now recall a result of interpolation which explains the link between the space $B^s_{p,1}$ and the space $B^s_{p,\infty}$ (see [2] sections 2.11 and 10.2.4):

Proposition 3 *There exists a constant C such that for all $s\in\mathbb{R}$, $\varepsilon>0$ and $1\leq p<+\infty$,*

$$\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^s_{p,1})}\leq C\frac{1+\varepsilon}{\varepsilon}\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^s_{p,\infty})}\log\Big(e+\frac{\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^{s-\varepsilon}_{p,\infty})}+\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^{s+\varepsilon}_{p,\infty})}}{\|u\|_{\widetilde{L}^\sigma_T(\dot{B}^s_{p,\infty})}}\Big).$$

Let us end this section by recalling the following estimates for the heat equation:

Proposition 4 *Let $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$ and $1\leq\rho_2\leq\rho_1\leq+\infty$. Assume that $u_0\in\dot{B}^s_{p,r}$ and $f\in\widetilde{L}^{\rho_2}_T(\dot{B}^{s-2+2/\rho_2}_{p,r})$. Let u be a solution of:*

$$\begin{cases}\partial_t u-\mathcal{A}u=f\\ u_{/t=0}=u_0\,.\end{cases}$$

Then there exists $C>0$ depending only on N,μ,ρ_1 and ρ_2 such that:

$$\|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s+2/\rho_1}_{p,r})}\leq C\big(\|u_0\|_{\dot{B}^s_{p,r}}+\|f\|_{\widetilde{L}^{\rho_2}_T(\dot{B}_p^{s-2+2/\rho_2})}\big)\,.$$

If in addition r is finite then u belongs to $C([0,T],\dot{B}^s_{p,r})$.

3 Existence of solution

3.1 A priori estimates

Let us emphasize that we use the fact that $P'(1)>0$ only in the global existence resul. Denoting by G the unique primitive of $x\mapsto P'(x)/x$ such that $G(1)=0$, recall that system (1.6) now reads:

$$\begin{cases}\partial_t q+u\cdot\nabla q+(1+q)\mathrm{div}u=0,\\ \partial_t u+u\cdot\nabla u-\mathcal{A}u-2D(u).\nabla(\ln(1+q))+\nabla(G(1+q))=0,\end{cases}\tag{3.12}$$

where the operator $\mathcal{A}$ is defined by: $\mathcal{A}u = \Delta u + \nabla \mathrm{div}\, u$.

Let u_L be the unique global solution of the following linear heat equation:

$$\begin{cases} \partial_t u_L - \mathcal{A}u_L = 0, \\ u_{L/t=0} = u_0, \end{cases} \tag{3.13}$$

Thanks to the classical heat estimates recalled in Proposition 4 (we refer for example to [2], lemma 2.4 and chapter 3), as $u_0 \in \dot{B}_{2,2}^{\frac{d}{2}-1} \cap \dot{B}_{\infty,1}^{1}$ we have for all time:

$$u_L \in \left(\widetilde{L}_t^\infty \dot{B}_{2,2}^{\frac{d}{2}-1} \cap \widetilde{L}_t^1 \dot{B}_{2,2}^{\frac{d}{2}+1}\right) \cap \left(\widetilde{L}_t^\infty \dot{B}_{\infty,1}^{-1} \cap L_t^1 \dot{B}_{\infty,1}^{1}\right), \tag{3.14}$$

and the corresponding energy estimates. Moreover, as $\mathrm{div}\, u_0 \in \dot{B}_{2,1}^{\frac{d}{2}-2}$, and as $\mathrm{div}\, u_L$ satisfies:

$$\begin{cases} \partial_t \mathrm{div}\, u_L - \mathcal{A}\mathrm{div}\, u_L = \partial_t \mathrm{div}\, u_L - 2\Delta \mathrm{div}\, u_L = 0, \\ \mathrm{div}\, u_{L/t=0} = \mathrm{div}\, u_0, \end{cases} \tag{3.15}$$

we also have that:

$$\mathrm{div}\, u_L \in \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-2} \cap L_t^1 \dot{B}_{2,1}^{\frac{d}{2}}, \tag{3.16}$$

which will be crucial in the study of the density equation. Then, if we denote by $\overline{u} = u - u_L$, we now need to study the following system:

$$\begin{cases} \partial_t q + (\overline{u} + u_L).\nabla q + (1+q)\mathrm{div}\,(\overline{u} + u_L) = 0, \\ \partial_t \overline{u} - \Delta \overline{u} - \nabla \mathrm{div}\, \overline{u} + (\overline{u} + u_L).\nabla \overline{u} + \overline{u}.\nabla u_L + u_L.\nabla u_L \\ \qquad\qquad - 2D(\overline{u} + u_L).\nabla\Big(\ln(1+q)\Big) + \nabla\Big(G(1+q)\Big) = 0, \end{cases} \tag{3.17}$$

The interest of introducing this system is that the most problematic term in the additionnal external force terms, namely $u_L \cdot \nabla u_L$, is in fact regular. Thanks to the paraproduct and Bernstein estimates, we have (even for $d = 2$):

$$\|u_L \cdot \nabla u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \le C\|u_L\|_{\dot{B}_{\infty,2}^{-1}} \|u_L\|_{\dot{B}_{2,2}^{\frac{d}{2}+1}} + \|R(u_L, \nabla u_L)\|_{\dot{B}_{1,1}^{d-1}}.$$

so we can obtain:

$$\|u_L \cdot \nabla u_L\|_{L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-1}} \le \|u_L\|_{\widetilde{L}_t^\infty \dot{B}_{2,2}^{\frac{d}{2}-1}} \|u_L\|_{\widetilde{L}_t^1 \dot{B}_{2,2}^{\frac{d}{2}+1}}$$

In this article we will prove existence and uniqueness of a local solution such that the velocity fluctuation $\overline{u}$ is in the space $\widetilde{L}_T^\infty \dot{B}_{2,1}^{\frac{d}{2}-1} \cap L_T^1 \dot{B}_{2,1}^{\frac{d}{2}+1}$ for some $T > 0$. Then, going back to the original functions, thanks to the following embeddings:

$$\dot{B}_{2,1}^{\frac{d}{2}-1} \hookrightarrow \dot{B}_{2,2}^{\frac{d}{2}-1}, \quad \dot{B}_{2,1}^{\frac{d}{2}-1} \hookrightarrow \dot{B}_{\infty,1}^{-1},$$

we will end with a velocity u with the same regularity as u_L (that is (3.14) and (3.16)).

Let us now state the following transport-diffusion estimates which are adaptations of the ones given in [2] section 3:

Lemma 1 *Let $T>0$, $-\frac{d}{2}<s<\frac{d}{2}$, $u_0\in \dot{B}^s_{2,1}$, $f\in L^1_T\dot{B}^s_{2,1}$ and $v,w\in \widetilde{L}^1_T\dot{B}^{\frac{d}{2}+1}_{2,2}\cap \dot{B}^1_{\infty,1}$. If u is a solution of:*

$$\begin{cases}\partial_t u+v\cdot\nabla u+u\cdot\nabla w-\mathcal{A}u=f,\\ u_{t=0}=u_0,\end{cases}$$

then, if $V(t)=\int_0^t\Big(\|\nabla v(\tau)\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap\dot{B}^0_{\infty,1}}+\|\nabla w(\tau)\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap\dot{B}^0_{\infty,1}}\Big)d\tau$, there exists a constant $C>0$ such that for all $t\in[0,T]$:

$$\|u\|_{\widetilde{L}^\infty_t\dot{B}^s_{2,1}}+\|u\|_{L^1_t\dot{B}^{s+2}_{2,1}}\leq Ce^{CV(t)}\Big(\|u_0\|_{\dot{B}^s_{2,1}}+\int_0^t\|f(\tau)\|_{\dot{B}^s_{2,1}}e^{-CV(\tau)}d\tau\Big).$$

Proof: We refer to [2] for details. Localizing in frequency, if for $j\in\mathbb{Z}$, $u_j=\dot{\Delta}_j u$, we have:

$$\partial_t u_j+v\cdot\nabla u_j-\mathcal{A}u_j=f_j-\dot{\Delta}_j(u\cdot\nabla w)+R_j,$$

where $R_j=[v.\nabla,\dot{\Delta}_j]u$. Taking the L^2 innerproduct we obtain:

$$\partial_t\|u_j\|_{L^2}+2^{2j}\|u_j\|_{L^2}\leq\|\mathrm{div}\,v\|_{L^\infty}\|u_j\|_{L^2}+\|f_j\|_{L^2}+\|\dot{\Delta}_j(u\cdot\nabla w)\|_{L^2}+\|R_j\|_{L^2}.$$

