From Gross-Pitaevskii equation to Euler Korteweg system, existence of global strong solutions with small irrotational initial data

Corentin Audiard, Boris Haspot

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Corentin Audiard [*][†] and Boris Haspot [‡]

Abstract

In this paper we prove the global well-posedness for small data for the Euler Korteweg system in dimension $N \geq 3$, also called compressible Euler system with quantum pressure. It is formally equivalent to the Gross-Pitaevskii equation through the Madelung transform. The main feature is that our solutions have no vacuum for all time. Our construction uses in a crucial way some deep results on the scattering of the Gross-Pitaevskii equation due to Gustafson, Nakanishi and Tsai in [28, 29, 30]. An important part of the paper is devoted to explain the main technical issues of the scattering in [29] and we give a detailed proof in order to make it more accessible. Bounds for long and short times are treated with special care so that the existence of solutions does not require smallness of the initial data in H^s, $s > \frac{N}{2}$. The optimality of our assumptions is also discussed.

Contents

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1 Introduction

The motion of a general Euler Korteweg compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div} K, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density and P the pressure. We shall work in the sequel with $P(\rho) = \rho^2/2$. Throughout the paper, we denote the space variable $x \in \mathbb{R}^N$. We restrict ourselves to the case $N \geq 3$. The general Korteweg tensor reads as follows:

$$\operatorname{div} K = \operatorname{div}\left(\left(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa^{'}(\rho))|\nabla\rho|^2 \right) Id - \kappa(\rho)\nabla\rho \otimes \nabla\rho \right). \tag{1.2}$$

Here κ is the capillary coefficient and in the sequel we shall deal with the specific case:

$$\kappa(\rho) = \frac{\kappa_1}{\rho} \quad \text{so that} \quad \operatorname{div} K = 2\kappa_1\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}), \ \kappa_1 \in \mathbb{R}^{+*}.$$

This case corresponds to the so called quantum pressure. We have in particular the energy estimate that we obtain by multiplying the momentum equation by $2u$:

$$\int_{\mathbb{R}^N} 4\kappa_1|\nabla\sqrt{\rho}|^2(t,x) + (\rho|u|^2)(t,x) + (\rho - 1)(t,x)^2)dx$$
$$= \int_{\mathbb{R}^N} 4\kappa_1|\nabla\sqrt{\rho_0}|^2(x) + (\rho_0|u_0|^2)(x) + (\rho_0 - 1)(x)^2)dx. \tag{1.3}$$

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When the velocity $u = \nabla\theta$ is irrotational, the Madelung transform $\psi = \sqrt{\rho}e^{i\frac{\theta}{2\sqrt{\kappa_1}}}$ allows formally to rewrite the Euler Korteweg system as the Gross-Pitaevski equation [1] (GP):

$$\begin{cases} 2i\sqrt{\kappa_1}\partial_t\psi + 2\kappa_1\Delta\psi = (|\psi|^2 - 1)\psi, \\ \psi(0,\cdot) = \psi_0. \end{cases} \tag{1.4}$$

with the boundary condition $\lim_{|x|\to+\infty}\psi = 1$. The Gross-Pitaevskii equation is the Hamiltonian evolution associated to the Ginzburg-Landau energy:

$$\begin{aligned} E(\psi) &= \int_{\mathbb{R}^N}\big(\kappa_1|\nabla\psi(t,x)|^2 + \frac{1}{4}(|\psi|^2 - 1)^2\big)dx \\ &= \int_{\mathbb{R}^N}\big(\kappa_1|\nabla\varphi(t,x)|^2 + \frac{1}{4}(2Re\varphi + |\varphi|^2)^2\big)dx. \end{aligned} \tag{1.5}$$

with $\psi = 1 + \varphi$. In the sequel we focus on the Gross-Pitaevski equation, if ψ is smooth and does not vanish the two systems are equivalent. The main goal of this paper is to prove the existence of global strong solution $\psi = 1 + \varphi$ with $\|\varphi\|_{L^\infty_{x,t}} < 1$ which will require a smallness assumption on the initial data. We first recall some results on Gross-Pitaevskii equation.

1.1 On the Gross Pitaevskii equation

Up to a change of variable we may take in (1.4) $2\sqrt{\kappa_1} = 2\kappa_1 = 1$ and we consider the equation on $\varphi = \psi - 1$:

$$\begin{cases} i\partial_t\varphi + \Delta\varphi - 2Re\varphi = F(\varphi), \\ F(\varphi) = (\varphi + 2\bar\varphi + |\varphi|^2)\varphi. \end{cases} \tag{1.6}$$

The previous equation is close at the main order of the nonlinearity to the defocusing cubic Schrödinger equation but the linearized system reads

$$i\partial_t\varphi + \Delta\varphi - 2Re\varphi = 0. \tag{1.7}$$

This system (on $Re(\varphi), Im(\varphi)$) can be diagonalized by the change of unknown (see [28])

$$v = V\varphi = Re\varphi + iU Im\varphi \quad \text{with } U = \sqrt{-\Delta(2 - \Delta)^{-1}},$$

and setting $H = \sqrt{-\Delta(2 - \Delta)}$ we get the linear Schrödinger-like equation:

$$i\partial_t v - Hv = 0.$$

Let us mention that the defocusing cubic Schrödinger equation is now well understood. In dimension $N \leq 3$ the corresponding NLS equation is globally well-posed in $H^1(\mathbb{R}^N)$ and has scattering property (see [16, 37]). The global existence and scattering for $N = 4$ corresponds to the energy critical case, it has been solved after intense efforts by Tao et al in [41] in the case of spherical initial data. For completeness we recall what we mean by "scattering".

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[1]It should be pointed out that for a general capillarity the Euler Korteweg system can also be rewritten as some degenerate quasi linear Schrödinger equation, see [5]. The change of variable does not involve the Madelung transform.

Definition 1.1. Consider the nonlinear Schrödinger-like equation

$$\begin{cases} i\partial_t u - A(-\Delta)u = f(u), \\ u(0) = u_0, \end{cases}$$

where $A(-\Delta)$ is a Fourier multiplier with real valued symbol. Assume that the problem has an unique solution $u \in X(\mathbb{R} \times \mathbb{R}^N) \supset C(\mathbb{R}, Y(\mathbb{R}^N))$ where X, Y are Banach spaces such that e^{-itA} acts continuously on Y. The solution u scatters to $u_+ \in Y$ if $\|e^{-itA}u(t) - u_+\|_Y \to_{t \to +\infty} 0$.

A very natural frame for scattering corresponds to the case where the equation has an energy and is globally well-posed in the energy space. As we mentioned the case of defocusing Schrödinger equations is relatively well understood even for large initial data.

The situation is more delicate for the Gross-Pitaevskii equation, where the natural energy space is not $H^1(\mathbb{R}^N)$. In particular the $L^2(\mathbb{R}^N)$ norm is not conserved (we will see that it is related to the low frequencies behavior of (1.7) which is similar to the wave equation). The natural energy space associated to the Gross-Pitaevskii equation is

$$E_1 = \{\psi \in H^1_{loc}(\mathbb{R}^N), \ \nabla\psi \in L^2(\mathbb{R}^N), \ |\psi|^2 - 1 \in L^2(\mathbb{R}^N)\}.$$

Global well-posedness in E_1 has been proved by C. Gallo and P. Gérard in [21, 22] in dimension $N \leq 3$ and by Kilipp et al in [36] in the critical case $N = 4$. It was also proved that for $s \geq 1$ the $H^s(\mathbb{R}^N)$ regularity is also propagated with a growth in time (in particular it does not imply any $L^\infty_{t,x}$ control).

A striking difference between (GP) and the cubic defocusing Schrödinger equation is the existence of traveling waves which prevent the scattering for arbitrary initial data (see cite [7, 9, 19, 38]). A traveling wave is a solution of the form (up to symetry):

$$\psi(t, x) = u_c(x_1 - ct, x_2, \cdots, x_N),$$

where u_c satisfies:

$$ic\partial_1 u_c - \Delta u_c - u_c(1 - |u_c|^2) = 0. \tag{1.8}$$

It was proved that such solutions of finite energy exist for small c (see [7, 9, 19]) and the full range $0 < |c| < \sqrt{2}$ was obtained by Maris in [38] in dimension $N \geq 3$. In dimension $N \geq 2$ there is no supersonic traveling waves ($c > \sqrt{2}$, see [26]). It is also proved in [7] that there is a lower bound on the energy of all possible traveling waves for (1.6) in dimension $N = 3$:

$$\mathcal{E}_0 = \inf\{E(\psi), \psi(t, x) = u_c(x_1 - ct, x_2, \cdots, x_N) \text{ solves (1.6) for } c > 0\}. \tag{1.9}$$

Let us mention that in the case $N = 2$ the situation is radically different since the energy of the traveling waves goes to zero when c goes to the subsonic limit $\sqrt{2}$, this has been conjectured by C.A. Jones, S. J. Putterman and P. H. Roberts (see [34])).This would imply that there is no scattering even for small energy initial data when $N = 2$.

Despite these issues, scattering has been obtained in a series of papers by Gustafson, Nakanishi and Tsai in [28, 29, 30]. For $N \geq 4$ they proved scattering for small initial data in $H^{\frac{N}{2}-1}(\mathbb{R}^N)$, the case $N = 3$ is much more intricate and requires the data to be small in weighted $H^1(\mathbb{R}^N)$ spaces (to which, nevertheless, traveling waves belong).

Let us briefly explain the ideas of these papers that we shall recall more in details in the sequel. After diagonalization the equation reads:

$$\begin{aligned} i\partial_t v - Hv =& U(3v_1^2 + (U^{-1}v_2)^2 + |v_1 + iU^{-1}v_2|^2 v_1) \\ &+ i(2v_1(U^{-1}v_2) + |v_1 + iU^{-1}v_2|^2(U^{-1}v_2)). \end{aligned} \tag{1.10}$$

It is interesting to compare it to the classical NLS equation (focusing or defocusing, indeed we consider small initial data so the sign of the nonlinearity should not play any role):

$$i\partial_t\varphi + \Delta\varphi = \lambda|\varphi|^\alpha\varphi \tag{1.11}$$

with $\lambda \in \mathbb{C}$. We know that there is global strong solution with small initial data in $H^1(\mathbb{R}^N)$ with an additional hypothesis in low frequencies on φ_0 for any $\alpha_0(N) < \alpha < \frac{4}{N-2}$ for $N \geq 3$ (see [16] chapter 6) with $\alpha_0(N)$ the Strauss exposant:

$$\alpha_0(N) = \frac{2 - N + \sqrt{N^2 + 12N + 4}}{2N}. \tag{1.12}$$

The difficulty in order to obtain the existence of global strong solution with small initial data is related to the smallness of α. Roughly speaking the decay in time is stronger with the size of the exponent α. In particular for $N = 3$ we have $\alpha_0(3) = 1$ so the quadratic nonlinearity corresponds exactly to the critical case of the Strauss exponent. This can be understood as follows : in dimension three the dispersion estimate reads :

$$\|e^{it\Delta}\varphi\|_{L^3(\mathbb{R}^3)} \lesssim |t|^{-\frac{1}{2}}\|\varphi\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \tag{1.13}$$

so that L^3 is mapped back to the dual $L^{\frac{3}{2}}$ by the quadratic nonlinearity, with the critical non integrable decay $\frac{1}{t}$. It explains why the case $N = 3$ is difficult for (GP) since there are quadratic nonlinearities (see [29]).

Before stating the ideas use in [29] in order to overcome this difficulty in the case of the Gross-Pitaevskii equation, let us briefly review the known results on the NLS for comparison in the case of a quadratic nonlinearity when $N = 3$. In this situation it is in particular necessary to take resonances into account. Global existence is known only if the nonlinearity has no resonance in space (we shall in the sequel give more details on the notion of non resonance in space and in time), in particular we mention the work of Hayashi and Naumkin [32], Hayashi Mizumachi and Naumkin [31], Kawahara [35] with the following nonlinearity $\lambda_1 u^2 + \lambda_2 \bar{u}^2$. For a nonlinearity $|u|^2$, the space time resonant set is three-dimensional. In this case almost global existence has been proved by Ginibre and Hayashi [24], but it is not clear at all if global existence is true. These results have been reformulated by using the notion of space-time resonance by Germain, Masmoudi and Shatah in [23].

The situation is even worse for (GP) because of the singular terms $U^{-1}v_2$. To overcome these difficulties Gustafson, Nakanishi and Tsai in [29] introduce a normal form in order to cancel both singular terms in low frequencies and the resonances. The functional settings combine Strichartz spaces and additional time decay via the introduction of weighted spaces (which are related to the pseudoconformal transformation). Indeed as we mention in the previous section the use of the Strichartz estimate is not sufficient in the case of the Strauss exponent, there is not enough time decay and they compensate this obstruction by additional decay due to the weight on initial data. The main difficulty of the proof consists in estimating the weight spaces and to do this they need to split the frequencies in non space and non time resonant regions. The case $N \geq 4$ is simpler as the normal form only need to cancel the singularities in low frequencies.

2 Main results

We start by recalling some important results due to Gustafson et al in [28, 29]. The first one deals with scattering in dimension $N \geq 4$.

Theorem 2.1 (Gustafson, Nakanishi, Tsai [28]). *Suppose that $N \geq 4$ and $|\sigma| \leq \frac{N-3}{2} - \frac{1}{N}$. If $U^\sigma U^{-1} V \varphi_0$ is sufficiently small in $H^{\frac{N}{2}-1}(\mathbb{R}^N)$, then $U^\sigma U^{-1} V \varphi(t)$ remains small in $H^{\frac{N}{2}-1}(\mathbb{R}^N)$ for all $t \in \mathbb{R}$. Moreover, there exists $v_\pm \in U^{-\sigma} H^{\frac{N}{2}-1}(\mathbb{R}^N)$ such that:*

$$\|U^\sigma(e^{itH} U^{-1} V \varphi(t) - v_\pm)\|_{H^{\frac{N}{2}-1}(\mathbb{R}^N)} \longrightarrow_{\pm\infty} 0, \tag{2.14}$$

and the wave operators $v_\pm \to U^{-1} V u(0)$ are local homeomorphisms around 0 in $U^{-\sigma} H^{\frac{N}{2}-1}(\mathbb{R}^N)$.

Below we denote $\langle x \rangle = \sqrt{2 + |x|^2}$ and $\langle x \rangle^{-1} H^1$ is the weighted space with norm $\|\langle x \rangle v\|_{H^1}$ and $\langle \nabla \rangle = \sqrt{2 - \Delta}$. In dimension $N = 3$ for small initial data, Gustafson, Nakanishi, Tsai obtain the following result.

Theorem 2.2 (Gustafson, Nakanishi, Tsai [29]). *There exists $\delta > 0$ such that for any $\varphi_0 \in H^1(\mathbb{R}^3)$ satisfying:*

$$\int_{\mathbb{R}^3} \langle x \rangle^2 \left(|Re(\varphi_0)|^2 + |\nabla \varphi_0|^2\right) < \delta, \tag{2.15}$$

then there exists a unique global strong solution $\psi = 1 + \varphi$ of (1.6) such that $v = V\varphi = Re\varphi + iU Im\varphi$ satisfies $e^{itH} v \in C(\mathbb{R}, H^1/\langle x \rangle)$ and for some $v_+ \in \langle x \rangle^{-1} H^1$

$$\|v(t) - e^{-itH} v_+\|_{H^1} = O_{+\infty}(t^{-1/2}), \ \|\langle x \rangle (v(t) - e^{-itH} v_+)\|_{H^1} \longrightarrow_{+\infty} 0. \tag{2.16}$$

Moreover we have $E_1(\psi) = \|\langle \nabla \rangle v_+\|_{L^2}^2$, and the correspondence $v(0) \to v_+$ defines a bi-Lipschitz map between 0 neighborhoods of $\langle x \rangle^{-1} H^1$.

Remark 1. It may be possible to slightly improve this result in two ways. The first one consist to weaken the regularity hypothesis on the initial data. Indeed we observe that after the use of the normal form we have to solve a Schrödinger equation with quartic nonlinearities. In particular it would be enough to choose initial data $\varphi_0 \in H^{\frac{5}{6}}$ (it corresponds to the critical case of local existence). In particular it would ensure the global existence of strong solution with infinite energy for (GP). Via the Madelung transform these solutions imply solutions of infinite energy for the Euler Korteweg system (1.1).

As it is mentioned by Gustafson, Nakanishi and Tsai in [29] a second way consists in applying only weight in $\langle x \rangle^{\frac{1}{2}+\epsilon}$ with $\epsilon > 0$. Indeed in this situation we have for $\alpha(\epsilon)$ small enough:

$$\|\epsilon^{itH} \varphi_0\|_{L^{3+\alpha(\epsilon)}} \leq \frac{1}{t^{\frac{1}{2}+\beta(\epsilon)}} \|J\varphi_0\|_{L^2}$$

and it should be sufficient to deal with the quadratic terms. Here J corresponds to $Ju = e^{-itH}(\langle x \rangle^{\frac{1}{2}+\epsilon} e^{itH})u$.

Remark 2. Let us mention that all the traveling wave with finite energy have (2.15) finite. In particular the condition on $Re u_0$ is necessary, indeed let us recall that in their work Jones, Putterman and Roberts [34] claim that the traveling wave have the following asymptotic expansion in space (see [26, 27] for rigorous mathematical results):

$$u_c(x_1, x_\perp) \sim 1 + \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})|x_\perp|^2)^{\frac{N}{2}}} + \cdots.$$

In particular ∇u_c is of the form $\frac{1}{|x|^N}$ which is in $\langle x \rangle L^2$.

Remark 3. Let us recall that setting ψ^{κ_1}:

$$\psi^{\kappa_1}(t,x) = \psi(\frac{t}{\kappa_1}, \frac{\sqrt{2}x}{\kappa_1}),$$

where ψ is the solution of theorem 2.2 then if ψ is solution of (1.4), ψ^{κ_1} is solution of the following Gross-Pitaevski equation:

$$i\kappa_1 \partial_t \psi^{\kappa_1} + \frac{\kappa_1^2}{2} \Delta \psi^{\kappa_1} = (|\psi^{\kappa_1}|^2 - 1)\psi^{\kappa_1}. \tag{2.17}$$

In particular the smallness assumption (3) corresponds to:

$$\int_{\mathbb{R}^3} \langle \sqrt{2}x/\kappa_1 \rangle^2 \big(|\mathrm{Re}(\varphi_0^{\kappa_1})|^2 + \frac{\kappa_1^2}{2}|\nabla \varphi_0^{\kappa_1}|^2\big) < \left(\frac{\kappa_1}{\sqrt{2}}\right)^3 \delta,$$

For the critical weight $|x|^{\frac{1}{2}}$ the smallness assumption becomes

$$\int_{\mathbb{R}^3} |x| \big(|\mathrm{Re}(\varphi_0^{\kappa_1})|^2 + \kappa_1^2 |\nabla \varphi_0^{\kappa_1}|^2\big) < \frac{\kappa_1^4}{4}\delta.$$

By using the Madelung transform $\psi^{\kappa_1}(t,x) = \sqrt{\rho}(t,x)e^{i\frac{\varphi(t,x)}{\kappa_1}}$, then $(\rho, \nabla\varphi)$ is solution of system (1.1) with κ_1. When κ_1 decreases, the condition becomes more and more restrictive.

Let us recall a conjecture proposed by Gustafson, Nakanishi and Tsai in [29].

Theorem 2.3. *[**Conjecture**] For any global solution $\psi \in C(\mathbb{R}, 1 + E_1)$ of (GP) satisfying $E(\psi) < \mathcal{E}_0$ there is a unique $z_+ \in H^1(\mathbb{R}^3)$ satisfying $E(\psi) = \|\langle\nabla\rangle z_+\|_{L^2}^2$ and*

$$\|M(\varphi(t)) - e^{-itH}z_+\|_{H^1(\mathbb{R}^3)} \to_{t\to+\infty} 0, \tag{2.18}$$

with $M(\varphi) = v + \langle\nabla\rangle^{-2}|\varphi|^2$. Moreover the map $\varphi(0) \to z_+$ is a homeomorphism between the open balls of radius $\mathcal{E}_0^{\frac{1}{2}}$ around 0 in E_1 and H^1.

Remark 1. This conjecture implies that the global solutions have scattering property when the initial energy is less than $\mathcal{E}_0$.

In this paper we are interested in proving the existence of global strong solutions for the system (1.1) with small initial data. Let us recall that the Euler Korteweg system has been studied by Benzoni, Danchin and Descombes in [5] where they prove for general capillary coefficient the existence of strong solution in finite time for large data when $(\rho_0 - 1, u_0)$ belong to $H^{s+1}(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ with $s > \frac{N}{2} + 1$. To do this they obtain energy inequalities using gauge transforms. The lack of global dispersive estimates does not allow to obtain global solutions. For general capillarities dispersive estimates can be obtained (see [2] for local smoothing properties). On the other hand Antonelli and Marcati in [1] proved the existence of global weak solution for the system (1.1) when $N \geq 2$ for initial data in energy space where the density is assumed to be close from the vacuum (see also [15]).
However uniqueness was left open in dimension $N \geq 2$, moreover no control of the vacuum for the Gross-Pitaevskii equation was proved so far (in the Gross-Pitaevskii community cancellation of ψ is usually called vortex). Our main result concerns the existence of global strong solution with small irrotational initial data for the Euler Korteweg system when $N \geq 3$. For the

solution that we construct $|\psi|$ remains bounded away from 0. Our approach consists in using the scattering in order to obtain $\|\varphi(t,\cdot)\|_{L^\infty)} \leq \frac{C\|\varphi_0\|_X}{t^\alpha}$ with X the set of initial data and $\alpha > 0$. It provides a bound of φ in L^∞ norm in long time. The proof requires the smallness of the initial data φ_0 in X. The second step corresponds to control the L^∞ norm in short time without assuming the smallness of $\|\varphi_0\|_{H^{\frac{N}{2}+\epsilon}}$. To do this we use a type of nonlinear Kato smoothing effect (see [12]). It enables us to get the existence of global weak solution with small initial data φ_0 in H^s with $s > \frac{N}{2} - \frac{1}{6}$ when $N \geq 3$. In order to get uniqueness we need to control the Lipschitz norm of the velocity u (it means $\nabla u \in L^1(L^\infty)$), to do this we assume that φ_0 belongs to H^s with $s > \frac{N}{2}+1$. In this case the regularity is propagated on u and using the Strichartz estimate u is in $L^2_{loc}(B^{\frac{N}{2}+\epsilon}_{\frac{2N}{N-2},2}(\mathbb{R}^N))$.

Let us start with the case $N \geq 4$.

Theorem 2.4. *Let $N \geq 4$, $q_0 = \rho_0 - 1$ and $u_0 = \nabla\theta_0$. We assume $\rho_0 \in L^\infty$ with $\rho_0 \geq c > 0$. Let $\psi_0 = \sqrt{\rho_0}e^{i\theta_0}$ and $U^{-1}V\varphi_0 = U^{-1}V(\psi_0 - 1) \in H^{\frac{N}{2}/2-1/6+\varepsilon}$, $\widehat{\varphi_0} \in L^1$, $\frac{1}{a'} = \frac{1}{2} + \frac{1}{3N}$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if:*

$$\|U^{-1}V\varphi_0\|_{H^{N/2-\frac{1}{6}+\epsilon}\cap B^{\frac{N}{2}-\frac{1}{6}+\epsilon}_{a',2}} + \|\widehat{\varphi_0}\|_{L^1} < \delta,$$

then there exists a global weak solution of the system (1.1) satisfying:

$$\sup_{x,t}|\rho - 1| \leq \frac{1}{2}, \quad \rho \in 1 + L^\infty(H^{\frac{N}{2}-\frac{1}{6}+\epsilon}(\mathbb{R}^N)) \quad and \quad u \in L^\infty(H^{\frac{N}{2}-\frac{4}{3}+2\epsilon}(\mathbb{R}^N)).$$

If in addition $\varphi_0 \in H^{\frac{N}{2}+1+\epsilon}$ then the global solution $(\rho - 1, u)$ belongs to $L^\infty(H^{\frac{N}{2}+1+\epsilon}(\mathbb{R}^N)) \times (L^\infty(H^{\frac{N}{2}+\epsilon}(\mathbb{R}^N)) \cap L^2(B^{\frac{N}{2}+\epsilon}_{\frac{2N}{N-2},2}(\mathbb{R}^N))$ and is unique in this space.

In dimension $N = 3$ the statement is more intricate.

Theorem 2.5. *Let $N = 3$ and $q_0 = \rho_0 - 1$ and $u_0 = \nabla\theta_0$. Furthermore $\rho_0 \in L^\infty$ with $\rho_0 \geq c > 0$. Assume that [2]$\langle x \rangle \nabla\sqrt{\rho_0} \in L^2$, $\langle x \rangle u_0 \in L^2$, $q_0 \in L^2$, $\cos\theta_0 - 1 \in L^2$ and $|x|(\sqrt{\rho_0}\cos\theta_0 - 1) \in L^2$. If $\varphi_0 = \sqrt{\rho_0}e^{i\theta_0} - 1$ is such that $\varphi_0 \in H^{\frac{4}{3}+\epsilon}$ with $\epsilon > 0$ and $\widehat{\varphi_0} \in L^1$. Then there exists $\delta > 0$ depending on $\|\varphi_0\|_{H^{\frac{4}{3}+\epsilon}}$ such that :*

$$\int_{\mathbb{R}^3}\langle x \rangle^2\big(|\nabla\rho_0|^2 + |u_0|^2\big) + \langle x \rangle^2(\sqrt{\rho_0}\cos\theta_0 - 1)^2 dx + \|\widehat{\varphi_0}\|_{L^1} + \|\varphi_0\|_{H^{\frac{4}{3}+\epsilon}} < \delta, \qquad (2.19)$$

then there exists a global weak solution (ρ, u) of the system (1.1) such that:

$$\max(\rho, \frac{1}{\rho}) \in L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^3)), \quad \rho \in 1 + L^\infty_{loc}(H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3)) \quad and \quad u \in L^\infty_{loc}(H^{\frac{1}{6}+2\epsilon}(\mathbb{R}^3)).$$

If $\varphi_0 \in H^{\frac{N}{2}+1+\epsilon}$ then the global solution is unique and the solution verifies the additional regularity:

$$\rho \in 1 + L^\infty_{loc}(H^{\frac{N}{2}+1+\epsilon}(\mathbb{R}^3)) \quad and \quad u \in L^\infty_{loc}(H^{\frac{N}{2}+\epsilon}(\mathbb{R}^3)) \cap L^2_{loc}(B^{\frac{N}{2}+\epsilon}_{6,2}(\mathbb{R}^3)).$$

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[2]These assumptions are the translation of the condition on φ_0 of the theorem 2.2, see (2.15)

Remark 2. This result extends [5] in the specific case $\kappa(\rho) = \frac{\kappa_1}{\rho}$ and small initial data inasmuch as it provides global strong solutions without vacuum thanks to global dispersive estimates. The Strichartz estimates also allow to weaken the assumptions on the initial data since we require one derivative less than in [5] .

Remark 3. Let us mention that the existence of global strong solution remains open in dimension $N = 2$ even for small initial data and seems really difficult. Indeed the scattering approach is very delicate to implement for at least two reasons : the dispersion is weaker in dimension two (quadratic nonlinearities are below the Strauss exponent), and there exists traveling waves with arbitrary low energy.

Remark 4. As we mentioned in the introduction, there exists traveling waves u_c with speed $0 < |c| < \sqrt{2}$. Such solutions do not cancel for a threshold $c_v < c < \sqrt{2}$, from numerical experiments it is expected that the traveling wave of minimal energy u_{c_r} satisfies $c_v < c_r < \sqrt{2}$. This would give an other kind of solution of Euler Korteweg (in the sense that they do not scatter) without vacuum via the Madelung transform. This depends on the regularity of u_c.

Remark 5. The assumption $\widehat{\varphi_0} \in L^1$ is somehow unavoidable since we want the linear part $e^{itH}\varphi_0$ to be bounded in L^∞. Essentially we prove in theorem 6.1 that if φ_0 is only in $L^\infty \cap H^s$ with $s < \frac{N}{2}$ then the solution of (GP) can blow up in L^∞ for arbitrary short time.

Following the same idea than in the previous proofs, we easily get the following results of local existence of strong solution with large initial data.

Corollary 2.0. *Let $N \geq 3$. Assuming that $\rho_0 \geq c > 0$ and $\varphi_0 = (\sqrt{\rho_0}e^{i\theta_0} - 1) \in H^{\frac{N}{2}+1+\epsilon}$ with $u_0 = \nabla\theta_0$ and $\epsilon > 0$ then there exists $T > 0$ and a local strong solution (ρ, u) on $[0, T)$ of the system (1.1) with the following regularity:*

$$(\rho - 1) \in C_T(H^{\frac{N}{2}+1+\epsilon}), \ u \in C_T(H^{\frac{N}{2}+\epsilon}) \cap L_T^2(B_{6,2}^{\frac{N}{2}+\epsilon}).$$

Plan of the paper

In the section 3 we introduce the main technical tools (functional spaces, Strichartz estimates, bilinear product and paraproduct) that are required for the proof of theorem 2.5 and 2.4. In section 4 we detail the proof of [29] and we highlight the main technical issues. We begin by explaining the choice of the normal form, for this choice we are reduced to estimate quadratic, cubic and quartic nonlinearities. A long section is devoted to estimate the worst term $Z\bar{Z}$ which is suitably decomposed in frequencies. This decomposition split the frequency space in non space resonant and non time resonant regions. Let us mention that 0 is space ann time resonant, the choice of the normal form allows to compensate in a subtle way the decay in time s. In section 5 we prove an alternative of the theorem 2.1 of Gustafson et al which is a bit simpler and sufficient for our purpose. In the section 6 we show the existence of global weak and strong solution with small initial data of the Euler Korteweg system (1.1) for $N = 3$. To do this, we prove a Kato smoothing effect and we discuss the optimality of our initial data in order to control the vacuum. In particular we prove the existence of initial data arbitrarily small in H^s with $s \leq \frac{N}{2}$ which blow up in L^∞ norm for arbitrary time. The uniqueness is also proved by using Strichartz estimate. In the section 7 we deal with the simpler case $N \geq 4$.

3 Main Tools

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X, we denote by $C([0,T], X)$ the set of continuous functions on $[0, T]$ with values in X. For $p \in [1, +\infty]$, $L^p(0, T, X)$ or $L_T^p(X)$ is for the set of measurable functions on $(0, T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0, T)$.

In this section we recall some notation, definitions and technical tools. We denote the Lebesgue, the Lorentz, the Sobolev and the Besov spaces as L^p, $L^{p,q}$, $H^{s,p}$ and $B_{p,q}^s$ respectively for $1 \leq p, q \leq +\infty$ and $s \in \mathbb{R}$. We denote the Fourier transform on $\mathbb{R}^N$ by:

$$
\begin{aligned}
\mathcal{F}\varphi(\xi) &= \int_{\mathbb{R}^N} \varphi(x) e^{-ix\xi} dx, \\
\mathcal{F}_x[f(x,y)](\xi) = \mathcal{F}_x f(\xi, y) &= \int_{\mathbb{R}^N} f(x,y) e^{-ix\xi} dx,
\end{aligned} \tag{3.20}
$$

and the Fourier multiplier of any function φ:

$$
\begin{aligned}
\varphi(-i\nabla) f &= \mathcal{F}^{-1}[\varphi(\xi)\mathcal{F}f(\xi)], \\
\varphi(-i\nabla)_x f(x,y) &= \mathcal{F}_x^{-1}[\varphi(\xi)\mathcal{F}_x f(\xi, y)].
\end{aligned} \tag{3.21}
$$

Notations

We are going to follow some notations of [30]. For any number or vector a we denote:

$$
\langle a \rangle = \sqrt{2 + |a|^2}, \ \hat{a} = \frac{a}{|a|}, \ U(a) = \frac{|a|}{\langle a \rangle}, \ H(a) = |a|\langle a \rangle, \tag{3.22}
$$

For any complex-valued function f, we often denote the complex conjugate by:

$$
f^+ = f, \ f^- = \bar{f}. \tag{3.23}
$$

This non standard notation will prove useful in section 4.

3.1 Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\varphi \in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, χ is supported in the ball $\{\xi \in \mathbb{R}^N / |\xi| \leq \frac{4}{3}\}$ such that:

$$
\forall \xi \in \mathbb{R}^N, \ \chi(\xi) = \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1 \ \text{if} \ \xi \neq 0.
$$

Denoting $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$, we define the dyadic blocks by:

$$
\begin{aligned}
&\Delta_l u = 0 \text{ if } l \leq -2, \\
&\Delta_{-1} u = \chi(D)u = \tilde{h} * u, \\
&\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y)u(x-y)dy \ \text{if } l \geq 0, \\
&S_l u = \sum_{k \leq l-1} \Delta_k u.
\end{aligned}
$$

One can write: $u = \sum_{k \in \mathbb{Z}} \Delta_k u$ for all temperate distribution. This decomposition is called nonhomogeneous Littlewood-Paley decomposition. The homogeneous dyadic blocks are defined by:

$$\dot{\Delta}_l u = \varphi(2^{-l}D)u \text{ for all } l \in \mathbb{Z}.$$

The above definition deserves two important preliminary remarks:

- For $u \in \mathcal{S}'$, we have $\sum_{l \in \mathbb{Z}} \dot{\Delta}_l u = u$ modulo a polynomial only.

- In contrast with the non homogeneous case we do not have $S_q u = \sum_{p \leq q-1} \dot{\Delta}_p u$.

Definition 3.1. Let $1 \leq p, r \leq +\infty$ and $s \in \mathbb{R}$. For $u \in \mathcal{S}'(\mathbb{R}^N)$, we set:

$$\|u\|_{B_{p,q}^s} = \left(\sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q\right)^{\frac{1}{q}}.$$

The Besov space $B_{p,q}^s$ is the set of temperate distribution u such that $\|u\|_{B_{p,q}^s} < +\infty$.

Proposition 3.1. *The following properties holds:*

1. $B_{p,r_1}^{s'} \hookrightarrow B_{p,r}^s$ *if* $s' > s$ *or if* $s = s'$ *and* $r_1 \leq r$.

2. $B_{p_1,r}^s \hookrightarrow B_{p_2,r}^{s-N(1/p_1-1/p_2)}$ *for* $p_2 \geq p_1$.