Classical commutator estimates (we refer to [2] section 2.10) then imply that there exists a summable positive sequence $c_j=c_j(t)$ whose sum is 1 such that:

$$\|R_j\|_{L^2}\leq c_j 2^{-js}\|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,\infty}\cap\dot{B}^0_{\infty,1}}\|u\|_{\dot{B}^s_{2,1}}.$$

Thanks to the paraproduct and remainder laws, we have:

$$\|u\cdot\nabla w\|_{\dot{B}^s_{2,1}}\leq C\|u\|_{\dot{B}^s_{2,1}}\|\nabla w\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap\dot{B}^0_{\infty,1}}.$$

so that we finally obtain the result. ■

Concerning the transport equation of the density fluctuation, localization implies:

$$\partial_t q_j+(\overline{u}+u_L)\cdot\nabla q_j=-\dot{\Delta}_j\Big((1+q)\mathrm{div}\,(\overline{u}+u_L)\Big)+R_j,$$

where $R_j=[(\overline{u}+u_L).\nabla,\dot{\Delta}_j]q$. Using the same method, we get the estimate:

$$\begin{aligned}\|q\|_{\widetilde{L}^\infty_t\dot{B}^{\frac{d}{2}}_{2,1}}\leq\|q_0\|_{\widetilde{L}^\infty_t\dot{B}^{\frac{d}{2}}_{2,1}}+\int_0^t\Big(&\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\nabla(\overline{u}+u_L)\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap\dot{B}^0_{\infty,1}}\\ &+(1+\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}})(\|\mathrm{div}\,u_L\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|\nabla\overline{u}\|_{\dot{B}^{\frac{d}{2}}_{2,1}})\Big)d\tau.\end{aligned}\quad(3.18)$$

Remark 6 *Note that here, due to the external force terms for the density, we need to have a $\dot{B}^{\frac{d}{2}}_{2,1}$-control of $\mathrm{div}\,u_L$.*

With a rough majoration,

$$\|q\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} \le C\Big(\|q_0\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} + \int_0^t (1+\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}})V'(\tau)d\tau\Big),$$

where

$$V(t) = \int_0^t \Big(\|\overline{u}\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|\nabla u_L\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap \dot{B}^0_{\infty,1}} + \|\operatorname{div} u_L\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)d\tau, \tag{3.19}$$

and then thanks to the Gronwall lemma,

$$\|q\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} \le Ce^{CV(t)}\Big(\|q_0\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} + \int_0^t V'(\tau)e^{-CV(\tau)}d\tau\Big),$$

which gives:

$$\|q\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{2}}_{2,1}} \le Ce^{CV(t)}\left(1+\|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right) - 1, \tag{3.20}$$

For the velocity, the same localization technique is used, and if we introduce: for $j\in\mathbb{Z}$, $\overline{u}_j = \dot{\Delta}_j \overline{u}$, we have:

$$\partial_t \overline{u}_j + (\overline{u}+u_L)\cdot\nabla\overline{u}_j - \mathcal{A}\overline{u}_j = \dot{\Delta}_j f - \dot{\Delta}_j(\overline{u}\cdot\nabla u_L) + R_j,$$

where

$$\begin{cases} R_j = [(\overline{u}+u_L).\nabla, \dot{\Delta}_j]\overline{u}, \\ f = -u_L.\nabla u_L + 2D(\overline{u}+u_L).\nabla\Big(\ln(1+q)\Big) - \nabla\Big(G(1+q)\Big). \end{cases}$$

Taking the L^2 innerproduct with $\overline{u}_j$ we obtain:

$$\partial_t\|\overline{u}_j\|_{L^2} + 2^{2j}\|\overline{u}_j\|_{L^2} \le \|\operatorname{div}(\overline{u}+u_L)\|_{L^\infty}\|\overline{u}_j\|_{L^2} + \|\dot{\Delta}_j f\|_{L^2} + \|\dot{\Delta}_j(\overline{u}\cdot\nabla u_L)\|_{L^2} + \|R_j\|_{L^2}.$$

After a time integration, the multiplication by $2^{j(\frac{d}{2}-1)}$ followed by a summation over $j\in\mathbb{Z}$ gives ($\overline{u}(0)=0$ and the commutator is estimated as above):

$$\begin{aligned} &\|\overline{u}\|_{\widetilde{L}_t^\infty \dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\overline{u}\|_{L_t^1 \dot{B}^{\frac{d}{2}+1}_{2,1}} \le \\ &\qquad \int_0^t \Big(\|f\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\overline{u}\cdot\nabla u_L\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\nabla(\overline{u}+u_L)\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap\dot{B}^0_{\infty,1}}\|\overline{u}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\Big)d\tau. \end{aligned} \tag{3.21}$$

Classical computations show that:

$$\|\overline{u}\cdot\nabla u_L\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le \|\overline{u}\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\|\nabla u_L\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap\dot{B}^0_{\infty,1}},$$

and from the definition of f:

$$\|f\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le \|u_L\|_{\dot{B}^{\frac{d}{2}-1}_{2,2}}\|u_L\|_{\dot{B}^{\frac{d}{2}+1}_{2,2}} + \|\nabla G(1+q)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + 2\|\nabla\ln(1+q)\cdot D(\overline{u}+u_L)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}.$$

The second term is estimated thanks to the composition lemma (see proposition 2):

$$\|\nabla G(1+q)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le \|G(1+q)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \le C(\|q\|_{L^\infty})\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}}.$$

The last term has to be treated carefully. As we cannot rely on the smallness of q (our data can be large), we will follow the same method as Danchin in [10] and thanks to a frequency cut-off we decompose this term into two parts which will both be small. For $m \in \mathbb{Z}$:

$$\nabla \ln(1+q)\cdot D(\overline{u}+u_L) = I + II \text{ with: } \begin{cases} I = \nabla\Big(\ln(1+q) - \ln(1+\dot{S}_m q)\Big)\cdot D(\overline{u}+u_L) \\ II = \nabla \ln(1+\dot{S}_m q)\cdot D(\overline{u}+u_L). \end{cases}$$

Thanks to the paraproduct and remainder laws we have,

$$\begin{aligned} \|I\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} &\leq \|\ln(1+q) - \ln(1+\dot{S}_m q)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|\nabla(\overline{u}+u_L)\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap \dot{B}^0_{\infty,1}} \\ &\leq C(\|q\|_{L^\infty})\|q - \dot{S}_m q\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|\nabla(\overline{u}+u_L)\|_{\dot{B}^{\frac{d}{2}}_{2,2}\cap \dot{B}^0_{\infty,1}}. \end{aligned} \tag{3.22}$$

We then use the estimate given in [10]: there exists a constant $C \geq 1$ such that (we refer to 3.19 for the expression of V):

$$\|q - \dot{S}_m q\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \leq \|q_0 - \dot{S}_m q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + (1 + \|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}})(e^{CV(t)} - 1)$$

If $0 < \alpha < 1$, the other term is estimated the following way:

$$\begin{aligned} \|II\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} &\leq \|\nabla \ln(1+\dot{S}_m q)\|_{\dot{B}^{\frac{d}{2}+\alpha-1}_{2,1}} \|D(\overline{u}+u_L)\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}} \\ &\leq C(\|q\|_{L^\infty}) 2^{m\alpha} \|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|\overline{u}+u_L\|_{\dot{B}^{\frac{d}{2}+1-\alpha}_{2,2}}. \end{aligned} \tag{3.23}$$

And thanks to the Gronwall lemma, we finally obtain that for all $0 < \alpha < 1$ (for example we can take $\alpha = 1/2$) and all $m \in \mathbb{Z}$:

$$\begin{aligned} &\|\overline{u}\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\overline{u}\|_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,1}} \leq \\ &e^{CV(t)}\Bigg[\|u_L\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,2}} \|u_L\|_{\widetilde{L}^1_t \dot{B}^{\frac{d}{2}+1}_{2,2}} + \int_0^t C(\|q\|_{L^\infty}) \Bigg(\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \Big(1 + 2^{\alpha m}\|\overline{u}+u_L\|_{\dot{B}^{\frac{d}{2}+1-\alpha}_{2,2}}\Big) \\ &\quad + \Big(\|q_0 - S_m q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + (1+\|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}})(e^{CV(\tau)} - 1)\Big)\|\overline{u}+u_L\|_{\dot{B}^{\frac{d}{2}+1}_{2,2}\cap \dot{B}^1_{\infty,1}} \Bigg) d\tau \Bigg], \end{aligned} \tag{3.24}$$