3. *Real interpolation: if* $u \in B_{p,\infty}^s \cap B_{p,\infty}^{s'}$ *and* $s < s'$ *then* u *belongs to* $B_{p,1}^{\theta s + (1-\theta)s'}$ *for all* $\theta \in (0,1$ *and there exists a universal constant* C *such that:*

$$\|u\|_{B_{p,1}^{\theta s + (1-\theta)s'}} \leq \frac{C}{\theta(1-\theta)(s'-s)} \|u\|_{B_{p,\infty}^s}^\theta \|u\|_{B_{p,\infty}^{s'}}^{1-\theta}.$$

Lemma 3.2. *Let* $s \in \mathbb{R}$ *and* $1 \leq p, r \leq +\infty$. *Let* $(u_q)_{q \geq -1}$ *be a sequence of functions such that:*

$$\|u\|_{B_{p,q}^s} = \left(\sum_{q \geq -1} (2^{qs} \|u_q\|_{L^p})^r\right)^{\frac{1}{r}}$$

- *If* $\operatorname{supp}\widehat{u_{-1}} \subset B(0, R_2)$ *and* $\operatorname{supp}\widehat{u_q} \subset C(0,, 2^q R_1, 2^q R_2)$ *for some* $0 < R_1 < R_2$ *then* $u = \sum_{q \geq -1} u_q$ *belongs to* $B_{p,r}^s$ *and there exists a universal constant* C *such that:*

$$\|u\|_{B_{p,r}^s} \leq C\left(\sum_{q \geq -1} (2^{qs} \|u_q\|_{L^p})^r\right)^{\frac{1}{r}}.$$

- *If* s *is positive and* $\operatorname{supp}\widehat{u_q} \subset B(0, 2^q R)$ *for some* $R > 0$ *then* $u = \sum_{q \geq -1} u_q$ *belongs to* $B_{p,r}^s$ *and there exists a universal constant* C *such that:*

$$\|u\|_{B_{p,r}^s} \leq C\left(\sum_{q \geq -1} (2^{qs} \|u_q\|_{L^p})^r\right)^{\frac{1}{r}}.$$

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [14, 3]). Indeed for u and v two temperate distributions we have the following formal decomposition $uv = \sum_{p,q} \Delta_p u \Delta_q v$ and in particular the following Bony decomposition:

$$uv = T_u v + T_v u + R(u,v),$$

11

with:

$$T_u v = \sum_{p \leq q-2} \Delta_p u \Delta_q v = \sum_q S_{q-1} u \Delta_q v,$$

$$R(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v \quad \text{with} \quad \widetilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

Proposition 3.2. *Let $p_1, p_2, r \in [1, +\infty]$, $(s_1, s_2) \in \mathbb{R}^2$ and $p \in [1, +\infty]$ then we have the following estimates:*

- *If $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$ and $s_1 + s_2 + N \inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$ then:*

$$\|R(u,v)\|_{B_{p,r}^{s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2}}} \lesssim \|u\|_{B_{p_1,r}^{s_1}} \|v\|_{B_{p_2,r}^{s_2}}. \tag{3.24}$$

- *If $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $s_1 + s_2 = 0$ then:*

$$\|R(u,v)\|_{B_{p,\infty}^{\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2}}} \lesssim \|u\|_{B_{p_1,1}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}. \tag{3.25}$$

- *If $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda} \leq 1$ with $\lambda \in [1, +\infty]$ and $p_1 \leq \lambda$ then:*

$$\|T_u v\|_{B_{p,r}^{s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2}}} \lesssim \|v\|_{B_{p_2,r}^{s_2}} \begin{cases} \|u\|_{B_{p_1,\infty}^{s_1}} & \text{if } s_1 + \dfrac{N}{\lambda} < \dfrac{N}{p_1}, \\[2mm] \|u\|_{B_{p_1,1}^{s_1}} & \text{if } s_1 + \dfrac{N}{\lambda} = \dfrac{N}{p_1}. \end{cases} \tag{3.26}$$

- *If $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda} \leq 1$ with $\lambda \in [1, +\infty]$ and $p_1 \leq \lambda$ then:*

$$\|T_u v\|_{B_{p,r}^{s_1+s_2+\frac{N}{p}-\frac{N}{p_1}-\frac{N}{p_2}}} \lesssim \|v\|_{B_{p_2,\infty}^{s_2}} \begin{cases} \|u\|_{B_{p_1,r}^{s_1}} & \text{if } s_1 + \dfrac{N}{\lambda} < \dfrac{N}{p_1}, \end{cases} \tag{3.27}$$

The study of non stationary PDE's requires space of type $L^\rho(0, T, X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0, T, X)$ (except if $r = p$). We are now going to define the spaces of Chemin-Lerner (see [18]) in which we will work, which are a refinement of the spaces $L_T^\rho(B_{p,r}^s)$.

Definition 3.3. *Let $\rho \in [1, +\infty]$, $T \in [1, +\infty]$ and $s_1 \in \mathbb{R}$. We set:*

$$\|u\|_{\widetilde{L}_T^\rho(B_{p,r}^{s_1})} = \Big(\sum_{l \in \mathbb{Z}} 2^{lrs_1} \|\Delta_l u(t)\|_{L_T^\rho(L^p)}^r \Big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}_T^\rho(B_{p,r}^{s_1})$ as the set of temperate distribution u over $(0, T) \times \mathbb{R}^N$ such that $\|u\|_{\widetilde{L}_T^\rho(B_{p,r}^{s_1})} < +\infty$.

We set $\widetilde{C}_T(\widetilde{B}_{p,r}^{s_1}) = \widetilde{L}_T^\infty(\widetilde{B}_{p,r}^{s_1}) \cap C([0, T], B_{p,r}^{s_1})$. Let us emphasize that, according to Minkowski's inequality, we have:

$$\|u\|_{\widetilde{L}_T^\rho(B_{p,r}^{s_1})} \leq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \geq \rho, \quad \|u\|_{\widetilde{L}_T^\rho(B_{p,r}^{s_1})} \geq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \leq \rho. \tag{3.28}$$

Remark 4. It is easy to generalize propositions 3.2 to $\widetilde{L}_T^\rho(B_{p,r}^{s_1})$ spaces. The indices s_1, p, r behave just as in the stationary case whereas the time exponent ρ behaves according to Hölder inequality.

In the sequel we will need of composition lemma in $\widetilde{L}_T^\rho(B_{p,r}^s)$ spaces (we refer to [3] for a proof).

Proposition 3.3. *Let* $s > 0$, $(p,r) \in [1,+\infty]$ *and* $u \in \widetilde{L}_T^\rho(B_{p,r}^s) \cap L_T^\infty(L^\infty)$.

1. *Let* $F \in W_{loc}^{[s]+2,\infty}(\mathbb{R}^N)$ *such that* $F(0) = 0$. *Then* $F(u) \in \widetilde{L}_T^\rho(B_{p,r}^s)$. *More precisely there exists a function* C *depending only on* s, p, r, N *and* F *such that:*

$$\|F(u)\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \leq C(\|u\|_{L_T^\infty(L^\infty)})\|u\|_{\widetilde{L}_T^\rho(B_{p,r}^s)}.$$

2. *Let* $F \in W_{loc}^{[s]+3,\infty}(\mathbb{R}^N)$ *such that* $F(0) = 0$. *Then* $F(u) - F'(0)u \in \widetilde{L}_T^\rho(B_{p,r}^s)$. *More precisely there exists a function* C *depending only on* s, p, r, N *and* F *such that:*

$$\|F(u) - F'(0)u\|_{\widetilde{L}_T^\rho(B_{p,r}^s)} \leq C(\|u\|_{L_T^\infty(L^\infty)})\|u\|_{\widetilde{L}_T^\rho(B_{p,r}^s)}^2.$$

Let us recall the useful lemma (see [3]).

Lemma 3.4. *Let* $R_l = [v, \Delta_l] \cdot \nabla f$, *let* $\sigma \in \mathbb{R}$. *Assume that* $\sigma > -N\min(\frac{1}{2}, \frac{1}{p})$. *There exists a constant* C *such that the following inequality is true: for all* s *such that* $-N\min(\frac{1}{p}, \frac{1}{2}) < s < \frac{N}{p} + 1$, *the following inequality holds true for some constant* $C > 0$

$$\|2^{l\sigma}\|R_l\|_{L^2}\|_{l^2} \leq C\|\nabla v\|_{B_{p,\infty}^{\frac{N}{p}} \cap L^\infty}\|f\|_{H^\sigma}. \tag{3.29}$$

3.2 Multilinear Fourier multiplier

For any function $B(\xi_1, \cdots, \xi_d)$ on $(\mathbb{R}^N)^d$, we associate the d-multilinear operator $B[f_1, \cdots, f_d]$ defined by:

$$\mathcal{F}_x B[f_1, \cdots, f_d] = \int_{\xi = \xi_1 + \cdots + \xi_d} B(\xi_1, \cdots, \xi_d)\mathcal{F}f_1(\xi_1) \cdots \mathcal{F}f_d(\xi_d)d\xi_2 \cdots d\xi_d, \tag{3.30}$$

which is called a **multilinear Fourier multiplier with symbol** B, thus we identify the symbol to the operator.

Remark 5. Whenever we write a symbol in the variables $(\xi, \xi_1, \cdots, \xi_d)$, it should be understood as a function of $(\xi_1, \cdots, \xi_d)$ by substuting $\xi = \xi_1 + \cdots + \xi_d$.

Remark 6. For any bilinear symbol $B(\xi_1, \xi_2)$ we assign the variable η and ζ such that $\eta = \xi_1$ and $\zeta = \xi_2$, but regarding (ξ, η) and (ξ, ζ) respectively as the independent variables. Hence the partial derivatives of the symbol B in each coordinates are given by:

$$\begin{aligned}
(\nabla_\xi^{(\eta)} B, \nabla_\eta B) &= (\nabla_{\xi_2} B(\eta, \xi - \eta), (\nabla_{\xi_1} - \nabla_{\xi_2})B(\eta, \xi - \eta)), \\
(\nabla_\xi^{(\zeta)} B, \nabla_\zeta B) &= (\nabla_{\xi_1} B(\xi - \zeta, \zeta), (\nabla_{\xi_2} - \nabla_{\xi_1})B(\xi - \zeta, \zeta)).
\end{aligned} \tag{3.31}$$

The fundamental theorem of Coifman and Meyer (see [20]) states that in the case of bilinear Fourier multiplier, these operators have the same boundedness properties as the ones given by Hölder's inequality for the standard product.

Theorem 3.5 (Coifman-Meyer)**.** *Suppose that B satisfies:*

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta B(\xi_1, \xi_2)| \lesssim \frac{1}{(|\xi_1| + |\xi_2|)^{|\alpha| + |\beta|}}, \tag{3.32}$$

for sufficiently many multi-indices (α, β). Then the operator:

$$B[\cdot, \cdot] : L^p \times L^q \to L^r,$$

is bounded for:

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \ 1 < p, q < +\infty \ \ and \ 1 \le r < +\infty. \tag{3.33}$$

Remark 7. For condition (3.32) to hold, it suffices for B to be homogeneous of degree 0 and of class C^∞ on the sphere.

Remark 8. As was shown in [25], one cannot generally replace the right hand side of (3.32) by $|\xi_1|^{-|\alpha|}|\xi_2|^{-|\beta|}$, but we can reduce one regular multipliers to the above case. Indeed due to divisors coming from the phase integrations in the main estimates of theorem 2.2, we shall as in [30] use another bilinear estimates (see [30]).

Proposition 3.4. *Let $k \in \mathbb{N}$ then we have :*

$$\sup_{0 \le a \le 1} \|\frac{\langle \xi_1 \rangle^{2k(1-a)} \langle \xi_2 \rangle^{2ka}}{\langle (\xi_1, \xi_2) \rangle^{2k}}[f, g]\|_{L^{p_0}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}, \tag{3.34}$$

for any $p_0, p, q \in (1, +\infty)$ satisfying $\frac{1}{p_0} = \frac{1}{p} + \frac{1}{q}$.

3.3 Strichartz and dispersive estimates

Lemma 3.6. *Let $2 \le p \le +\infty$, $0 \le \theta \le 1$, $s \in \mathbb{R}$, and $\sigma = \frac{1}{2} - \frac{1}{p}$. Then we have:*

$$\|e^{-itH}v\|_{B^s_{p,2}} \lesssim |t|^{-(N-\theta)\sigma} \|U^{(N-2+3\theta)\sigma} \langle \nabla \rangle^{2\theta\sigma} v\|_{B^s_{p',2}}, \tag{3.35}$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate. For $2 \le p < +\infty$, we have also:

$$\|e^{-itH}v\|_{L^{p,2}} \lesssim |t|^{-(N-\theta)\sigma} \|U^{(N-2+3\theta)\sigma} \langle \nabla \rangle^{2\theta\sigma} v\|_{L^{p',2}}. \tag{3.36}$$

Let us recall the Strichartz estimate for the operator H, we recall here a proposition due to Gustafson et al in [28, 30].

Proposition 3.5. *For $j = 1, 2$, let $2 \le p_j, q_j \le +\infty$, $\frac{2}{q_j} + \frac{N}{p_j} = \frac{N}{2}$ and $s_j = \frac{N-2}{2}(\frac{1}{2} - \frac{1}{p_j})$ but $(q_j, p_j) \ne (2, +\infty)$. Then we have:*

$$\|e^{-iHt}\Delta_j \varphi\|_{L^{q_1}(L^{p_1})} \le \|U^{s_1} \Delta_j \varphi\|_{L^2},$$

$$\|e^{-iHt}\varphi\|_{L^{q_1}(B^s_{p_1,2})} \lesssim \|U^{s_1} \varphi\|_{B^s_{2,2}},$$

$$\|\int_0^t e^{-i(t-s)H} \Delta_j f\|_{L^{q_1}(L^{p_1})} \le \|U^{s_1+s_2} \Delta_j f\|_{L^{q'_2}(L^{p'_2})},$$

$$\|\int_0^t e^{-i(t-s)H} f\|_{L^{q_1}(B^s_{p_1,2})} \lesssim \|U^{s_1+s_2} f\|_{L^{q'_2}(B^s_{p'_2,2})},$$

Remark 9. These Strichartz estimates are very close from the classical one for Schrödinger equations except in low frequencies.

We recall here the space of Chemin and Lerner:

$$\|f(\xi,\eta)\|_{\widetilde{L}^p_\xi(\dot{B}^s_{q,r,\eta})} = \Big(\sum_{l\in\mathbb{Z}} 2^{lsr}\|f\|^r_{L^p_\xi(L^q_\eta)}\Big)^{\frac{1}{r}}.$$

Let us recall a crucial lemma due to Gustafson et al in [30].

Lemma 3.7. Let $0 \le s \le \frac{N}{2}$, (p,q) any dual Strichartz exponent except for the endpoint it means:

$$\frac{2}{p} + \frac{N}{q} = 2 + \frac{N}{2} \tag{3.37}$$

Let $(p_1 q_1)$ and (p_2,q_2) satisfy:

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \ \frac{1}{q} + \frac{1}{q(s)} = \frac{1}{q_1} + \frac{1}{q_2}, \ \frac{1}{q(s)} = \frac{1}{2} - \frac{s}{N},$$
$$p \le p_1, p_2 \le +\infty, \ q \le q_1, q_2 \le +\infty$$

Then for any bilinear Fourier multiplier we have:

$$\Big\|\int e^{itH}B[u,v]dt\Big\|_{L^2} \lesssim \|B\|_{\widetilde{L}^\infty_\xi \dot{B}^s_{2,1,\eta}+\widetilde{L}^\infty_\xi \dot{B}^s_{2,1,\zeta}}\|u\|_{L^{p_1}L^{q_1}}\|v\|_{L^{p_2}L^{q_2}}.$$

with the first norm of B is in the (ξ,η) coordinates and the second in the $(\xi,\zeta) = (\xi,\xi-\eta)$. And we have if $1/q_1 + 1/q_2 = 1/2 + 1/q(s)$, $2 \le q_1, q_2 \le q(s)$, $\frac{1}{q(s)} = \frac{1}{2} - \frac{s}{N}$, we have for any bilinear Fourier multiplier:

$$\|B[\varphi,\psi]\|_{L^2} \lesssim \|B\|_{\widetilde{L}^\infty_\xi B^s_{2,1,\eta}+\widetilde{L}^\infty_\xi B^s_{2,1,\zeta}}\|\varphi\|_{L^{q_1}}\|\psi\|_{L^{q_2}}.$$

Notation. We set:

$$[B^s] = \widetilde{L}^\infty_\xi B^s_{2,1,\eta} + \widetilde{L}^\infty_\xi B^s_{2,1,\zeta}, \tag{3.38}$$

and:

$$[H^s] = \widetilde{L}^\infty_\xi \dot{H}^s_{2,\eta} + \widetilde{L}^\infty_\xi \dot{H}^s_{2,\zeta}. \tag{3.39}$$

Remark 10. When $s = \frac{N}{2}$ lemma 3.7 is similar to a classical Strichartz estimates where $B[u,v]$ behaves like a classical product $u\,v$. Indeed in this situation we have by Hölder's inequality $uv \in L^p L^q$ and $(p,q) = (p_1', q_1')$ with (p_1,q_1) a Strichartz pair since $\frac{2}{p} + \frac{N}{q} = 2 + \frac{N}{2}$. In high frequencies we shall work in the particular case $s = \frac{N}{2}$ since in high frequencies the classical Strichartz estimates are sufficient.
Let us observe that $q(s)$ is the Sobolev exponent for the embedding $\dot{B}^s_{2,1} \subset L^{q(s)}$ and $\frac{1}{q(s)}$ gives the precise loss compared with Holder inequality. In particular it implies that when $0 \le s < \frac{N}{2}$ we need a better long time decay to use this result compared with the classical Strichartz estimate (indeed we need that p_1 or p_2 is less than the classical case for Strichartz estimate). We will see this more in details in the sequel when we will be interested in dealing with the low frequencies.

Remark 11. In the sequel we will control the $\widetilde{L}^\infty(H^s)$ norm by the $L^\infty(H^{s+\epsilon})$ norm with $\epsilon > 0$.

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4 Proof of the theorem 2.2 when $N = 3$

In this section we are going to recall the main steps of the proof of the theorem 2.2 of Gustafson, Nakanishi and Tsai in [30]. We will start by explaining the normal form that they use.

4.1 Normal form

We consider the Gross Pitaevskii equation that we can write under the following form:

$$i\partial_t\varphi + \Delta\varphi - 2\mathrm{Re}\varphi = (3\varphi_1^2 + \varphi_2^2 + |\varphi|^2\varphi_1) + i(2\varphi_1\varphi_2 + |\varphi|^2\varphi_2). \qquad (4.40)$$

Following Gustafson et al in [30] we can diagonalize the previous equation in setting $v = \varphi_1 + iU\varphi_2$:

$$i\partial_t v - Hv = U(3\varphi_1^2 + \varphi_2^2 + |\varphi|^2\varphi_1) + i(2\varphi_1\varphi_2 + |\varphi|^2\varphi_2). \qquad (4.41)$$

The term $2\varphi_1\varphi_2$ is delicate to deal with since it is quadratic and (to the opposite of the real part) there is no low frequency regularization due to U. A classical method to overcome this difficulty is to transform the system (4.41) by applying a normal form (Shatah in [40] was the first to use this type of idea in the PDE framework for the Klein Gordon equation. A normal form allows to increase the order of the nonlinearity which is essential regarding the Strauss exponant). We also underline that since $\varphi_2 = U^{-1}v_2$, we have loss of derivative at low frequencies, this is particularly bad for the term $U(\varphi_2^2)$. This must be taken into account for the choice of the normal form.

Let us introduce the normal form:

$$w = \varphi + B_1'[\varphi_1, \varphi_1] + B_2'[\varphi_2, \varphi_2],$$

where B_j' are real valued symmetric bilinear Fourier multipliers.
Some tedious computations give:

$$(i\partial_t w + \Delta - 2Re)w = B_3'[\varphi_1, \varphi_1] + B_4'[\varphi_2, \varphi_2] + iB_5'[\varphi_1, \varphi_2] + |\varphi|^2\varphi_1 + iC_3'[\varphi_1, \varphi_1, \varphi_2]$$
$$+ iC_4'[\varphi_2, \varphi_2, \varphi_2] + iQ_1(\varphi), \qquad (4.42)$$

where B_j' ($j = 3, 4, 5$) are bilinear multipliers, C_j' cubic multipliers and Q_1 a quartic multiplier defined as follows:

$$\begin{aligned}
&B_3' = 3 - \langle\xi\rangle^2 B_1', \ B_4' = 1 - \langle\xi\rangle^2 B_2', \ B_5' = 2 + 2|\xi_2|^2 B_1' - 2\langle\xi_1\rangle^2 B_2', \\
&C_3'(\xi_1, \xi_2, \xi_3) = 1 + 4B_1'(\xi_1, \xi_2 + \xi_3) - 6B_2'(\xi_1 + \xi_2, \xi_3), \\
&C_4'(\xi_1, \xi_2, \xi_3) = 1 - 2B_2'(\xi_1 + \xi_2, \xi_3), \\
&Q_1(\varphi) = 2B_1'[\varphi_1, |\varphi|^2\varphi_2] - 2B_2'[\varphi_2, |\varphi|^2\varphi_1].
\end{aligned} \qquad (4.43)$$

Next by applying the diagonalization on (4.42), we have for:

$$Z = V(w) = w_1 + iUw_2 = v + B_1'[\varphi_1, \varphi_1] + B_2'[\varphi_2, \varphi_2],$$

the following system:

$$i\partial_t Z - HZ = U(B_3'[\varphi_1, \varphi_1] + B_4'[\varphi_2, \varphi_2] + |\varphi|^2\varphi_1) + i(B_5'[\varphi_1, \varphi_2] + C_3'[\varphi_1, \varphi_1, \varphi_2]$$
$$+ C_4'[\varphi_2, \varphi_2, \varphi_2] + Q_1(\varphi)), \qquad (4.44)$$

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The worst term is $B'_5[\varphi_1, \varphi_2]$ since it is quadratic and there is no low frequency smoothing by U as it is the case for B'_3 and B'_4. The U factor will be a key point in order to deal with the region where the phase is space and time resonant (see section ??). Following these considerations Gustafson, Nakanishi and Tsai chose $B'_5 = 0$ by setting:

$$- B'_1 = B'_2 = \langle (\xi_1, \xi_2) \rangle^{-2} = \frac{1}{2 + |\xi_1|^2 + |\xi_2|^2}, \tag{4.45}$$

this gives:

$$B'_4 = \frac{-2\xi_1\xi_2}{2 + |\xi_1|^2 + |\xi_2|^2}, \ B'_5 = 0, \ C'_4 = \frac{|\xi_1 + \xi_2|^2 + |\xi_3|^2}{2 + |\xi_1 + \xi_2|^2 + |\xi_3|^2}. \tag{4.46}$$

We rewrite the equation as follows:

$$i\partial_t Z - HZ = \mathcal{N}_Z(v), \tag{4.47}$$

with:

$$\mathcal{N}_Z(v) = B_3[v_1, v_1] + B_4[v_2, v_2] + C_1[v_1, v_1, v_1] \quad + \quad C_2[v_2, v_2, v_1] + iC_3[v_1, v_1, v_2]$$
$$+ \quad iC_4[v_2, v_2, v_2] + iQ_1(u),$$

where:

$$B_3 = -U(\xi) B'_3 = 2U(\xi)\widetilde{B}_3 = 2U(\xi)\frac{4 + 4|\xi_1|^2 + 4|\xi_2|^2 - \xi_1\xi_2}{2 + |\xi_1|^2 + |\xi_2|^2},$$

$$B_4 = U(\xi) U(\xi_1)^{-1} U(\xi_2)^{-1} B'_4 = -2U(\xi)\widetilde{B}_4 = -2U(\xi)\frac{\langle \xi_1 \rangle \langle \xi_2 \rangle \widehat{\xi_1}\widehat{\xi_2}}{2 + |\xi_1|^2 + |\xi_2|^2},$$

$$C_1 = U(\xi), \ ; C_2 = U(\xi) U(\xi_1)^{-1} U(\xi_2)^{-1},$$
$$C_3 = U(\xi_3)^{-1} C'_3, \ ; C_4 = U(\xi_1)^{-1} U(\xi_2)^{-1} U(\xi_3)^{-1} C'_4,$$
$$Q_1(u) = -2\langle (\xi_1, \xi_2) \rangle^{-2} [u_1, |u|^2 u_2] - 2\langle (\xi_1, \xi_2) \rangle^{-2} [u_2, |u|^2 u_1]. \tag{4.48}$$

The following bounds are crucial:

$$|B_3| + |B_4| \lesssim U(\xi),$$
$$|C_j| \lesssim U(\xi_1)^{-1} U(\xi_2)^{-1} + U(\xi_2)^{-1} U(\xi_3)^{-1} + U(\xi_3)^{-1} U(\xi_1)^{-1}. \tag{4.49}$$

Roughly speaking B_3 and B_4 have the same smoothing effect than U, it will be essential in order to deal with the lack of time decay of the quadratic terms (in section ?? we will see that the region $\xi = 0$ is both time and space resonant and the term $U(\xi)$ will compensate these bad effects).

4.2 Fixed point and Functional Space

Our aim is to solve the equation (4.47) using the Duhamel formula this is equivalent to obtain a fixed point to:

$$Z \mapsto e^{itH} Z(0) + \int_0^t e^{i(t-s)H} \mathcal{N}_Z(v)(s) ds. \tag{4.50}$$

Let us define the functional space in which we are going to work. As we explain in the introduction the Strichartz estimate are not sufficient since we are dealing with the critical case of the Strauss exponent. It is then natural to choose a space such that we have additional

dispersion in longtime (or better decay estimate in time). Following Gustafson, Nakanishi and Tsai in [29] we define an equivalent of the pseudoconformal transform for Schrödinger equation by setting :

$$J(t) = e^{-itH} x e^{itH}$$

As in [29] We now take a functional space which combins Strichartz estimate and control of the pseudoconformal transform J, namely we define $X(t)$ by

$$\|Z\|_{X(t)} = \|Z\|_{H^1} + \|J(t)Z\|_{H^1}, \ \|Z\|_X = \sup_t \|Z(t)\|_{X(t)}.$$

and the "dispersive" functional space S with:

$$\|Z\|_S = \|Z\|_{L^\infty_t H^1} + \|U^{-1/6} Z\|_{L^2_t H^{1,6}} < +\infty.$$

The main difficulty will be to prove the stability of the solution in $X \cap S$ and more particularly the stability of $X(t)$, this will be done by using the concept of non space-time resonance. In the remaining of section 4 we prove that the map (4.50) is stable and contractive in the following space for α small enough:

$$E(\alpha) = \{Z \in X \cap S(0, +\infty) \text{ with } \|Z\|_{X \cap S(0,+\infty)} \le \alpha\}. \tag{4.51}$$

Actually in [29] the authors obtain even the following time decay :

$$\|Z - e^{-i(t-T_0)H} Z(T_0)\|_{X \cap S(T_0,+\infty)} \lesssim \langle T_0 \rangle^{-\varepsilon} \|Z\|^2_{X \cap S(T_0,\infty)} \big(1 + \|Z\|_{X \cap S(T_0,\infty)}\big).$$

4.3 Long time dispersion and $\|\cdot\|_X$

The point of working with the space X is to give a stronger time decay compared with Strichartz estimates in long time, in particular it shall be enough to conserve the dispersive regularity for the small nonlinearities. More precisely we have the following proposition (see [29]), it is essentially obtained by using a combination of the dispersive estimate for (GP) (see lemma 3.6) and standard analysis.

Proposition 4.6. *We have the following estimates with $0 \le \theta \le 1$:*

$$\|v(t)\|_{\dot{H}^{-1}} \lesssim \|v(t)\|_{X(t)}, \tag{4.52}$$

$$\|U^{-2}v\|_{L^6} \lesssim \|v(t)\|_{X(t)} \lesssim \|v(t)\|_{X(t)}, \tag{4.53}$$

$$\||\nabla|^{-2+\frac{5\theta}{3}} v_{<1}(t)\|_{L^6} \lesssim \min(1, t^{-\theta}) \|v(t)\|_{X(t)},$$
$$\||\nabla|^{\theta} v_{\ge 1}(t)\|_{L^6} \lesssim \min(t^{-\theta}, t^{-1}) \|v(t)\|_{X(t)}. \tag{4.54}$$

We have the following Strichartz estimate on $U^{-1}v$:

$$\|U^{-1}v(t)\|_{L^6} \lesssim \langle t \rangle^{-\frac{3}{5}} \|v(t)\|_{X(t)},$$
$$\|U^{-1}v\|_{L^2_t(H^{1,6})} \lesssim \|v\|_{X(t) \cap S(t)}, \tag{4.55}$$

$$\|\langle \nabla \rangle^{\frac{2}{3}} U^{-1} v(t)\|_{L^4} \lesssim t^{-\frac{7}{12}} \|v(t)\|_{X(t)}. \tag{4.56}$$

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We have also Strichartz estimates for φ in terms of the norm of v in $X \cap S$.

Proposition 4.7. *We have the following estimates:*

$$\begin{aligned}
\|\varphi\|_{L^\infty(H^1)} &\lesssim \|v\|_X, \\
\|\varphi\|_{L^2(H^{1,6})} &\lesssim \|v\|_{X \cap S}.
\end{aligned} \tag{4.57}$$

Proof: Let us recall that $[\nabla_j, J_k] = \delta_{j,k}$ and $[\langle\nabla\rangle, J] = -\langle\nabla\rangle^{-1}\nabla$ imply that:

$$\begin{aligned}
\|J\nabla v(t)\|_{L^2} &\leq \|[\nabla, J]v(t)\|_{L^2} + \|\nabla Jv(t)\|_{L^2} \leq \|v(t)\|_{L^2} + \|Jv(t)\|_{H^1}, \\
\|J\langle\nabla\rangle v(t)\|_{L^2} &\leq \|v(t)\|_{L^2} + \|Jv(t)\|_{H^1}.
\end{aligned}$$

Control on $\|U^{-1}v\|_{L^2}$ or $\|v\|_{H^{-1}}$

By Sobolev embedding and Hölder inequality for Lorentz space we have since $\frac{1}{|x|} \in L^{3,\infty}$ ($\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$):

$$\|v(t)\|_{\dot{H}^{-1}} = \|e^{itH}v(t)\|_{\dot{H}^{-1}} \leq \|e^{itH}v(t)\|_{L^{\frac{6}{5}}} \lesssim \|xe^{itH}v(t)\|_{L^2} = \|Jv(t)\|_{L^2} \lesssim \|v(t)\|_{X(t)}. \tag{4.58}$$

In particular it implies a gain of regularity on v in low frequencies.

Control on $\|U^{-2}v\|_{L^6}$

$$\|U^{-2}v\|_{L^6} \lesssim \|v\|_{H^{-1}} + \|v\|_{H^1} \lesssim \|U^{-1}v\|_{H^1} \lesssim \|v(t)\|_{X(t)}. \tag{4.59}$$

Gain of time decay on v via X

Using the lemma 3.6 with $p = 6$ and $\theta = 0$ ($\sigma = \frac{1}{3}$), Hölder's inequality in Lorentz space , the fact that $J\langle\nabla\rangle = \langle\nabla\rangle J + \langle\nabla\rangle^{-1}\nabla$ and $L^p = L^{p,p}$ for $1 < p < +\infty$ we obtain:

$$\begin{aligned}
\|\langle\nabla\rangle U^{-\frac{1}{3}} e^{-itH} e^{itH}v(t)\|_{L^6} &\lesssim \|e^{-itH} e^{itH} \langle\nabla\rangle U^{-\frac{1}{3}}v(t)\|_{L^6}, \\
&\lesssim t^{-1}\|\langle\nabla\rangle e^{itH}v(t)\|_{L^{\frac{6}{5},2}}, \\
&\lesssim t^{-1}\|\frac{1}{x}\|_{L^{3,\infty}}\|x\langle\nabla\rangle e^{itH}v(t)\|_{L^2}, \\
&\lesssim t^{-1}\|xe^{itH}\langle\nabla\rangle v(t)\|_{L^2}, \\
&\lesssim t^{-1}\|J\langle\nabla\rangle v(t)\|_{L^2} \lesssim t^{-1}(\|Jv(t)\|_{H^1} + \|v(t)\|_{L^2}).
\end{aligned} \tag{4.60}$$

Combining (4.53) and (4.60) we get by interpolation ($\theta(-\frac{1}{3}) - 2(1-\theta) = -2 + \frac{5\theta}{3}$) and the fact that $\||\nabla|^{-2+\frac{5\theta}{3}}v_{<1}(t)\|_{L^6} \leq \|U^{-2}v\|_{L^6}$:

$$\||\nabla|^{-2+\frac{5\theta}{3}}v_{<1}(t)\|_{L^6} \lesssim \min(1, t^{-\theta})\|v(t)\|_{X(t)}, \tag{4.61}$$

for $0 \leq \theta \leq 1$ where $v_{<1}$ and $v_{\geq 1}$ denote the smooth separation of the frequency. Similarly we have using the lemma 3.6 with $p = 6$ and $\theta = 0$ ($\sigma = \frac{1}{3}$), we have:

$$
\begin{aligned}
\||\nabla| v_{\geq 1}(t)\|_{L^6} &\lesssim \|e^{-itH} |\nabla| e^{itH} \langle \nabla \rangle v_{\geq 1}(t)\|_{L^6}, \\
&\lesssim t^{-1} \|U^{\frac{1}{3}} |\nabla| e^{itH} v_{\geq 1}(t)\|_{L^{\frac{6}{5},2}}, \\
&\lesssim t^{-1} \||\nabla| e^{itH} v(t)\|_{L^{\frac{6}{5},2}}, \\
&\lesssim t^{-1} \|\frac{1}{x}\|_{L^{3,\infty}} \|x \langle \nabla \rangle e^{itH} v(t)\|_{L^2}, \\
&\lesssim t^{-1} \|x e^{itH} \langle \nabla \rangle v(t)\|_{L^2}, \\
&\lesssim t^{-1} \|J \langle \nabla \rangle v(t)\|_{L^2} \lesssim t^{-1} (\|Jv(t)\|_{H^1} + \|v(t)\|_{L^2}).
\end{aligned}
\tag{4.62}
$$

Using (4.62) and the fact that $\|v_{t \geq 1}\|_{L^6} \leq \|v\|_{X(t)}$ by Sobolev embedding, by interpolation and since $\||\nabla|^\theta v_{\geq 1}(t)\|_{L^6} \lesssim \||\nabla| v_{\geq 1}(t)\|_{L^6}$ we obtain:

$$
\||\nabla|^\theta v_{\geq 1}(t)\|_{L^6} \lesssim \min(t^{-\theta}, t^{-1}) \|v(t)\|_{X(t)},
\tag{4.63}
$$

for any $0 \leq \theta \leq 1$. In particular we deduce (4.55) by using (4.63) with $\theta = \frac{3}{5}$ and (4.62) with $\theta = 0$:

$$
\begin{aligned}
\|U^{-1} v(t)\|_{L^6} &\lesssim \langle t \rangle^{\frac{-3}{5}} \|v(t)\|_{X(t)}, \\
\|U^{-1} v(t)\|_{L^2(H^{1,6})} &\lesssim \|v\|_{X \cap S}.
\end{aligned}
\tag{4.64}
$$

We obtain in a similar way the Strichartz bound on $\varphi = v_1 + i U^{-1} v_2$.

$$
\begin{aligned}
\|\varphi\|_{L^\infty(H^1)} &\lesssim \|U^{-1} v\|_{H^1} \lesssim \|v\|_{X(t)}, \\
\|\varphi\|_{L^2(H^{1,6})} &\lesssim \|U^{-1} v\|_{L^2(H^{1,6})} \lesssim \|v\|_{X \cap S}.
\end{aligned}
\tag{4.65}
$$

4.4 Initial condition on $\varphi(0)$ and equivalence of $\|v\|_{X \cap S}$ and $\|Z\|_{X \cap S}$ near zero

The goal now is to show that smallness on $Z(0)$ in $X(0) \cap S(0)$ ($\langle x \rangle Z(0)$ small in $H^1(\mathbb{R}^N)$) is equivalent to (2.15). On the other hand we will also prove that the following map

$$
\begin{aligned}
&X \cap S \to X \cap S, \\
&v \to Z = v + b_1'[\varphi_1, \varphi_1] + B_2'[\varphi_2, \varphi_2] = v + b(\varphi),
\end{aligned}
$$

is a diffeomorphism in a neighbourhood of 0 (in particular it will be ensure the equivalence of the norms $\|v\|_{X \cap S}$ and $\|Z\|_{X \cap S}$ when they are small). This last property will be useful to work indifferently with $\|v\|_{X \cap S}$ or $\|Z\|_{X \cap S}$.