Thanks to the Hölder estimate, we have:

$$\|\overline{u}+u_L\|_{L^1_t \dot{B}^{\frac{d}{2}+1-\alpha}_{2,2}} \leq t^{\frac{\alpha}{2}} \|\overline{u}+u_L\|_{\widetilde{L}_t^{\frac{1}{1-\frac{\alpha}{2}}} \dot{B}^{\frac{d}{2}+1-\alpha}_{2,2}},$$

and by interpolation,

$$\|f\|_{\widetilde{L}_t^{\frac{1}{1-\frac{\alpha}{2}}} \dot{B}^{\frac{d}{2}+1-\alpha}_{2,2}} \leq \|f\|^{\frac{\alpha}{2}}_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,2}} \|f\|^{1-\frac{\alpha}{2}}_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,2}} \leq \frac{\alpha}{2}\|f\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,2}} + (1 - \frac{\alpha}{2})\|f\|_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,2}},$$

so that,

$$\|\overline{u}+u_L\|_{L^1_t \dot{B}^{\frac{d}{2}+1-\alpha}_{2,2}} \leq t^{\frac{\alpha}{2}}(\|\overline{u}+u_L\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,2}} + \|\overline{u}+u_L\|_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,2}}),$$

If we denote $\beta(t)=\|\overline{u}\|_{\widetilde{L}_t^\infty\dot{B}_{2,2}^{\frac{d}{2}-1}}+\|\overline{u}\|_{L_t^1\dot{B}_{2,2}^{\frac{d}{2}+1}}$, the estimates on the velocity turn into:

$$\begin{aligned}\beta(t)\leq e^{CV(t)}\Bigg(&\|u_L\|_{\widetilde{L}_t^\infty\dot{B}_{2,2}^{\frac{d}{2}-1}}\|u_L\|_{\widetilde{L}_t^1\dot{B}_{2,2}^{\frac{d}{2}+1}}\\
&+C(\|q\|_{\widetilde{L}_t^\infty\dot{B}_{2,1}^{\frac{d}{2}}})\Big[t\|q\|_{\widetilde{L}_t^\infty\dot{B}_{2,1}^{\frac{d}{2}}}+\Big(\|q_0-S_mq_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}+(1+\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}})(e^{CV(t)}-1)+2^{\alpha m}t^{\frac{\alpha}{2}}\|q\|_{\widetilde{L}_t^\infty\dot{B}_{2,1}^{\frac{d}{2}}}\Big)\\
&\times\Big(\beta(t)+\|u_L\|_{\widetilde{L}_t^\infty\dot{B}_{2,2}^{\frac{d}{2}-1}\cap\dot{B}_{\infty,1}^{-1}}+\|u_L\|_{L_t^1\dot{B}_{2,2}^{\frac{d}{2}+1}\cap\dot{B}_{\infty,1}^{1}}\Big)\Big]\Bigg).\end{aligned}\tag{3.25}$$

Let us now state and prove the following lemma:

Lemma 2 *Let (q,u) satisfying (SW) on $[0,T]\times\mathbb{R}^d$. Assume that $q\in\mathcal{C}^1([0,T],\dot{B}_{2,1}^{\frac{d}{2}})$ and u, $u_L\in\mathcal{C}^1([0,T],\dot{B}_{2,2}^{\frac{d}{2}-1}\cap\dot{B}_{\infty,1}^{-1}\cap\dot{B}_{2,2}^{\frac{d}{2}+1}\cap\dot{B}_{\infty,1}^{1}))^d$, where u_L satisfies:*

$$\partial_tu_L-\mathcal{A}u_L=0,\quad u_{L|t=0}=u_0.\tag{3.26}$$

If we denote $\overline{u}=u-u_L$, there exist three positive constants η, C (only depending on d), C' (depending on C, d and q_0) and $m\in\mathbb{Z}$ $(=m(\eta,q_0))$ such that if $\eta\in]0,1]$ satisfies:

$$\eta e^{3C\eta}(1+C')(1+\|u_0\|_{\dot{B}_{2,2}^{\frac{d}{2}-1}\cap\dot{B}_{\infty,1}^{-1}})\leq\frac{1}{2},\tag{3.27}$$

and m is chosen such that $\|q_0-S_mq_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\leq\eta^2$. Then if T is small enough so that:

$$\begin{cases}\int_0^T\Big(\|\nabla u_L\|_{\dot{B}_{2,2}^{\frac{d}{2}}\cap\dot{B}_{\infty,1}^{0}}+\|\mathrm{div}\,u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\Big)d\tau\leq\eta^2,\\ TC'+(1+C')(e^{CV(T)}-1)+2^{\alpha m}T^{\frac{\alpha}{2}}C'\leq\eta^2.\end{cases}\tag{3.28}$$

then we have, for all $t\in[0,T]$,

$$\begin{cases}\|q\|_{\widetilde{L}_T^\infty\dot{B}_{2,1}^{\frac{d}{2}}}\leq e^{3C\eta}\Bigg(1+\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\Bigg)-1,\\ \|\overline{u}\|_{\widetilde{L}_t^\infty\dot{B}_{2,1}^{\frac{d}{2}-1}}+\|\overline{u}\|_{L_t^1\dot{B}_{2,1}^{\frac{d}{2}+1}}\leq 2\eta.\end{cases}\tag{3.29}$$

Proof: We recall that $(q,\overline{u})$ satisfies on the interval $[0,T]$ the following system:

$$\begin{cases}\partial_tq+(\overline{u}+u_L).\nabla q+(1+q)\mathrm{div}\,(\overline{u}+u_L)=0,\\ \partial_t\overline{u}-\Delta\overline{u}-\nabla\mathrm{div}\,\overline{u}+(\overline{u}+u_L).\nabla\overline{u}+\overline{u}.\nabla u_L+u_L.\nabla u_L\\ \qquad\qquad -2D(\overline{u}+u_L).\nabla\Big(\ln(1+q)\Big)+\nabla\Big(G(1+q)\Big)=0,\end{cases}$$

Assume that T is small enough so that we have for some $\eta\in[0,1]$ (to be precised later):

$$\int_0^T\Big(\|\nabla u_L\|_{\dot{B}_{2,2}^{\frac{d}{2}}\cap\dot{B}_{\infty,1}^{0}}+\|\mathrm{div}\,u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\Big)d\tau\leq\eta^2\leq\eta,$$

and let us define

$$T^* = \sup\{t \in [0,T], \quad \int_0^t \|\overline{u}\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \le 2\eta\}.$$

Then for all $t \in [0, T^*[$, from (3.20) we have ($\eta \le 1$):

$$\|q\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}}_{2,1}} \le e^{C(2\eta+\eta^2)}\left(1 + \|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right) - 1 \le e^{3C}\left(1 + \|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right) - 1 \overset{\text{def}}{=} c_0. \tag{3.30}$$

Concerning the velocity, using the previous estimate on q and the fact that

$$\|u_L\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}} + \|u_L\|_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,2} \cap \dot{B}^{1}_{\infty,1}} \le \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}}$$

allows us to write that:

$$\beta(t) \le e^{3C\eta}\Bigg(\|u_0\|_{\dot{B}^{-1}_{\infty,1}}\eta^2 + C(c_0)\Big[tc_0 + \Big(\|q_0 - S_m q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + (1+c_0)(e^{CV(t)}-1) + 2^{\alpha m} t^{\frac{\alpha}{2}} c_0\Big) \times \Big(\beta(t) + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}}\Big)\Big]\Bigg). \tag{3.31}$$

Let us first fix $m = m(\eta)$ such that $\|q_0 - S_m q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \le \eta^2$. Then let us take T so small that:

$$Tc_0 + (1+c_0)(e^{CV(T)} - 1) + 2^{\alpha m} T^{\frac{\alpha}{2}} c_0 \le \eta^2.$$

The estimate turns into:

$$\beta(t) \le e^{3C\eta}\left(\|u_0\|_{\dot{B}^{-1}_{\infty,1}}\eta^2 + C(c_0)\eta^2\Big[1 + \beta(t) + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}}\Big]\right).$$

Then, if η is taken so small that

$$\eta e^{3C\eta}\Big(\|u_0\|_{\dot{B}^{-1}_{\infty,1}} + C(c_0)(1 + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}})\Big) \le \frac{1}{2},$$

then as $\eta \in]0,1]$, we obtain that:

$$\beta(t) \le \frac{1}{2}(\beta(t) + \eta),$$

which implies:

$$\beta(t) \le \eta.$$

Then by contradiction this leads to $T^* = T$ and for all $t \in [0,T]$,

$$\|u\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,1}} + \|u\|_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,1}} \le 2\eta.$$

3.2 Existence

We use a standard scheme for proving the existence of the solutions:

1. We smooth out the data and get a sequence of smooth solutions $(q^n, u^n)_{n\in\mathbb{N}}$ of an approximated system of (1.6), on a bounded interval $[0, T^n]$ which may depend on n.
2. We exhibit a positive lower bound T for T^n, and prove uniform estimates on $(q^n, \bar{u}^n)$ (we refer to the next subsection for the definition of $\bar{u}^n$) in the space
$$E_T = \widetilde{C}_T(\dot{B}^{\frac{d}{2}}_{2,1}) \times \big(\widetilde{C}_T(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})\big). \tag{3.32}$$
3. We use compactness to prove that the sequence (q^n, u^n) converges, up to extraction, to a solution of (1.6).

3.2.1 Step 1: Friedrichs approximation

In order to construct approximated solutions of system (1.6) we shall use the classical Friedrichs approximation where we define the frequency truncation operator J_n by:

$$\text{for all } n \in \mathbb{N} \text{ and for all } g \in L^2(\mathbb{R}^d), J_n g = \mathcal{F}^{-1}\left(\mathbf{1}_{\frac{1}{n}\le|\xi|\le n}(\xi)\widehat{g}(\xi)\right),$$

and we define the following approximated system:

$$\begin{cases} \partial_t q_n + J_n(J_n u_n\cdot\nabla J_n q_n) + J_n\Big((1+J_n q_n)\mathrm{div} J_n u_n\Big) = 0,\\ \partial_t u_n + J_n(J_n u_n\cdot\nabla J_n u_n) - \mathcal{A}J_n u_n - 2J_n\Big(D(J_n u_n).\nabla(\ln(1+J_n q_n))\Big)\\ \qquad\qquad\qquad\qquad\qquad\qquad + J_n\Big(\nabla(G(1+J_n q_n))\Big) = 0,\\ (q_n, u_n)_{t=0} = (J_n q_0, J_n u_0). \end{cases}$$

we recall the operator $\mathcal{A}$ is defined by: $\mathcal{A}u = \Delta u + \nabla\mathrm{div}\, u$.

We can easily check that it is an ordinary differential equation in $L^2_n \times (L^2_n)^d$, where $L^2_n = \{u \in L^2(\mathbb{R}^d),\ J_n u = u\}$. Then for every $n \in \mathbb{N}$, by Cauchy-Lipschitz theorem there exists a unique maximal solution in the space $\mathcal{C}^1([0, T^*_n[, L^2_n)$ and this system can be rewritten into:

$$\begin{cases} \partial_t q_n + J_n(u_n\cdot\nabla q_n) + J_n\Big((1+q_n)\mathrm{div} u_n\Big) = 0,\\ \partial_t u_n + J_n(u_n\cdot\nabla u_n) - \mathcal{A}u_n - 2J_n\Big(D(u_n).\nabla(\ln(1+q_n))\Big)\\ \qquad\qquad\qquad\qquad\qquad + J_n\Big(\nabla(G(1+q_n))\Big) = 0. \end{cases} \tag{3.33}$$

3.2.2 Step 2: Uniform estimates

In the sequel, we will split u_n into the solution of a linear system with initial data $J_n u_0$, and the discrepancy to that solution. More precisely, we define by u^n_L the solution of the following heat equation:

$$\begin{cases} \partial_t u^n_L - \mathcal{A}u^n_L = 0\\ (u^n_L)_{/t=0} = J_n u_0. \end{cases} \tag{3.34}$$

We now set $\bar{u}_n = u_n - u_L^n$. Obviously, the definition of $\bar{u}_n$ leads to the following system:

$$\begin{cases} \partial_t q_n + J_n\Big((\overline{u}_n + u_L^n).\nabla q_n\Big) + J_n\Big((1+q_n)\mathrm{div}\,(\overline{u}_n + u_L^n)\Big) = 0, \\ \partial_t \overline{u}_n - \Delta\overline{u}_n - \nabla\mathrm{div}\,\overline{u}_n + J_n\Big((\overline{u}_n + u_L^n).\nabla\overline{u}_n\Big) + J_n\Big(\overline{u}_n.\nabla u_L^n\Big) + J_n\Big(u_L^n.\nabla u_L^n\Big) \\ \qquad\qquad - 2J_n\Big(D(\overline{u}_n + u_L^n).\nabla\ln(1+q_n)\Big) + \nabla J_n\Big(G(1+q_n)\Big) = 0, \\ (q_n, \bar{u}_n)_{t=0} = (J_n q_0, 0). \end{cases} \tag{3.35}$$

We would like to obtain uniform estimates on $(q_n, \bar{u}_n)$ in the space E_T (see 3.32). Before doing this, let us recall that thanks to proposition 4 and as $J_n u_0$ uniformly belongs (for all n) to $\dot{B}^{-1}_{\infty,1} \cap \dot{B}^{\frac{d}{2}-1}_{2,2}$ we obtain that for all $T > 0$:

$$\|u_L^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,2}\cap\dot{B}^{-1}_{\infty,1})} + \|u_L^n\|_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,2}\cap\dot{B}^{1}_{\infty,1})} \leq C(\|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2}\cap\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,2}\cap\dot{B}^{-1}_{\infty,1})}). \tag{3.36}$$

In particular by Besov embedding, we can remark that ∇u_L^n belongs to $L^1_T(L^\infty)$, this property will be crucial in the sequel in order to estimate $\bar{u}_n$. We would also point out that as $\mathrm{div}\, J_n u_0$ uniformly belongs (for all n) to $\dot{B}^{\frac{d}{2}-1}_{2,1}$ we obtain that for all $T > 0$:

$$\|\mathrm{div}\ u_L^n\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-2}_{2,1})} + \|\mathrm{div}\ u_L^n\|_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} \leq C(\|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,2}\cap\dot{B}^{-1}_{\infty,1}} + \|f\|_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,2}\cap\dot{B}^{-1}_{\infty,1})}). \tag{3.37}$$

Thanks to the apriori estimates from lemma 2 (as $J_n q_n = q_n$ and $J_n\overline{u}_n = \overline{u}_n$ the proof of this lemma, which is based on L^2-scalar products, remains true) there exists $\eta > 0$ and a time $T > 0$ (all of them independant of n) such that for any $n \in \mathbb{N}$ and any $t \in [0, \min(T_n^*, T)]$,

$$\begin{cases} \|q_n\|_{\widetilde{L}^\infty_T \dot{B}^{\frac{d}{2}}_{2,1}} \leq e^{3C\eta}\left(1 + \|q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right) - 1, \\ \|\overline{u}_n\|_{\widetilde{L}^\infty_t \dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\overline{u}_n\|_{L^1_t \dot{B}^{\frac{d}{2}+1}_{2,1}} \leq 2\eta. \end{cases} \tag{3.38}$$

From this we deduce that the L^2_n-norm of $(q_n, \overline{u}_n)$ is bounded. As J_n is the truncation operator in $\{\xi \in \mathbb{R}^d, \frac{1}{n} \leq |\xi| \leq n\}$ the bound blows up as n goes to infinity, but all that is important is that it implies (by contradiction) that for all n, the maximal lifespan $T_n^* \geq T$.