Equivalence of $\|v\|_{X \cap S}$ and $\|Z\|_{X \cap S}$
In this section, we derive decay estimates on $b(\varphi)$ with:

$$
b(\varphi) = -\langle (\xi_1, \xi_2) \rangle^{-2} [\varphi_1, \varphi_1] - \langle (\xi_1, \xi_2) \rangle^{-2} [\varphi_2, \varphi_2].
$$

First we have:

$$
(Id - \Delta) \langle (\xi_1, \xi_2) \rangle)^{-2} [f, g] = \frac{1 + |\xi_1|^2 + |\xi_2|^2 + 2\xi_1 \cdot \xi_2}{\langle (\xi_1, \xi_2) \rangle)^2} [f, g].
$$

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Using lemma 3.4 and the fact that:

$$\frac{\xi_1 \cdot \xi_2}{\langle (\xi_1, \xi_2) \rangle\rangle^2}[f, g] = \frac{\langle \xi_1 \rangle U(\xi_1) \frac{\xi_1}{|\xi_1|} \cdot \langle \xi_2 \rangle U(\xi_2) \frac{\xi_2}{|\xi_2|}}{\langle (\xi_1, \xi_2) \rangle\rangle^2}[f, g] = \frac{\langle \xi_1 \rangle \langle \xi_2 \rangle}{\langle (\xi_1, \xi_2) \rangle\rangle^2}[U \, Rf, U \, Rg],$$

hence using the bilinear estimate (3.34), by interpolation and (4.52) and (4.55) we get:

$$\begin{aligned}
\|b(\varphi)\|_{H^{2,p}} &\lesssim \|U^{-1}v\|_{L^{p_1}} \|U^{-1}v\|_{L^{p_2}} \lesssim \|U^{-1}v\|_{L^2}^{2-\theta} \|U^{-1}v\|_{L^6}^\theta, \\
&\lesssim \langle t \rangle^{-\frac{3\theta}{5}} \|v(t)\|_{X(t)}^2,
\end{aligned} \tag{4.66}$$

for $0 < \theta \le 2$ where:

$$\frac{1}{p} = 1 - \frac{\theta}{3} = \frac{1}{p_1} + \frac{1}{p_2}, \ 2 \le p_1, p_2 \le 6. \tag{4.67}$$

In particular when we choose $\theta = \frac{3}{2}$ we have:

$$\|b(\varphi)(t)\|_{H^1} \lesssim \|b(\varphi)(t)\|_{H^{2,p}} \lesssim \langle t \rangle^{-\frac{9}{10}} \|v\|_{X(t)}^2. \tag{4.68}$$

Furthermore we have:

$$\|U^{-\frac{1}{6}}b(\varphi)\|_{L^2(H^{1,6})} \lesssim \|v\|_{X(t)}^2. \tag{4.69}$$

We consider $Jb(\varphi)$ in Fourier space, using the notation in (3.23). It is given by a linear combination of terms of the form:

$$\begin{aligned}
\mathcal{F}(Jb(\varphi)(\xi) &= e^{-itH(\xi)} \nabla_\xi \int_{\xi=\eta+\zeta} e^{it(H(\xi)\pm H(\eta)\pm H(\zeta))} B(\eta, \zeta) \mathcal{F}v^\pm(\eta) \mathcal{F}v^\pm(\zeta) d\eta \\
&= \mathcal{F}\big[(\nabla_\xi^{(\eta)} B + it \nabla_\xi^{(\eta)} \Omega \cdot B)[v^\pm, v^\pm] + B[v^\pm, (Jv)^\pm]\big],
\end{aligned} \tag{4.70}$$

with $B = \langle (\xi_1, \xi_2) \rangle^{-2}$ (when we consider the term $-\langle (\xi_1, \xi_2) \rangle^{-2}[\varphi_1, \varphi_1]$) or $\langle (\xi_1, \xi_2) \rangle^{-2} U(\xi_1)^{-1} U(\xi_2)^{-1}$ (when we consider the term $-\langle (\xi_1, \xi_2) \rangle^{-2}[\varphi_2, \varphi_2]$) and:

$$\Omega = H(\xi) \pm H(\eta) \pm H(\zeta), \ \nabla_\xi^{(\eta)} \Omega = \nabla H(\xi) \pm \nabla H(\zeta). \tag{4.71}$$

By the bilinear estimate (3.34) and the L^p decay (4.53), (4.55) and (4.56) we have:

$$\begin{aligned}
\|\nabla_\xi^{(\eta)} B[v^\pm, v^\pm]\|_{H^1} &\lesssim \|U^{-1}v\|_{L^3} \|U^{-2}v\|_{L^6} \lesssim \langle t \rangle^{-\frac{3}{10}} \|v\|_X^2, \\
\|t \nabla_\xi^{(\eta)} \Omega \cdot B[v^\pm, v^\pm]\|_{H^1} &\lesssim \|t^{\frac{1}{2}} U^{-1}v\|_{L^4}^2 \lesssim \langle t \rangle^{-\frac{1}{6}} \|v\|_X^2, \\
\|B[v^\pm, (Jv)^\pm]\|_{H^1} &\lesssim \|U^{-1}v\|_{L^3} \|U^{-1}Jv\|_{L^6} \lesssim \langle t \rangle^{-\frac{3}{10}} \|v\|_X^2.
\end{aligned} \tag{4.72}$$

Thus we obtain:

$$\|Jb(\varphi)(t)\|_{H^1} \lesssim \langle t \rangle^{-\frac{1}{6}} \|v\|_{X(t)}^2. \tag{4.73}$$

This shows that $\|v\|_{X \cap S} \sim \|Z\|_{X \cap S}$ provided that these norms are small.

Smallness assumption on φ_0 in terms of Z_0

It suffices here to verify that we can translate the smallness condition on $Z(0)$ in terms of smallness condition on φ_0 as in (2.15). We have seen that the smallness on $Z(0)$ in $X(0) \cap S(0)$ is equivalent to the smallness on $v(0)$ in $X(0) \cap S(0)$. It suffices then to show that (2.15) implies smallness on $v(0)$ in $X(0) \cap S(0)$.

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In particular since we have seen that $\|Z(0)\|_{X(0)}$ is equivalent to $\|v(0)\|_{X(0)}$, and by Sobolev and Hölder inequality in Lorentz spaces, we have:

$$\|\varphi(0)\|_{L^2} \lesssim \|\nabla\varphi(0)\|_{L^{\frac{6}{5}}} \lesssim \|x\nabla\varphi(0)\|_{L^2} \lesssim \delta. \tag{4.74}$$

Hence we deduce that:

$$\|\varphi(0)\|_{H^1} + \|x\varphi_1(0)\|_{L^2} + \|x\nabla\varphi_1(0)\|_{L^2} + \|x\nabla\varphi_2\|_{L^2} \lesssim \delta. \tag{4.75}$$

Furthermore by using the commutator $[x, U] = \mathcal{F}^{-1}[i\nabla_\xi, U(\xi)]\mathcal{F} = i\nabla_\xi U(D)$:

$$\begin{aligned}\|\langle\nabla\rangle U\varphi_2\|_{L^2} &\leq \|\langle\nabla\rangle Ux\varphi_2\|_{L^2} + \|\langle\nabla\rangle[x, U]\varphi_2\|_{L^2},\\ &\lesssim \|\langle\nabla\rangle Ux\varphi_2\|_{L^2} + \|\langle\nabla\rangle\nabla_\xi U(D)\varphi_2\|_{L^2}.\end{aligned} \tag{4.76}$$

Indeed we use the fact that: $\partial_i U(\xi) = \xi_i\left(\frac{1}{|\xi|(2+|\xi|^2)^{\frac{1}{2}}} - \frac{|\xi|}{(2+|\xi|^2)^{\frac{3}{2}}}\right)$ and $\langle\xi\rangle\partial_i U(\xi) = \frac{\xi_i}{|\xi|}\frac{2}{2+|\xi|^2}$ which implies that $\langle\nabla\rangle\nabla_\xi U(D) = 2R_i\langle\nabla\rangle^{-1}$ which is a bounded operator in L^2. Furthermore we recall that $\langle\nabla\rangle U \sim \nabla_{\leq 1} + \nabla_{\geq 1}$. Hence we have:

$$\|\langle x\rangle v(0)\|_{H^1} \lesssim \delta. \tag{4.77}$$

This is sufficient in our context since it ensures the smallness condition on $v(0)$ in terms of δ.

4.5 Dispersive estimates, Stability of the space S

Proposition 4.8. *We have the following estimate for all $0 \leq T$:*

$$\|\int_T^t e^{-i(t-s)H}\mathcal{N}_Z ds\|_{S(T,\infty)} \lesssim \langle T\rangle^{-1/2}(\|v\|_{X\cap S}^2 + \|v\|_{X\cap S}^4), \tag{4.78}$$

Proof: We recall that:

$$\begin{aligned}\mathcal{N}_Z(v) &= B_3[v_1, v_1] + B_4[v_2, v_2] + C_1[v_1, v_1, v_1] + C_2[v_2, v_2, v_1] + iC_3[v_1, v_1, v_2]\\ &\quad + iC_4[v_2, v_2, v_2] + iQ_1(u),\end{aligned}$$

$$S = L^\infty(H^1) \cap U^{\frac{1}{6}}L^2(H^{1,6}).$$

For the sake of conciseness we are going only to deal with the term $\int_T^t e^{-i(t-s)H}B_4[v_2, v_2]ds$ for $T \geq 1$ as it contains most of the difficulties. Using the Strichartz estimate of proposition 3.5 gives:

$$\begin{aligned}\|U^{\frac{-1}{6}}\int_T^t e^{-i(t-s)H}B_4[v_2, v_2]ds\|_{L^2_{t>T}(H^{1,6})} &\lesssim \|B_4[v_2, v_2]\|_{L^1_{t>T}(H^1)},\\ \|\int_T^t e^{-i(t-s)H}B_4[v_2, v_2]ds\|_{L^\infty_{t>T}(H^1)} &\lesssim \|B_4[v_2, v_2]\|_{L^1_{t>T}(H^1)}.\end{aligned} \tag{4.79}$$

We recall that $B_4[v_2, v_2] = UB_4'[u_2, u_2]$ and more precisely we have:

$$B_4 = -2U(\xi)\left(\frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\langle(\xi_1, \xi_2)\rangle^2}\right)\widehat{\xi_1}\widehat{\xi_2}.$$

It means that in particular $B_4[v_2, v_2] = -2UB_4''[Rv_2, Rv_2]$, with R the Riesz operator and $B_4'' = \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\langle(\xi_1, \xi_2)\rangle^2}$ a continuous bilinear operator according the lemma 3.4. We shall use in a crucial

way the additional decay estimates due to the control of the pseudoconformal transformation Ju. We have by Hölder's inequality, interpolation and (4.55)

$$
\begin{aligned}
\|B_4[v_2,v_2]\|_{L^1_{t>T}(H^1)} &\lesssim \|B_4''[Rv_2,Rv_2]\|_{L^1_{t>T}(H^1)} \\
&\lesssim \|Rv_2\|_{L^4_{t>T}(H^{1,3})}\|Rv_2\|_{L^{\frac{4}{3}}_{t>T}(L^6)} \\
&\lesssim \|v_2\|^{\frac{1}{2}}_{L^\infty_{t>T}(H^1)}(\int_T^{+\infty}\|v_2\|^2_{H^{1,6}}ds)^{\frac{1}{4}}(\int_T^{+\infty}\|v_2\|^{\frac{4}{3}}_{L^6}ds)^{\frac{3}{4}} \\
&\lesssim \|v_2\|^{\frac{1}{2}}_{L^\infty_{t>T}(H^1)}(\int_T^{+\infty}\frac{1}{s^2}\|sv_2\|^2_{H^{1,6}}ds)^{\frac{1}{4}}(\int_T^{+\infty}\frac{1}{s^{\frac{4}{3}}}\|sv_2\|^{\frac{4}{3}}_{L^6}ds)^{\frac{3}{4}} \\
&\lesssim \frac{1}{T^{\frac{1}{2}}}\|v_2\|^{\frac{1}{2}}_{L^\infty_{t>T}(H^1)}\|sv_2\|^{\frac{3}{2}}_{L^\infty_{t>T}(H^{1,6})}.
\end{aligned}
\tag{4.80}
$$

Remark 12. In order to estimate the term $\|Rv_2\|_{L^{\frac{4}{3}}_{t>T}(L^6)}$ it requires to have additional time decay which are given by the space X. Indeed Strichartz estimates allow only a control in $L^2_{t>T}$ which is roughly speaking weaker in long time.

And we have by (4.54) for $t > T$:

$$
\||(v_2)_{\leq 1}(t)\|_{L^6} \lesssim \||\nabla|^{-\frac{1}{3}}(v_2)_{\leq 1}\|_{L^6} \lesssim \frac{1}{t}\|v\|_{X(t)},
$$

$$
\|(v_2)_{\geq 1}(t)\|_{L^6} \lesssim \frac{1}{t}\|v\|_{X(t)}\|v\|_{X(t)},
$$

$$
\||\nabla|(v_2)_{\leq 1}(t)\|_{L^6} \lesssim \|(v_2)_{\leq 1}(t)\|_{L^6} \lesssim \frac{1}{t}\|v\|_{X(t)},
$$

$$
\||\nabla|(v_2)_{\geq 1}(t)\|_{L^6} \lesssim \frac{1}{t}\|v\|_{X(t)}.
$$

In particular it implies that:
$$
\|sv_2\|^{\frac{3}{2}}_{L^\infty_{t>T}(H^{1,6})} \lesssim \|v\|^{\frac{3}{2}}_{X(t)}.
$$

We have finally:
$$
\|B_4[v_2,v_2]\|_{L^1_{t>T}(H^1)} \lesssim \frac{1}{T^{\frac{1}{2}}}\|v\|^2_{X(t)}.
\tag{4.81}
$$

In the case $0 \leq T \leq 1$ we can apply a classical Strichartz estimate (indeed we know that we have existence of strong solution in finite time, see [16]). We have then by using proposition 3.4, (4.54) and the fact that the Riesz operator is continuous in L^p with $1 < p < +\infty$:

$$
\begin{aligned}
\|U^{\frac{1}{12}}B_4[v_2,v_2]\|_{L^{\frac{4}{3}}_{T\leq t\leq 1}(H^{1,\frac{3}{2}})} &\lesssim \|v\|_{L^\infty(H^1)}\|v\|_{L^{\frac{4}{3}}_{T\leq t\leq 1}(L^6)}, \\
&\lesssim \|v\|_{L^\infty(H^1)}\|v\|_{L^2_{T\leq t\leq 1}(L^6)}, \\
&\lesssim \|v\|^2_X.
\end{aligned}
\tag{4.82}
$$

And we proceed as in the previous case for $t \geq 1$.

4.6 Weight estimate on X

Following Gustafson, Nakanishi and Tsai in [29] we are left to estimate $\int_0^t e^{i(t-s)H} \mathcal{N}_Z ds$ in the weighted space X, we start by rewriting the system (4.41) in terms of Z, it gives :

$$i\partial_t Z - HZ = \mathcal{N}_Z(v) \tag{4.83}$$

where:

$$v = Z - B'[u_1, u_1] - B'[u_2, u_2] = Z - b(\varphi),$$

with:

$$\begin{aligned}\mathcal{N}_Z(v) =\, &B_3[Z_1, Z_1] + B_4[Z_2, Z_2] + C_1[v_1, v_1, v_1] + C_2[v_2, v_2, v_1] + iC_3[v_1, v_1, v_2] \\ &+ iC_4[v_2, v_2, v_2] + C_5 + iQ_1(\varphi) + Q_2(\varphi).\end{aligned} \tag{4.84}$$

Here we have:

$$C_5(v, v, Z) = -2B_3[b(\varphi), Z_1], \ Q_2(\varphi) = B_3[b(\varphi), b(\varphi)]. \tag{4.85}$$

let us start with the bilinear terms.

Bilinear terms

We have:

$$B_3[Z_1, Z_1] = \frac{1}{4}(B_3[Z, Z] + 2B_3[Z, \bar{Z}] + B_3[\bar{Z}, \bar{Z}]). \tag{4.86}$$

Applying J to the bilinear term $B_3[Z, \bar{Z}]$, we have in Fourier space:

$$\mathcal{F}\big(J \int_0^t e^{-i(t-s)H} B_3[Z, \bar{Z}]\big) =$$

$$e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^d} \nabla_\xi \Big(e^{is(H(\xi)+H(\eta)-H(\xi-\eta)} B_3(\eta, \xi-\eta)(e^{-isH(\eta)}\widehat{Z}(\eta))(e^{isH(\zeta)}\widehat{\bar{Z}}(\zeta))\Big)d\zeta,$$

$$= e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^d} \nabla_\xi \Big(e^{is(H(\xi)+H(\eta)-H(\xi-\eta)} B_3(\eta, \xi-\eta)\check{Z}(s)\widetilde{Z}(s)\Big)d\zeta \, ds,$$

with $\check{Z}(s) := \mathcal{F}(e^{-isH}Z)(s)$, $\widetilde{Z}(s) = \mathcal{F}(\overline{e^{-isH}Z})(s)$. The same computation for the other terms in formula 4.86 gives an expression of $\mathcal{F}\big(J \int_0^t e^{-i(t-s)H} B_3[Z_1, Z_1]\big)$. We are reduced for the bilinear terms to study terms of the form:

$$e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^d} \nabla_\xi \Big(e^{is(H(\xi)+H(\eta)-H(\xi-\eta)} B_j(\eta, \xi-\eta)\check{Z}(s)\widetilde{Z}(s)\Big)d\zeta \, ds, \tag{4.87}$$

$$e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^d} \nabla_\xi \Big(e^{is(H(\xi)+H(\eta)+H(\xi-\eta)} B_j(\eta, \xi-\eta)\check{Z}(s)\check{Z}(s)\Big)d\zeta \, ds, \tag{4.88}$$

$$e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^d} \nabla_\xi \Big(e^{is(H(\xi)-H(\eta)-H(\xi-\eta)} B_j(\eta, \xi-\eta)\widetilde{Z}(s)\widetilde{Z}(s)\Big)d\zeta \, ds \tag{4.89}$$

There is three different phases to study:

$$\Omega = H(\xi) \pm H(\eta) \pm H(\xi - \eta).$$

Each exhibits a different behavior in terms of space and time resonance. Since the case $Z\bar{Z}$ is the worst, we will only consider the phase $\Omega_1 = H(\xi) + H(\eta) - H(\xi - \eta)$. Depending on which term ∇_ξ lands, the following integrals arise:

$$\mathcal{F}I_1 = e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left(e^{is(H(\xi)+H(\eta)-H(\xi-\eta))} \nabla_\xi^{(\eta)} B_j(\eta, \xi - \eta) \check{Z}(s, \eta) \widetilde{Z}(s, \xi - \eta) \right) d\eta ds,$$

$$\mathcal{F}I_2 = e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left(e^{is(H(\xi)+H(\eta)-H(\xi-\eta))} B_j(\eta, \xi - \eta) \check{Z}(s, \eta) \nabla_\xi^{(\eta)} \widetilde{Z}(s, \xi - \eta) \right) d\eta ds,$$

$$\mathcal{F}I_3 = e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left((\nabla_\xi e^{is(H(\xi))+H(\eta)-H(\xi-\eta))} B_j(\eta, \xi - \eta) \check{Z}(s, \eta) \widetilde{Z}(s, \xi - \eta) \right) d\eta ds.$$

with:

$$\left(J \int_0^t e^{-i(t-s)H} B_j[Z, \bar{Z}] \right) = I_1 + I_2 + I_3.$$

How to deal with I_1

Let us observe that:

$$I_1 = \int_0^t e^{i(s-t)H} \nabla_2 B_j[Z, \bar{Z}](s) ds.$$

We want to estimate I_1 in $L^\infty(H^1)$, since $\nabla_2 B_j$ is a bounded multiplier, we can use the Strichartz estimate as in the previous section.

How to deal with I_2

We observe that in this case a short computation shows that:

$$\mathcal{F}I_2 = e^{-itH(\xi)} \int_0^t e^{isH(\xi)} \mathcal{F}(B_j[Z, \overline{JZ}])(\xi) ds$$

$$= \mathcal{F} \int_0^t e^{i(s-t)H} B_j[Z, \overline{(JZ)}](s) ds.$$

It suffices then to apply Strichartz estimates as in the previous section, for example using estimate (4.54):

$$\|B_j[Z, \overline{(JZ)}]\|_{L_{t>T}^{\frac{4}{3}}(H^{1,\frac{3}{2}})} \lesssim \|Z\|\|_{L_{t>T}^{\frac{4}{3}}(H^{1,6})} \|\overline{JZ}\|_{L_{t>T}^\infty(H^1)}$$

$$\lesssim \|Z\|_X^2 \left(\int_T^{+\infty} \frac{1}{s^{\frac{4}{3}}} ds \right)^{\frac{3}{4}} \qquad (4.90)$$

$$\lesssim \frac{1}{T^{\frac{1}{4}}} \|Z\|_X^2.$$

We have still used the fact that $\|sZ\|_{L^\infty(H^{1,6})} \lesssim \|Z\|_{X(t)}$.

We shall deal with I_3 in the section 4.7 which is the heart of the proof of the result of [29]. This is the part where we use in a crucial way the structure of the nonlinearity which will compensate the space-time resonance in $\xi = 0$ of the phase $\Omega_1 = H(\xi) + H(\eta) - H(\xi - \eta)$.

Cubic terms

Let us deal now with the cubic terms. We have different terms of the form $C_j[v,v,\bar v], C_j[v,v,v]$ which can be treated essentially in the same way (see for more details the section 4.5). Let us deal with the case $C_j[v,v,\bar v]$ which can be expressed as follows:

$$
\begin{aligned}
\mathcal{F}\Big(J\int_0^t e^{-i(t-s)H}C_j[v,v,\bar v]\Big) &= \\
&= e^{-itH(\xi)}\nabla_\xi \int_0^t \int\!\!\int_{\xi=\xi_1+\xi_2+\xi_3} e^{is\Omega}C_j(\xi_1,\xi_2,\xi_3)\check v(s,\xi_1))\check v(s,\xi_2))\widetilde v(s,\xi_3)d\xi_1 d\xi_2 ds,
\end{aligned}
\tag{4.91}
$$

for $1\le j\le 4$, where $\Omega = H(\xi)+H(\xi_1)+H(\xi_2)-H(\xi_3)$. In the same way than before we have for $j=1\cdots 4$ three case:

$$
\Big(J\int_0^t e^{-i(t-s)H}C_j[v,v,\bar v]\Big)ds = J_1 + J_2 + J_3,
$$

with:

$$
\begin{aligned}
J_1 &= e^{-itH(\xi)}\int_0^t e^{-i(t-s)H}\nabla_3 C_j[v,v,\bar v],\\
J_2 &= e^{-itH(\xi)}\int_0^t e^{-i(t-s)H}C_j[Jv,v,\bar v]ds + \int_0^t e^{-i(t-s)H}C_j[v,Jv,\bar v]ds,\\
\mathcal{F}J_3 &= e^{-itH(\xi)}\int_0^t \int\!\!\int_{\xi=\xi_1+\xi_2+\xi_3}\nabla_\xi(e^{is\Omega})C_j(\xi_1,\xi_2,\xi_3)\check v(s,\xi_1))\check v(s,\xi_2))\widetilde v(s,\xi_3)d\xi_1 d\xi_2 ds.
\end{aligned}
\tag{4.92}
$$

For the same reason than in the previous case, we are just going to deal with the cases J_1 and J_2 (we refer to the section 4.10 for the term J_3). We can bound J_1 in the same way as before by using the previous estimates on the dispersive part and the fact that $\nabla_3 C_j$ is a bounded multiplier.

In order to estimate J_2, we have via the Strichartz estimate to control $\|C_j[Jv,v,\bar v]\|_{L^2(H^{1,\frac{6}{5}})}$ and $\|C_j[v,Jv,\bar v]\|_{L^2(H^{1,\frac{6}{5}})}$. Its contribution is estimated by using the estimate (3.34) and the estimate (4.49) on the cubic terms:

$$
\begin{aligned}
\|C_j[v,Jv,\bar v]]\|_{L^2(H^{1,\frac{6}{5}})} &\lesssim \|Jv\|_{L^\infty(H^1)}\|U^{-1}v\|_{L^2(L^6)}\|U^{-1}v\|_{L^\infty L^6}\\
&+ \|\nabla U^{-1}Jv\|_{L^\infty(L^2)}\|v\|_{L^2(L^6)}\|U^{-1}v\|_{L^\infty L^6} + \|v\|_{L^\infty H^1}\|U^{-1}v\|_{L^2(L^6)}\|U^{-1}Jv\|_{L^\infty L^6}\\
&+ \|U^{-1}v\|_{L^\infty H^1}\|v\|_{L^2(L^6)}\|U^{-1}Jv\|_{L^\infty L^6} + \|U^{-1}v\|_{L^\infty H^1}\|U^{-1}v\|_{L^2(L^6)}\|Jv\|_{L^\infty L^6},
\end{aligned}
\tag{4.93}
$$

$$
\begin{aligned}
&\lesssim \|Jv\|_{L^\infty(H^1)}\|U^{-1}v\|_{L^2(H^{1,6})}\|U^{-1}v\|_{L^\infty L^6} + \|U^{-1}v\|_{L^\infty H^1}\|v\|_{L^2(H^{1,6})}\|U^{-1}Jv\|_{L^\infty L^6},\\
&\lesssim \|v\|_{X\cap S}^3.
\end{aligned}
$$

where if the derivative in the $H^{1,\frac{6}{5}}$ norm lands on Jv (with large frequency), it is dominated by the first term on the right, otherwise we use the other term. We refer to (4.55) for the $L^2(H^{1,6})$ norm on $U^{-1}v$.

For C_5 we have just to replace the last v with Z in (4.91) (indeed we have $b(u) = Z - v$), hence the final bound in (4.93) is replaced by $\|v\|_{X\cap S}^2\|Z\|_{X\cap S}$. To see this it suffices just to use the equivalence of norm between v and Z which is proved in section 4.4.

Quartic terms

The quartic term $Q_j(\varphi)$ with $j = 1, 2$ are smooth, it is then natural to use the physical space. We are only going to deal with $B[\varphi, |\varphi|^2\varphi] = \frac{1}{2+|\xi|^2+|\xi_2|^2}[\varphi, |\varphi|^2\varphi]$. Since we have:

$$J \int_0^t e^{-i(t-s)H} Q_j(\varphi) ds = \int_0^t e^{-i(t-s)H} \big(\nabla_2 B[\varphi, |\varphi|^2\varphi] + B[J\varphi, |\varphi|^2\varphi] + B[\varphi, J(|\varphi|^2\varphi)] \big) ds. \tag{4.94}$$

By Strichartz estimates it is sufficient to control $\big(\nabla_2 B[\varphi, |\varphi|^2\varphi] + B[J\varphi, |\varphi|^2\varphi] + B[\varphi, J(|\varphi|^2\varphi)] \big)$ in $L^2(H^{1,\frac{6}{5}})$. The term $\int_0^t e^{-i(t-s)H} \nabla_2 B[\varphi, |\varphi|^2\varphi] ds$ can be treated as in the section 4.5 and the fact that $\nabla_2 B$ is a bounded multiplier. Let us focus on the second term $\int_0^t e^{-i(t-s)H} B[J\varphi, |\varphi|^2\varphi] ds$. Since $H(\xi) = \sqrt{|\xi|^2(2+|\xi|^2)}$ and $J\varphi = x\varphi + it\nabla H(D)\varphi$ we have $J_j\varphi = x_j\varphi + itR_j(2Id - \Delta)^{\frac{1}{2}}\varphi + t\partial_j U\varphi$. We have to estimate the following term by using Strichartz estimate and by (3.34) and propositions 4.6 and 4.7:

$$\| \int_0^t e^{-i(t-s)H} B[it\nabla H(D)\varphi, |\varphi|^2\varphi] ds \|_{L^\infty(H^1)} \lesssim \| B[it\nabla H(D)\varphi, |\varphi|^2\varphi] \|_{L^2(H^{1,\frac{6}{5}})},$$

$$\lesssim \| \frac{\langle\xi_1\rangle^2}{2+|\xi|^2+|\xi_2|^2} [it(2-\Delta)^{-1}\langle\nabla\rangle H(D)\varphi, |\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})} \tag{4.95}$$

$$+ \| \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{2+|\xi|^2+|\xi_2|^2} [it(2-\Delta)^{-\frac{1}{2}}\nabla H(D)\varphi, \langle\nabla\rangle(2-\Delta)^{-\frac{1}{2}}|\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})},$$

$$\lesssim \| t^{\frac{1}{2}}\varphi \|_{L^\infty L^6}^2 \|\varphi\|_{L^\infty L^3} \|\varphi\|_{L^2 L^6} \lesssim \|v\|_{X\cap S}^4.$$

Let us deal now with the term involving $x\varphi$. Since $x\varphi = J\varphi - it\nabla H(D)\varphi$ we have:

$$\| B[x\varphi, |\varphi|^2\varphi] \|_{L^2(H^{1,\frac{6}{5}})} \lesssim$$

$$\| \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{2+|\xi|^2+|\xi_2|^2} [\langle\nabla\rangle(2-\Delta)^{-\frac{1}{2}}x\varphi, (2-\Delta)^{-\frac{1}{2}}|\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})} \tag{4.96}$$

$$+ \| \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{2+|\xi|^2+|\xi_2|^2} [(2-\Delta)^{-\frac{1}{2}}x\varphi, \langle\nabla\rangle(2-\Delta)^{-\frac{1}{2}}|\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})}$$

On the other hand by using proposition (4.6) we have:

$$\|J\varphi\|_{\dot{H}^1} + \|(\nabla_\xi H)\varphi\|_{L^2} \lesssim \|J\varphi\|_{\dot{H}^1} + \|U^{-1}v\|_{H^1} \lesssim \|v(t)\|_{X(t)}. \tag{4.97}$$

so that by proposition 4.6 and 4.7:

$$\| \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{2+|\xi|^2+|\xi_2|^2} [\langle\nabla\rangle(2-\Delta)^{-\frac{1}{2}}J\varphi, (2-\Delta)^{-\frac{1}{2}}|\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})}$$

$$\lesssim \|J\varphi\|_{L^\infty(L^6)} \|\langle t\rangle^{\frac{1}{2}}\varphi\|_{L^\infty L^6}^2 \|\frac{1}{\langle t\rangle}u\|_{L^2(L^3)}$$

$$\lesssim \|J\varphi\|_{L^\infty(\dot{H}^1)} \|\langle t\rangle^{\frac{1}{2}}\varphi\|_{L^\infty L^6}^2 \|\frac{1}{\langle t\rangle}u\|_{L^2(L^3)} \lesssim \|v\|_{S\cap X}^4, \tag{4.98}$$

$$\| \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{2+|\xi|^2+|\xi_2|^2} [\langle\nabla\rangle(2-\Delta)^{-\frac{1}{2}}t\nabla H(D)\varphi, (2-\Delta)^{-\frac{1}{2}}|\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})}$$

$$\lesssim \|\nabla H(D)\varphi\|_{L^\infty(L^2)} \|\langle t\rangle^{\frac{1}{2}}u\|_{L^\infty L^6}^2 \|u\|_{L^2(L^\infty)} \lesssim \|v\|_{S\cap X}^4$$

The same arguments work for $\| \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{2+|\xi|^2+|\xi_2|^2} [(2-\Delta)^{-\frac{1}{2}}x\varphi, \langle\nabla\rangle(2-\Delta)^{-\frac{1}{2}}|\varphi|^2\varphi] \|_{L^2(L^{\frac{6}{5}})}$.

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4.7 Estimate on $Z\bar{Z}$

We are going to deal with the bilinear terms by using integrations by parts and decomposing the space in non space and non time resonance regions. We will only focus on the term $B_j[Z, \bar{Z}]$ which corresponds to the phase $\Omega_1 = H(\xi) + H(\eta) - H(\xi - \eta)$ (the other terms can be treated in a similar way):

$$I_3 = e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left(is\nabla_\xi \Omega(\xi,\eta) e^{is(H(\xi)) + H(\eta) - H(\xi-\eta)} B_j(\eta, \xi - \eta) \check{Z}^\pm(\eta) \widetilde{Z}(\xi - \eta) \right) d\eta ds.$$

This term is quite hard to deal with because of the factor s, in particular we lose some decay in time. In order to "kill" this factor we should exploit the non-resonance property through integration by part on the phase in space η and in time s.

We will use integration by part in space and time by rewriting $e^{is\Omega}$ under the form:

$$e^{is\Omega} = \frac{\nabla_\eta \Omega}{is|\nabla_\eta \Omega|^2} \cdot \nabla_\eta e^{is\Omega},$$

$$e^{is\Omega} = \frac{1}{i\Omega} \partial_s e^{is\Omega}.$$

Definition 4.1. A region is space non-resonant if $\nabla_\eta \Omega$ does not vanish. Similarly a region of the space is time non-resonant if Ω does not vanish.

Remark 13. Let us emphasize in particular that the intersection of the spatially resonant and temporally resonant regions is only at $\xi = 0$. This is generally a major issue, in our case it will be compensated by the cancellation at $\xi = 0$ of the symbol of the bilinear forms B_j (let us recall that $|B_j(\xi_1, \xi_2)| \lesssim U(\xi_1 + \xi_2) = U(\xi)$).

To do this we have to decompose the bilinear terms B_j in two parts:

$$B_j = B_j^X + B_j^T,$$

such that B_j^X is supported in (ξ, η) in a spatially non-resonant region, and B_j^T is supported in a temporally non-resonant region.

More precisely we are going decompose each B_j ($j = 3, 4$) by using the Littlewood-Paley decomposition such that:

$$B_j^{abc} = \chi^a(\xi)\chi^b(\eta)\chi^c(\zeta)B_j(\eta,\zeta) = B_j^{a,b,c,X} + B_j^{a,b,c,T} \tag{4.99}$$

with $a, b, c \in 2^{\mathbb{Z}}$ with $|\xi| \sim a$, $\eta \sim b$ and $|\xi - \eta| \sim c$. The smooth decomposition into $B_j^{a,b,c,X}$ and $B_j^{a,b,c,T}$ will be given in the section **??**.

Spatial and time integration of phase

In this section we treat I_3 in the non space and non time resonant regions $|\nabla_\eta \Omega| > 0$ and $|\Omega| > 0$ with $\Omega = H(\xi) + H(\eta) - H(\xi - \eta)$ the phase. Let us deal with $B_j^{a,b,c,X}$ that we decompose in dyadic shell by setting:

$$B_j^{a,b,c,X} = \chi^a(\xi)\chi^b(\eta)\chi^c(\zeta)B_j^X(\eta,\zeta),$$

Let us recall that:

$$e^{is\Omega} = \frac{\nabla_\eta \Omega}{is|\nabla_\eta \Omega|^2} \cdot \nabla_\eta e^{is\Omega}, \tag{4.100}$$

we must estimate

$$\mathcal{F}I_3^{a,b,c,X} = e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left((\nabla_\xi e^{is(H(\xi))+H(\eta)-H(\xi-\eta)} B_j^{a,b,c,X}(\eta,\xi-\eta) \check{Z}(s,\eta) \widetilde{Z}(s,\xi-\eta) \right) d\eta ds.$$

An integration by part in space gives:

$$\begin{aligned} I_3^{a,b,c,X} = -\mathcal{F}^{-1}(& \left(e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left(e^{is\Omega(\xi,\eta)} \mathcal{B}_{1,j}^{a,b,c,X}(\eta,\xi-\eta) \cdot \nabla_\eta[\check{Z}(\eta)\widetilde{Z}(\xi-\eta)] \right. \right. \\ & \left. \left. + \mathcal{B}_{2,j}^{a,b,c,X}(\eta,\xi-\eta)\check{Z}(\eta)\widetilde{Z}(\xi-\eta) \right) ds \right) \end{aligned} \qquad (4.101)$$

$$= -\int_0^t e^{isH}\left(\mathcal{B}_{1,j}^{a,b,c,X}[JZ,\bar{Z}] - \mathcal{B}_{1,j}^{a,b,c,X}[Z,\overline{JZ}] + \mathcal{B}_{2,j}^{a,b,c,X}[Z,\bar{Z}] \right) ds.$$

with:

$$\mathcal{B}_{1,j}^{a,b,c,X} = \frac{\nabla_\xi \Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} B_j^{a,b,c,X}, \quad \mathcal{B}_{2,j}^{a,b,c,X} = \nabla_\eta \left(\frac{\nabla_\xi \Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} B_j^{a,b,c,X} \right),$$

We are now interested in estimating $I_3^{a,b,c,T}$ in the non resonant time region, it means when Ω does not vanish. Let us recall that:

$$\frac{1}{i\Omega} \partial_s e^{is\Omega} = e^{is\Omega}.$$

By an integration by part in time on I_3 we have:

$$\begin{aligned} I_3^{a,b,c,T} = & -\mathcal{F}^{-1}\left(\left(e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left(\frac{i\nabla_\xi \Omega}{i\Omega} e^{is\Omega} B_j^{a,b,c,T}(\eta,\xi-\eta) \check{Z}(\eta)\widetilde{Z}(\xi-\eta) \right) d\eta ds \right) \right) \\ & -\mathcal{F}^{-1}\left(\left(e^{-itH(\xi)} \int_0^t \int_{\mathbb{R}^N} \left(\frac{is\nabla_\xi \Omega}{i\Omega} e^{is\Omega} B_j^{a,b,c,T}(\eta,\xi-\eta) \partial_s\big(\check{Z}(\eta)\widetilde{Z}(\xi-\eta)\big) \right) d\eta ds \right) \right) \\ & + \left[\mathcal{F}^{-1}\left(\left(e^{-itH(\xi)} \int_{\mathbb{R}^N} \left(\frac{is\nabla_\xi \Omega}{i\Omega} e^{is\Omega(\xi,\eta)} B_j^{a,b,c,T}(\eta,\xi-\eta) \partial_s\big(\check{Z}(\eta)\widetilde{Z}(\xi-\eta)\big) \right) d\eta ds \right) \right) \right]_0^t \\ = & -\int_0^t e^{isH}\left(\mathcal{B}_3^{a,b,c,T}[Z,\bar{Z}] ds + \mathcal{B}_3^{a,b,c,T}[s\mathcal{N}Z,\bar{Z}] + \mathcal{B}_3^{a,b,c,T}[Z,s\overline{\mathcal{N}Z}] \right) ds \\ & + \left[e^{isH}\mathcal{B}_3^{a,b,c,T}[sZ,\bar{Z}] \right]_0^t, \end{aligned} \qquad (4.102)$$

with:

$$\mathcal{B}_3^{a,b,c,T} = \frac{\nabla_\xi \Omega}{\Omega} B_j,$$

with $j = 3, 4$.

Remark 14. Let us mention that the second and the third term are due to the normal form. While they seem delicate to deal with because the factor s is still present, the nonlinearities are now at least cubic. In particular we know that in low frequencies the behavior of the cubic term is better for long time estimate so it will cancel out the s growth.

We are now interested in dealing with the terms $\mathcal{B}_j[\bar{Z}, Z]$ with $j = 3, 4$. Classically we use paraproduct estimates to distinguish the three region:

$$a << b \sim c, \ b << a \sim c \ c << a \sim b. \qquad (4.103)$$

In the sequel we are interested in using the lemma 3.7 in order to estimate the terms on the right hand side of (4.101) and (4.102). To do this Gustafson, Nakanishi and Tsai proved the following lemma in [29].

Lemma 4.2. *Denoting $M = \max(a, b, c)$, $m = \min(a, b, c)$ and $l = \min(b, c)$ we have:*

- *If $M << 1$ then for $\epsilon > 0$ small:*

$$\|\mathcal{B}_1^{a,b,c,X}\|_{H^{1+\epsilon}} \lesssim l^{\frac{1}{2}-2\epsilon}, \quad \|\mathcal{B}_2^{a,b,c,X}\|_{H^{1+\epsilon}} \lesssim l^{\frac{1}{2}-2\epsilon} M^{-1}, \tag{4.104}$$

- *If $M \geq 1$ then for $|\epsilon| > 0$ small:*

$$\|\mathcal{B}_1^{a,b,c,X}\|_{H^{\frac{3}{2}+\epsilon}} \lesssim l^{1-\epsilon}\langle a\rangle^{-1}, \quad \|\mathcal{B}_2^{a,b,c,X}\|_{H^{\frac{3}{2}+\epsilon}} \lesssim l^{-\epsilon}\langle a\rangle^{-1}, \tag{4.105}$$

Lemma 4.3.
$$\|\mathcal{B}_3\|_{[H^s]} \lesssim (\frac{\langle M\rangle}{M})^s l^{\frac{3}{2}-s}\langle a\rangle^{-1}. \tag{4.106}$$

Remark 15. As mentioned in the remark 10 we are going to use lemma 3.7, in high frequencies we observe that we are going to work with $s = \frac{3}{2}$ which corresponds to the classical Strichartz estimates (it means that we can deal with the problem in high frequencies only with Strichartz estimates which corresponds in some way to the local existence of strong solution). In low frequencies the things are different, indeed roughly speaking the existence of global strong solution is related to the behavior in low frequencies (low frequencies control the behavior in long time). In this case the situation is different since we have a loss of one demi derivative, we work with $s = 1 + \epsilon$. To comp ensate this loss, we need better long time decay, it is exactly at this point that we shall work with the $\|\cdot\|_X$ norm.

A simple way to explain this fact is to observe that the Strichartz estimate are very good in finite time since for $N = 3$ we get a control on $L_t^2(L^6)$ whereas when we use punctual estimate we have only a control on $u(t)$ in L^6 with a time decay in $\frac{1}{t}$ (what is very bad in finite time, modulo that $u_0 \in L^{\frac{6}{5}}$). Conversely in long time this decay is very good compared with the L_t^2 decay. It will be exactly what we shall use for low frequencies, Strichartz estimate in finite time and $\|\cdot\|_X$ norm for long time estimate (indeed we recall to use weight is a way to come back to the classical estimate $\|u(t)\|_{L^p} \lesssim t^{-\frac{N}{2}(\frac{1}{p'}-\frac{1}{p})}\|u_0\|_{L^{p'}}$).

Before giving some arguments of the proof of the lemmas 4.2 and 4.3, let us setimate I_3 in the case $B_j[Z, \bar{Z}]$ with $j = 3, 4$.

Estimate of I_3 in spatial non resonant or time non resonant region

Spatial non resonant

We have now to distinguish the regions when $a \lesssim b \sim c$, $b \lesssim c \sim a$ or $c \lesssim a \sim b$. Let us start with the case $b \lesssim c \sim a$, it gives by Minkowski inequality and Bernstein lemma:

$$\begin{aligned}
&\|\int_0^t e^{isH}\big(\sum_{b\lesssim c\sim a} \mathcal{B}_1^{a,b,c}[JZ, \bar{Z}] - \mathcal{B}_{1,j}[Z, \overline{JZ}]\big)ds\|_{H^1} \\
&\lesssim \sum_{b\lesssim c\sim a} \langle a\rangle\|\int_0^t e^{isH}\big(\mathcal{B}_1^{a,b,c}[JZ, \bar{Z}] - \mathcal{B}_1[Z, \overline{JZ}]\big)ds\|_{L^2}.
\end{aligned} \tag{4.107}$$

Since $c \sim a$ means that for some $K > 0$ we have $\frac{c}{K} \leq a \leq Kc$, for fixed a the sum over c involves a finite number of terms that have all a similar contribution, and thus $\sum_{b \lesssim c \sim a} \|B^{a,b,c}\|_{[H]^s} \lesssim \sum_a \sum_{b \lesssim a} \|B^{a,b,a}\|_{[H]^s}$. And we have for $0 < \frac{\epsilon}{2} \leq \frac{1}{6}$:

$$\sum_{b \lesssim c \sim a} \langle a \rangle \| \int_0^t e^{isH} \big(\mathcal{B}_1^{a,b,c}[JZ^\pm, Z^\pm] - \mathcal{B}_1[Z^\pm, JZ^\pm] \big) ds \|_{L^2} \lesssim$$

$$\lesssim \sum_{b \lesssim a << 1} \langle a \rangle \big(\| \int_0^t e^{isH} \mathcal{B}_{1,j}^{a,b,a}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} JZ, \bar{Z}] \big) ds \|_{L^2}$$

$$+ \sum_{b \lesssim a << 1} \langle a \rangle \big(\| \int_0^t e^{isH} \mathcal{B}_{1,j}^{a,b,a}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} Z, \overline{JZ}] \big) ds \|_{L^2} \big)$$

$$+ \sum_{b \lesssim a, a \gtrsim 1} \langle a \rangle \big(\| \int_0^t e^{isH} \mathcal{B}_{1,j}^{a,b,a}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} \langle \nabla \rangle^{-1+\frac{\epsilon}{2}} \langle \nabla \rangle JZ, U^{\frac{1}{6}} U^{-\frac{1}{6}} \langle \nabla \rangle^{-1} \langle \nabla \rangle \bar{Z}] \big) ds \|_{L^2}$$

$$+ \sum_{b \lesssim a, a \gtrsim 1} \langle a \rangle \big(\| \int_0^t e^{isH} \mathcal{B}_{1,j}^{a,b,a}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} \langle \nabla \rangle^{-1} \langle \nabla \rangle Z, \langle \nabla \rangle^{-1+\frac{\epsilon}{2}} \langle \nabla \rangle^{1-\frac{\epsilon}{2}} \overline{JZ}] \big) ds \|_{L^2} \big) \tag{4.108}$$

$$\lesssim \sum_{b \lesssim a << 1} m^{\frac{\epsilon}{2}} \|\mathcal{B}_{1,j}^{a,b,a}\|_{[B^{1+\epsilon}]} \|1_{\{|\xi| \lesssim 1\}} U^{-\frac{\epsilon}{2}} JZ\|_{L_t^\infty L^{\frac{1}{\frac{1}{2}-\frac{\epsilon}{6}}}} \|1_{\{|\xi| \lesssim 1\}} U^{-1/6} Z\|_{L^{\frac{1}{1-\frac{\epsilon}{4}}}(L^6)}$$

$$+ \sum_{b \lesssim a, a \gtrsim 1} U(m)^{\frac{\epsilon}{2}} \|\mathcal{B}_{1,j}^{a,b,a}\|_{[B^{\frac{3}{2}}]} \langle a \rangle \langle b \rangle^{-1+\frac{\epsilon}{2}} \langle a \rangle^{-1+\frac{\epsilon}{2}} \|U^{-\frac{\epsilon}{2}} \langle \nabla \rangle^{1-\frac{\epsilon}{2}} JZ\|_{L_t^\infty L^{\frac{1}{\frac{1}{2}-\frac{\epsilon}{6}}}}$$

$$\times \|U^{-1/6} \langle \nabla \rangle Z\|_{L^{\frac{1}{\frac{3}{4}+\frac{\epsilon}{4}}}(L^6)}.$$

Indeed we are in the conditions of the lemma 3.7 since:

$$\frac{1}{+\infty} + 1 - \frac{\epsilon}{4} = 1 - \frac{\epsilon}{4} = \frac{1}{p},$$
$$\frac{1}{q(s)} = \frac{1}{2} - \frac{1}{3} - \frac{\epsilon}{3} = \frac{1}{6} - \frac{\epsilon}{3},$$
$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_1} - \frac{1}{q(s)} = \frac{1}{2} - \frac{\epsilon}{6} + \frac{1}{6} - \frac{1}{6} + \frac{\epsilon}{3} = \frac{1}{2} + \frac{\epsilon}{6},$$
$$\frac{1}{q} = \frac{1}{2} + \frac{2}{N} - \frac{2}{Np} = \frac{1}{2} + \frac{2}{3} - \frac{2}{3} + \frac{\epsilon}{6} = \frac{1}{2} + \frac{\epsilon}{6}.$$

Next we have by Sobolev embedding:

$$\|1_{\{|\xi| \lesssim 1\}} U^{-\frac{\epsilon}{2}} JZ\|_{L^\infty L^{\frac{1}{\frac{1}{2}-\frac{\epsilon}{6}}}} \lesssim \|JZ\|_{L^\infty(L^2)}. \tag{4.109}$$

and by Sobolev embedding:

$$\|\langle \nabla \rangle^{1-\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} JZ\|_{L^\infty L^{\frac{1}{\frac{1}{2}-\frac{\epsilon}{6}}}} \lesssim \|JZ\|_{L^\infty(L^2)}. \tag{4.110}$$

Next we have by using (4.60) and the fact that $\frac{1}{t^{\frac{1}{1-\frac{\epsilon}{4}}}}1_{t>1}$ be integrable:

$$\|U^{-\frac{1}{6}}Z\|\|_{L^{\frac{1}{1-\frac{\epsilon}{4}}}(L^6)} + \|U^{-1/6}\langle\nabla\rangle Z\|_{L^{\frac{1}{\frac{3}{4}+\frac{\epsilon}{4}}}(L^6)} \lesssim$$
$$\lesssim \|U^{-\frac{1}{6}}Z\|_{L^2_{t<1}(H^{1,6})} + \|U^{-\frac{1}{3}}Z\|_{L^{\frac{1}{1-\frac{\epsilon}{4}}}_{t>1}(H^{1,6})} + \|U^{-\frac{1}{3}}Z\|_{L^{\frac{1}{\frac{3}{4}+\frac{\epsilon}{4}}}_{t>1}(H^{1,6})}, \tag{4.111}$$
$$\lesssim \|U^{-\frac{1}{6}}Z\|_{L^2(H^{1,6})} + \|tU^{-\frac{1}{3}}Z\|_{L^\infty(H^{1,6})} \lesssim \|Z\|_{X\cap S}.$$

In the same way we deduce that:

$$\|U^{-\frac{1}{6}}Z\|_{L^{\frac{1}{1-\frac{\epsilon}{4}}}_{t>T}(L^6)} + \|U^{-1/6}\langle\nabla\rangle Z\|_{L^{\frac{1}{\frac{3}{4}+\frac{\epsilon}{4}}}_{t>T}(L^6)} \lesssim \|Z\|_X\left(\frac{1}{T^{\frac{\epsilon}{4}}} + \frac{1}{T^{\frac{1}{4}-\frac{\epsilon}{4}}}\right).$$

Remark 16. We observe that for the long time, we need to use the control in X in order to bound the estimate. Roughly speaking the symbol is less regular in low frequencies which implies to have better time decay in long time in order to bound the estimates.

We start with the case $M << 1$, we will use interpolation estimate $\|\cdot\|_{[B^{1+\epsilon}]} \lesssim \|\cdot\|_{[H^{1+\frac{\epsilon}{2}}]}^{\frac{1}{2}}\|\cdot\|_{[H^{1+\frac{3\epsilon}{2}}]}^{\frac{1}{2}}$ so that it is sufficient to perform estimate in $[H^{1+\epsilon'}]$ for any small enough ϵ'. It gives by using the lemma 4.3:

$$\sum_{b\lesssim a<<1} m^{\frac{\epsilon}{2}}\|\mathcal{B}_{1,j}^{a,b,a}\|_{[H^{1+\epsilon'}]} \lesssim \sum_{a<<1}\sum_{b\lesssim a} b^{\frac{\epsilon}{2}}b^{\frac{1}{2}-2\epsilon'}$$
$$\lesssim \sum_{a<<1} a^{\frac{\epsilon}{2}}a^{\frac{1}{2}-2\epsilon'} \lesssim 1. \tag{4.112}$$

Let us deal now with the case $M \geq 1$, we have for $|\epsilon'| > 0$:

$$\sum_{b\lesssim a,a\gtrsim 1} U(m)^{\frac{\epsilon}{2}}\|\mathcal{B}_{1,j}^{a,b,a}\|_{[H^{\frac{3}{2}+\epsilon'}]}\langle a\rangle\langle b\rangle^{-1+\frac{\epsilon}{2}}\langle a\rangle^{-1+\frac{\epsilon}{2}} \lesssim \sum_{a\gtrsim 1}\sum_{b\lesssim a} U(b)^{\frac{\epsilon}{2}}\langle b\rangle^{-1+\frac{\epsilon}{2}}b^{1-\epsilon'}\langle a\rangle^{-1+\frac{\epsilon}{2}},$$
$$\lesssim \sum_{a\gtrsim 1}\langle a\rangle^{-1+\frac{\epsilon}{2}}\left(\sum_{b\leq 1} b^{1-\epsilon'+\frac{\epsilon}{2}} + \sum_{1\leq b\lesssim a} b^{\frac{\epsilon}{2}-\epsilon'}\right) \lesssim \sum_{a\gtrsim 1}\langle a\rangle^{-1+\frac{\epsilon}{2}} + \sum_{a\gtrsim 1}\langle a\rangle^{-1+\frac{\epsilon}{2}}a^{\frac{\epsilon}{2}-\epsilon'} \lesssim 1.$$
$$\tag{4.113}$$

We proceed similarly in the region $c \lesssim b \sim a$. Let us deal now with the case $a \lesssim b \sim c$.

Using Bernstein inequalities and interpolation in Besov spaces we have for $\epsilon', \epsilon > 0$:

$$\| \sum_{a \lesssim b \sim c} \int_0^t e^{isH} \big(\mathcal{B}_{1,j}^{a,b,c}[JZ,\bar{Z}] - \mathcal{B}_1[Z,\overline{JZ}]\big) ds \|_{H^1} \lesssim \sum_b \sum_{a \lesssim b} \| \int_0^t e^{isH} \big(\mathcal{B}_{1,j}^{a,b,b}[JZ,\bar{Z}] - \mathcal{B}_1[Z,\overline{JZ}]\big) ds \|_{H^1}$$

$$\lesssim \sum_b \sum_{a \lesssim b} \langle a \rangle \| \int_0^t e^{isH} \big(\mathcal{B}_{1,j}^{a,b,a}[JZ,\bar{Z}] - \mathcal{B}_1[Z,\overline{JZ}]\big) ds \|_{L^2}$$

$$\lesssim \sum_b \| \int_0^t e^{isH} \sum_{a \lesssim b} \langle a \rangle \big(\mathcal{B}_{1,j}^{a,b,a}[JZ,\bar{Z}] - \mathcal{B}_1[Z,\overline{JZ}]\big) ds \|_{B_{2,1}^0}$$

$$\lesssim \sum_b \| \int_0^t e^{isH} \sum_{a \lesssim b} \langle a \rangle \big(\mathcal{B}_{1,j}^{a,b,a}[JZ,\bar{Z}] - \mathcal{B}_1[Z,\overline{JZ}]\big) ds \|_{H^{\epsilon'}}$$

$$\lesssim \sum_b \langle b \rangle^{\epsilon'} \| \int_0^t e^{isH} \sum_{a \lesssim b} \langle a \rangle \big(\mathcal{B}_{1,j}^{a,b,a}[JZ,\bar{Z}] - \mathcal{B}_1[Z,\overline{JZ}]\big) ds \|_{L^2}$$

$$\lesssim \sum_{b<<1} \langle b \rangle^{\epsilon'} \big(\| \int_0^t e^{isH} \big(\sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} JZ, \bar{Z}]\big) ds \|_{L^2}$$

$$+ \| \int_0^t e^{isH} \big(\sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} Z, \overline{JZ}]\big) ds \|_{L^2}\big)$$

$$+ \sum_{b \gtrsim 1} \langle b \rangle^{\epsilon'} \big(\| \int_0^t e^{isH} \big(\sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} \langle \nabla \rangle^{-1} \langle \nabla \rangle JZ, U^{\frac{1}{6}} U^{-\frac{1}{6}} \langle \nabla \rangle^{-1} \langle \nabla \rangle \bar{Z}]\big) ds \|_{L^2}$$

$$+ \| \int_0^t e^{isH} \big(\sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b}[U^{\frac{\epsilon}{2}} U^{-\frac{\epsilon}{2}} \langle \nabla \rangle^{-1} \langle \nabla \rangle Z, \langle \nabla \rangle^{-1} \langle \nabla \rangle \overline{JZ}]\big) ds \|_{L^2}\big)$$

$$\lesssim \sum_{b<<1} M^{\frac{\epsilon}{2}} \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[B^{1+\epsilon}]} \| U^{-\frac{\epsilon}{2}} JZ \|_{L_t^\infty L^{\frac{1}{\frac{1}{2} - \frac{\epsilon}{6}}}} \| U^{-1/6} Z \|_{L^{\frac{1}{1 - \frac{\varepsilon}{4}}}(L^6)}$$

$$+ \sum_{b \gtrsim 1} \langle M \rangle^{\epsilon'} U(M)^{\frac{\epsilon}{2}} \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[B^{\frac{3}{2}}]} \langle b \rangle^{-1} \langle b \rangle^{-1} \| U^{-\frac{\epsilon}{2}} \langle \nabla \rangle JZ \|_{L_t^\infty L^{\frac{1}{\frac{1}{2} - \frac{\epsilon}{6}}}} \| U^{-1/6} \langle \nabla \rangle Z \|_{L^{\frac{1}{\frac{3}{4} + \frac{\varepsilon}{4}}}(L^6)}. \tag{4.114}$$

Next we have when $M << 1$ and for ϵ' small enough by using lemma 4.3 (indeed we recall the interpolation estimate $\| \cdot \|_{[B^{1+\epsilon}]} \lesssim \| \cdot \|_{[H^{1+\frac{\epsilon}{2}}]}^{\frac{1}{2}} \| \cdot \|_{[H^{1+\frac{3\epsilon}{2}}]}^{\frac{1}{2}}$)

$$\sum_{b<<1} M^{\frac{\epsilon}{2}} \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[H^{1+\epsilon'}]} \lesssim \sum_{b<<1} b^{\frac{\epsilon}{2}} \sup_{a \lesssim b} \langle a \rangle \| \mathcal{B}_{1,j}^{a,b,b} \|_{[H^{1+\epsilon'}]}$$
$$\lesssim \sum_{b<<1} b^{\frac{\epsilon}{2}} b^{\frac{1}{2} - 2\epsilon'} \lesssim 1. \tag{4.115}$$

Indeed we recall that since the χ_a have disjoint supports:

$$\| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[H^{1+\epsilon'}]} = \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{L_\xi^\infty(H_\eta^{1+\epsilon'}) + L_\xi^\infty(H_\zeta^{1+\epsilon'})}$$

$$\lesssim \| \sum_{a \lesssim b} \langle a \rangle \chi_a \, \chi_b \, \chi_b \mathcal{B}_{1,j} \|_{L_\xi^\infty(H_\eta^{1+\epsilon'}) + L_\xi^\infty(H_\zeta^{1+\epsilon'})}$$

$$\lesssim \sup_{a \lesssim b} \langle a \rangle \| \mathcal{B}_{1,j}^{a,b,b} \|_{[H^{1+\epsilon'}]}.$$

In the same way we have for $|\epsilon''| > 0$ by using lemma 4.3:

$$\sum_{b \geq 1} M^{\epsilon'} U(M)^{\frac{\epsilon}{2}} \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[H^{\frac{3}{2}+\epsilon''}]} \langle b \rangle^{-1} \langle b \rangle^{-1}$$

$$\lesssim \sum_{b \geq 1} M^{\epsilon'} U(M)^{\frac{\epsilon}{2}} \sup_{a \lesssim b} \langle a \rangle \| \mathcal{B}_{1,j}^{a,b,b} \|_{[H^{\frac{3}{2}+\epsilon'}]} \langle b \rangle^{-1} \langle b \rangle^{-1} \tag{4.116}$$

$$\lesssim \sum_{b \geq 1} M^{\epsilon'} U(M)^{\frac{\epsilon}{2}} b^{1-\epsilon''} \langle b \rangle^{-1} \langle b \rangle^{-1} \lesssim \sum_{b \geq 1} b^{\epsilon'+|\epsilon''|} \langle b \rangle^{-1}$$

$$\lesssim 1.$$

Let us deal now with the term $\mathcal{B}_2$, we are going to proceed in a similar way than the previous case. Let us start with the case $b \lesssim a \sim c$, we have then using Bernstein inequality and lemma 3.7:

$$\| \int_0^T e^{isH} \sum_{b \lesssim a \sim c} \mathcal{B}_2^{a,b,c}[Z, \bar{Z}] ds \|_{H^1} \lesssim \sum_{b \lesssim a} \langle a \rangle \| \int_0^T e^{isH} \mathcal{B}_2^{a,b,a}[Z, \bar{Z}] ds \|_{L^2},$$

$$\lesssim \sum_{b \lesssim a << 1} \langle a \rangle U(b) U(a)^{\frac{1}{6}} \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_2^{a,b,a} \|_{[B^{1+\epsilon}]} \| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{L^{\frac{1}{1-\epsilon/2}}(H^{1,6})} \tag{4.117}$$

$$+ \sum_{b \lesssim, a \gtrsim 1} \langle a \rangle U(b) U(a)^{\frac{1}{6}} \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_2^{a,b,a} \|_{[B^{\frac{3}{2}}]} \| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{L^{4/3}(H^{1,6})}.$$

By using proposition 4.6, we observe that:

$$\| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{(L^{\frac{1}{1-\epsilon/2}} \cap L^{\frac{4}{3}})(H^{1,6})} \lesssim \| Z \|_{X \cap S}^2. \tag{4.118}$$

Let us deal now with the case $M << 1$, by interpolation and lemma 4.2 we have:

$$\sum_{b \lesssim a << 1} \langle a \rangle U(b) U(a)^{\frac{1}{6}} \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_2^{a,b,a} \|_{[H^{1+\epsilon'}]} \lesssim \sum_{a << 1} U(a)^{\frac{1}{6}} a^{-1} \sum_{b \lesssim a} U(b) \langle b \rangle^{-1} b^{\frac{1}{2}-2\epsilon'}$$

$$\lesssim \sum_{a << 1} U(a)^{\frac{1}{6}} a^{-1} \sum_{b \lesssim a << 1} b^{\frac{3}{2}-2\epsilon'} \lesssim \sum_{a << 1} U(a)^{\frac{1}{6}} a^{-1} a^{\frac{3}{2}-2\epsilon'} \lesssim 1. \tag{4.119}$$

In the case $M \geq 1$ we get by interpolation for $|\epsilon'| > 0$ and lemma 4.2:

$$\sum_{b \lesssim, a \gtrsim 1} \langle a \rangle U(b) U(a)^{\frac{1}{6}} \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_2^{a,b,c} \|_{[H^{\frac{3}{2}+\epsilon'}]} \lesssim \sum_{a \geq 1} U(a)^{\frac{1}{6}} \langle a \rangle^{-1} \sum_{b \lesssim a} U(b) \langle b \rangle^{-1} b^{-\epsilon'}$$

$$\lesssim \sum_{a \gtrsim 1} U(a)^{\frac{1}{6}} \langle a \rangle^{-1} \left(\sum_{b \lesssim 1} b^{1-\epsilon'} + \sum_{1 \leq b \lesssim a} b^{-\epsilon'-1} \right) \lesssim \sum_{a \gtrsim 1} U(a)^{\frac{1}{6}} \langle a \rangle^{-1} \lesssim 1. \tag{4.120}$$

The case $c \lesssim a \sim b$ can be treated similarly. Let us deal now with the case $a \lesssim b \sim c$, we have

then:

$$\|\sum_{a\lesssim b\sim c}\int_0^t e^{isH}\big(\mathcal{B}_2^{a,b,c}[Z^\pm,Z^\pm]\big)ds\|_{H^1}\lesssim\sum_{a\lesssim b\sim c}\langle a\rangle\|\int_0^t e^{isH}\big(\mathcal{B}_2^{a,b,c}[Z^\pm,Z^\pm]\big)ds\|_{L^2}$$

$$\lesssim\sum_{a\lesssim b\sim c}\|\int_0^t e^{isH}\big(\langle a\rangle\mathcal{B}_2^{a,b,c}[Z^\pm,Z^\pm]\big)ds\|_{L^2}\lesssim\|\int_0^t e^{isH}\sum_{a\lesssim b\sim c}\langle a\rangle\big(\mathcal{B}_2^{a,b,c}[Z^\pm,Z^\pm]\big)ds\|_{L^2}$$

$$\lesssim\|\sum_{a\lesssim b\sim c,M<<1}\langle a\rangle\int_0^t e^{isH}\big(\mathcal{B}_2^{a,b,c}[Z^\pm,Z^\pm]\big)ds\|_{L^2}$$

$$+\|\sum_{a\lesssim b\sim c,M\geq 1}\langle a\rangle\int_0^t e^{isH}\big(\mathcal{B}_2^{a,b,c}[Z^\pm,Z^\pm]\big)ds\|_{L^2}$$

$$\lesssim\sum_{b<<1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}\|\sum_{a\lesssim b}\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[B^{1+\epsilon}]}\|U^{-1}Z\|_{L^\infty(H^1)}\|U^{-1/6}Z\|_{L^{\frac{1}{1-\epsilon/2}}(H^{1,6})}$$

$$+\sum_{b\geq 1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}\|\sum_{a\lesssim b}\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[B^{\frac32}]}\|U^{-1}Z\|_{L^\infty(H^1)}\|U^{-1/6}Z\|_{L^{\frac43}(H^{1,6})}.$$

$$(4.121)$$

Let us deal now with the case $M\lesssim 1$ and working in $[H^{1+\epsilon'}]$ by interpolation. We have then:

$$\sum_{b<<1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}\|\sum_{a\lesssim b}\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{1+\epsilon'}]}$$

$$\lesssim\sum_{b<<1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}\sup_{a\lesssim b}\|\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{1+\epsilon'}]}$$

$$\lesssim\sum_{b<<1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}b^{\frac12-2\epsilon'}b^{-1}\langle b\rangle \qquad (4.122)$$

$$\lesssim\sum_{b<<1}b^{\frac76}b^{-\frac12-2\epsilon'}$$

$$\lesssim 1.$$

Let us deal now with the case $M\geq 1$ and we are going to work in $[H^{1+\epsilon'}]$ by interpolation. We have then:

$$\sum_{b\geq 1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}\|\sum_{a\lesssim b}\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[B^{\frac32+\epsilon'}]}\lesssim\sum_{b\geq 1}U(b)U(b)^{\frac16}\langle b\rangle^{-2}b^{-\epsilon'}\lesssim\sum_{b\geq 1}\langle b\rangle^{-2}\lesssim 1.$$

$$(4.123)$$

This conclude the part on the non resonant space region, let us deal now with the non resonant time region.

Remark 17. Let us observe that in the previous estimates, we need to distinguish the case of the low frequencies and of the high frequencies. Indeed it is due to the fact that in low frequencies when $M<<1$ then the symbol is less regular that it is why we fork only in $[H^{1+\epsilon}]$ but this is compensate by an additional regularity on Z and JZ due to the control of the norm X. It is of course the opposite in high frequencies. Let us mention in addition that the case $\frac{N}{2}$ corresponds to the classical of Strichartz estimate for un product.