3.2.3 Step 3: Time derivatives

Once the uniform time T is obtained the rest of the method is very classical. Using the previous uniform estimates to bound the time derivatives of the approximated solutions, we obtain that:

Lemma 3 *With the same notations, $(\partial_t q_n)_n$ is (uniformly in n) bounded in $L^2_T \dot{B}^{\frac{d}{2}-1}_{2,1}$ and $(\partial_t \overline{u}_n)_n$ is bounded in $L^{\frac{4}{3}}_T \dot{B}^{\frac{d}{2}-\frac{3}{2}}_{2,1} + L^\infty_T \dot{B}^{\frac{d}{2}-1}_{2,1}$ and then in $L^{\frac{4}{3}}_T(\dot{B}^{\frac{d}{2}-1}_{2,1} + \dot{B}^{\frac{d}{2}-\frac{3}{2}}_{2,1})$.*

This result allows to get that:

- $q_n - q_n(0)$ is (uniformly in n) bounded in $\mathcal{C}_T \dot{B}^{\frac{d}{2}}_{2,1} \cap \mathcal{C}^{\frac{1}{2}}_T \dot{B}^{\frac{d}{2}-1}_{2,1}$,
- $\overline{u}_n$ is bounded in $\mathcal{C}_T \dot{B}^{\frac{d}{2}-1}_{2,1} \cap \mathcal{C}^{\frac{1}{4}}_T (\dot{B}^{\frac{d}{2}-1}_{2,1} + \dot{B}^{\frac{d}{2}-\frac{3}{2}}_{2,1})$.

3.2.4 Step 4: compactness and convergence

This part is also classical and we refer for example to [10] (chapter 10) for details: using the previous result and the Ascoli theorem, we can extract a subsequence that weakly converges towards some couple $(q, \overline{u})$, which is proved to be a solution of the original system and to satisfy the energy estimates. This concludes the existence part of the theorem.

4 Uniqueness

Once more, system (SW) is very close to (NSC) and the uniqueness is dealt the same way except, obviously, that here the external force terms that have to be plugged into the apriori estimates are different from the ones in (NSC) and we will focus on it in this section. As for (NCS) we will have (due to endpoints in the paradifferential remainder) to treat separately the cases $d = 2$ and $d \geq 3$. The second difficulty is that, as we present a local result for large data, we will have to make frequency cut-off (as in [2]) in order to bound some external force terms.

Let us introduce for $s \in \mathbb{R}$ the following space:

$$E_d(t) = \widetilde{L}^\infty_t (\dot{B}^{\frac{d}{2}-1}_{2,2} \cap \dot{B}^{-1}_{\infty,1}) \cap L^1_t (\dot{B}^{\frac{d}{2}+1}_{2,2} \cap \dot{B}^{1}_{\infty,1}).$$

Theorem 3 *Let $d \geq 2$ and assume that (q_i, u_i) ($i \in \{1,2\}$) are two solutions of (SW) with the same initial data on the same interval $[0,T]$ and both belonging to the space $E_d(T)$ with $(q_i, u_i - u_L) \in E_T$ (see (3.32)). Then $(q_1, u_1) \equiv (q_2, u_2)$ on $[0,T]$.*

Proof: for $i \in \{1,2\}$, let us introduce $\overline{u}_i = u_i - u_L$ (see (3.26) for the definition of u_L.), then $(q_i, \overline{u}_i)$ satisfy the system:

$$\begin{cases} \partial_t q_i + (\overline{u}_i + u_L).\nabla q_i + (1+q_i) \mathrm{div}\,(\overline{u}_i + u_L) = 0, \\ \partial_t \overline{u}_i - \Delta \overline{u}_i - \nabla \mathrm{div}\, \overline{u}_i + (\overline{u}_i + u_L).\nabla \overline{u}_i + \overline{u}_i.\nabla u_L + u_L.\nabla u_L \\ \qquad\qquad\qquad - 2D(\overline{u}_i + u_L).\nabla\Big(\ln(1+q_i)\Big) + \nabla\Big(G(1+q_i)\Big) = 0, \end{cases}$$

and if we denote by $\delta q = q_1 - q_2$ and $\delta \overline{u} = \overline{u}_1 - \overline{u}_2$, then $(\delta q, \delta \overline{u})$ satisfy the following system:

$$\begin{cases} \partial_t \delta q + (u_L + \overline{u}_2).\nabla \delta q = \delta F_1 + \delta F_2 + \delta F_3, \\ \partial_t \delta \overline{u} - \Delta \delta \overline{u} - \nabla \mathrm{div}\, \delta \overline{u} + (u_L + \overline{u}_1).\nabla \delta \overline{u} + \delta \overline{u}.\nabla (u_L + \overline{u}_2) = \delta G_1 + \delta G_2 + \delta G_3, \end{cases} \tag{4.39}$$

with:
$$\begin{cases} \delta F_1 = -\delta\overline{u}.\nabla q_1, \\ \delta F_2 = -(1+q_1)\mathrm{div}\,\delta\overline{u}, \\ \delta F_3 = -\delta q.\mathrm{div}\,(\overline{u}_2 + u_L), \\ \delta G_1 = 2D(u_L + \overline{u}_1).\nabla\Big(\ln(1+q_1) - \ln(1+q_2)\Big), \\ \delta G_2 = -D(\delta\overline{u}).\nabla\ln(1+q_2), \\ \delta G_3 = -\nabla\Big(G(1+q_1) - G(1+q_2)\Big). \end{cases}$$

4.1 The case $d \geq 3$

We wish to prove (as for (NSC)) the uniqueness in the following space:
$$F_T = \mathcal{C}_T\dot{B}_{2,1}^{\frac{d}{2}-1} \times \Big(\mathcal{C}_T\dot{B}_{2,1}^{\frac{d}{2}-2} \cap L_T^1\dot{B}_{2,1}^{\frac{d}{2}}\Big)^d$$
Due to endpoint estimates for the paradifferential remainder, the case $d = 2$ has to be treated in a different space. We refer to the following section. As for the classical (NSC) system, we prove that $(\delta q, \delta\overline{u}) \in F_T$ (the proof is left to the reader and the computations are the same as the ones done in the following).

Let us begin with δq. As q_1 and q_2 have the same initial data, doing the same computations as for the transport estimates (see [2] theorem 3.14) leads to:
$$\begin{aligned} &\|\delta q\|_{\widetilde{L}_t^\infty\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\leq \int_0^t \Bigg(\Big(\|\mathrm{div}\,(u_L+\overline{u}_2)\|_{L^\infty} + \|\nabla(\overline{u}_2+u_L)\|_{\dot{B}_{2,\infty}^{\frac{d}{2}}\cap L^\infty}\Big)\|\delta q\|_{\widetilde{L}_\tau^\infty\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\delta F\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\Bigg)d\tau, \end{aligned} \tag{4.40}$$
where $\delta F = \delta F_1 + \delta F_2 + \delta F_3$. These terms are estimated thanks to the paraproduct and remainder estimates recalled in section 2 (see (2.11)):

- Thanks to the Bernstein lemma we have $\dot{B}_{1,1}^{d-1} \hookrightarrow \dot{B}_{2,1}^{\frac{d}{2}-1}$ so that:
$$\begin{aligned} \|\delta F_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} &\leq \|T_{\delta\overline{u}}\nabla q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|T_{\nabla q_1}\delta\overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|R(\delta\overline{u},\nabla q_1)\|_{\dot{B}_{1,1}^{d-1}}, \\ &\leq \|\delta\overline{u}\|_{L^\infty}\|\nabla q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \|\nabla q_1\|_{\dot{B}_{\infty,\infty}^{-1}}\|\delta\overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\delta\overline{u}\|_{\dot{B}_{2,2}^{\frac{d}{2}}}\|\nabla q_1\|_{\dot{B}_{2,2}^{\frac{d}{2}-1}} \\ &\leq \Bigg(\|\delta\overline{u}\|_{\dot{B}_{\infty,1}^0} + \|\delta\overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\Bigg)\|q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \|\delta\overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\|q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned} \tag{4.41}$$
- Similarly we get that
$$\begin{cases} \|\delta F_2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \Big(1 + \|q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\Big)\|\delta\overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|\delta F_3\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \|\delta q\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\|u_L + \overline{u}_2\|_{\dot{B}_{2,2}^{\frac{d}{2}+1}\cap\dot{B}_{\infty,1}^1} \end{cases},$$

then we obtain:

$$\|\delta q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \int_0^t \left(\|u_L+\overline{u}_2\|_{\dot{B}_{2,\infty}^{\frac{d}{2}+1}\cap \dot{B}_{\infty,1}^1} \|\delta q\|_{\widetilde{L}_\tau^\infty \dot{B}_{2,1}^{\frac{d}{2}-1}} + \left(1+\|q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\right) \|\delta \overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) d\tau. \quad (4.42)$$