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Time non resonant

Let us start with dealing with the first term on the right hand side of (4.102), we consider for beginning the case $b \lesssim a \sim c$ and we have then by using lemma 3.7:

$$\| \int_0^T e^{isH} \sum_{b \lesssim a \sim c} \mathcal{B}_3[Z, \bar{Z}] ds \|_{H^1} \lesssim$$

$$\lesssim \sum_{b \lesssim a, a << 1} U(b) U(a)^{\frac{1}{6}} \langle a \rangle \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_3^{a,b,a} \|_{[B^{1+\epsilon}]} \| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{L^{\frac{1}{1-\epsilon/2}}(H^{1,6})}$$

$$+ \sum_{b \lesssim a \sim c, a \gtrsim 1} U(b) U(a)^{\frac{1}{6}} \langle a \rangle \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_3^{a,b,a} \|_{[B^{\frac{3}{2}}]} \| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{L^{4/3}(H^{1,6})}.$$
$$(4.124)$$

Let us treat now the case $M << 1$ we have then by interpolation:

$$\sum_{b \lesssim a, a << 1} U(b) U(a)^{\frac{1}{6}} \langle a \rangle \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_3^{a,b,a} \|_{[H^{1+\epsilon}]} \lesssim \sum_{a << 1} \sum_{b \lesssim a} b a^{\frac{1}{6}} (\frac{\langle a \rangle}{a})^{1+\epsilon} b^{\frac{1}{2}-\epsilon} \langle a \rangle^{-1},$$

$$\lesssim \sum_{a << 1} \sum_{b \lesssim a} b a^{\frac{1}{6}} a^{-1-\epsilon} b^{\frac{1}{2}-\epsilon} \lesssim \sum_{a << 1} a^{-\frac{5}{6}-\epsilon} \sum_{b \lesssim a} b^{\frac{3}{2}-\epsilon} \lesssim \sum_{a << 1} a^{-\frac{5}{6}-\epsilon} a^{\frac{3}{2}-\epsilon} \lesssim 1.$$
$$(4.125)$$

We are now going to treat the case $M \geq 1$ using again interpolation and we have with $|\epsilon| > 0$ small enough:

$$\sum_{b \lesssim a, a \gtrsim 1} U(b) U(a)^{\frac{1}{6}} \langle a \rangle \langle b \rangle^{-1} \langle a \rangle^{-1} \| \mathcal{B}_3^{a,b,a} \|_{[H^{\frac{3}{2}+\epsilon}]} \lesssim \sum_{a \geq 1} \sum_{b \lesssim a} U(b) (\frac{\langle a \rangle}{a})^{\frac{3}{2}+\epsilon} \langle b \rangle^{-1} b^{-\epsilon} \langle a \rangle^{-1}$$

$$\lesssim \sum_{a \gtrsim 1} \langle a \rangle^{-1} (\sum_{b \lesssim 1} b^{1-\epsilon} + \sum_{1 \leq b \lesssim a} b^{-1-\epsilon}) \lesssim 1.$$
$$(4.126)$$

We proceed similarly to deal with the case $c \lesssim a \sim b$. Let us treat now the case $a \lesssim b \sim c$ which follows the same lines than in the previous case $\mathcal{B}_1$ with $\epsilon' > 0$:

$$\| \sum_{a \lesssim b \sim c} \int_0^t e^{isH} (\mathcal{B}_3^{a,b,c}[Z^\pm, Z^\pm]) ds \|_{H^1} \lesssim \sum_b \sum_{a \lesssim b} \langle a \rangle \| \int_0^t e^{isH} (\mathcal{B}_3^{a,b,b}[Z, \bar{Z}]) ds \|_{L^2}$$

$$\lesssim \sum_b \sum_{a \lesssim b} \| \int_0^t e^{isH} (\langle a \rangle \mathcal{B}_3^{a,b,b}[Z, \bar{Z}]) ds \|_{L^2}$$

$$\lesssim \sum_b \langle b \rangle^{\epsilon'} \| \int_0^t e^{isH} \sum_{a \lesssim b} \langle a \rangle (\mathcal{B}_3^{a,b,b}[Z, \bar{Z}]) ds \|_{L^2}$$

$$\lesssim \| \sum_{a \lesssim b \sim c, M << 1} \langle a \rangle \int_0^t e^{isH} (\mathcal{B}_3^{a,b,b}[Z, \bar{Z}]) ds \|_{L^2} + \| \sum_{a \lesssim b \sim c, M \geq 1} \langle a \rangle \int_0^t e^{isH} (\mathcal{B}_3^{a,b,b}[Z, \bar{Z}]) ds \|_{L^2}$$

$$\lesssim \sum_{b << 1} U(b) U(b)^{\frac{1}{6}} \langle b \rangle^{-2} \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[B^{1+\epsilon}]} \| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{L^{\frac{1}{1-\epsilon/2}}(H^{1,6})}$$

$$+ \sum_{b \geq 1} U(b) U(b)^{\frac{1}{6}} \langle b \rangle^{-2} \| \sum_{a \lesssim b} \langle a \rangle \mathcal{B}_{1,j}^{a,b,b} \|_{[B^{\frac{3}{2}}]} \| U^{-1} Z \|_{L^\infty(H^1)} \| U^{-1/6} Z \|_{L^{4/3}(H^{1,6})}.$$
$$(4.127)$$

Let us deal now with the case $M << 1$, we have then:

$$\sum_{b<<1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}\|\sum_{a\lesssim b}\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{1+\epsilon}]} \lesssim \sum_{b<<1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}\sup_{a\lesssim b}\|\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{1+\epsilon}]},$$

$$\lesssim \sum_{b<<1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}\sup_{a\lesssim b}\|\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{1+\epsilon}]} \lesssim \sum_{b<<1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}(\frac{\langle b\rangle}{b})^{1+\epsilon}b^{\frac{1}{2}-\epsilon},$$

$$\lesssim \sum_{b<<1} b^{\frac{7}{6}}b^{-\frac{1}{2}-2\epsilon} \lesssim 1.$$

(4.128)

In the case $M \geq 1$ we have for $|\epsilon| > 0$ small enough:

$$\sum_{b\geq 1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}\|\sum_{a\lesssim b}\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{\frac{3}{2}+\epsilon}]} \lesssim \sum_{b\geq 1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}\sup_{a\lesssim b}\|\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{\frac{3}{2}+\epsilon}]},$$

$$\lesssim \sum_{b\geq 1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}\sup_{a\lesssim b}\|\langle a\rangle\mathcal{B}_{1,j}^{a,b,b}\|_{[H^{\frac{3}{2}+\epsilon}]} \lesssim \sum_{b\geq 1} U(b)U(b)^{\frac{1}{6}}\langle b\rangle^{-2}(\frac{\langle b\rangle}{b})^{\frac{3}{2}+\epsilon}b^{-\epsilon},$$

(4.129)

$$\lesssim \sum_{b<<1} \langle b\rangle^{-\frac{1}{2}+\epsilon}b^{-\frac{3}{2}} \lesssim 1.$$

Let us deal with the second and third term on the right handside of (4.102), we have then by applying lemma 3.7 and Bernstein lemma in the case $b \lesssim a \sim c$:

$$\|\int_0^T e^{isH}\Big(\sum_{b\lesssim a\sim c}\mathcal{B}_3^{a,b,c}[s\mathcal{N}Z,Z] + \mathcal{B}_3[Z,s\mathcal{N}Z]\Big)ds\|_{H^1}$$

$$\lesssim \sum_{b\lesssim a\sim c} U(l)^{\frac{1}{2}}U(M)\langle a\rangle\langle b\rangle^{-1}\langle c\rangle^{-1}\|\mathcal{B}_3^{a,b,a}\|_{[B^{\frac{1}{2}+\epsilon}]+[B^{3/2}]}\|U^{-\frac{1}{2}}\langle\nabla\rangle t\mathcal{N}Z\|_{(L^{\frac{1}{1-\epsilon/2}}\cap L^2)(L^3)}$$

$$\times \|U^{-1}\langle\nabla\rangle Z\|_{L^\infty(L^2)}$$

(4.130)

$$\lesssim \sum_{b\lesssim a\sim c} U(b)^{\frac{1}{2}}U(a)\langle b\rangle^{-1}\|\mathcal{B}_3^{a,b,a}\|_{[B^{\frac{1}{2}+\epsilon}]+[B^{3/2}]}\|U^{-\frac{1}{2}}\langle\nabla\rangle t\mathcal{N}Z\|_{(L^{\frac{1}{1-\epsilon/2}}\cap L^2)(L^3)}$$

$$\times \|U^{-1}\langle\nabla\rangle Z\|_{L^\infty(L^2)}.$$

Let us deal now with the case $M << 1$ which gives:

$$\sum_{b\lesssim a<<1} U(b)^{\frac{1}{2}}U(a)\langle b\rangle^{-1}\|\mathcal{B}_3^{a,b,a}\|_{[H^{1+\epsilon}]} \lesssim \sum_{a<<1}\sum_{b\lesssim a} U(b)^{\frac{1}{2}}U(a)\langle b\rangle^{-1}(\frac{\langle a\rangle}{a})^{1+\epsilon}b^{\frac{1}{2}-\epsilon}\langle a\rangle^{-1},$$

$$\lesssim \sum_{a<<1}\sum_{b\lesssim a} b^{\frac{1}{2}}a\,a^{-1-\epsilon}b^{\frac{1}{2}-\epsilon} \lesssim \sum_{a<<1} a^{-\epsilon}\sum_{b\lesssim a} b^{1-\epsilon} \lesssim \sum_{a<<1} a^{-\epsilon}a^{1-\epsilon} \lesssim 1.$$

(4.131)

and for $M \geq 1$ we get with $|\epsilon| > 0$ small enough:

$$\sum_{b\lesssim a,\, a\geq 1} U(b)^{\frac{1}{2}}U(a)\langle b\rangle^{-1}\|\mathcal{B}_3^{a,b,a}\|_{[H^{\frac{3}{2}+\epsilon}]} \lesssim \sum_{a\geq 1}\sum_{b\lesssim a} U(b)^{\frac{1}{2}}U(a)\langle b\rangle^{-1}(\frac{\langle a\rangle}{a})^{\frac{3}{2}+\epsilon}b^{-\epsilon}\langle a\rangle^{-1},$$

$$\lesssim \sum_{a\geq 1}\langle a\rangle^{-1}\sum_{b\lesssim a} U(b)^{\frac{1}{2}}\langle b\rangle^{-1}b^{-\epsilon} \lesssim 1.$$

(4.132)

We proceed similarly in the case $c \lesssim a \sim b$ and $a \lesssim b \sim c$.

It remains to estimate the normal form $\mathcal{N}_Z$, it means the part $U^{-\frac{1}{2}}\langle\nabla\rangle t\mathcal{N}_Z$. Let us start with

the bilinear terms with $j = 3, 4$, we have then by using proposition (3.4) (indeed we recall that the bilinear Fourier multiplier $U^{-1}B_j$ verify the hypothesis of proposition (3.4))

$$\|U^{-\frac{1}{2}}\langle\nabla\rangle tB_j[v^\pm, v^\pm]\|_{(L_{t<T}^{\frac{1}{1-\epsilon/2}} \cap L_{t<T}^2)(L^3)} \lesssim \|U^{-\frac{1}{2}}B_j[v^\pm, v^\pm]\|_{L^2(H^{1,3})},$$
$$\lesssim \|v\|_{L^2(H^{1,6})}\|v\|_{L^\infty(L^6)}. \tag{4.133}$$

and:

$$\|U^{-\frac{1}{2}}\langle\nabla\rangle tB_j[v^\pm, v^\pm]\|_{(L_{t\geq T}^{\frac{1}{1-\epsilon/2}} \cap L_{t\geq T}^2)(L^3)} \lesssim \|U^{-\frac{1}{2}}t^2 B_j[v^\pm, v^\pm]\|_{L^\infty(H^{1,3})}\|\frac{1}{t}\|_{L_{t\geq T}^{\frac{1}{1-\epsilon/2}} \cap L_{t\geq T}^2},$$
$$\lesssim \frac{1}{T^{\frac{\epsilon}{2}}}\|tv\|_{L^\infty(H^{1,6})}^2, \tag{4.134}$$

It remains to deal with the cubic terms and the quartic terms. We have then for $j = 1, 2, 3, 4$ using (4.49), the proposition 3.4, the fact that by Sobolev embedding $U^{-\frac{1}{2}}(H^1 \cap H^{1,3}) \subset H^{1,3}$ (indeed we have $U^{-\frac{1}{2}}H^1_{<1} \subset H^{1,3}_{<1}$ by Sobolev embedding en low frequencies and $U^{-\frac{1}{2}}H^{1,3}_{>1} = H^{1,3}_{>1}$)and by Sobolev embedding:

$$\|U^{-\frac{1}{2}}\langle\nabla\rangle tC_j[v^\pm, v^\pm, v^\pm]\|_{(L^{\frac{1}{1-\epsilon/2}} \cap L^2)(L^3)} \lesssim \|U^{-\frac{1}{2}}tC_j[v^\pm, v^\pm, v^\pm]\|_{(L^1 \cap L^2)(H^{1,3})}$$
$$\lesssim \|tC_j[v^\pm, v^\pm, v^\pm]\|_{(L^1 \cap L^2)(H^1 \cap H^{1,3})}, \tag{4.135}$$
$$\lesssim \|tv\|_{L^\infty(H^{1,6})}\|U^{-1}v\|_{L^\infty(H^1) \cap L^2(H^{1,6})}^2$$

For the quartic term we have using that $U^{-\frac{1}{2}}(H^1 \cap H^{1,3}) \subset H^{1,3}$ and the proposition 3.4 since we have

$$Q_1(u) = -2\frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\langle(\xi_1, \xi_2)\rangle^2}[(2Id - \Delta)^{-\frac{1}{2}}u_1, (2Id - \Delta)^{-\frac{1}{2}}(|u|^2 u_2)]$$
$$- 2\frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\langle(\xi_1, \xi_2)\rangle^2}[(2Id - \Delta)^{-\frac{1}{2}}u_2, (2Id - \Delta)^{-\frac{1}{2}}(|u|^2 u_1)],$$

we get by Sobolev embedding:

$$\|U^{-\frac{1}{2}}\langle\nabla\rangle tQ_1(u)\|_{(L^{\frac{1}{1-\epsilon/2}} \cap L^2)(L^3)} \lesssim \|Q_1(u)\|_{(L^{\frac{1}{1-\epsilon/2}} \cap L^2)(H^1 \cap H^{1,3})}$$
$$\|Q_1(u)\|_{(H^1 \cap H^{1,3})} \lesssim \|u\|_{L^6}^4 \lesssim \langle t\rangle^{-2}\|v\|_X^4. \tag{4.136}$$

It concludes the estimates on $\mathcal{N}_Z$. Let us finish now by dealing with the time boundary term. Using the corollary 3.37 with $s = 1$ and $s = \frac{3}{2}$ we have in the case $b \lesssim a \sim c$:

$$\|\sum_{b \lesssim a \sim c} e^{itH}\mathcal{B}_3^{a,b,b}[tZ^\pm, Z^\pm]\|_{H^1} \lesssim \sum_{b \lesssim a \sim c} U(b)U(a)^{\frac{1}{3}}\langle a\rangle\langle b\rangle^{-\frac{1}{2}}\langle a\rangle^{-1}\|\mathcal{B}^{a,b,b}\|_{[B^1]+[B^{\frac{3}{2}}]}$$
$$\times \|U^{-\frac{1}{3}}tZ\|_{H^{1,6}}\|U^{-1}Z\|_{H^1}, \tag{4.137}$$

4.8 Decomposition in non space-time resonant region, Proof of the lemma 4.2 and 4.3

In this section, the most important of the proof of the theorem 2.2 of [29], we describe the decomposition of the space in non space and non time resonant regions used by Gustafson,

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Nakanishi and Tsai in [29] for the phase $\Omega = H(\xi) + H(\eta) - H(\xi - \eta)$.

More precisely we are interested in proving the lemma 4.2 and 4.3, let us recall that $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ depend on the phase $\Omega = H(\xi) + H(\eta) - H(\xi - \eta)$. We are going to deal with the bilinear term of the form $e^{itH}B_j[\bar{Z}, Z]$ with $j = 3, 4$. Let us recall that:

$$e^{itH}B_j[\bar{Z}, Z] = \mathcal{F}^{-1} \int e^{it\Omega} B_j(\eta, \xi - \eta) \mathcal{F}(Z)(\eta) \mathcal{F}(\bar{Z})(\xi - \eta) d\eta. \tag{4.138}$$

In this case we have:

$$\begin{aligned} \nabla_\xi^{(\eta)} \Omega &= \nabla H(\xi) - \nabla H(\xi - \eta), \\ \nabla_\eta \Omega &= \nabla H(\eta) + \nabla H(\xi - \eta). \end{aligned} \tag{4.139}$$

Recall the notations:

$$\begin{aligned} &|\xi| \sim a, \ |\eta| \sim b \ |\zeta| \sim c, \\ &M = \max(a, b, c), \ m = \min(a, b, c), \ l = \min(b, c). \end{aligned} \tag{4.140}$$

We note as in [29]:

$$\alpha = |\widehat{\zeta} - \widehat{\xi}|, \ \beta = |\widehat{\zeta} + \widehat{\eta}|, \ \eta^\perp = -\widehat{\xi} \times \widehat{\xi} \times \eta. \tag{4.141}$$

Remark 18. α corresponds here to an angular estimate between ξ and ζ.

Gustafson et al in [28] decompose the (ξ, η, ζ) region (with $\zeta = \xi - \eta$) into the following five cases where each later case excludes the previous ones:

1. $|\eta| \sim |\xi| >> |\zeta|$ (or $c << b \sim a$) temporally non-resonant.

2. $\alpha > \sqrt{3}$ temporally non-resonant.

3. $|\zeta| \geq 1$ spatially non-resonant.

4. $|\eta^\perp| << M|\eta|$ temporally non resonant.

5. Otherwise spatially non-resonant.

Basic backgrounds

Let us recall now that for $H(\xi) = G(|\xi|)$ and we shall note the radial derivative $H' = G'$. Furthermore elementary computations show that:

$$G'(r) = \frac{2(1 + r^2)}{\langle r \rangle}, \ G''(r) = \frac{2r(3 + r^2)}{\langle r \rangle^3} \tag{4.142}$$

For the difference we have for any $r \geq s \geq 0$

$$G(r) - G(s) \sim \langle r \rangle (r - s), \ G'(r) - G'(s) \sim \frac{r}{\sqrt{1 + r^2}}(r - s). \tag{4.143}$$

Furthermore we have using (4.143) for $|\xi| \geq |\eta|$:

$$\begin{aligned} |\nabla H(\xi) - \nabla H(\eta)| &= \Big(\sum_i \Big| G'(|\xi|)\frac{\xi_i}{|\xi|} - G'(|\eta|)\frac{\eta_i}{|\eta|}\Big|^2\Big)^{\frac{1}{2}} \\ &= |G'(|\xi|)\widehat{\xi} - G'(|\eta|)\widehat{\eta}| \\ &\sim |G'(|\xi|) - G'(|\eta|)| + G'(|\eta|)|\widehat{\xi} - \widehat{\eta}|, \\ &\sim \frac{|\xi|}{\langle \xi \rangle}\big||\xi| - |\eta|\big| + \langle \eta \rangle |\widehat{\xi} - \widehat{\eta}|. \end{aligned} \tag{4.144}$$

We have used the fact that for any $r \geq s \geq 0$ and unit vector α, β we have $|r\alpha - s\beta| \sim |r - s| + s|\alpha - \beta|$ which follows from the identity:

$$|r\alpha - s\beta|^2 = r^2 + s^2 - 2rs\alpha \cdot \beta = r^2 + s^2 - 2sr + sr|\alpha - \beta|^2 = (r - s)^2 + + sr|\alpha - \beta|^2.$$

Furthermore we verify easily that for the higher derivatives, we have:

$$|\nabla^k H(\xi)| \lesssim \frac{\langle \xi \rangle}{|\xi|^{k-1}}, \tag{4.145}$$

for any $k \in \mathbb{N}$. And by Taylor formula when $|\xi| \geq |\eta|$ we have:

$$|\nabla^k H(\xi) - \nabla^k H(\eta)| \lesssim \frac{\langle \xi \rangle |\xi - \eta|}{|\xi| |\eta|^{k-1}}. \tag{4.146}$$

Indeed we have:

$$|\nabla^k H(\xi) - \nabla^k H(\eta)| \leq \int_{|\eta|}^{|\xi|} |\nabla^{k+1} H(s)| ds \leq \int_{|\eta|}^{|\xi|} \frac{s+1}{s^k} ds,$$

$$\leq (|\xi| - |\eta|) \frac{|\eta| + 1}{|\xi| |\eta|^{k-1}} \leq (|\xi| - |\eta|) \frac{\langle \xi \rangle}{|\xi| |\eta|^{k-1}}$$

Finally we have:

$$\alpha^2 = \frac{|\eta|^2 - (|\xi| - |\zeta|)^2}{|\xi| |\zeta|}. \tag{4.147}$$

and:

$$\beta^2 = \frac{(|\zeta| + |\eta|)^2 - |\zeta - \eta|^2}{|\xi| |\zeta|}. \tag{4.148}$$

Let us recall that in the sequel we will have:

$$\nabla_\eta B_j(\eta, \zeta) = (\nabla_{\xi_1} B_j - \nabla_{\xi_2} B_j)(\eta, \zeta) = U(\xi) \nabla_\eta B_j'(\eta, \zeta),$$
$$\nabla_\zeta B_j(\eta, \zeta) = (\nabla_{\xi_2} B_j - \nabla_{\xi_1} B_j)(\xi - \zeta, \zeta) = U(\xi) \nabla_\zeta B_j'(\eta, \zeta). \tag{4.149}$$

Furthermore we deduce that:

$$|\nabla_\eta^k B_j'(\eta, \zeta)| \lesssim \frac{1}{\min(b, c)^k}. \tag{4.150}$$

Temporally non resonant cases, estimates for $\mathcal{B}_3$ in these cases

Let us recall that in this case we have:

$$\mathcal{B}_3 = \frac{\nabla_\xi \Omega}{\Omega} B_j.$$

1. Let us start with the first case $|\eta| \sim |\xi| >> |\zeta|$, we shall verify in a first time that Ω does not vanish in this region. Indeed we have:

$$|\Omega| = \Omega = H(\xi) + H(\eta) - H(\zeta) \geq H(M) \sim M \langle M \rangle. \tag{4.151}$$

We verify that:

$$\nabla_\zeta^k \Omega = -\nabla^k H(\zeta) + \nabla^k H(-\eta), \quad \nabla_\zeta^k \nabla_\xi^{(\eta)} \Omega = -\nabla^{1+k} H(\zeta), \tag{4.152}$$

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and we have from (4.145):

$$|\nabla_\zeta \Omega| \lesssim |\nabla H(\eta)| \lesssim \langle M \rangle,$$
$$|\nabla_\zeta^2 \Omega| \lesssim \frac{\langle m \rangle}{m} + \frac{\langle M \rangle}{M} \lesssim \frac{\langle m \rangle}{m},$$
$$|\nabla_\xi^{(\eta)} \Omega| \lesssim |\nabla H(\xi) - \nabla H(\zeta)| \lesssim \langle M \rangle, \tag{4.153}$$
$$|\nabla_\zeta^k \nabla_\xi^{(\eta)} \Omega| \lesssim \frac{\langle m \rangle}{m^k}.$$

We deduce from (4.49) and (4.153):

$$\|\frac{\nabla_\xi^{(\eta)} \Omega}{\Omega} B_j^{a,b,c}\|_{L_\xi^\infty(L_\zeta^2)} \lesssim \|\frac{\nabla_\xi \Omega}{\Omega} \chi^a(\xi) \chi^b(\eta) \chi^c(\zeta) B_j(\eta, \zeta)\|_{L_\xi^\infty(L_\zeta^2)} \lesssim \frac{\langle M \rangle}{M \langle M \rangle} \frac{M}{\langle M \rangle} m^{\frac{3}{2}}, \tag{4.154}$$

where $m^{\frac{3}{2}}$ corresponds to the square root of the measure of the dyadic shell of size m. To estimate the H_ζ^1 norm we compute:

$$\nabla_\zeta(\frac{\nabla_\xi^{(\eta)} \Omega}{\Omega} B_j^{a,b,c}) = \frac{\nabla_\zeta \nabla_\xi^{(\eta)} \Omega}{\Omega} B_j^{a,b,c} - \frac{\nabla_\xi^{(\eta)} \Omega \cdot \nabla_\zeta \Omega}{\Omega^2} B_j^{a,b,c} + (\frac{\nabla_\xi^{(\eta)} \Omega}{\Omega} \nabla_\zeta B_j^{a,b,c}).$$

We have then:

$$\|\frac{\nabla_\zeta \nabla_\xi^{(\eta)} \Omega}{\Omega} B_j^{a,b,c}\|_{L_\xi^\infty(L_\zeta^2)} \lesssim \frac{\langle m \rangle}{m} \frac{1}{M \langle M \rangle} \frac{M}{\langle M \rangle} m^{\frac{3}{2}} \lesssim \frac{\langle m \rangle}{\langle M \rangle} m^{\frac{3}{2}-1} \langle M \rangle^{-1} \lesssim m^{\frac{3}{2}-1} \langle M \rangle^{-1},$$

$$\|\frac{\nabla_\xi^{(\eta)} \Omega \cdot \nabla_\zeta \Omega}{\Omega^2} B_j^{a,b,c}\|_{L_\xi^\infty(L_\zeta^2)} \lesssim \langle M \rangle \frac{\langle M \rangle}{M^2 \langle M \rangle^2} m^{\frac{3}{2}} \frac{M}{\langle M \rangle} \lesssim m^{\frac{3}{2}} \langle M \rangle^{-1} M^{-1} \lesssim m^{\frac{3}{2}-1} \langle M \rangle^{-1},$$

$$\|(\frac{\nabla_\xi^{(\eta)} \Omega}{\Omega} \nabla_\zeta B_j^{a,b,c})\|_{L_\xi^\infty(L_\zeta^2)} \lesssim \frac{\langle M \rangle}{M \langle M \rangle} \frac{U(M)}{m} m^{\frac{3}{2}} \lesssim m^{\frac{3}{2}-1} \langle M \rangle^{-1}, \tag{4.155}$$

We proceed similarly in the case of $\nabla_\zeta^2(\frac{\nabla_\xi \Omega}{\Omega} B_j^{a,b,c})$ and we obtain finally:

$$\|\frac{\nabla_\xi \Omega}{\Omega} B_j^{a,b,c}\|_{L_\xi^\infty(\dot{H}_\zeta^s)} \lesssim \frac{\langle M \rangle}{M \langle M \rangle} \frac{m^{\frac{3}{2}}}{m^s} \frac{M}{\langle M \rangle} = m^{\frac{3}{2}-s} \langle M \rangle^{-1}, \tag{4.156}$$

for $j = 3, 4$ and $s = 0, 1, 2$ which implies(4.106) in this case by interpolation.

2. Let us start with the second case $\alpha > \sqrt{3}$ where we exclude the case 1.
We cutoff the multipliers by:
$$\chi_{[\alpha]} = \Gamma(\widehat{\xi} - \widehat{\zeta}), \tag{4.157}$$

for a fixed $\Gamma \in C^\infty(\mathbb{R}^3)$ satisfying $\Gamma(x) = 1$ for $|x| \geq \sqrt{3}$ and $\Gamma(x) = 0$ for $|x| \leq \frac{3}{2}$. By excluding the previous case we must have $|\zeta| \sim M$ since we are in the case $a \lesssim b \sim c$ or $b \lesssim c \sim a$. Since $\alpha > \sqrt{3} > \frac{3}{2}$ we have by using (4.147):

$$|\eta|^2 - |\zeta|^2 > |\xi|^2 + \frac{1}{4}|\xi| |\zeta|. \tag{4.158}$$

and we deduce that we are working in a region of the form $D_1(\xi) = \{\eta \in \mathbb{R}^N; \ |\eta| - |\zeta| > (1 - 2\delta)|\xi|\}$. We have then:

$$|\nabla_\eta \chi_{[\alpha]}| = |\nabla_\eta^k \chi_{[\alpha]}(\widehat{\xi} - \frac{\xi - \eta}{|\xi - \eta|})| = |\nabla \chi_{[\alpha]}(\widehat{\xi} - \frac{\xi - \eta}{|\xi - \eta|})\nabla_\eta(-\frac{\xi - \eta}{|\xi - \eta|})|$$
$$\lesssim \frac{1}{|\zeta|} \lesssim M^{-1}, \tag{4.159}$$

and similarly we have:

$$|\nabla_\eta^k \chi_{[\alpha]}| \sim M^{-k} \tag{4.160}$$

for any $k \in \mathbb{N}$.

Let us prove now in this case (4.106). We have since $\alpha > \sqrt{3} > \frac{3}{2}$ we have by using (4.158) $|\eta| - |\zeta| \gtrsim |\xi|$ and since $|\zeta| \sim M$ we deduce that $m \sim |\xi|$. We have then to check that Ω does not vanish:

$$|\Omega| = \Omega = H(\xi) + H(\eta) - H(\zeta) \geq H(\eta) - H(\zeta) = (|\eta| - |\zeta|)\langle|\eta|\rangle + |\zeta|(\langle|\eta|\rangle - \langle|\zeta|\rangle),$$
$$\geq \langle M\rangle|\xi| \sim \langle M\rangle m, \tag{4.161}$$

and from (4.144) and the fact that $\nabla_\eta \Omega = \nabla H(\eta) + \nabla H(\zeta) = \nabla H(\eta) - \nabla H(-\zeta)$ we have:

$$|\nabla_\xi^{(\eta)} \Omega| \lesssim \frac{M}{\langle M\rangle}|\eta| + \langle\xi\rangle\alpha,$$
$$|\nabla_\eta \Omega| \lesssim \frac{|\zeta|}{\langle\zeta\rangle}||\zeta| - |\eta|| + \langle\eta\rangle\beta \lesssim \frac{M}{\langle M\rangle}|\xi| + \langle\eta\rangle\beta. \tag{4.162}$$

From $\alpha > \frac{3}{2}$, it yields $\beta < c < 1$ indeed we have:

$$\beta^2 = 2 + 2\frac{\zeta \cdot \eta}{|\zeta||\eta|},$$
$$\alpha^2 = 2 - 2\frac{\xi \cdot \zeta}{|\xi||\zeta|} = 2 - 2\frac{\zeta \cdot \zeta}{|\xi||\zeta|} - 2\frac{\eta \cdot \zeta}{|\xi||\zeta|},$$

and we have:

$$2\frac{\zeta \cdot \eta}{|\zeta||\eta|} = (2 - \alpha^2 - 2\frac{|\zeta|}{|\xi|})\frac{|\xi|}{|\eta|},$$
$$\beta^2 = 2(1 - \frac{|\zeta|}{|\eta|}) + \frac{|\xi|}{|\eta|}(2 - \alpha^2).$$

Since $\beta < c < 1$ it implies that $\sin(\frac{\widehat{(\zeta,\eta)}}{2}) \sim 1$ and we recall that:

$$\alpha = 2\sin(\frac{\widehat{(\xi,\zeta)}}{2}), \ \beta = 2\cos(\frac{\widehat{(\eta,\zeta)}}{2}).$$

By using the sine theorem and the fact that $\sin(\frac{\widehat{(\zeta,\eta)}}{2}) \sim 1$ we have:

$$\frac{a}{\sin(\widehat{(\eta,\zeta)})} = \frac{b}{\sin(\widehat{(\xi,\zeta)})},$$
$$= \frac{a}{2\sin(\frac{\widehat{(\eta,\zeta)}}{2})\cos(\frac{\widehat{(\eta,\zeta)}}{2})} \sim \frac{a}{\beta}$$
$$= \frac{b}{2\sin(\frac{\widehat{(\xi,\zeta)}}{2})\cos(\frac{\widehat{(\xi,\zeta)}}{2})} \gtrsim \frac{b}{\alpha}$$

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thus $\beta \lesssim \frac{a\alpha}{b} \sim \frac{m}{M}$. Thus we get by using (4.162), (4.145) and the fact that $\beta \lesssim \frac{m}{M}$:

$$|\nabla_\xi^{(\eta)}\Omega| \lesssim \frac{M^2}{\langle M \rangle} + \langle m \rangle \lesssim \langle M \rangle, \ |\nabla_\eta \Omega| \lesssim \frac{Mm}{\langle M \rangle} + \frac{\langle M \rangle m}{M} \lesssim \frac{|\Omega|}{M},$$

$$|\nabla_\eta^k \nabla_\xi \Omega| = |\nabla^{1+k} H(\zeta)| \lesssim \frac{\langle M \rangle}{M^k}. \tag{4.163}$$

By (4.146) we get:

$$|\nabla_\eta^2 \Omega| = |\nabla^2 H(\eta) - \nabla^2 H(\zeta)| = |\nabla^2 H(\eta) - \nabla^2 H(-\zeta)| \lesssim \frac{\langle M \rangle m}{M^2} \lesssim \frac{|\Omega|}{M^2}. \tag{4.164}$$

We get finally:

$$\left\| \frac{\nabla_\xi^{(\eta)}\Omega}{\Omega} \chi_{[\alpha]} B_j^{a,b,c} \right\|_{L_\xi^\infty(\dot{H}_\eta^s)} \lesssim \|\mathcal{B}_3^{a,b,c}\|_{L_\xi^\infty(\dot{H}_\zeta^s)} \lesssim \frac{\langle M \rangle}{m \langle M \rangle} \frac{M^{\frac{3}{2}}}{M^s} \frac{m}{\langle m \rangle} = \frac{M^{\frac{3}{2}-s}}{\langle m \rangle} \lesssim \frac{l^{\frac{3}{2}-s}}{\langle a \rangle}, \tag{4.165}$$

for $j = 3, 4$ and $s = 0, 1, 2$. Indeed let us deal with the case $s = 0, 1$. We have then from (4.161), (4.163):

$$\left\| \frac{\nabla_\xi^{(\eta)}\Omega}{\Omega} \chi_{[\alpha]} B_j^{a,b,c} \right\|_{L_\xi^\infty(L_\eta^2)} \lesssim \frac{\langle M \rangle}{m \langle M \rangle} M^{\frac{3}{2}} U(m) \lesssim \frac{l^{\frac{3}{2}}}{\langle a \rangle}, \tag{4.166}$$

$$\left\| \frac{\nabla_\xi^{(\eta)}\Omega}{\Omega} \nabla_\eta(\chi_{[\alpha]}) B_j^{a,b,c} \right\|_{L_\xi^\infty(L_\eta^2)} \lesssim \frac{\langle M \rangle}{m \langle M \rangle} \frac{1}{M} M^{\frac{3}{2}} U(m) \lesssim \frac{l^{\frac{3}{2}-1}}{\langle a \rangle}, \tag{4.167}$$

We proceed similarly for the other term of H_η^1. Finally this shows (4.106). The case $\dot{H}_\eta^2$ is similar and we conclude by interpolation for $\dot{H}_\eta^s$.

Remark 19. The choice of the normal form is essential here as for general $B_j^{a,b,c}$ we would obtain in equation (4.166):

$$\left\| \frac{\nabla_\xi^{(\eta)}\Omega}{\Omega} \chi_{[\alpha]} B_j^{a,b,c} \right\|_{L_\xi^\infty(L_\eta^2)} \lesssim \frac{l^{\frac{3}{2}}}{m} \|B_j^{a,b,c}\|_{L_\xi^\infty(L_\eta^\infty)}, \tag{4.168}$$

and the term $\frac{1}{m}$ may blow up in L_ξ^∞ norm when $m = a$ which prevents any $[H^s]$ control. In our case $B_j^{a,b,c}$ compensates this issue by exhibiting a $U(m)$.

3. Let us deal with the case $M \sim |\zeta| \gtrsim 1$ and $\alpha < \sqrt{3}$ where we exclude the case 1 and 2. We will prove estimates (4.105) on:

$$\mathcal{B}_{1,j}^{a,b,c} = \frac{\nabla_\xi \Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} B_j^{a,b,c},$$

$$\mathcal{B}_{2,j}^{a,b,c} = \nabla_\eta \cdot \frac{\nabla_\eta \Omega \cdot \nabla_\xi \Omega \cdot B_j^{a,b,c}}{|\nabla_\eta \Omega|^2}. \tag{4.169}$$

From (4.144), we get by (4.144):

$$|\nabla_\xi^{(\eta)}\Omega| = |\nabla H(\xi) - \nabla H(\zeta)| \sim ||\zeta| - |\xi|| + \langle \xi \rangle \alpha. \tag{4.170}$$

Furthermore we have:

$$\nabla_\eta \Omega = \nabla H(\eta) + \nabla H(\zeta) = \nabla H(\eta) - \nabla H(-\zeta). \tag{4.171}$$

Using (4.144) we have:

$$|\nabla_\eta \Omega| \sim \frac{|\zeta|}{\langle \zeta \rangle}||\zeta| - |\eta|| + \langle \eta \rangle \beta \sim ||\zeta| - |\eta|| + \langle \eta \rangle \beta. \tag{4.172}$$

Furthermore, we have:

$$\begin{aligned}|\xi|^2 = ||\zeta| - |\eta||^2 + 2|\zeta||\eta|(1 + \widehat{\zeta} \cdot \widehat{\eta}) = ||\zeta| - |\eta||^2 + |\zeta||\eta|\beta^2, \\ |\eta|^2 = ||\zeta| - |\xi||^2 + 2|\zeta||\xi|(1 - \widehat{\zeta} \cdot \widehat{\xi}) = ||\zeta| - |\xi||^2 + |\xi||\zeta|\alpha^2. \end{aligned} \tag{4.173}$$

From the second equality of (4.173) we have $||\zeta| - |\xi|| \lesssim |\eta|$ and if $|\xi| \gtrsim 1$ then $\langle \xi \rangle^2 \alpha^2 \lesssim |\xi||\zeta|\alpha^2 \le |\eta|$ while for $|\xi|$ small $|\eta| \sim |\zeta| \gtrsim 1$, so that again $\langle \xi \rangle \alpha \lesssim |\eta|$. Using (4.170) we get :

$$|\nabla_\xi^\eta \Omega| \lesssim |\eta|. \tag{4.174}$$

To bound from below $|\nabla_\eta \Omega|$ we use the first line of (4.173): if $|\eta| << |\zeta|$ then $|\xi| \sim ||\zeta| - |\eta||$, if $|\eta| \sim |\zeta|$ then $|\xi|^2 \sim ||\zeta| - |\eta||^2 + |\zeta||\eta|\beta^2 \sim ||\zeta| - |\eta||^2 + \langle \eta \rangle^2 \beta^2$, in either case

$$|\nabla_\eta \Omega| \sim ||\zeta| - |\eta|| + \langle \eta \rangle \beta \gtrsim |\xi|. \tag{4.175}$$

For the higher derivatives, we have from (4.145),

$$|\nabla_\xi^k \nabla_\xi^{(\eta)} \Omega| \lesssim \langle \zeta \rangle |\zeta|^{-k} \sim M^{1-k}. \tag{4.176}$$

We recall now that:

$$|\nabla_\eta^k \Omega| = |\nabla^k H(\eta) - \nabla^k H(-\zeta)|.$$

We deduce from (4.146) and the fact that $|\zeta| \gtrsim 1$:

$$|\nabla_\eta^k \Omega| \lesssim \frac{\langle \zeta \rangle}{|\zeta|} |\xi| \, |\eta|^{1-k} \lesssim |\xi| \, |\eta|^{1-k}. \tag{4.177}$$

We are going to prove:

$$\begin{aligned} &\|\frac{\nabla_\xi^{(\eta)} \Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} \chi_{[\alpha]}^C B_j\|_{\dot{H}_\eta^s} \sim \|\mathcal{B}_{1,j} \chi_{[\alpha]}^C\|_{\dot{H}_\eta^s} \lesssim \frac{b^{1+\frac{3}{2}}}{ab^s} U(a) = l^{\frac{5}{2}-s} \langle a \rangle^{-1}, \\ &\|\nabla_\eta \cdot \left(\frac{\nabla_\eta \Omega \cdot \nabla_\xi^{(\eta)} \Omega}{|\nabla_\eta \Omega|^2} \cdot B_j \chi_{[\alpha]}^C \right)\|_{\dot{H}_\eta^s} \sim \|\mathcal{B}_{2,j} \chi_{[\alpha]}^C\|_{\dot{H}_\eta^s} \\ &\lesssim \|\left(\frac{\nabla_\eta \nabla_\xi^{(\eta)} \Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} \cdot B_j \chi_{[\alpha]}^C \right)\|_{\dot{H}_\eta^s} + \|\left(\frac{\nabla_\xi^{(\eta)} \Omega (\nabla_\eta \Omega)^2 \nabla_\eta^2 \Omega}{|\nabla_\eta \Omega|^4} \cdot B_j \chi_{[\alpha]}^C \right)\|_{\dot{H}_\eta^s} \\ &+ \|\left(\frac{\nabla_\xi^{(\eta)} \Omega \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} \nabla_\eta [B_j \chi_{[\alpha]}^C] \right)\|_{\dot{H}_\eta^s}, \\ &\lesssim \frac{b^{\frac{3}{2}}}{ab^s} U(a) = l^{\frac{3}{2}-s} \langle a \rangle^{-1}, \end{aligned} \tag{4.178}$$

for $j = 3, 4$ and $s \in [0, 2]$. Let us deal just with the case $s = 0, 1$, $s = 2$ is similar and general s is obtained by interpolation. Here using (4.150), (4.159) and the fact that $\chi_{[\alpha]}^C = 1 - \chi_{[\alpha]}$ we have:

$$
\|\frac{\nabla_\xi^{(\eta)}\Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2}\chi_{[\alpha]}^C B_j\|_{L_\eta^2} \lesssim \frac{b}{a}U(a)b^{\frac{3}{2}} = l^{\frac{5}{2}}\langle a \rangle^{-1}
$$

$$
\|(\frac{\nabla_\eta \nabla_\xi^{(\eta)}\Omega \cdot \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2} \cdot B_j \chi_{[\alpha]}^C)\|_{L_\eta^2} + \|(\frac{\nabla_\xi^{(\eta)}\Omega(\nabla_\eta \Omega)^2 \nabla_\eta^2 \Omega}{|\nabla_\eta \Omega|^4} \cdot B_j \chi_{[\alpha]}^C)\|_{L_\eta^2}
$$

$$
+ \|(\frac{\nabla_\xi^{(\eta)}\Omega \nabla_\eta \Omega}{|\nabla_\eta \Omega|^2}\nabla_\eta[B_j \chi_{[\alpha]}^C])\|_{L_\eta^2},
$$

$$
\lesssim \frac{1}{a}U(a)b^{\frac{3}{2}} + \frac{1}{a^2}b\frac{a}{b}U(a)b^{\frac{3}{2}} + \frac{b}{a}\frac{U(a)}{M}b^{\frac{3}{2}} \lesssim \frac{b^{\frac{3}{2}}}{\langle a \rangle}.
$$

(4.179)

Remark 20. As for the second zone, the term $|\nabla_\eta \Omega|^{-1} \lesssim 1/|\xi|$ is compensated by the estimate $|B_j(\xi, \eta)| \lesssim U(\xi)$.

4. Let us deal now with the case $|\eta^\perp| << M|\eta|$ where we exclude the case 1, 2 and 3 which implies that:
$$
1 >> M \sim |\zeta|, \ \alpha < \sqrt{3}. \tag{4.180}
$$

Furthermore we have:
$$
|\eta^\perp| = |\eta||\sin(\widehat{(\eta, \xi)})| << M|\eta|,
$$

which gives:
$$
|\sin(\widehat{(\eta, \xi)})| << M.
$$

Remark 21. These conditions give that $\widehat{(\eta, \xi)}$ is close from 0 or π, and that $\widehat{(\zeta, \xi)}$ is included between $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$. It gives essentially to case $\widehat{(\eta, \xi)}$ is close from 0 and in this case $|\eta| \gtrsim |\xi| \sim |\zeta|$ and the second case $\widehat{(\eta, \xi)}$ is close from π with $|\xi| \lesssim |\eta| \sim |\zeta|$.

To do this we cut-off the multipliers by :
$$
\chi_{[\perp]} = \chi(\frac{\langle M \rangle |\eta \times \widehat{\xi}|}{100Mb}), \tag{4.181}
$$

with $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(u) = 1$ for $|u| \leq 1$ and $\chi(u) = 0$ for $|u| \geq 2$. Furthermore it gives:

$$
|\nabla_\eta^k \chi_{[\perp]}| \sim |\nabla^k \chi_{[\perp]}|(\frac{\langle M \rangle}{Mb})^k|\nabla_\eta(|\eta \times \widehat{\eta}|)|^k + \text{smaller terms} \lesssim (\frac{\langle M \rangle}{Mb})^k, \tag{4.182}
$$

for all $k \geq 1$. Let us prove now (4.106) in this region. First if $M \neq |\zeta|$ then $\Omega = H(\xi) + H(\eta) - H(\zeta) \geq H(m) \gtrsim m$ and:

$$
|\nabla_\eta^k \nabla_\xi^{(\eta)}\Omega| \lesssim \frac{\langle \zeta \rangle}{|\zeta|^k} \lesssim \frac{|\eta|^{1-k}}{M},
$$

$$
|\nabla_\eta^{k+1}\Omega| = [\nabla^{k+1}H(\eta) - \nabla^{k+1}H(-\zeta)| \lesssim \frac{|\xi|}{M|\eta|^k},
$$

(4.183)

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for $k \geq 0$ by (4.145) and (4.146). For the case $k = 0$ we have $|\nabla_\xi^{(\eta)} \Omega| = |\nabla H(\xi) - \nabla H(\zeta)| \lesssim \frac{|\eta|}{M}$ by the mean value theorem.

We obtain in this region:

$$\|\frac{\nabla_\xi^{(\eta)} \Omega}{\Omega} \chi_{[\perp]} \chi_{[\alpha]}^C B_j^{a,b,c}\|_{\dot{H}_\eta^s} \lesssim \frac{b^{1+\frac{3}{2}}}{mM(Mb)^s} a \lesssim l^{\frac{3}{2}}(Ml)^{-s}, \qquad (4.184)$$

for $j = 3, 4$ and $s = 0, 1, 2$ where $\chi_{[\alpha]}^C$ is due to exclusion of the case 2. We are going only to deal with the case $s = 0, 1$ (the case $s = 2$ is similar, and we conclude for general s by interpolation):

$$\|\frac{\nabla_\xi^\eta \Omega}{\Omega} \chi_{[\perp]} \chi_{[\alpha]}^C B_j^{a,b,c}\|_{L_\eta^2} \lesssim \frac{b}{Mm} b^{\frac{3}{2}} U(a) \lesssim \frac{b^{\frac{5}{2}}}{mM} a \lesssim b^{\frac{3}{2}}. \qquad (4.185)$$

To see this, it suffices to consider the case where $m = a$ or b. Similarly since $M << 1$

$$\|\frac{\nabla_\eta \nabla_\xi^{(\eta)} \Omega}{\Omega} \chi_{[\perp]} \chi_{[\alpha]}^C B_j^{a,b,c}\|_{L_\eta^2} \lesssim \frac{1}{m\,M} b^{\frac{3}{2}} a \lesssim \frac{l^{\frac{3}{2}}}{M\,l}. \qquad (4.186)$$

We assume now $M = |\zeta|$ which implies since $\widehat{(\xi, \eta)}$ is close from 0 or π that $\widehat{(\xi, \eta)}$ is near from π. We denote by:

$$\lambda = |\xi| + |\eta| - |\zeta|. \qquad (4.187)$$

Then we have since $(\widehat{(\xi, \eta)}) = \pi - (\widehat{(\zeta, \eta)}) - (\widehat{(\xi, \zeta)})$ and the fact that $\widehat{(\xi, \eta)}$ is near from π and $\widehat{(\zeta, \eta)}$ near from $-\pi$ we have:

$$
\begin{aligned}
|\eta^\perp| &= |\sin((\widehat{(\xi, \eta)}))||\eta| = |\eta||\sin(\pi - (\widehat{(\zeta, \eta)}) - (\widehat{(\xi, \zeta)}))| \\
&= |\eta||\sin(((\widehat{(\zeta, \eta)}) + (\widehat{(\xi, \zeta)}))| = |\eta|(|\sin(((\widehat{(\zeta, \eta)})))\cos(\widehat{(\xi, \zeta)}) + \cos(((\widehat{(\zeta, \eta)})))\sin(\widehat{(\xi, \zeta)})| \\
&\sim |\eta|(|\sin(((\widehat{(\zeta, \eta)})))| + |\sin(\widehat{(\xi, \zeta)})|) \\
&\sim |\eta|(2|\sin(\frac{(\widehat{(\zeta, \eta)})}{2})||\cos(\frac{(\widehat{(\zeta, \eta)})}{2})| + 2\sin(\frac{\widehat{(\xi, \zeta)}}{2})\cos(\frac{\widehat{(\xi, \zeta)}}{2})) \\
&\sim |\eta|(\alpha + \beta).
\end{aligned}
\qquad (4.188)
$$

As a byproduct, the argument gives $\widehat{(\xi, \eta)} - \pi \sim \alpha + \beta$, we deduce

$$
\lambda = \frac{|\xi|^2 + |\eta|^2 - |\xi - \eta|^2}{|\xi| + |\eta| + |\zeta|} \sim \frac{2|\xi||\eta|(1 + \cos(\widehat{(\xi, \eta)}))}{M} \sim \frac{2|\xi||\eta|(\alpha + \beta)^2}{M} \quad \sim \quad m(\alpha^2 + \beta^2)
$$
$$
\sim \quad m\left(\frac{|\eta^\perp|}{|\eta|}\right)^2,
$$

so that by assumption $\lambda << M^2 m$.

Now Ω can be estimated using the identity:

$$-\Omega = [H(|\xi| + |\eta|) - H(\xi) - H(\eta)] + [H(\xi - \eta) - H(|\xi| + |\eta|)], \qquad (4.189)$$

for the first part of the right hand side Taylor expansion gives

$$H(|\xi| + |\eta|) - H(\xi) - H(\eta) \sim \frac{|\xi||\eta|(|\xi| + |\eta|)}{\langle \xi \rangle + \langle \eta \rangle} \sim M^2 m.$$

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Secondly, using $|\xi - \eta| = |\xi| + |\eta| - \lambda$ implies

$$H(\xi - \eta) - H(|\xi| + |\eta|) \lesssim \lambda << M^2 m,$$

so that $-\Omega \sim M^2 m$.

We have then, using the sine theorem $|\eta|\beta \sim |\xi|\alpha$, so that

$$\max(|\xi|\alpha, |\eta|\beta) \sim \min(|\xi|\alpha, |\eta|\beta) \lesssim m(\alpha + \beta) << mM \sim |\xi||\eta|,$$

we deduce from (4.144)

$$\begin{aligned}
|\nabla_\xi^{(\eta)}\Omega| &= |\nabla H(\xi) - \nabla H(\zeta)| \\
&\lesssim |\zeta||\,|\zeta| - \xi|\,| + |\widehat{\zeta} - \widehat{\xi}| \sim M|\eta| + \alpha << |\eta|, \qquad (4.190) \\
|\nabla_\eta\Omega| &\lesssim M|\xi| + \beta << |\xi|.
\end{aligned}$$

Concerning the higher derivatives we have due to $(4.145, 4.146)$

$$|\nabla_\eta^k \nabla_\xi^{(\eta)}\Omega| \lesssim M^{-k}, \quad |\nabla_\eta^{1+k}\Omega| = |\nabla^{k+1}H(\eta) - \nabla^{k+1}H(\zeta)| \lesssim \frac{|\xi|}{M|\eta|^k}. \qquad (4.191)$$

Moreover, since η is confined in a region where $\sin\widehat{\xi,\eta} << M$, this is essentially a cone of angle smaller than M and length b, which is of volume $M^2 b^3$, we get by integration

$$\left\| \frac{\nabla_\xi\Omega}{\Omega}\chi_{[\perp]}\chi_{[\alpha]}^C B_j^{a,b,c} \right\|_{L_\eta^2} \lesssim \frac{ab(M^2 b^3)^{1/2}}{M^2 m} \lesssim l^{\frac{3}{2}}, \qquad (4.192)$$

for $j = 3, 4$. As previously, the estimates for $s = 1, 2$ are done thanks to (4.191) and (4.106) follows by interpolation.

5. Let us finish with the last case, for which:

$$1 >> M \sim |\zeta|, \; \alpha < \sqrt{3} \Rightarrow -\frac{\pi}{3} \leq \frac{\widehat{(\xi, \zeta)}}{2} \leq \frac{\pi}{3}, \; |\eta^\perp| \gtrsim M|\eta|. \qquad (4.193)$$

For the first derivatives, we have from (4.144):

$$|\nabla_\eta\Omega| = |\nabla H(\eta) - \nabla H(-\zeta)| \sim M|\,|\zeta| - |\eta|\,| + \beta, \; |\nabla_\xi^{(\eta)}\Omega| \sim M|\,|\zeta| - |\xi|\,| + \alpha. \qquad (4.194)$$

By using the sine theorem and the fact that $\cos(\frac{\widehat{(\xi,\zeta)}}{2}) \gtrsim 1$ we have:

$$\begin{aligned}
\frac{a}{\sin(\widehat{(\eta, \zeta)})} &= \frac{b}{\sin(\widehat{(\xi, \zeta)})}, \\
&= \frac{a}{2\sin(\frac{\widehat{(\eta,\zeta)}}{2})\cos(\frac{\widehat{(\eta,\zeta)}}{2})} = \frac{a}{\beta\sin(\frac{\widehat{(\eta,\zeta)}}{2})} \\
&= \frac{b}{2\sin(\frac{\widehat{(\xi,\zeta)}}{2})\cos(\frac{\widehat{(\xi,\zeta)}}{2})} \sim \frac{b}{\alpha}
\end{aligned}$$

this gives the estimate on α

$$\frac{\alpha}{\beta} \sim \frac{|\eta|}{|\xi|}\sin\left(\frac{\widehat{(\eta, \zeta)}}{2}\right) \leq \frac{|\eta|}{|\xi|}. \qquad (4.195)$$

Clearly $||\zeta| - |\xi|| \leq |\eta|$ and from (4.173):

$$M^2|\xi|^2 = M^2||\zeta| - |\eta||^2 + M^2|\zeta||\eta|\beta^2 \lesssim (M||\zeta| - |\eta|| + \beta)^2 \sim |\nabla_\eta\Omega|^2. \tag{4.196}$$

Collecting these estimates yields

$$\frac{|\nabla_\xi^{(\eta)}\Omega|}{|\nabla_\eta\Omega|} \lesssim \frac{|\eta|}{|\xi|} + \frac{\alpha}{|\nabla\eta\Omega|} \lesssim \frac{|\eta|}{|\xi|}. \tag{4.197}$$

For the higher derivatives, we use $\alpha \leq \sqrt{3} \Rightarrow |\eta^\perp| = |\zeta \wedge \widehat{\xi}| \sim |\zeta||\widehat{\zeta} - \widehat{\xi}|$ so that

$$|\xi||\eta^\perp| \lesssim |\zeta||\xi|\alpha \lesssim |\zeta||\eta|\beta. \tag{4.198}$$

and (4.191) gives

$$\frac{|\nabla_\eta^k\Omega|}{|\nabla_\eta\Omega|} \lesssim \frac{|\xi|}{M|\eta|^{k-1}\beta} \lesssim \frac{1}{|\eta|^{k-2}|\eta^\perp|}, \quad \frac{|\nabla_\eta^k\nabla_\xi^{(\eta)}\Omega|}{|\nabla_\eta\Omega|} \lesssim \frac{1}{M^k\beta} \lesssim \frac{|\eta|}{M^{k-1}|\xi||\eta^\perp|}, \tag{4.199}$$

We split the domain with the dyadic decomposition

$$|\eta^\perp| \sim \mu \in \{k \in 2^{\mathbb{Z}}, k \geq Mb\}, \tag{4.200}$$

the volume $|\{\eta : |\eta| \lesssim b, |\eta^\perp| \sim \mu\}|$ is of order $\mu^2 b$ (it is contained in a cone of length b and angle $\sim \mu/b$). By integration this gives:

$$\|\mathcal{B}_{1,j}\chi_{[\alpha]}^C\chi_\perp^C\|_{L_\eta^2} = \|\frac{\nabla_\xi^{(\eta)}\Omega \cdot \nabla_\eta\Omega}{|\nabla_\eta\Omega|^2}\chi_{[\alpha]}^C\chi_\perp^C B_j\|_{\dot{H}_\eta^s} \lesssim \sum_{1 \gtrsim \mu \geq Mb} \frac{b\mu b^{\frac{1}{2}}}{a} a \lesssim l^{\frac{5}{2}}M. \tag{4.201}$$

Similarly[3], $\|\mathcal{B}_{1,j}\chi_{[\alpha]}^C\chi_\perp^C\|_{\dot{H}_\eta^2} \lesssim l^{1/2}M^{-1}$ and (4.104) follows by interpolation.

For $\mathcal{B}_2$ we have

$$\|\mathcal{B}_{2,j}\chi_\perp^C\chi_{[\alpha]}^C\|_{\dot{H}_\eta^1} = \|\nabla_\eta^2 \cdot \left(\frac{\nabla_\eta\Omega \cdot \nabla_\xi^{(\eta)}\Omega}{|\nabla_\eta\Omega|^2} \cdot B_j\chi_{[\alpha]}^C\chi_\perp^C\right)\|_{L_\eta^2}$$

$$\lesssim \|\frac{|\nabla_\eta^2\nabla_\xi^{(\eta)}\Omega|}{|\nabla_\eta\Omega|}B_j\chi_{[\alpha]}^C)\|_{L_\eta^2} + \|\frac{|\nabla_\xi^{(\eta)}\Omega||\nabla_\eta^3\Omega|}{|\nabla_\eta\Omega|^2}B_j\chi_{[\alpha]}^C\chi_\perp^C\|_{L_\eta^2}$$

$$+ \|\frac{|\nabla_\eta\nabla_\xi^{(\eta)}\Omega||\nabla_\eta^2\Omega|}{|\nabla_\eta\Omega|^2} \cdot B_j\chi_{[\alpha]}^C\chi_\perp^C)\|_{L_\eta^2} + \|\frac{|\nabla_\xi^{(\eta)}\Omega||\nabla_\eta\Omega|}{|\nabla_\eta\Omega|}\nabla_\eta^2[B_j\chi_{[\alpha]}^C\chi_\perp^C])\|_{L_\eta^2},$$

$$+ \|\frac{|\nabla_\xi^{(\eta)}\Omega||\nabla_\eta^2\Omega|}{|\nabla_\eta\Omega|}\nabla_\eta[B_j\chi_{[\alpha]}^C\chi_\perp^C])\|_{L_\eta^2} + \|\frac{|\nabla_\eta\nabla_\xi^{(\eta)}\Omega||\nabla_\eta\Omega|}{|\nabla_\eta\Omega|}\nabla_\eta[B_j\chi_{[\alpha]}^C\chi_\perp^C]\|_{L_\eta^2}$$

$$\lesssim \sum_{1 \gtrsim \mu \geq Mb} \frac{b\mu b^{\frac{1}{2}}}{a\mu^2} a \lesssim l^{\frac{1}{2}}M^{-1}, \tag{4.202}$$

for $j = 3, 4$ and similarly $\|\mathcal{B}_{2,j}\chi_\perp^C\chi_{[\alpha]}^C\|_{\dot{H}_\eta^2} \lesssim l^{-1/2}M^{-2}$, so that (4.104) follows by interpolation. As before $\chi_{[\alpha]}^C = 1 - \chi_{[\alpha]}$ and $\chi_{[\perp]}^C = 1 - \chi_{[\perp]}$ are excluding the previous case.

---

[3]The argument does not work directly in $\dot{H}^1$ since the sum $\sum_\mu b^{3/2}$ appears, which can not be bounded by $b^{3/2}$

4.9 Multiplier estimates for ZZ and $\bar{Z}\bar{Z}$

We are now interested in dealing with the bilinear terms which involve the product ZZ and $\bar{Z}\bar{Z}$. The main difference with the previous case is that the phase change. Indeed the phase corresponds to the two following cases:

$$\begin{aligned}
\Omega_1 &= H(\xi) + H(\eta) + H(\xi - \eta), \\
\Omega_2 &= H(\xi) - H(\eta) - H(\xi - \eta),
\end{aligned} \tag{4.203}$$

The proof to estimate I_3 is similar to the previous section (that it why we will not recall the computation, see [29]) and use the two crucial lemma showed by Gustafson, Nakanishi and Tsai in [29].

Lemma 4.4. *Denoting $M = \max(a, b, c)$, $m = \min(a, b, c)$ and $l = \min(b, c)$ we have:*

- *If $M << 1$ then for $\epsilon > 0$ small:*

$$\|\mathcal{B}_1^{a,b,c}\|_{H^{1+\epsilon}} \lesssim l^{\frac{1}{2}-2\epsilon}, \quad \|\mathcal{B}_2^{a,b,c}\|_{H^{1+\epsilon}} \lesssim l^{\frac{1}{2}-2\epsilon}M^{-1}, \tag{4.204}$$

- *If $M \geq 1$ then for $\epsilon > 0$ small with $m \neq a$:*

$$\|\mathcal{B}_1^{a,b,c}\|_{H^{\frac{3}{2}+\epsilon}} \lesssim l^{1-\epsilon}\langle a \rangle^{-1} + m^{-\epsilon}, \quad \|\mathcal{B}_2^{a,b,c}\|_{H^{\frac{3}{2}+\epsilon}} \lesssim l^{-\epsilon}\langle a \rangle^{-1} + m^{-\epsilon}, \tag{4.205}$$

Remark 22. Let us mention that in the previous lemma we are never in high frequencies in the situation where $a \lesssim b \sim c$ is spatially non resonant, indeed in this situation we could not deal with the term $m^{-\epsilon}$. In fact we are in this case time non resonant.

Lemma 4.5.
$$\|\mathcal{B}_3\|_{[H^s]} \lesssim \left(\frac{\langle M \rangle}{M}\right)^s l^{\frac{3}{2}-s}\langle a \rangle^{-1}. \tag{4.206}$$

4.10 Estimates of the Cubic term J_3

Let us prove the control of the cubic terms in the part $\|JZ\|_X$. We are going to use paraproduct, more precisely we have $v = \sum_l \Delta_l v$, thus in Fourier space we have to estimate:

$$\sum_{l_1, l_2, l_3} \int_0^t s \nabla_\xi \Omega e^{isH} C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm] ds \tag{4.207}$$

where:
$$\Omega = H(\zeta) \pm H(\xi_1) \pm H(\xi_2) \pm H(\xi_3), \ \nabla_\xi \Omega = \nabla H(\xi) \pm \nabla H(\xi_1), \tag{4.208}$$

In the case $j = 5$ we have to replace one of the v by Z.

Remark 23. In the cubic terms we have in general a large region of interactions which are both space and time resonant, that it is why it is not possible in general to integrate by parts as for the bilinear terms.

However when we are in a region which is both time and space resonant, it is possible to bound simply $|\nabla_\xi \Omega|$ in L^∞ norm and this is sufficient since $sC_j[v^\pm, v^\pm, v^\pm]$ have sufficient time decay, essentially because we have a cubic nonlinearity and not only a quadratic.

In the other case we can apply a integration by part since the region is non time resonant or in other word $|\Omega|$ dominates $|\nabla_\xi \Omega|$.

We start with the case $l_1 \geq \max(l_2, l_3)$ and we work with the smaller ξ_2, ξ_3 as integral variables. The other terms are treated similarly.

If $l_1 \lesssim 1$

If $l_1 \lesssim 1$ then $|\nabla H(\xi_k)|$ are obviously bounded, hence using the lemma 3.2 since the Fourier transform of $\sum_{\max(l_2, l_3) \lesssim l_1} \int_0^t s e^{isH} \nabla\Omega(\xi) C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm]$ has his support in dyadic shell and Strichartz estimate, we have (here we put just consider the case where C_j has a $U^{-1}(\xi_1)$) :

$$\| \int_0^t s e^{isH} \sum_{\max(l_2,l_3) \lesssim l_1 \lesssim 1} \nabla\Omega(\xi) C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm] ds \|_{L^\infty(H^1)}$$

$$\lesssim \| \sum_{l_1 \lesssim 1} \int_0^t s e^{isH} \nabla\Omega(\xi) C_j[\Delta_{l_1} v^\pm, S_{l_1} v^\pm, S_{l_1} v^\pm] ds \|_{L^\infty(H^1)},$$

$$\lesssim \Big(\sum_{l_1 \lesssim 1} l_1^2 \| \sum_{\max(l_2,l_3) \lesssim l_1} \int_0^t s e^{isH} \nabla\Omega(\xi) C_j[\Delta_{l_1} v^\pm, S_{l_1} v^\pm, S_{l_1} v^\pm] ds \|_{L^\infty(L^2)}^2 \Big)^{\frac{1}{2}},$$

$$\lesssim \Big(\sum_{l_1 \lesssim 1} \| \int_0^t s e^{isH} C_j[\Delta_{l_1} v^\pm, S_{l_1} v^\pm, S_{l_1} v^\pm] ds \|_{L^\infty(L^2)}^2 \Big)^{\frac{1}{2}}, \tag{4.209}$$

$$\lesssim \Big(\sum_{l_1 \lesssim 1} \| t C_j[\Delta_{l_1} v^\pm, S_{l_1} v^\pm, S_{l_1} v^\pm] \|_{L^2(L^{\frac{6}{5}})}^2 \Big)^{\frac{1}{2}},$$

$$\lesssim \Big(\sum_{l_1 \lesssim 1} \| U^{-1} \Delta_{l_1} v \|_{L^2(L^6)}^2 \| U^{-1} S_{l_1} v \|_{L^\infty(L^2)}^2 \| t S_{l_1} v \|_{L^\infty(L^6)}^2 \Big)^{\frac{1}{2}},$$

$$\lesssim \| U^{-1} v_{<1} \|_{L^2(B_{6,2}^0)} \| U^{-1} v \|_{L^\infty(L^2)} \| t v \|_{L^\infty(L^6)},$$

Using (4.54) we observe that by Besov embedding in low frequencies $B_{6,6}^{-\epsilon} \hookrightarrow B_{6,2}^0$:

$$\| t v_{<1} \|_{\widetilde{L}^\infty(B_{6,2}^0)} \lesssim \| v(t) \|_{X(t)},$$

$$\| U^{-1} v_{<1} \|_{\widetilde{L}^2(B_{6,2}^0)} \lesssim \| v(t) \|_{X(t)},$$

$$\| U^{-1} v_{<1} \|_{\widetilde{L}^\infty(L^2)} \lesssim \| v(t) \|_{X(t)}.$$

If $1 << l_1 \sim \max(l_2, l_3)$

In this situation we have $|\nabla_\xi \Omega| \lesssim \max(l_2, l_3)$. Hence $\nabla_\xi \Omega$ has to be considered as un multiplier of order one in high frequencies, that is why by using the fact that the Fourier support of $\int_0^t s \nabla_\xi \Omega e^{isH} C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm] ds$ in included in a ball, using lemma 3.2 and Besov

embeddingg via proposition 3.1 we have

$$\| \sum_{1<<l_1\sim\max(l_2,l_3)} \int_0^t s\nabla_\xi\Omega e^{isH} C_j[\Delta_{l_1}v^\pm,\Delta_{l_2}v^\pm,\Delta_{l_3}v^\pm]ds\|_{L^\infty(H^1)}$$

$$\lesssim \| \sum_{1<<l_1\sim l_2} \int_0^t s\nabla_\xi\Omega e^{isH} C_j[\Delta_{l_1}v^\pm,\Delta_{l_1}v^\pm,S_{l_1}v^\pm]ds\|_{L^\infty(H^1)}$$

$$+ \| \sum_{1<<l_1\sim l_3} \int_0^t s\nabla_\xi\Omega e^{isH} C_j[\Delta_{l_1}v^\pm,S_{l_1}v^\pm,\Delta_{l_1}v^\pm]ds\|_{L^\infty(H^1)}$$

$$\lesssim \Big(\sum_{1<<l_1} l_1^4 \| \int_0^t s e^{isH} C_j[\Delta_{l_1}v^\pm,\Delta_{l_1}v^\pm,S_{l_1}v^\pm]ds\|^2_{L^\infty(L^2)} \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{1<<l_1} l_1^4 \| \int_0^t s e^{isH} C_j[\Delta_{l_1}v^\pm,S_{l_1}v^\pm,\Delta_{l_1}v^\pm]ds\|^2_{L^\infty(L^2)} \Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\sum_{1<<l_1} l_1^4 \| s C_j[\Delta_{l_1}v^\pm,\Delta_{l_1}v^\pm,S_{l_1}\Delta_{l_3}v^\pm]ds\|^2_{L^2(L^{\frac{6}{5}})} \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{1<<l_1} l_1^4 \| s C_j[\Delta_{l_1}v^\pm,S_{l_1}v^\pm,\Delta_{l_1}v^\pm]ds\|^2_{L^2(L^{\frac{6}{5}})} \Big)^{\frac{1}{2}} \tag{4.210}$$

$$\lesssim \Big(\sum_{1<<l_1} \| s C_j[l_1\Delta_{l_1}v^\pm,l_1\Delta_{l_1}v^\pm,S_{l_1}v^\pm]\|^2_{L^2(L^{\frac{6}{5}})} \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{1<<l_1} \| s C_j[l_1\Delta_{l_1}v^\pm,S_{l_1}v^\pm,l_1\Delta_{l_1}v^\pm]\|^2_{L^2(L^{\frac{6}{5}})} \Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\sum_{1<<l_1} \| s l_1\Delta_{l_1}v^\pm\|^2_{L^\infty(L^6)} \| l_1\Delta_{l_1}v^\pm\|^2_{L^2(L^6)} \| U^{-1}S_{l_1}v^\pm\|^2_{L^\infty(L^2)} \Big)^{\frac{1}{2}}$$

$$+ \Big(\sum_{1<<l_1} \| s l_1\Delta_{l_1}v^\pm\|^2_{L^\infty(L^6)} \| l_1\Delta_{l_1}v^\pm\|^2_{L^2(L^6)} \| U^{-1}S_{l_1}v^\pm\|^2_{L^\infty(L^2)} \Big)^{\frac{1}{2}}$$

$$\lesssim \Big(\sum_{1<<l_1} \| s v^\pm\|^2_{\widetilde{L}^\infty(B^1_{6,\infty})} \| l_1\Delta_{l_1}v^\pm\|^2_{L^2(L^6)} \| U^{-1}v^\pm\|_{L^\infty(L^2)} \Big)^{\frac{1}{2}}$$

$$\lesssim \| s v^\pm\|_{\widetilde{L}^\infty(B^1_{6,\infty})} \| v^\pm\|_{L^2(B^1_{6,2})} \| U^{-1}v^\pm\|_{L^\infty(L^2)}$$

The only point here is that we do not control a priori the term $\|v^\pm\|_{L^2(B^1_{6,2})}$. In fact it suffices to come back to the proof of the estimate (4.54) in applying the same idea in frequencies, in particular we verify easily that doing the same we have:

$$\|\Delta_l\langle\nabla\rangle U^{-\frac{1}{3}}v(t)\|_{L^6} \lesssim t^{-1}\|x\Delta_l e^{itH}\langle\nabla\rangle v(t)\|_{L^2},$$

But we have $x\Delta_l = \Delta_l x + \frac{1}{2^l}\nabla\varphi(\frac{D}{2^l})$ and we deduce using the fact that $J\langle\nabla\rangle = \langle\nabla\rangle J + \langle\nabla\rangle^{-1}\nabla$

$$\|\Delta_l t\langle\nabla\rangle U^{-\frac{1}{3}}v(t)\|_{L^6} \lesssim \|\Delta_l J v(t)\|_{L^2} + \|\Delta_l v(t)\|_{L^2}.$$

Taking the norm l^2 then the L^∞ norm we have:

$$\|t\langle\nabla\rangle U^{-\frac{1}{3}}v\|_{L^\infty(B^0_{6,2})} \lesssim \|J v\|_{L^\infty(H^1)} + \|v\|_{L^\infty(L^2)}.$$

In order to show that $\|v^\pm\|_{L^2(B^1_{6,2})}$ is bounded it suffices to follows the same proof than the proof of proposition 4.6.
When $j=5$, we just replace v by Z.

If $l_1 >> \max(1, l_2, l_3)$

Let us start with dealing with the phase of the form: $\Omega = H(\xi) - H(\xi_1) \pm \cdots$, then we have:

$$\nabla_\xi \Omega = F(\xi_1, \xi_2 + \xi_3) \cdot (\xi_2 + \xi_3), \quad F(\xi, \eta) = \int_0^1 \nabla^2 H(\xi - \theta\eta) d\theta. \tag{4.211}$$

In the region $|\xi_1| >> \langle\eta\rangle$, following [29] we observe that:

$$|\nabla_\xi^k \nabla_\eta^l F(\xi, \eta)| \lesssim \int_0^1 |\nabla^{2+k+l} H(\xi - \theta\eta)| d\theta \lesssim |\xi_1|^{-k-l}. \tag{4.212}$$

It implies that F is a Coifman-Meyer multiplier. In particular we have for example for C_1:

$$\mathcal{F}(\int_0^t s\nabla_\xi \Omega e^{isH} C_1[\Delta_{l_1} v, \Delta_{l_2} v, \Delta_{l_3} v] ds)(\xi)$$

$$= U(\xi) \int_0^t e^{isH(\xi)} \int_{\mathbb{R}^N} F(\xi_1, \eta) \cdot \eta \widehat{\Delta_{l_1} v}(\xi_1) \mathcal{F}(\Delta_{l_2} v \Delta_{l_3} v)(\eta) d\eta \, ds,$$

$$= U(\xi) \int_0^t e^{isH(\xi)} \int_{\mathbb{R}^N} F(\xi_1, \eta) \cdot \eta \widehat{\Delta_{l_1} v}(\xi_1) \mathcal{F}(\Delta_{l_2} v \Delta_{l_3} v)(\eta) d\eta \, ds, \tag{4.213}$$

$$= \int_0^t U(\xi) e^{isH(\xi)} \mathcal{F} F \cdot [\nabla(\Delta_{l_2} v \Delta_{l_3} v)), \Delta_{l_1} v](\xi) ds.$$

$$\int_0^t s\nabla_\xi \Omega e^{isH} C_1[\Delta_{l_1} v, \Delta_{l_2} v, \Delta_{l_3} v] ds = \int_0^t se^{isH} U \, F \cdot [\nabla(\Delta_{l_2} v \Delta_{l_3} v)), \Delta_{l_1} v] ds.$$

Similarly for C_2 we have:

$$\int_0^t s\nabla_\xi \Omega e^{isH} C_1[\Delta_{l_1} v, \Delta_{l_2} v, \Delta_{l_3} v] ds = \int_0^t se^{isH} U \, F \cdot [\nabla(U^{-1}(\Delta_{l_2} v) \Delta_{l_3} v)), U^{-1}\Delta_{l_1} v] ds. \tag{4.214}$$

We proceed similarly for C_3, C_4, C_5.
Hence using the Coifman-Meyer theorem since 3.32 is verified for F, Strichartz estimate and lemma 3.2 allow us to deal with the following term

$$\| \sum_{\max(1, l_2, l_3) << l_1} \int_0^t s\nabla_\xi \Omega e^{isH} C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm] ds \|_{L^\infty(H^1)}.$$

Indeed roughly speaking we put the l^1 which comes from the derivative H^1 with $\Delta_l 1 v^\pm$ and we use the fact that $\nabla_\xi \Omega$ has a behavior of Coifman Meyer type (in particular the term $\nabla_\xi \Omega C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm]$ behaves like a cubic product in L^p spaces).