Finally, thanks to the Gronwall estimate:

$$\|\delta q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-1}} \leq e^{C\int_0^t \|\overline{u}_2+u_L\|_{\dot{B}_{2,\infty}^{\frac{d}{2}+1}\cap \dot{B}_{\infty,1}^1} d\tau} \int_0^t \left(1+\|q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\right) \|\delta \overline{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau. \quad (4.43)$$

Remark 7 *Note that this estimate on δq is valid for all $d \geq 2$.*

Concerning the velocity, using the a priori estimate for the transport-diffusion equation provided in the present article, we can write that (we recall that $\overline{u}_1$ and $\overline{u}_2$ have the same initial data):

$$\|\delta \overline{u}\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-2}} + \|\delta \overline{u}\|_{L_t^1 \dot{B}_{2,1}^{\frac{d}{2}}} \leq e^{C\int_0^t \left(\|\nabla \overline{u}_1+\nabla u_L\|_{\dot{B}_{2,2}^{\frac{d}{2}}\cap \dot{B}_{\infty,1}^0} + \|\nabla \overline{u}_2+\nabla u_L\|_{\dot{B}_{2,2}^{\frac{d}{2}}\cap \dot{B}_{\infty,1}^0} \right) d\tau} \times \int_0^t \|\delta G_1+\delta G_2+\delta G_3\|_{\dot{B}_{2,1}^{\frac{d}{2}-2}} d\tau. \quad (4.44)$$

- The last term is dealt the usual way:

$$\begin{aligned}\|\delta G_3\|_{\dot{B}_{2,1}^{\frac{d}{2}-2}} &\leq C(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}) \left(1+\|q_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|q_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}\right) \|\delta q\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \\ &\leq C(\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\delta q\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \quad (4.45)\end{aligned}$$

- Without surprise, the first term is estimated by:

$$\begin{aligned}\|\delta G_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-2}} &\leq 2\|D(u_L+\overline{u}_1)\|_{\dot{B}_{2,2}^{\frac{d}{2}}\cap \dot{B}_{\infty,1}^0} \|\nabla \Big(\ln(1+q_1)-\ln(1+q_2)\Big)\|_{\dot{B}_{2,1}^{\frac{d}{2}-2}}, \\ &\leq C(\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|u_L+\overline{u}_1\|_{\dot{B}_{2,2}^{\frac{d}{2}+1}\cap \dot{B}_{\infty,1}^1} \|\delta q\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \quad (4.46)\end{aligned}$$

  Note that here, after using the injection $\dot{B}_{1,1}^{d-2} \hookrightarrow \dot{B}_{2,1}^{\frac{d}{2}-2}$, we needed $d-2>0$ in the remainder. In the case $d=2$, this term will have to be dealt differently in the following subsection.

- As in (3.22), we have to decompose δG_2 into two parts:

$$\delta G_2 = -D(\delta \overline{u}).\nabla \Big(\ln(1+q_2)-\ln(1+S_m q_2)\Big) - D(\delta \overline{u}).\nabla \ln(1+S_m q_2) = R_1+R_2.$$

Here the second paraproduct for R_2 requires that $-1+\alpha<0$. The remainders require that $d-2>0$). We obtain that:

$$\begin{aligned}\|R_1\|_{\dot B^{\frac d2-2}_{2,1}} &\leq \|\delta\overline{u}\|_{\dot B^{\frac d2}_{2,1}} C(\|q_2\|_{L^\infty})\Big(1+\|q_2\|_{\dot B^{\frac d2}_{2,1}}\Big)\|q_2-S_m q_2\|_{\dot B^{\frac d2}_{2,1}}\\ &\leq C(\|q_0\|_{\dot B^{\frac d2}_{2,1}})\|q_2-S_m q_2\|_{\dot B^{\frac d2}_{2,1}}\|\delta\overline{u}\|_{\dot B^{\frac d2}_{2,1}}\\ &\leq C(\|q_0\|_{\dot B^{\frac d2}_{2,1}})\Big(\|q_0-S_m q_0\|_{\dot B^{\frac d2}_{2,1}}+(e^{CV(t)}-1)\Big)\|\delta\overline{u}\|_{\dot B^{\frac d2}_{2,1}}, \end{aligned}\tag{4.47}$$

and

$$\begin{aligned}\|R_2\|_{\dot B^{\frac d2-2}_{2,1}} \leq \|\delta\overline{u}\|_{\dot B^{\frac d2-\alpha}_{2,1}}\|\ln(1+S_m q_2)\|_{\dot B^{\frac d2+\alpha}_{2,1}} &\leq \|\delta\overline{u}\|_{\dot B^{\frac d2-\alpha}_{2,1}} C(\|S_m q_2\|_{L^\infty})\|S_m q_2\|_{\dot B^{\frac d2+\alpha}_{2,1}}\\ &\leq C(\|q_0\|_{\dot B^{\frac d2}_{2,1}})2^{\alpha m}\|\delta\overline{u}\|_{\dot B^{\frac d2-\alpha}_{2,1}}.\end{aligned}\tag{4.48}$$

The first term is small if m is large enough and T small enough, and the second term introduces a nonnegative power of t.

Finally we have:

$$\begin{aligned}\|\delta\overline{u}\|_{\widetilde L^\infty_t \dot B^{\frac d2-2}_{2,1}}+\|\delta\overline{u}\|_{L^1_t \dot B^{\frac d2}_{2,1}} &\leq e^{C\int_0^t\Big(\|\nabla\overline{u}_1\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla\overline{u}_2\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla u_L\|_{\dot B^{\frac d2}_{2,2}\cap\dot B^0_{\infty,1}}\Big)d\tau}\\ &\times C(\|q_0\|_{\dot B^{\frac d2}_{2,1}})\int_0^t\Bigg((1+\|\nabla\overline{u}_1+\nabla u_L\|_{\dot B^{\frac d2}_{2,2}\cap\dot B^0_{\infty,1}})\|\delta q\|_{\dot B^{\frac d2-1}_{2,1}}+2^{\alpha m}\|\delta\overline{u}\|_{\dot B^{\frac d2-\alpha}_{2,1}}\\ &+\Big(\|q_0-S_m q_0\|_{\dot B^{\frac d2}_{2,1}}+(e^{C\int_0^t\Big(\|\nabla\overline{u}_2\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla u_L\|_{\dot B^{\frac d2}_{2,2}\cap\dot B^0_{\infty,1}}\Big)d\tau}-1)\Big)\|\delta\overline{u}\|_{\dot B^{\frac d2}_{2,1}}\Bigg)d\tau.\end{aligned}\tag{4.49}$$

Introducing c_0 a constant only depending on $\|q_0\|_{\dot B^{\frac d2}_{2,1}}$ and:

$$V(t)=\int_0^t\Big(\|\nabla\overline{u}_1\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla\overline{u}_2\|_{\dot B^{\frac d2}_{2,1}}+\|\nabla u_L\|_{\dot B^{\frac d2}_{2,2}\cap\dot B^0_{\infty,1}}\Big)d\tau.$$

$$\begin{aligned}\|\delta\overline{u}\|_{\widetilde L^\infty_t \dot B^{\frac d2-2}_{2,1}}+\|\delta\overline{u}\|_{L^1_t \dot B^{\frac d2}_{2,1}} &\leq c_0 e^{CV(t)}\Bigg((t+V(t))\|\delta q\|_{\widetilde L^\infty_t \dot B^{\frac d2-1}_{2,1}}\\ &+2^{\alpha m}\|\delta\overline{u}\|_{L^1_t\dot B^{\frac d2-\alpha}_{2,1}}+\Big(\|q_0-S_m q_0\|_{\dot B^{\frac d2}_{2,1}}+(e^{CV(t)}-1)\Big)\|\delta\overline{u}\|_{L^1_t\dot B^{\frac d2}_{2,1}}\Bigg).\end{aligned}\tag{4.50}$$

If $\beta(t)=\|\overline{u}\|_{\widetilde L^\infty_t \dot B^{\frac d2-1}_{2,2}}+\|\overline{u}\|_{L^1_t \dot B^{\frac d2+1}_{2,2}}$, as in section 3.1 we have $\|\delta\overline{u}\|_{L^1_t\dot B^{\frac d2-\alpha}_{2,2}}\leq t^{\frac\alpha2}\beta(t)$, then as,

$$\|\delta q\|_{\widetilde L^\infty_t \dot B^{\frac d2-1}_{2,1}}\leq c_0 e^{CV(t)}\|\delta\overline{u}\|_{L^1_t\dot B^{\frac d2}_{2,1}},\tag{4.51}$$

we obtain:

$$\beta(t) \le c_0 e^{2CV(t)}\Big(t + V(t) + 2^{\alpha m} t^{\frac{\alpha}{2}} + \|q_0 - S_m q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + (e^{CV(t)} - 1)\Big)\beta(t).$$

When $\eta \in]0,1]$ satisfies:

$$2c_0 \eta e^{2C\eta} \le \frac{1}{2}, \tag{4.52}$$

and m is chosen such that $\|q_0 - S_m q_0\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \le \eta$. Then if T is small enough so that:

$$\begin{cases} \int_0^T \Big(\|\nabla u_L\|_{\dot{B}^{\frac{d}{2}}_{2,2} \cap \dot{B}^0_{\infty,1}} + \|\mathrm{div}\, u_L\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big) d\tau \le \eta, \\ T + V(T) + (e^{CV(T)} - 1) + 2^{\alpha m} T^{\frac{\alpha}{2}} \le \eta. \end{cases} \tag{4.53}$$

then we have, for all $t \in [0,T]$,

$$0 \le \beta(t) \le 2c_0 \eta e^{2C\eta} \beta(t) \le \frac{\beta(t)}{2}.$$

So $\beta(t) = 0$ for all $t \in [0,T]$ and the same goes for δq, which proves the uniqueness on $[0,T]$.

Remark 8 *Note that these conditions are implied by the ones from the apriori estimates.*

To end the proof when T is not small, let us introduce (as in [10], section 10.2.4) the set:

$$I \overset{\text{def}}{=} \{t \in [0,T] / (q_1(t'), \overline{u}_1(t')) = (q_2(t'), \overline{u}_2(t')),\ \forall t' \in [0,t]\}.$$

This is a nonempty closed subset of $[0,T]$. Using the same method as above allows to prove it is also open and then $I = [0,T]$.

4.2 The case $d = 2$

In this case, the estimates on δq remain correct, but the paradifferential remainders, when estimating the external forces terms in the velocity equation, are modified. Indeed, in the case $d = 2$ we reach the following endpoint where for all $1/p_1 + 1/p_2 = 1 = 1/r_1 + 1/r_2$:

$$\|R(f,g)\|_{\dot{B}^0_{1,\infty}} \le C \|f\|_{\dot{B}^s_{p_1,r_1}} \|g\|_{\dot{B}^{-s}_{p_2,r_2}}.$$

Estimate (4.44) is then replaced by

$$\|\delta \overline{u}\|_{\widetilde{L}^\infty_t \dot{B}^{-1}_{2,\infty}} + \|\delta \overline{u}\|_{L^1_t \dot{B}^1_{2,\infty}} \le e^{C \int_0^t \Big(\|\nabla \overline{u}_1 + \nabla u_L\|_{\dot{B}^1_{2,2} \cap \dot{B}^0_{\infty,1}} + \|\nabla \overline{u}_2 + \nabla u_L\|_{\dot{B}^1_{2,2} \cap \dot{B}^0_{\infty,1}}\Big) d\tau} \times \int_0^t \|\delta G_1 + \delta G_2 + \delta G_3\|_{\dot{B}^{-1}_{2,\infty}} d\tau, \tag{4.54}$$

with

$$\begin{aligned} \|\delta G_3\|_{\dot{B}^{-1}_{2,\infty}} &\le C(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty})\Big(1 + \|q_1\|_{\dot{B}^1_{2,1}} + \|q_2\|_{\dot{B}^1_{2,1}}\Big) \|\delta q\|_{\dot{B}^0_{2,\infty}} \\ &\le C(\|q_0\|_{\dot{B}^1_{2,1}}) \|\delta q\|_{\dot{B}^0_{2,1}}, \end{aligned} \tag{4.55}$$

and (here we reach the endpoint $d-2=0$ in the remainder)

$$\begin{aligned}
\|\delta G_1\|_{\dot{B}^{-1}_{2,\infty}} &= 2\|D(u_L+\overline{u}_1).\nabla\Big(\ln(1+q_1)-\ln(1+q_2)\Big)\|_{\dot{B}^{-1}_{2,\infty}}\\
&\le 2(\|T_D\nabla\|_{\dot{B}^{-1}_{2,\infty}}+\|T_\nabla D\|_{\dot{B}^{-1}_{2,\infty}}+\|R(D,\nabla)\|_{\dot{B}^{-1}_{1,\infty}})\\
&\le 2(\|D\|_{L^\infty}\|\nabla\|_{\dot{B}^{-1}_{2,\infty}}+\|\nabla\|_{\dot{B}^{-2}_{\infty,\infty}}\|D\|_{\dot{B}^{1}_{2,\infty}}+\|D\|_{\dot{B}^{1}_{2,2}}\|\nabla\|_{\dot{B}^{-1}_{2,2}})\\
&\le C(\|q_0\|_{\dot{B}^{1}_{2,1}})\|u_L+\overline{u}_1\|_{\dot{B}^{2}_{2,2}\cap\dot{B}^{1}_{\infty,1}}\|\delta q\|_{\dot{B}^{0}_{2,1}}. \qquad (4.56)
\end{aligned}$$

Concerning the last term, with the same decomposition, $\delta G_2=R_1+R_2$ and when we choose some $\alpha\in]0,1[$ (for R_2) we obtain that:

$$\begin{aligned}
\|R_1\|_{\dot{B}^{-1}_{2,\infty}} &\le \|\delta\overline{u}\|_{\dot{B}^{1}_{2,\infty}}C(\|q_2\|_{L^\infty})\Big(1+\|q_2\|_{\dot{B}^{1}_{2,1}}\Big)\|q_2-S_mq_2\|_{\dot{B}^{1}_{2,1}}\\
&\le C(\|q_0\|_{\dot{B}^{1}_{2,1}})\Big(\|q_0-S_mq_0\|_{\dot{B}^{1}_{2,1}}+(e^{CV(t)}-1)\Big)\|\delta\overline{u}\|_{\dot{B}^{1}_{2,\infty}}, \qquad (4.57)
\end{aligned}$$

and

$$\|R_2\|_{\dot{B}^{-1}_{2,\infty}} \le \|\delta\overline{u}\|_{\dot{B}^{1-\alpha}_{2,\infty}}\|\ln(1+S_mq_2)\|_{\dot{B}^{1+\alpha}_{2,1}} \le C(\|q_0\|_{\dot{B}^{1}_{2,1}})2^{\alpha m}\|\delta\overline{u}\|_{\dot{B}^{1-\alpha}_{2,\infty}}. \qquad (4.58)$$

As in the previous section, we collect the estimates and obtain:

$$\|\delta q\|_{\widetilde{L}^\infty_t\dot{B}^{0}_{2,1}} \le c_0e^{CV(t)}\|\delta\overline{u}\|_{L^1_t\dot{B}^{1}_{2,1}},$$

and with obvious notations,

$$\beta(t)\le c_0e^{CV(t)}\Bigg(\int_0^t(1+V'(\tau))\|\delta q\|_{\widetilde{L}^\infty_\tau\dot{B}^{0}_{2,1}}d\tau+\Big(2^{\alpha m}t^{\frac{\alpha}{2}}+\|q_0-S_mq_0\|_{\dot{B}^{1}_{2,1}}+(e^{CV(t)}-1)\Big)\beta(t)\Bigg).$$

under the conditions from the previous section, we get:

$$\beta(t)\le c_0e^{CV(t)}\Bigg(\int_0^t(1+V'(\tau))\|\delta\overline{u}\|_{L^1_t\dot{B}^{1}_{2,1}}d\tau\Bigg)+\frac{1}{2}\beta(t).$$