Following [29] let us deal with the phase of the form: $\Omega = H(\xi) + H(\xi_1) \pm \cdots$ when $l_1 >> \max(1, l_2, l_3)$. We have in this situation:

$$|\Omega| \gtrsim l_1^2, \quad |\nabla_\xi \Omega| \lesssim l_1, \quad |\frac{\nabla_\xi \Omega}{\Omega}| \lesssim \frac{1}{l_1}. \tag{4.215}$$

so that we can integrate on $e^{is\Omega}$ in s since we are non time resonant. Thus we get terms in $\Delta_{l_1}, \Delta_{l_2}, \Delta_{l_3}$ like:

$$\int_0^t \frac{\nabla_\xi \Omega}{\Omega} e^{isH}(C_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm] + sC_j[\Delta_{l_1} \mathcal{N}_v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm]) ds$$

$$+ [\frac{\nabla_\xi \Omega}{\Omega} e^{isH} sC_j[\Delta_{l_1} v^\pm, \Delta_{l_2} v^\pm, \Delta_{l_3} v^\pm]]_0^t, \tag{4.216}$$

Following [29], by using the Coifman-Meyer estimate, Strichartz estimate we can control each term of the following form (because $\max(l_2, l_3) << l_1$ and the fact that we get a gain of one derivative since $|\frac{\nabla_\xi \Omega}{\Omega}| \lesssim \frac{1}{l_1}$):

$$\|C_j[\Delta_{l_1} v^\pm, S_{\frac{l_1}{1000}} v^\pm, S_{\frac{l_1}{1000}} v^\pm]\|_{L^2 L^{\frac{6}{5}}} + \|tC[\Delta_{l_1} \mathcal{N}_v^\pm, S_{\frac{l_1}{1000}} v^\pm, S_{\frac{l_1}{1000}} v^\pm]\|_{L^2 L^{\frac{6}{5}} + L^{\frac{4}{3}} L^{\frac{3}{2}}}$$

$$+ \|tC[\Delta_{l_1} v^\pm, S_{\frac{l_1}{1000}} v^\pm, S_{\frac{l_1}{1000}} v^\pm]\|_{L^\infty L^2},$$

$$\lesssim \|\Delta_{l_1} v^\pm\|_{L^2 L^6} \| U^{-1} v^\pm\|_{L^\infty L^3}^2 + (\|\Delta_{l_1} \mathcal{N}_v^\pm\|_{L^2 L^2 + L^{\frac{4}{3}} L^3} + \|\Delta_{l_1} v^\pm\|_{L^\infty L^6}) \|t^{\frac{1}{2}} U^{-1} v^\pm\|_{L^\infty L^6}^2,$$

$$(4.217)$$

with $i \in \{2, 3\}$ where the norm on $\mathcal{N}_v$ is bounded by $L^2(H^{1,\frac{6}{5}}) + L^{\frac{4}{3}}(H^{1,\frac{3}{2}}) \hookrightarrow L^2 L^2 + L^{\frac{4}{3}} L^3$ (see the section 4.5). In particular since $L^2(H^{1,\frac{6}{5}}) \hookrightarrow \widetilde{L}^2(B^1_{\frac{6}{5},2})$ and $L^{\frac{4}{3}}(H^{1,\frac{3}{2}}) \hookrightarrow \widetilde{L}^{\frac{4}{3}}(B^1_{\frac{3}{2},2})$, it implies that:

$$\|\Delta_{l_1} \mathcal{N}_v^\pm\|_{L^2 L^2 + L^{\frac{4}{3}} L^3} \lesssim c_{l_1},$$

with $(c_{l_1})_{l_1 \in \mathbb{Z}}$ belongs in l^2. It is enough to conclude using the lemma 3.2 as in the previous case.

4.11 Proof of the theorem 2.2

Following [29], we recall that Bethuel and Saut have proved in [9] the existence of global strong solution with initial data in H^1 with ψ belonging in $C(\mathbb{R}, 1 + H^1(\mathbb{R}^3))$. We just want to choose an initial data such that the solution remains in X in order to apply the previous estimates of stability. To do this we are going to take $\psi_0^n \in H^2 \cap \langle x \rangle^{-1} H^1$ such that ψ_0^n converge to ψ_0 in norm defined by (2.15). It suffices then to apply a fixed point argument in order to show that the solution ψ^n remain in X on an interval $[0, T]$. We can apply the fixed point argument using the following estimates (indeed the pseudo conformal transform ensures that $x + 2it\nabla = e^{it\Delta} x e^{-it\Delta}$):

$$\|\varphi\|_{L^\infty(H^2)} \lesssim (T^{\frac{1}{2}} + T) \|\varphi\|_{L^\infty(H^2)} (1 + \|\varphi\|_{L^\infty(H^1)})^2,$$

$$\|(x + 2it\nabla)\varphi\|_{H^1} \lesssim \|(x + 2it\nabla)\mathcal{N}_\varphi\|_{L^2(H^{1,\frac{6}{5}} + L^1(H^1)},$$

$$\lesssim (T^{\frac{1}{2}} + T^2)(\|xu\|_{L^\infty(H^1)} + \|u\|_{L^\infty(H^2)})(1 + \|u\|_{L^\infty(H^1)})^2,$$

$$(4.218)$$

where the nonlinearity $\mathcal{N}_\varphi = \varphi^2 + 2|\varphi|^2 + |\varphi|^2 \varphi$. Thus we control on $[0, T_n]$ the $\|\varphi_n\|_X$, by Gronwall argument we can extend globally this control. Now we can apply the previous estimate to φ^n on any interval $[0, T]$ such that we have with $v^n = \text{Re}(\varphi^n) + iU\text{Im}\varphi^n$ we have:

$$\|v^n\|_{X \cap S} + \|Z^n\|_{X \cap S} \lesssim \|v^n(0)\|_{X(0)} \lesssim \delta.$$

Indeed the initial data $v^n(0)$ is uniformly bounded in $X(0)$ by δ. We conclude the proof of the theorem 2.2 by passing to the limit via compactness argument (see for more details the section 6.5).

5 A subcritical version of 2.1 ($N \geq 4$)

As for 2.2, the proof of 2.1 relies on a normal form, however it is remarkably simpler as the time decay is stronger for $N > 3$. This section describes the argument from [29], however we

will only prove well-posedness and scattering in H^s, $s > \frac{N}{2} - 1$ (subcritical regularity) as it is sufficient for our purpose. This allows us to use simpler spaces with more usual product rules. We recall the same diagonalization process as in (4.41): $v = \varphi_1 + iU\varphi_2 = V\varphi$, so that v is solution of

$$
\begin{aligned}
i\partial_t v - Hv &= U(3\varphi_1^2 + \varphi_2^2 + |\varphi|^2\varphi_1) + i(2\varphi_1\varphi_2 + |\varphi|^2\varphi_2) \\
&= U(3v_1^2 + (U^{-1}v_2)^2 + |V^{-1}v|^2v_1) + i(2v_1U^{-1}v_2 + |V^{-1}v|^2V^{-1}v_2).
\end{aligned}
$$

Since $N \geq 4$, quadratic terms are not supposed to be an issue (the Strauss exponent in dimension 4 is $\simeq 1.78$) and the only thing to be handled is the singular multiplier U^{-1}. The remedy was given in [29] thanks to a change of normal form $z = v + P|u|^2$ where P is a Fourier multiplier localized near $\xi = 0$. The choice of P was refined in [30] by taking $P = 2(2 - \Delta)^{-1} = 2/\langle\nabla\rangle^2$ (this leads to $B_4' = 0$ in 4.44). We follow this choice and set

$$
z = v + \frac{1}{\langle\nabla\rangle^2}|\varphi|^2 = \varphi_1 + U\varphi_2 + \frac{1}{\langle\nabla\rangle^2}|\varphi|^2,
$$

which satisfies the equation

$$
\begin{aligned}
i\partial_t z - Hz &= i\partial_t v - Hv + \frac{2i}{\langle\nabla\rangle^2}\mathrm{Re}(\overline{u}\partial_t u) - \frac{H}{\langle\nabla\rangle^2}|u|^2 \\
&= U(2\varphi_1^2 + |\varphi|^2\varphi_1) - i\frac{\mathrm{div}}{\langle\nabla\rangle^2}\big[4\varphi_1\nabla\varphi_2 + \nabla(|\varphi|^2\varphi_2)\big] \\
&= U(2v_1^2 + |V^{-1}v|^2v_1) - i\frac{\mathrm{div}}{\langle\nabla\rangle^2}\big[4v_1U^{-1}\nabla v_2 + \nabla(|U^{-1}v|^2U^{-1}v_2)\big].
\end{aligned}
$$

We underline that the only terms involving singularities at low frequencies are now cubic. This will be sufficient to perform a fixed point argument. Finally, since H^s bounds on $z = \varphi_1 + iU\varphi_2 + \langle\nabla\rangle^{-2}|\varphi|^2$ do not imply bounds on φ (even for small data), we will work on $Z = U^{-1}z = U^{-1}\varphi_1 + i\varphi_2 + U^{-1}\langle\nabla\rangle^{-2}|\varphi|^2$, solution of

$$
i\partial_t Z - HZ = 2\varphi_1^2 + |\varphi|^2\varphi_1 - i\frac{U^{-1}\mathrm{div}}{\langle\nabla\rangle^2}\big[4\varphi_1\nabla\varphi_2 + \nabla(|\varphi|^2\varphi_2)\big] = P(\varphi). \tag{5.219}
$$

Functional spaces   As we work in subcritical settings $s > \frac{N}{2} - 1$ we will not need to separate low and high frequencies. Set $b = 1/q = \frac{1}{2} - \frac{1}{N}$, $1/p = 1/2 - 1/(2N)$ so that $(2, q)$, $(4, p)$ are admissible Strichartz pairs. For $s > N/2 - 1$ we define the space

$$
X = L^\infty(\mathbb{R}^+,\ H^s(\mathbb{R}^N)) \cap L^2(\mathbb{R}^+,\ B_{q,2}^s(\mathbb{R}^N)).
$$

In particular we note that by interpolation

$$
\|u\|_{L^4 B_{p,2}^s} \lesssim \|u\|_X. \tag{5.220}
$$

Mapping $\varphi \to U^{-1}V\varphi \to Z$:   Using Bernstein's inequalities we have

$$
\|U^{-1}f\|_{B_{r,2}^s} \lesssim \|f\|_{B_{\widetilde{r},2}^{s+1}} \text{ if } \frac{1}{\widetilde{r}} = \frac{1}{r} + \frac{1}{N} \leq 1, \tag{5.221}
$$

so that the product estimates from proposition 3.2 yield

$$\|U^{-1}\langle\nabla\rangle^{-2}|f|^2\|_{B^{s-1}_{q',2}} \quad \lesssim \quad \|f\|_{H^{s-1}}\|f\|_{B^0_{N,1}} \lesssim \|f\|^2_{H^s}, \tag{5.222}$$

$$\|U^{-1}\langle\nabla\rangle^{-2}|f|^2\|_{B^s_{q,2}} \quad \lesssim \quad \|f^2\|_{B^{s-1}_{6N/(3N+2),2}} \lesssim \|\varphi\|_{H^{s-1}}\|f\|_{B^0_{3N,1}} \lesssim \|f\|_{H^s}\|f\|_{B^s_{q,2}}, \tag{5.223}$$

From a fixed point argument, the application $w \to w+U^{-1}|V^{-1}Uw|^2$ is Lipschitz with Lipschitz inverse $H^s \to H^s$ on a neighbourhood of 0. In particular for some $\delta > 0$,

$$\|Z\|_{H^s} < \delta \Rightarrow \|Z\|_{H^s} \sim \|U^{-1}V\varphi\|_{H^s},$$

and provided $\|Z\|_X$ is small enough

$$\|Z\|_X \sim \|U^{-1}V\varphi\|_X \gtrsim \|\varphi\|_X. \tag{5.224}$$

Fixed point argument (sketch of)   Scattering for Z is equivalent to solve

$$Z(t) = e^{-itH}Z_0 - i\int_0^t e^{-i(t-\tau)H}P(\varphi(\tau))d\tau.$$

From the Strichartz estimates 3.5, we have

$$\|e^{-itH}Z_0 - i\int_0^t e^{-i(t-\tau)H}P(\varphi(\tau))d\tau\|_X \lesssim \|Z_0\|_{H^s} + \|P(\varphi)\|_{L^2 B^s_{q'}},$$

and we first check $\|P(\varphi)\|_{L^2 B^s_{q'}}\| \lesssim \|Z\|^2_X$. For example, the term φ_1^2 can be estimated as follows

$$\|\varphi_1^2\|_{L^2 B^s_{q'}} \lesssim \|\varphi_1\|_{L^\infty H^s}\|\varphi_1\|_{L^2 B^s_q} \le \|\varphi_1\|^2_X \lesssim \|Z\|^2_X.$$

The cubic term $U^{-1}\Delta\langle\nabla\rangle^{-2}(|\varphi|^2\varphi_2)$ is handled thanks to the embedding (5.220) (note that the multiplier $U^{-1}\Delta\langle\nabla\rangle^{-2}$ is not singular at $\xi = 0$)

$$\|U^{-1}\Delta\langle\nabla\rangle^{-2}(|\varphi|^2\varphi_2)\|_{L^2 B^s_{q'}} \lesssim \||\varphi|^2\varphi_2\|_{L^2 B^s_{q'}} \quad \lesssim \quad \||\varphi|^2\|_{L^2 H^s}\|\varphi\|_{L^\infty H^s}$$
$$\lesssim \quad \|\varphi\|^2_{L^4 B^s_p}\|\varphi\|_{L^\infty H^s}$$
$$\lesssim \quad \|Z\|^3_X.$$

The other terms can be dealt with similarly, this gives

$$\|e^{-itH}Z_0 - i\int_0^t e^{-i(t-\tau)H}P(\varphi(\tau))d\tau\|_{X_0} \le \|Z_0\|_{H^s} + \|Z\|^2_X + \|Z\|^3_X.$$

Contractivity can be obtained by a similar argument since $P(\varphi)$ is essentially polynomial, and the fixed point theorem can be applied to obtain a unique global scattering solution.

6   Proof of the theorem 2.5

In order to prove the theorem 2.5, it is enough to prove that the global strong solution ψ of (GP) obtained by Gustafson et al in [29] does not vanish, or in other term that φ remains small enough in $L^\infty_{t,x}$ norms. It will be the objective of the first part of the proof, in the sequel via the Madelung transform we should propagate the regularity of φ on $\rho - 1$ and $u = \nabla\theta$. It will be enough to obtain the existence of global weak solution and by adding regularity hypothesis on the initial data existence of global strong solution to the system (1.1).

Control L^∞ norm of φ which should remains small

It may seem peculiar to bound the L^∞ norm by using the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ for $s > 3/2$, especially since the Gross-Pitaevskii equation is known to be globally well-posed in the energy space $\{u \in H^1_{loc}(\mathbb{R}^3) : \nabla u \in L^2, |u|^2 - 1 \in L^2(\mathbb{R}^3)\}$ (Gérard [22]).
Nevertheless, the L^∞ regularity is not propagated by the semi-group $e^{it\Delta}$ which explains why we need to use stronger spaces than the naive choice $L^\infty \cap H^1$. However, in order to exploit the time decay we will split the analysis between short and long time. In long time using weighted data ensures a L^∞ control In short time we will use a smoothing effect on the Duhamel part in order to control the L^∞ norm with initial data in $H^{N/2-1/6+\epsilon}$ rather than $H^{\frac{N}{2}+\epsilon}$.

6.1 L^∞ control on φ in long time $t \geq 1$

Let us recall that we have via (4.54):

$$
\begin{aligned}
&\||\nabla|^{-2+\frac{5\theta}{3}} v_{<1}(t)\|_{L^6} \lesssim \min(1, t^{-\theta}) \|v(t)\|_{X(t)}, \\
&\||\nabla|^\theta v_{\geq 1}(t)\|_{L^6} \lesssim \min(t^{-\theta}, t^{-1}) \|v(t)\|_{X(t)},
\end{aligned}
\tag{6.225}
$$

with $0 \leq \theta \leq 1$. In particular it implies that:

$$
\|U^{-1} v_{<1}(t)\|_{L^6} + \|\nabla U^{-1} v_{\geq 1}(t)\|_{L^6} \lesssim (\min(1, t^{-\frac{3}{5}}) + \frac{1}{t}) \|v(t)\|_{X(t)}.
\tag{6.226}
$$

Since $\varphi = V^{-1}v = \mathrm{Re}v + iU^{-1}\mathrm{Im}v$ we have for $t \geq \alpha$ (which will be fixed in section 6.3) and by Sobolev embedding:

$$
\begin{aligned}
&\|\varphi\|_{L^\infty_{t\geq\alpha}(H^{1,6})} \lesssim \|V^{-1}v\|_{L^\infty_{t\geq 1}(H^{1,6})} \lesssim \frac{1}{t^{\frac{3}{5}}} \|v\|_{X(t)}, \\
&\|\varphi\|_{L^\infty_{t\geq\alpha}(L^\infty)} \lesssim \frac{1}{t^{\frac{3}{5}}} \|v\|_{X(t)} \lesssim \frac{\delta}{t^{\frac{3}{5}}}.
\end{aligned}
\tag{6.227}
$$

This implies that φ remains small for large $t \geq 1$.

6.2 L^∞ control on φ in short time $t \leq 1$

It remains now to show that ψ does not cancel on the time interval $[0, 1]$. To do this we are going to show that φ is small in L^∞ by using a Kato smoothing property which gives us a gain of one $\frac{1}{2}$ derivatives on the nonlinear evolution term. This is a relatively well-known property that seems to have been explicitly stated only recently by Bona et al in [12]. This was used to prove a dispersive blow up result for Schrödinger and Gross-Pitaevskii equations. We include their result in this section as it enlightens the fact that L^∞ is a "bad" space for initial data.

Theorem 6.1. *(Bona-Ponce-Saut-Sparber [12])*
For $N \geq 3$, $s > \frac{N}{2} - 1/6$, $\varphi_0 \in H^s(\mathbb{R}^N)$, let φ be the solution of (1.6) satisfying:

$$
\begin{aligned}
\begin{pmatrix} Re(\varphi) \\ Im(\varphi) \end{pmatrix}(t) &= A(t) \begin{pmatrix} Re(\varphi_0) \\ Im(\varphi_0) \end{pmatrix} + \int_0^t A(t-s) \begin{pmatrix} Re(F(\varphi(s))) \\ Im(F(\varphi(s))) \end{pmatrix} ds \\
&= A(t) \begin{pmatrix} Re(\varphi_0) \\ Im(\varphi_0) \end{pmatrix} + I(t, x).
\end{aligned}
$$

with
$$A(t) = \begin{pmatrix} \cos(Ht) & U\sin(Ht) \\ -U^{-1}\sin(Ht) & \cos(Ht) \end{pmatrix}.$$

For any $T > 0$ there exists $\varepsilon_0 > 0$ such that if $\|\varphi_0\|_{H^s} < \varepsilon_0$

1. *the solution is defined on $[0, T]$ and*

$$\varphi(t) = e^{it(\Delta - 1)}\varphi_0 + \int_0^t e^{i(t-s)(\Delta - 1)} F(\varphi) ds + g(t), \ \|g\|_{L_T^\infty H^{s+1}} \lesssim \|\varphi_0\|_{H^s}$$

2. $\int_0^t e^{i(t-s)(\Delta - 1)} F(\varphi) ds \in C([0, T], H^{s+\frac{1}{2}})$, *in particular $I \in C_b([0, T] \times \mathbb{R}^N)$ and there exists $\alpha > 0$:*

$$\|I\|_{C_b([0,T]\times\mathbb{R}^N)} \lesssim T^\alpha(\|u_0\|_{H^s}^2 + \|u_0\|_{H^s}^3). \tag{6.228}$$

3. *Moreover for any $\varepsilon < \varepsilon_0$, there exists $\varphi_0 \in H^s(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ such that $\|\varphi_0\|_{H^s \cap L^\infty} \leq \varepsilon$ and $\|\varphi(\cdot, T)\|_{L^\infty(\mathbb{R}^3)} = +\infty$.*

Remark 24. • This is essentially is a linear result. The blow up of the L^∞ norm is due to the linear evolution part while nonlinear terms are controlled. It can be proved (using invariances of the equations) that for any $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}^*$ there exists an initial data leading to a solution blowing precisely at (x_0, t_0).

• Actually the result from [12] only applies to $N \leq 3$, we include here a slightly modified proof that works for larger dimensions. Also we chose to work directly on the original system rather than the diagonalized one in order to avoid issues with the singular multiplier U^{-1}.

Proof. We first prove 1). Consider the Cauchy problem

$$\begin{cases} i\partial_t \varphi + \Delta \varphi - 2\mathrm{Re}(\varphi) = (\varphi + 1)|\varphi|^2 + 2\mathrm{Re}(\varphi)\varphi = F(\varphi), \\ \varphi|_{t=0} = \varphi_0. \end{cases}$$

Since $s > N/2 - 1$ (the critical index for cubic NLS) the existence of a solution in $S_T := C([0, T], H^s) \cap L^2([0, T], H^{s, \frac{2N}{N-2}})$ follows from the standard theory (see [16]). Indeed taking φ_0 small enough in H^s ensures the existence of a strong solution on $[0, T]$ (T depends on $\|\varphi_0\|_{H^s}$ since we are subcritical) with in addition: $\|\varphi\|_{S_T} \lesssim \|\varphi_0\|_{H^s}$.
We remind the notations:

$$H = \sqrt{-\Delta/(2 - \Delta)}, \ U = \sqrt{-\Delta/(2 - \Delta)}, \ \text{and } A(t) = \begin{pmatrix} \cos(Ht) & U\sin(Ht) \\ -U^{-1}\sin(Ht) & \cos(Ht) \end{pmatrix},$$

the Duhamel formula reads

$$\begin{pmatrix} \mathrm{Re}(\varphi) \\ \mathrm{Im}(\varphi) \end{pmatrix}(t) = A(t) \begin{pmatrix} \mathrm{Re}(\varphi_0) \\ \mathrm{Im}(\varphi_0) \end{pmatrix} + \int_0^t A(t - s) \begin{pmatrix} \mathrm{Re}(F(\varphi(s))) \\ \mathrm{Im}(F(\varphi(s))) \end{pmatrix} ds.$$

The linear evolution operator $A(t)$ can be compared with the Schrödinger evolution group $A_S = \begin{pmatrix} \cos(-\Delta t) & \sin(-\Delta t) \\ -\sin(-\Delta t) & \cos(-\Delta t) \end{pmatrix}$ by using a Taylor expansion

$$|H(\xi) - |\xi|^2 - 1| + |V(\xi) - 1)| = O((|\xi|^2 + 1)^{-1}), \ |V^{-1}(\xi) - 1| = O(\min(|\xi|^{-1}, |\xi|^{-2})),$$

We can deduce directly

$$R(\xi, t) := A(\xi, t) - \begin{pmatrix} \cos\left((|\xi|^2 + 1)t\right) & \sin\left((|\xi|^2 + 1)t\right) \\ -\sin\left(|\xi|^2 + 1)t\right) & \cos\left((|\xi|^2 + 1)t\right) \end{pmatrix} = O\left(1/(|\xi|^2 + 1)\right)$$

(the singularity of $U^{-1} - 1$ is harmless since there is a factor $\sin(Ht)$ which cancels at the same order). The associated operator $R(\Delta, t)$ is thus continuous $H^s \to H^{s+2}$, and setting $M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ the solution rewrites

$$\begin{aligned}
\begin{pmatrix} \mathrm{Re}(\varphi) \\ \mathrm{Im}(\varphi) \end{pmatrix}(t) &= A_S(t)M(t)\begin{pmatrix} \mathrm{Re}(\varphi_0) \\ \mathrm{Im}(\varphi_0) \end{pmatrix} + \int_0^t A_S(t-s)M(t-s)\begin{pmatrix} \mathrm{Re}(F(\varphi(s))) \\ \mathrm{Im}(F(\varphi(s))) \end{pmatrix} ds \\
&\quad + R(t)\begin{pmatrix} \mathrm{Re}(\varphi_0) \\ \mathrm{Im}(\varphi_0) \end{pmatrix} + \int_0^t R(t-s)\begin{pmatrix} \mathrm{Re}(F(\varphi(s))) \\ \mathrm{Im}(F(\varphi(s))) \end{pmatrix} ds,
\end{aligned} \tag{6.229}$$

with

$$\left\| R(t)\begin{pmatrix} \mathrm{Re}(\varphi_0) \\ \mathrm{Im}(\varphi_0) \end{pmatrix} \right\|_{H^{s+2}} \lesssim \|\varphi_0\|_{H^s}.$$

For the nonlinear term of the second line we have $\varphi \in C_T H^s \cap L_T^p H^{s,q}$, for any admissible (p, q). Fix $\frac{6N}{3N-1} = q < \frac{2N}{N-2}$ so that $p > 2$ and $H^{s,q} \hookrightarrow L^\infty$. Using the rules on products (all embeddings are far from being sharp here)

$$\begin{aligned}
\|u^2\|_{H^{s-1}} &\lesssim \|u\|_{H^{s-1}}\|u\|_{L^\infty} \lesssim \|u\|_{H^{s-1}}\|u\|_{H^{s,q}}, \\
\|u^3\|_{H^{s-1}} &\lesssim \|u\|_{H^s}\|u\|_{H^{s,q}}^2,
\end{aligned}$$

so that putting these estimates in $\int R(t-s)F(\varphi)ds$

$$\begin{aligned}
\left\| \int_0^t R(t-s)\begin{pmatrix} \mathrm{Re}(F(\varphi(s))) \\ \mathrm{Im}(F(\varphi(s))) \end{pmatrix} ds \right\|_{H^{s+1}} &\lesssim \|F(\varphi)\|_{L_T^1 H^{s-1}} \\
&\lesssim T^{(p-2)/p}\|\varphi\|_{L_T^\infty H^s}(\|\varphi\|_{L_T^p H^s} + \|\varphi\|_{L_T^p H^{N/2-1,q}}^2) \\
&\lesssim T^{(p-2)/p}(\|\varphi\|_{S_T}^2 + \|\varphi\|_{S_T}^3).
\end{aligned}$$

The estimate $\|\varphi\|_{S_T} \lesssim \|\varphi_0\|_{H^s}$ ends the proof of 1).

The first line of (6.229) rewrites in complex coordinates as $e^{it(\Delta-1)}u_0 + \int_0^t e^{i(t-s)(\Delta-1)}F(\varphi(s))ds$, so that points 2) and 3) are precisely proposition 4.1 and lemma 2.1 from [13][4].

For completeness we mention the argument for lemma 2.1: the function $\varphi_0 = \dfrac{e^{-ix^2/4}}{(1+x^2)^{m/2}}$ belongs to H^s for $m > s + N/2$ (and obviously $C^\infty(\mathbb{R}^3) \cap L^\infty$) and the linear solution to the corresponding Cauchy problem $e^{it\Delta}u_0$ blows up precisely at time $t_b = 1$, and at the point $x_b = 0$ if $m \leq 3$. This follows from the explicit formula

$$e^{it\Delta}\varphi_0(x) = \frac{1}{(4i\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy = \frac{1}{(4i\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{i\frac{|x-y|^2}{4t}} dy \frac{e^{-iy^2/4}}{(1+y^2)^{m/2}}dy,$$

---

[4]The smoothing property of from [13] is actually stated for nonlinearities of type $|u|^\alpha u$ but can be carried out without any change to our case. There is also a condition $[\alpha + 1] \geq s + 1/2$ however it is only required to differentiate the function $u \to |u|^p u$, in our case the nonlinearity is polynomial so this is not an issue

which holds for all $(x,t) \neq (0,1)$ (it can be rigorously justified by oscillating integrals arguments). For $(x,t) = (0,1)$, $1/(1+|x|^2)^{m/2}$ is not integrable iff $m \leq N$. This gives the condition $s + N/2 < m < N$ and as expected L^∞ blow up can be obtained while we remain below the critical index $s = N/2$.

$\square$

6.3 Global L^∞ control of φ

For $\varepsilon > 0$, $\varphi_0 \in H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3)$ and if $\|e^{it\Delta}\varphi_0\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \leq \|\widehat{\varphi_0}\|_{L^1} \leq 1/4$ by smallness assumption on the initial data, the application of theorem 6.1 gives:

$$\|\varphi\|_{L^\infty([0,1],L^\infty(\mathbb{R}^N))} \leq \frac{1}{2}. \tag{6.230}$$

In particular it implies that:

$$|\psi(t,x)| \geq \frac{1}{2} \quad \forall 0 \leq t \leq 1 \text{ and } \forall x \in \mathbb{R}^3. \tag{6.231}$$

Combining (6.226) and (6.231), and for small enough initial data we obtain

$$|\psi(t,x)| \geq \frac{1}{2} \quad \forall t \geq 0 \text{ and } \forall x \in \mathbb{R}^3. \tag{6.232}$$

Remark 25. It will sufficient in the following to proves that the solution (ρ, u) of system (1.1) has no vacuum for $t \geq 0$.

6.4 How to propagate the regularity from φ to ρ and u

The theorem 2.2 ensures the existence of a unique global solution φ to the system (1.6) with in addition φ belonging in $C(\mathbb{R}, H^1(\mathbb{R}^3)) \cap L^2(H^{1,6})$. Indeed we have seen (see (4.55)) that $U^{-1}v \in L^2(H^{1,6})$.

Remark 26. In this section in order to prove the existence of global strong solution for the system (1.1) we propagate high regularity of φ on ρ and u. In particular we recall a some classical results, assume that in addition of the assumptions of theorem 2.2, φ_0 belongs in $H^{\frac{N}{2}+1+\epsilon}$. Then using [16] we can prove that φ belongs in $C((0,T), H^2(\mathbb{R}^N))$ (see theorem 5.3.1 p 146) for any $T > 0$ and next that φ belongs in $C((0,T), H^{\frac{N}{2}+1+\epsilon}(\mathbb{R}^3))$ (see theorem 5.4.1 p 146) for any $T > 0$. In particular by using Strichartz estimate we prove in addition that $\varphi \in L^2_T(B_{6,2}^{\frac{N}{2}+1+\epsilon})$ for any $T > 0$.

Proposition 6.9. *Assume that φ the solution of (1.6) is belongs in $C([0,+\infty), H^s(\mathbb{R}^N)) \cap L^2_T(B_{6,2}^s)$ with $s \geq \frac{N}{2}$ and that $|\psi| = |1+\varphi| \geq \frac{1}{2}$ then we have:*

$$q = \rho - 1 \in L^\infty_T(H^s(\mathbb{R}^N)) \cap L^2_T(B_{6,2}^s) \quad \text{and} \quad u \in L^\infty(H^{s-1}(\mathbb{R}^N)) \cap L^2_T(B_{6,2}^{s-1}). \tag{6.233}$$

More precisely we have:

$$\begin{aligned}
\|q\|_{L^\infty(H^s)\cap L^2_T(B_{6,2}^s)} &\lesssim (1 + \|\varphi\|_{L^\infty_{t,x}})\|\varphi\|_{L^\infty(H^s)\cap L^2_T(B_{6,2}^s)}, \\
\|u\|_{L^\infty(H^{s-1})} &\lesssim \big(1 + C_1(\|\varphi\|_{L^\infty_{t,x}})(1 + \|\varphi\|_{L^\infty(H^s)})\big)\|\varphi\|_{L^\infty(H^s)}, \\
\|u\|_{L^2(B_{6,2}^{s-1})} &\lesssim \big(1 + C_1(\|\varphi\|_{L^\infty_{t,x}})(1 + \|\varphi\|_{L^\infty(H^s)})\big)\|\varphi\|_{L^2(B_{6,2}^{s-1})}
\end{aligned} \tag{6.234}$$

Proof: We are now interested in translating the regularity of φ on ρ and u via the Madelung transform which corresponds to:

$$\psi(t,x) = 1 + \varphi(t,x) = \sqrt{\rho}(t,x) e^{\frac{i\theta(t,x)}{2\kappa_1}} \quad \text{with } u = \nabla\theta. \tag{6.235}$$

In particular we are going to use a polar decomposition (used also in [1]):

$$
\begin{aligned}
\tau(t,x) &= \frac{\psi(t,x)}{|\psi(t,x)|} = \left(\frac{1+\varphi(t,x)}{|1+\varphi(t,x)|} - 1\right) + 1, \\
q(t,x) &= 2\text{Re}(\varphi) + |\varphi|^2, \\
u(t,x) &= \nabla\theta(t,x) = 2\kappa_1 \text{Im}\Big(\big[(\frac{1+\bar{\varphi}(t,x)}{|1+\varphi(t,x)|^2} - 1)\nabla\varphi(t,x) + \nabla\varphi(t,x)\big]\Big).
\end{aligned}
\tag{6.236}
$$

Let us point out that u is well defined on $(0,+\infty) \times \mathbb{R}^N$ since we have assumed that $|\psi| \geq \frac{1}{2}$. We have now by proposition 3.2 and 3.3 (since $2s - 1 - \frac{N}{2} \geq s - 1$):

$$
\begin{aligned}
\|q\|_{L^\infty(H^s)\cap L_T^2(B_{6,2}^s)} &\lesssim (1 + \|\varphi\|_{L_{t,x}^\infty})\|\varphi\|_{L^\infty(H^s)\cap L_T^2(B_{6,2}^s)}, \\
\|u\|_{L^\infty(H^{s-1})} &\lesssim (1 + \|(\frac{1+\bar{\varphi}(t,x)}{|1+\varphi(t,x)|^2} - 1)\|_{L_{t,x}^\infty})\|\nabla\varphi\|_{L^\infty(H^{s-1})} \\
&\quad + \|(\frac{1+\bar{\varphi}(t,x)}{|1+\varphi(t,x)|^2} - 1)\|_{L^\infty(H^s)}\|\nabla\varphi\|_{L^\infty(H^{s-1})}, \\
&\lesssim \big(1 + C_1(\|\varphi\|_{L_{t,x}^\infty})(1 + \|\varphi\|_{L^\infty(H^s)})\big)\|\varphi\|_{L^\infty(H^s)}.
\end{aligned}
\tag{6.237}
$$

In the same way we have:

$$\|u\|_{L^2(B_{6,2}^{s-1})} \lesssim \big(1 + C_1(\|\varphi\|_{L_{t,x}^\infty})(1 + \|\varphi\|_{L^\infty(H^s)})\big)\|\varphi\|_{L^2(B_{6,2}^{s-1})} \tag{6.238}$$

$\square$

6.5 Existence of global weak solution when $N = 3$

Regular initial data

It suffices to follows the arguments developed in [1] and [15], let us first show the theorem 2.5 in the case of regular initial data (ρ_0, u_0) implying that φ_0 is in $H^s(\mathbb{R}^3)$ with s sufficiently large. Sufficiently large in the sense that involving the results of preservation of the initial regularity (see Cazenave [16] Chapter 5) the solution φ constructed in the theorem 2.2 remains in $C([-T,T], H^s(\mathbb{R}^3))$ for any $T > 0$. In particular taking s large enough by Sobolev embedding we show that φ is in $C^k(\mathbb{R} \times \mathbb{R}^3)$. It implies that φ is a classical solution of Gross-Pitaevskii equation and we are going to exhibit a solution (ρ, u) of the system (1.1) using the Madelung transform. Indeed it suffices to set:

$$\rho = |1 + \varphi|^2 \quad \text{and} \quad u = 2\kappa \text{Im}(\nabla\varphi \frac{1+\bar{\varphi}}{|1+\varphi|^2}).$$

Let us point out that u is well defined since $|\psi| \geq \frac{1}{2}$ via the section 6.3. In addition (ρ, u) is in $C^k(\mathbb{R} \times \mathbb{R}^3) \times C^{k-1}(\mathbb{R} \times \mathbb{R}^3)$. In particular by using computation related to the Madelung transform we see that (ρ, u) is a classical solution of the system (1.1) and in particular a global weak solution.