Thanks to Proposition 3, we can use the following logarithmic estimates ($d=2$)

$$\|\delta\overline{u}\|_{L^1_t\dot{B}^{\frac{d}{2}}_{2,1}}\le C\|\delta\overline{u}\|_{L^1_t\dot{B}^{\frac{d}{2}}_{2,\infty}}\log\Big(e+\frac{\|\delta\overline{u}\|_{L^1_t\dot{B}^{\frac{d}{2}-1}_{2,\infty}}+\|\delta\overline{u}\|_{L^1_t\dot{B}^{\frac{d}{2}+1}_{2,\infty}}}{\|\delta\overline{u}\|_{L^1_t\dot{B}^{\frac{d}{2}}_{2,\infty}}}\Big)$$

As $\delta u=\overline{u}_1-\overline{u}_2$, we can write:

$$\|\delta\overline{u}\|_{L^1_t\dot{B}^{0}_{2,\infty}}+\|\delta\overline{u}\|_{L^1_t\dot{B}^{2}_{2,\infty}}\le W(t)=W_1(t)+W_2(t),$$

with $W_i(t)=\|\overline{u}_i\|_{L^1_t\dot{B}^{0}_{2,\infty}}+\|\overline{u}_i\|_{L^1_t\dot{B}^{2}_{2,\infty}}$. This function is bounded on $[0,T]$ and the estimates turn into:

$$\beta(t)\le C_T\int_0^t(1+V'(\tau))\beta(\tau)\log\Big(e+\frac{W(T)}{\beta(\tau)}\Big)d\tau.$$

As we have,

$$\int_0^1 \frac{dr}{r\log(e+\frac{W(T)}{r})}=\infty,$$

The Osgood lemma allows us to conclude that $\beta(t)=0$ for all $t\in[0,T]$ (we refer for example to [2], section 3.1.1). Then the density fluctuation is also zero on this intervall. Then the conclusion is the same as in the case $d\geq 3$.

5 Global well-posedness

In this section we are interested in proving the global well-posedness of (1.1) when we assume smallness on the initial data. The proof follows the same lines than in the sections 3 and 4. The only difficulty consists in getting damped effects on the density in order to deal with the pressure in the remainder terms. To do this we have just to use the estimates in Besov spaces from [7] or [5] on the following linear system associated to (1.1):

$$\begin{aligned}&\partial_t q+v\cdot\nabla q+\mathrm{div}u=F,\\&\partial_t u+v\cdot\nabla u-\mathcal{A}u+\nabla q=G,\end{aligned}\tag{5.59}$$

There, they exhibit the parabolic smoothing effect on u and on the low frequencies of q, and a damping effect on the high frequencies of q. To do this, the authors need to introduce a paralinearisation in order to deal with the convection terms $u\cdot\nabla q$. More precisely they obtain the following proposition:

Proposition 5 *Let (q,u) a solution of the system (5.59) on $[0,T[$, $1-\frac{d}{2}<s<1+\frac{d}{2}$ and $V(t)=\int_0^t\|\nabla v(\tau)\|_{L^\infty\cap\dot{B}^{\frac{d}{2}}_{2,2}}\,d\tau$. We have then the following estimate for any $T>0$:*

$$\begin{aligned}&\|(q,u)\|_{\widetilde{L}^\infty_T(\widetilde{\dot{B}}^{s-1,s}_{2,1})\times\widetilde{L}^\infty_T(\dot{B}^{s-1}_{2,1})}+\|(q,u)\|_{\widetilde{L}^1_T(\widetilde{\dot{B}}^{s+1,s}_{2,1})\times\widetilde{L}^\infty_T(\dot{B}^{s+1}_{2,1})}\\&\qquad\leq Ce^{CV(t)}\big(\|(q_0,u_0)\|_{\widetilde{B}^{s-1,s}\times B^{s-1}}+\int_0^T e^{-CV(\tau)}\|(F,G)(\tau)\|_{\widetilde{\dot{B}}^{s-1,s}\times\dot{B}^{s-1}}d\tau\big),\end{aligned}$$

where C depends only on N and s and $\widetilde{\dot{B}}^{s_1,s_2}_{2,1}$ denotes the hybrid Besov space with regularity s_1 for low frequencies and s_2 for high frequencies (we refer to [7, 5] or [15] for details).

The rest of the proof consists in searching a solution of the form $(q,u_L+\overline{u})$ with u_L defines as in the previous section. We can verifies that $(q,\overline{u})$ check the following system:

$$\begin{aligned}&\partial_t q+u\cdot\nabla q+\mathrm{div}\overline{u}=F,\\&\partial_t\overline{u}+u\cdot\nabla\overline{u}-\mathcal{A}\overline{u}+G'(1)\nabla q=G,\end{aligned}\tag{5.60}$$

with:

$$\begin{aligned}&F=-\mathrm{div}u_L-q\mathrm{div}u,\\&G=-\overline{u}\nabla u_L-u_L\cdot\nabla u_L+2(Du\cdot\nabla\ln\rho)+\nabla(G(1+q)-G'(1)q)\end{aligned}$$

Using the proposition 5 on $\overline{u}$, the rest of the proof simply consists in getting estimates on $(q, \overline{u})$ in H with:

$$H = \big(\widetilde{C}(\mathbb{R}^+, B^{\frac{d}{2}}_{2,1}) \cap \widetilde{L}^1(B^{\frac{d}{2}}_{2,1} \cap B^{\frac{d}{2}-1}_{2,1})\big) \times \big((\widetilde{C}(\mathbb{R}^+, B^{\frac{d}{2}-1}_{2,1}) \cap \widetilde{L}^1(B^{\frac{d}{2}+1}_{2,1})\big).$$

We should to treat the remainder terms as in section 3. We would like in particular to mention that $\mathrm{div} u_L$ remains small in $\widetilde{L}^1(B^{\frac{d}{2}}_{2,1})$ with the smallness condition on $\mathrm{div} u_0$, by the Gronwall lemma we can conclude. For the uniqueness the method follows the same approach as in section 4.

6 Proof of theorem 2

Our method from section 3 and 4 may be adapted to the study of incompressible density dependent Navier-Stokes equations. This is just a matter of replacing the parabolic model below by a nonstationary Stokes system. More precisely we define u_L as the solution of the following system:

$$\begin{cases} \partial_t u_L - \Delta u_L + \nabla \Pi_L = 0, \\ \mathrm{div} u_L = 0, \\ (u_L)_{\ t=0} = u_0. \end{cases} \tag{6.61}$$

In the same way than in section 3, we are searching solution of the form $u = u_L + \bar{u}$ with:

$$\begin{cases} \partial_t q + (\overline{u} + u_L).\nabla q + (1+q)\mathrm{div}\,(\overline{u} + u_L) = 0, \\ \partial_t \overline{u} - \Delta \overline{u} + (\overline{u} + u_L).\nabla \overline{u} + \overline{u}.\nabla u_L + u_L.\nabla u_L \\ \qquad\qquad - 2D(\overline{u} + u_L).\nabla\Big(\ln(1+q)\Big) + \dfrac{1}{1+q}\nabla(\bar{\Pi} + \Pi_L) = 0, \\ \mathrm{div}\bar{u} = 0. \end{cases} \tag{6.62}$$

By applying the operator curl to the momentum equation, we obtain that:

$$\begin{cases} \partial_t q + (\overline{u} + u_L).\nabla q + (1+q)\mathrm{div}\,(\overline{u} + u_L) = 0, \\ \partial_t \mathrm{curl}\overline{u} - \Delta \mathrm{curl}\overline{u} + \mathrm{curl}\Big(\overline{u} + u_L).\nabla \overline{u} + \overline{u}.\nabla u_L + u_L.\nabla u_L \\ \qquad\qquad - 2D(\overline{u} + u_L).\nabla\Big(\ln(1+q)\Big)\Big) + \nabla(\dfrac{1}{1+q}) : \nabla(\bar{\Pi} + \Pi_L) = 0, \\ \mathrm{div}\bar{u} = 0, \end{cases} \tag{6.63}$$

where:

$$(\nabla f : \nabla g)_{i,j} = \partial_j f \partial_i g - \partial_i f \partial_j g.$$

By following the same idea as in section 3, we are able to estimate $\bar{u}$ in $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1}) \cap \widetilde{L}^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})$ by proving estimate on $\mathrm{curl} u$ in $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-2}_{2,1}) \cap \widetilde{L}^1_T(\dot{B}^{\frac{d}{2}}_{2,1})$, in order to deal with the pressure $\bar{\Pi}$ it is sufficient to adapt the idea of [1, 16] as $\bar{\Pi}$ verifies an elliptic equation.

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