General case

Let us now treat the general case where (ρ_0, u_0) verify the assumption of the theorem 2.5 with $\varphi_0 \in H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3)$ in particular. We set $\varphi_0^n = \varphi_0 * \psi_n$ with ψ_n a regularizing kernel (with $\int \psi_n(y)dy = 1$, $0 \le \psi_n \le 1$ and $\text{supp}\psi_n \subset B(0, \frac{1}{n})$) such that φ_0^n belongs in H^s for any s large enough. We are interesting in showing that φ_0^n verify the assumptions of the theorem 2.2. Assume that $\langle x \rangle f \in L^2(\mathbb{R}^3)$ then we have by Hölder's inequality and Fubini theorem:

$$\int \langle x \rangle^2 | \int f(x-y)\varphi_n(y)dy|^2 dx \le \int \langle x \rangle^2 (\int |f(x-y)|^2 \varphi_n(y)dy)dx,$$

$$\le 2(\int |f(x)|^2 dx)^2 + \int 2(|x-y|^2 + |y|^2)(\int |f(x-y)|^2 \varphi_n(y)dy)dxdy,$$

$$\le 2\big(2(\int |f(x)|^2 dx)^2 + (\int |x|^2 |f(x)|^2 dx)^2\big).$$

In particular we have:

$$\int_{\mathbb{R}^3} \langle x \rangle (|\text{Re}\varphi_0^n|^2 + |\nabla\varphi_0^n|^2)dx \le 2 \int_{\mathbb{R}^3} \langle x \rangle (|\text{Re}\varphi_0|^2 + |\nabla\varphi_0|^2)dx = \delta_1. \tag{6.239}$$

In addition we have φ_0^n which is uniformly bounded in $H^{\frac{5}{4}+\epsilon}(\mathbb{R}^3)$ then taking δ_1 small enough it exists some sequence φ_n solution of the Gross-Pitaevskii equation via the theorem 2.2 with $|\psi_n| \ge \frac{1}{2}$ (because we have $\|\varphi_n\|_{L^\infty} \le \frac{1}{2}$ see the section 6.3). Let us point out that the control of the vacuum $|\psi_n| \ge \frac{1}{2}$ is uniform in n, it is obvious since it depends only of the quantity δ_1 and $\|\varphi_0^n\|_{H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3)}$ which is uniformly bounded in n (see theorem 6.1 and 2.2) .

In addition setting $\rho_n = |1 + \varphi_n|^2$ and $u_n = 2\kappa\text{Im}(\nabla\varphi_n \frac{1+\bar{\varphi}_n}{|1+\varphi_n|^2})$ then (ρ_n, u_n) is a global weak solution of the system (1.1) with initial data (ρ_0^n, u_0^n) and this by using the previous section (indeed φ_0^n is in any $H^s(\mathbb{R}^3)$). From the proposition 6.9 we deduce that $(\rho_n - 1, u_n)$ is uniformly bounded in $L^\infty(H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3)) \times L^\infty(H^{\frac{1}{6}+\epsilon}(\mathbb{R}^3))$. Indeed it is easy to verify by proposition 3.2 that u_n belongs to $L^\infty(H^{\frac{1}{6}+\epsilon}(\mathbb{R}^3))$. From section 6.9 $(\rho_n, \frac{1}{\rho_n})$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}^3))$. Since the solution φ and φ^n are obtained by a fixed point argument, we have continuity with respect of initial data in $H^{\frac{4}{3}+\epsilon}$ and then we get for all $T > 0$:

$$\lim_{n \to +\infty} \|\varphi^n - \varphi\|_{C([0,T], H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3))} = 0.$$

We deduce in particular that:

$$\lim_{n \to +\infty} \|(\rho^n - 1) - (\rho - 1)\|_{C([0,T], H^{\frac{4}{3}+\epsilon}(\mathbb{R}^3))} = 0,$$

$$\lim_{n \to +\infty} \|u^n - u\|_{C([0,T], H^{\frac{1}{6}+\epsilon}(\mathbb{R}^3))} = 0.$$

It implies easily by compactness that $(\nabla\sqrt{\rho_n})_{n \in \mathbb{N}}$ converges strongly in $C([0,T], H^{\frac{1}{3}+\epsilon}(\mathbb{R}^3))$ by using the proposition 3.3 and the fact $(\frac{1}{\rho^n}, \rho^n)$ is uniformly bounded in $L^\infty(L^\infty)$. In particular it yields the strong convergence of $(\nabla\sqrt{\rho_n})_{n \in \mathbb{N}}$ in $L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^3)$ to $\nabla\sqrt{\rho}$ which is sufficient to pass to the limit in the sense of distribution for the capillary term. We proceed similarly in order to deal with the terms $\rho^n u^n = u^n + (\rho^n - 1)u^n$ and $\rho^n u^n \times u^n = u^n \times u^n + (\rho^n - 1)u^n \times u^n$. It finishes the proof.

6.6 Existence of global strong solution when $N \geq 3$

In the previous section we have proved the existence of global weak solution for the Euler-Korteweg system (1.1) when $N = 3$ under the assumption of smallness (2.19) and the fact that $\varphi_0 \in H^{\frac{5}{4}+\epsilon}(\mathbb{R}^3)$. With additional assumption on $\varphi_0 \in H^{\frac{N}{2}+1+\epsilon}$ we obtain new control on $q = \rho - 1$ and u since we have shown by using proposition 6.233 that they belong respectively in $L^\infty_{loc}(H^{\frac{N}{2}+1+\epsilon}) \cap L^2_{loc}(B^{\frac{N}{2}+1+\epsilon}_{q,2})$ (with $q = \frac{2N}{N-2}$) and in $L^\infty_{loc}(H^{\frac{N}{2}+\epsilon}) \cap L^2_{loc}(B^{\frac{N}{2}+\epsilon}_{q,2})$.

Our main taste is to prove now that under these control the global weak solution that we have constructed is unique. Let us start with rewriting the system (1.1) following the change of unknown introduced by Benzoni et al in [5]:

$$\begin{cases} \partial_t \ln \rho + u \cdot \nabla \ln \rho + \mathrm{div} u = 0, \\ \partial_t z + u \cdot \nabla z + i\nabla z \cdot w + i\sqrt{\kappa_1}\nabla \mathrm{div} z = \dfrac{1}{\sqrt{\kappa_1}}\rho w, \end{cases} \tag{6.240}$$

with:

$$w = \sqrt{\kappa_1}\nabla \ln \rho, \; a(\rho) = \sqrt{\kappa_1}, \; z = u + iw.$$

It is proven in [5] (see the proposition 4.2 p 20) the following proposition using a gauge method.

Proposition 6.10. *Let (L, z) be a H^s solution with $s > 0$ of:*

$$\begin{cases} \partial_t L + v \cdot \nabla L + \mathrm{div} v = 0, \\ \partial_t z + v \cdot \nabla z + i\nabla z \cdot w + i\sqrt{\kappa_1}\nabla \mathrm{div} z = f, \end{cases} \tag{6.241}$$

on $[0, T] \times \mathbb{R}^N$. Assume that $w = \nabla L$ and $L = \ln \rho$ with ρ verifying $0 < c \leq \rho(t, x) \leq M < +\infty$ on $[0, T] \times \mathbb{R}^3$ then the following estimates hold true for all $t \in [0, T]$ and $\alpha \in [0, 1]$:

$$\|z(t)\|^2_{H^s} \lesssim \|z_0\|^2_{H^s} + \int_0^t (\|f\|_{H^s}\|z\|_{H^s} + A(\tau)\|z\|^2_{H^s})d\tau + \|w(t)\|^2_{C^{-\alpha}}\|z(t)\|^2_{H^{s-1+\alpha}}. \tag{6.242}$$

and:

$$\|(\sqrt{\rho}z)(t)\|^2_{L^2} \lesssim \|\sqrt{\rho}z(0)\|^2_{L^2} + \int_0^t \|\sqrt{\rho}z\|_{L^2}\|\sqrt{\rho}f\|_{L^2}d\tau. \tag{6.243}$$

with:

$$A(t) = 1 + \|Dz(t)\|_{L^\infty}.$$

The authors of [5] obtain the following corollary (see 4.2 p 23 in [5])

Corollary 6.0. *Let (L, z) satisfy the assumptions of proposition 6.10, then we have for $C > 0$:*

$$\|z\|_{L^\infty_T(H^s)} \lesssim e^{C\int_0^T A(\tau)d\tau}(1 + \|w\|^{\max(1,s)}_{L^\infty_T(L^\infty)})(\|z_0\|_{H^s} + \|f\|_{L^1_T(H^s)}). \tag{6.244}$$

Remark 27. Let us mention that if z is irrotational, then we can extend the range of s to $s > -\frac{N}{2}$ (see the remark 4.1 p 24 of [5]).

In the spirit of the proposition 5.1 p 29 of [5] we obtain the following proposition.

Proposition 6.11. *Let* $N \geq 3$. *Let* $(L_1 = \ln \rho_1, z_1)$ *and* $(L_2 = \ln \rho_2, z_2)$ *be two solutions of (6.240) on* $[0,T] \times \mathbb{R}^N$ *in* $(L^\infty(H^{s+1}) \cap L^2(B^{s+1}_{q,2})) \times (L^\infty(H^s) \cap L^2(B^s_{q,2})$ *with* $\frac{N}{2} < s$, $s \neq 3 + \frac{N}{2}$ *and* $q = \frac{2N}{N-2}$. *Assume in addition that* L_i $(i=1,2)$ *is bounded in* L^∞. *Let us denote* $\delta L = L_2 - L_1$ *and* $\delta z = z_2 - z_1$. *Then the following estimate hold true for all* $t \in [0,T]$:

$$\|(\delta L(t), \delta z(t))\|_{H^{s-2}} \lesssim (1 + \|w_1\|^{\max(1,s-2)}_{L^\infty_t(L^\infty)}) \|(\delta L(0), \delta z(0))\|_{H^{s-2}}$$

$$\times e^{C \int_0^t (1 + \|Dz_1\|_{L^\infty \cap B^{\frac{N}{2}-1}_{\frac{2N}{N-2},\infty}} + (1 + \|Dw_1\|_{L^\infty \cap B^{\frac{N}{2}-1}_{\frac{2N}{N-2},\infty}} + \|Dw_2\|_{L^\infty \cap B^{\frac{N}{2}-1}_{\frac{2N}{N-2},\infty}}) \|Dz_2\|_{L^\infty \cap B^{\frac{N}{2}-1}_{\frac{2N}{N-2},\infty}} d\tau}, \tag{6.245}$$

where $w_i = Imz_i$.

Remark 28. Let us mention that this proposition extends the proposition 5.1 p 29 of [5] inasmuch as we need only a Lipschitz control on u, indeed ∇u is in $L^1_{loc}(L^\infty)$.

Proof: The equation satisfied by δz reads:

$$\partial_t \delta z + u_1 \cdot \nabla \delta z + i \nabla \delta z \cdot w_1 + i \nabla \text{div} \delta z = \nabla \delta \rho - (\delta u) \cdot \nabla z_2 - i \nabla z_2 \cdot \delta w,$$

with $\delta u = u_2 - u_1$, $\delta \rho = \rho_2 - \rho_1$ and $\delta w = w_2 - w_1$. We observe that δz solves an equation of type (6.241) since ρ_1 verifies the mass equation $\partial_t \rho_1 + \text{div}(\rho_1 u_1) = 0$. Besides $\rho_1 = L^{-1}(L_1)$ satisfies the mass conservation equation. Hence if $s > 2$ and $s \neq \frac{N}{2} + 3$, applying corollary 6.0 we can estimate δz in $H^{s'}$ as follows:

$$\|\delta z(t)\|_{H^{s'}} \lesssim \gamma(t) e^{C \int_0^t (1 + \|Dz_1\|_{L^\infty}) d\tau} \left(\|\delta z(0)\|_{H^{s'}} + \int_0^t (\|\delta \rho\|_{H^{s'+1}} + \|\delta u \cdot \nabla z_2\|_{H^{s'}} \right.$$

$$\left. + \|\nabla z_2 \cdot \delta w\|_{H^{s'}} d\tau) \right), \tag{6.246}$$

with $\gamma(t) = 1 + \|w_1\|^{\max(1,s')}_{L^\infty_t(L^\infty)}$.

It remains now to estimate the integrand in the right hand side of (6.246). By proposition 3.2 since we have $\frac{1}{2} \leq \frac{N-2}{2N} + \frac{1}{\lambda} \leq 1$ with $\lambda = N \in [1, +\infty]$, $2 \leq N$ and $s' + 1 < \frac{N}{2}$ then:

$$\|T_{\delta u} \nabla z_2\|_{H^{s'}} \lesssim \|\delta u\|_{H^{s'}} \|\nabla z_2\|_{B^{\frac{N}{2}-1}_{\frac{2N}{N-2},\infty}}, \tag{6.247}$$

$$\|T_{\nabla z_2} \delta u\|_{H^{s'}} \lesssim \|\nabla z_2\|_{L^\infty} \|\delta u\|_{H^{s'}},$$

Similarly we have since $s' + \frac{N-2}{2} = s' + \frac{N}{2} - 1 > 0$ then:

$$\|R(\nabla z_2, \delta u)\|_{H^{s'}} \lesssim \|\nabla z_2\|_{B^{\frac{N}{2}-1}_{\frac{2N}{N-2},\infty}} \|\delta u\|_{H^{s'}}. \tag{6.248}$$

We deduce that:

$$\|\delta u \cdot \nabla z_2\|_{H^{s'}} + \|\nabla z_2 \cdot \delta w\|_{H^{s'}} \lesssim \|Dz_2\|_{L^\infty \cap B^{\frac{N-2}{2}}_{\frac{2N}{N-2},\infty}} \|\delta z\|_{H^{s'}}.$$

Let us deal now with the term $\delta \rho$, we have then:

$$\|\delta \rho\|_{H^{s'+1}} \lesssim \int_0^1 \|\delta L \exp(L_1 + \tau \delta L)\|_{H^{s'+1}} d\tau,$$

$$\lesssim \int_0^1 \|\delta L (\exp(L_1 + \tau \delta L) - 1)\|_{H^{s'+1}} + \|\delta L\|_{H^{s'+1}}) d\tau$$

Next for $\tau \in [0,1]$ we have using proposition 3.2, 3.3, $s' + 1 - \frac{N}{2} < 0$ and the fact that $L_1 = \ln(1 + q_1), L_2 = \ln(1 + q_2) \in L^\infty$:

$$\|T_{(\exp(L_1 + \tau \delta L) - 1)} \delta L\|_{H^{s'+1}} \lesssim \|\delta L\|_{H^{s'+1}} \|(\exp(L_1 + \tau \delta L) - 1)\|_{L^\infty},$$

$$\|T_{\delta L}(\exp(L_1 + \tau \delta L) - 1)\|_{H^{s'+1}} \lesssim \|\delta L\|_{B_{\infty,2}^{s'+1-\frac{N}{2}}} \|(\exp(L_1 + \tau \delta L) - 1)\|_{B_{2,\infty}^{\frac{N}{2}}},$$

$$\lesssim \|\delta L\|_{H^{s'+1}} \|(\exp(L_1 + \tau \delta L) - 1)\|_{B_{2,\infty}^{\frac{N}{2}}},$$

$$\lesssim \|\delta L\|_{H^{s'+1}} (1 + \|L_1\|_{L^\infty \cap B_{2,\infty}^{\frac{N}{2}}} + \|L_2\|_{L^\infty \cap B_{2,\infty}^{\frac{N}{2}}})^2,$$

$$\lesssim \|\delta L\|_{H^{s'+1}} (1 + \|q_1\|_{L^\infty \cap B_{2,\infty}^{\frac{N}{2}}} + \|q_2\|_{L^\infty \cap B_{2,\infty}^{\frac{N}{2}}})^2$$

$$\|R(\exp(L_1 + \tau \delta L) - 1), \delta L)\|_{H^{s'+1}} \lesssim \|\delta L\|_{H^{s'+1}} \|\exp(L_1 + \tau \delta L) - 1\|_{B_{2,\infty}^{\frac{N}{2}}},$$

$$\lesssim \|\delta L\|_{H^{s'+1}} (1 + \|q_1\|_{L^\infty \cap B_{2,\infty}^{\frac{N}{2}}} + \|q_2\|_{L^\infty \cap B_{2,\infty}^{\frac{N}{2}}})^2.$$

Plugging all these inequalities in (6.246), using the fact that $\|\delta L\|_{H^{s'+1}} \le \|\delta z\|_{H^{s'}} + \|\delta L\|_{L^2} \le \|\delta z\|_{H^{s'}} + \|\delta L\|_{H^{s'}}$ if $s' \ge 0$ (here we have to choose $s' \in [0, \frac{N}{2} - 1[$ since we have seen that s' is in $] - \frac{N}{2} + 1, \frac{N}{2} - 1[$) and applying Gronwall's inequality, we end up with:

$$\begin{aligned}
\|\delta z(t)\|_{H^{s'}} \lesssim \\
\int_0^t \gamma(s) \|Dz_2(s)\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}} \\
\times \exp(C \int_s^t (1 + \|L_1(\tau)\|_{H^{\frac{N}{2}}} + \|L_2(\tau)\|_{H^{\frac{N}{2}}} + \|Dz_2(\tau)\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}) d\tau) ds \\
+ \gamma(t) \Big(\|\delta z(0)\|_{H^{s'}} + \int_0^t \|\delta L(\tau)\|_{H^{s'}}(1 + \|L_1(\tau)\|_{H^{\frac{N}{2}}} + \|L_2(\tau)\|_{H^{\frac{N}{2}}}) d\tau\Big)
\end{aligned} \tag{6.249}$$

In order to close the estimate we have to deal with the term $\|\delta L\|_{H^{s'}}$ which appears in the right hand side of (6.249). In order to do this, we use the fact that $\delta L = \ln \rho_2 - \ln \rho_1$ satisfies the following equation:

$$\partial_t \delta L + u_2 \cdot \nabla \delta L + \delta u \cdot \nabla L_1 + \mathrm{div} \delta u = 0,$$

and:

$$\partial_t \Delta_l \delta L + u_2 \cdot \nabla \Delta_l \delta L + \Delta_l(\delta u \cdot \nabla L_1 + \mathrm{div} \delta u) = [u_2^j, \Delta_l] \partial_j \delta L.$$

Taking the L^2 inner product of the above equation with $\Delta_l \delta L$, performing several integration by parts and integrate in time we get:

$$\begin{aligned}
\|\Delta_l \delta L(t)\|_{L^2}^2 \lesssim \|\Delta_l \delta L_0\|_{L^2}^2 + \int_0^t \big(\|\Delta_l(\delta u \cdot \nabla L_1)(\tau)\|_{L^2} \|\Delta_l \delta L(\tau)\|_{L^2} \\
+ \|\Delta_l \delta u\|_{L^2} \|\Delta_l(\delta w)(\tau)\|_{L^2}\big) d\tau + \int_0^t \big(\|\Delta_l \delta L(\tau)\|_{L^2} \|R_l(\tau)\|_{L^2} \\
+ \|\mathrm{div} u_2(\tau)\|_{L^\infty} \|\Delta_l \delta L(\tau)\|_{L^2}^2\big) d\tau.
\end{aligned} \tag{6.250}$$

We have set $R_l = [u_2^j, \Delta_l]\partial_j \delta L$. Using Lemma 3.4 (since $s' < \frac{N}{\frac{2N}{N-2}} = \frac{N}{2} - 1$), multiplying the previous equation by $2^{2ls'}$ and summing we have:

$$\|\delta L(t)\|_{H^{s'}}^2 \lesssim \|\delta L_0\|_{H^{s'}}^2 + \int_0^t \|\nabla u_2(\tau)\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}} \|\delta L(\tau)\|_{H^{s'}}^2 d\tau,$$
$$+ \int_0^t \left(\|\delta u \cdot \nabla L_1(\tau)\|_{H^{s'}} \|\delta L(\tau)\|_{H^{s'}} + \|\delta u\|_{H^{s'}} \|\delta w(\tau)\|_{H^{s'}}\right) d\tau. \tag{6.251}$$

Next by using proposition 3.2 we have since $s' < \frac{N}{2} - 1$:

$$\|\delta u \cdot \nabla L_1\|_{H^{s'}} \lesssim \|\delta u\|_{H^{s'}} \|\nabla L_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}},$$

Plugging this inequalities in (6.251) we have:

$$\|\delta L(t)\|_{H^{s'}}^2 \lesssim \|\delta L_0\|_{H^{s'}}^2 + \int_0^t \left(\|\nabla u_2(\tau)\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}} + \|\nabla L_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}\right) \|\delta L(\tau)\|_{H^{s'}}^2 d\tau$$
$$+ \int_0^t \left(1 + \|\nabla L_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}\right) \|\delta z(\tau)\|_{H^{s'}}^2 d\tau. \tag{6.252}$$

Using Gronwall's lemma we obtain :

$$\|\delta L(t)\|_{H^{s'}}^2 \lesssim e^{C \int_0^t \left(1 + \|\nabla u_2(\tau)\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}} + \|\nabla L_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}\right) d\tau}$$
$$\times \left(\|\delta L(0)\|_{H^{s'}}^2 + C \int_0^t (1 + \|\nabla L_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}) \|\delta z\|_{H^{s-2}}^2 d\tau\right).$$

Inserting (6.249); it yields after computations:

$$\|\delta L(t)\|_{H^{s'}}^2 \lesssim \gamma(t)^2 e^{C \int_0^t A(\tau) d\tau} \Bigg[\|\delta L(0)\|_{H^{s'}}^2$$
$$+ \left(\int_0^t A(\tau) e^{C \int_0^t A(\tau') d\tau'} d\tau\right) \left(\|\delta z(0)\|_{H^{s'}}^2 + \int_0^t A(\tau) \|\delta L\|_{H^{s'}} d\tau\right)^2 \Bigg],$$

with:

$$A(t) = (1 + \|Dz_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}} + (1 + \|Dw_1\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}} + \|Dw_2\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}) \|Dz_2\|_{L^\infty \cap B_{\frac{2N}{N-2},\infty}^{\frac{N}{2}-1}}.$$

Taking the square root and applying Gronwall's inequality, we conclude that:

$$\|\delta L(t)\|_{H^{s'}} \leq C\gamma(t)^2 e^{C \int_0^t A(\tau) d\tau} \left(\|\delta L(0)\|_{H^{s'}} + \|\delta z(0)\|_{H^{s'}}\right). \tag{6.253}$$

Finally, plugging (4.52) in (6.249) yields the desired inequality. $\qquad \square$

Lipschitz control on the velocity u

We recall that φ belongs in $L_T^2(B_{6,2}^s(\mathbb{R}^3))$ (with $s > \frac{3}{2} + 1 + \epsilon$) by propagation of the regularity for an initial data in H^s. In particular we deduce easily that u belongs in $L_T^2(B_{6,2}^{s-1})$ for any $T > 0$ and $\nabla u \in L_T^2(B_{6,2}^{s-2})$. By Sobolev embedding we have ∇u is in $L_T^2(L^\infty(\mathbb{R}^3))$ for any $T > 0$ since $\frac{1}{6} - \frac{s-2}{3} < 0$.

7 Proof of the theorem 2.4

The proof relies on the same arguments as for theorem 2.5 but is quite simpler.

Global control of $\|\varphi\|_{L^\infty}$

As a first step we prove time decay on $Z = U^{-1}(\varphi_1 + \langle\nabla\rangle^{-2}|\varphi|^2) + i\varphi_2$ (see section 5) which satisfies (5.219)

$$i\partial_t Z - HZ = 2\varphi_1^2 + |\varphi|^2\varphi_1 - i\frac{U^{-1}\text{div}}{\langle\nabla\rangle^2}\big[4\varphi_1\nabla\varphi_2 + \nabla(|\varphi|^2\varphi_2)\big] = P(\varphi).$$

We set $1/q = 1/2 - 2/(3N)$, $1/a = 1/2 - 1/(3N)$, $s > N/2 - 1$ and note that for $\|Z\|_{H^s \cap B^s_{a',2}}$ small enough, respectively $\|Z\|_{H^s \cap B^s_{q,2}}$ small enough,

$$\|Z\|_{H^s \cap B^s_{a',2}} \sim \|U^{-1}V\varphi\|_{H^s \cap B^s_{a',2}} \gtrsim \|\varphi\|_{H^s \cap B^s_{a',2}}, \tag{7.254}$$

$$\text{respectively } \|Z\|_{H^s \cap B^s_{q,2}} \sim \|U^{-1}V\varphi\|_{H^s \cap B^s_{q,2}} \gtrsim \|\varphi\|_{H^s \cap B^s_{q,2}}. \tag{7.255}$$

This follows from the estimates (see also (5.222, 5.223):

$$\|U^{-1}\langle\nabla\rangle^{-2}|\varphi|^2\|_{H^s} \lesssim \|\varphi\|_{H^s}^2,$$
$$\|U^{-1}\langle\nabla\rangle^{-2}|\varphi|^2\|_{B^s_{a',2}} \lesssim \|\varphi^2\|_{B^{s-1}_{6N/(3N+8),2}} \lesssim \|\varphi\|_{H^{s-1}}\|\varphi\|_{B^0_{3N/4,1}} \lesssim \|\varphi\|_{H^s}^2,$$
$$\|U^{-1}\langle\nabla\rangle^{-2}|\varphi|^2\|_{B^s_{q,2}} \lesssim \|\varphi\|_{H^s}\|\varphi\|_{B^s_{q,2}},$$

and a fixed point argument. Also from $\|Z\|_{B^s_{q,2}} \geq \|U^{-1}V\varphi\|_{B^s_{q,2}} - \|U^{-1}\langle\nabla\rangle^{-2}|\varphi|^2\|_{B^s_{q,2}}$,

$$\|Z\|_{B^s_{q,2}} \gtrsim \|U^{-1}V\varphi\|_{B^s_{q,2}}. \tag{7.256}$$

Proposition 1. Let $Z \in L^\infty(H^s) \cap L^2(B^s_{\frac{2N}{N-2},2})$ be the global solution of (5.219). Set $1/q = 1/2 - 2/(3N)$ ($q = \frac{6N}{3N-4}$). There exists $\varepsilon_0 > 0$ such that

$$\forall \varepsilon \leq \varepsilon_0, \ \sup_{t\geq 0} t^{1/3}\|e^{itH}Z_0\|_{B^s_{q,2}} \leq \varepsilon \Rightarrow \sup_{t\geq 0} t^{1/3}\|Z(t)\|_{B^s_{q,2}} \leq 2\varepsilon.$$

Proof. We set $m(t) = \sup_{0\leq\tau\leq t} \tau^{1/3}\|Z\|_{B^s_{q,2}}$. From

$$\|Z(t)\|_{B^s_{q,2}} \lesssim \|e^{itH}Z_0\|_{B^s_{q,2}} + \|\int_0^t e^{i(t-\tau)H}P(\varphi)ds\|_{B^s_{q,2}}$$

it is only a matter of bounding the Duhamel term, plugging the norm $\|\cdot\|_{B^s_{q,2}}$ inside the integral, using the dispersion estimate (see lemma (3.6) with $\theta = 0$, $\sigma = \frac{2}{3N}$) we obtain

$$\left\|\int_0^t e^{i(t-\tau)H}P(\varphi)ds\right\|_{B^s_{q,2}} \lesssim \int_0^t \frac{\|P(\varphi)(\tau,\cdot)\|_{B^s_{q',2}}}{(t-\tau)^{2/3}}d\tau \tag{7.257}$$

It remains to estimate $\|P(\varphi)(\tau,\cdot)\|_{B^s_{q',2}}$. Arguing as in section 5 it is sufficient to estimate $\|\varphi^2\|_{B^s_{q',2}} + \|\varphi^3\|_{B^s_{q',2}}$, this will be done by using paraproduct laws in Besov spaces. For quadratic

terms, proposition 3.2 with $q \leq \lambda = \frac{3N}{4}$ gives

$$\|T_\varphi \varphi\|_{B^s_{q',2}} \;\lesssim\; \|\varphi\|_{B^s_{q,2}} \|\varphi\|_{B^{\frac{N}{2}-2}_{q,1}},$$

$$\|R(\varphi,\varphi)\|_{B^s_{q',2}} \;\lesssim\; \|\varphi\|_{B^s_{q,2}} \|\varphi\|_{B^{\frac{N}{2}-2}_{q,1}},$$

$$\Rightarrow \|\varphi^2\|_{B^s_{q',2}} \;\lesssim\; \|\varphi\|^2_{B^s_{q,2}}.$$

For cubic terms we start with an estimate on $\|\varphi^2\|_{B^s_{p,2}}$ with $1/p \leq 1/q + 1/q$ to be precised later. Proposition 3.2 gives:

$$\|T_\varphi \varphi\|_{B^s_{p,2}} \lesssim \|\varphi\|_{B^s_{q,2}} \|\varphi\|_{B^{N-\frac{4}{3}-\frac{N}{p}}_{q,1}}.$$

We chose p such that $N - \frac{4}{3} - \frac{N}{p} < s$, this condition implies:

$$1 - \frac{4}{3N} - \frac{s}{N} \leq \frac{1}{p} < 1 - \frac{4}{3N}, \ (0 < 1/p \text{ if } N - 4/3 - s < 0). \tag{7.258}$$

In particular using proposition 3.1 we deduce that:

$$\|T_\varphi \varphi\|_{B^s_{p,2}} \lesssim \|\varphi\|^2_{B^s_{q,2}}. \tag{7.259}$$

Using again proposition 3.2 and the condition (7.258) we observe that $N - \frac{4}{3} - \frac{N}{p} + s > 0$ which implies:

$$\|R(\varphi,\varphi)\|_{B^s_{p,2}} \lesssim \|\varphi\|_{B^s_{q,2}} \|\varphi\|_{B^{N-\frac{4}{3}-\frac{N}{p}}_{q,\infty}}.$$

And we obtain finally using the previous inequality, (6.247) and proposition 3.1:

$$\|\varphi^2\|_{B^s_{p,2}} \lesssim \|\varphi\|^2_{B^s_{q,2}}, \tag{7.260}$$

with p verifying the condition (7.258).

Next using again proposition 3.2 with $\frac{1}{q'} = \frac{1}{p} + \frac{1}{\lambda}$, $2 \leq \lambda$ and $s_1 = \frac{N}{2} + \frac{N}{p} - \frac{N}{q'} = \frac{N}{p} - \frac{2}{3}$ we have:

$$\|T_\varphi \varphi^2\|_{B^s_{q',2}} \lesssim \|\varphi^2\|_{B^s_{p,2}} \|\varphi\|_{B^{\frac{N}{p}-\frac{2}{3}}_{2,1}} \lesssim \|\varphi^2\|_{B^s_{p,2}} \|\varphi\|_{B^{\frac{N}{2}-1}_{2,2}}. \tag{7.261}$$

Here we need p to satisfy

$$\frac{2}{3N} < \frac{1}{p} < \frac{s}{N} + \frac{2}{3N}. \tag{7.262}$$

For p to satisfy both conditions (7.262) and (7.258), we need only $1 - 3/(4N) - s/N < s/N + 2/(3N)$, which is satisfied since $s > \frac{N}{2} - 1$.

In the same way we estimate $T_{\varphi^2}\varphi$, by proposition 3.2 since $\frac{1}{q'} = \frac{1}{2} + \frac{1}{\lambda}$, $p \leq \lambda$ and $s_1 = \frac{N}{p} - \frac{N}{\lambda} = \frac{N}{p} + \frac{N}{2} - \frac{N}{q'} = \frac{N}{p} - \frac{2}{3}$ we have:

$$\|T_{\varphi^2}\varphi\|_{B^s_{q',2}} \lesssim \|\varphi\|_{B^s_{2,2}} \|\varphi^2\|_{B^{\frac{N}{p}-\frac{2}{3}}_{q,1}} \lesssim \|\varphi\|_{B^s_{2,2}} \|\varphi\|^2_{B^s_{q,2}}. \tag{7.263}$$

Similarly since $\frac{N}{p} - \frac{2}{3} + s > 0$ and $\frac{1}{q'} \leq \frac{1}{2} + \frac{1}{q}$ because $N \geq 4$ we have:

$$\|R(\varphi^2,\varphi)\|_{B^s_{q',2}} \lesssim \|\varphi\|_{B^s_{2,2}} \|\varphi^2\|_{B^{\frac{N}{p}-\frac{2}{3}}_{q,\infty}} \lesssim \|\varphi\|_{B^s_{2,2}} \|\varphi\|^2_{B^s_{q,2}}. \tag{7.264}$$

Using (7.261), (7.263) and (7.264) we have:

$$\|\varphi^3\|_{B^s_{q',2}} \lesssim \|\varphi\|_{B^s_{2,2}} \|\varphi\|^2_{B^s_{q,2}}. \tag{7.265}$$

Finally $\|\varphi\|_{B^s_{q,2}} \lesssim \|Z\|_{B^s_{q,2}}$, $\|\varphi\|_{H^s} \lesssim \|Z\|_{H^s}$, the estimate (7.257) yield

$$\left\| \int_0^t e^{i(t-\tau)H} P(\varphi)(\tau,\cdot) ds \right\|_{B^s_{q,2}} \lesssim \int_0^t \frac{m(\tau)^2}{(t-\tau)^{2/3}\tau^{2/3}} d\tau = \frac{m(t)^2}{t^{1/3}} \int_0^1 \frac{1}{(1-\tau)^{2/3}\tau^{2/3}} d\tau \lesssim \frac{m(t)^2}{t^{1/3}}.$$

Using the Duhamel formula we get $m(t) \leq \varepsilon + Cm(t)^2$. As it is not clear wether the solution is $L^\infty_{\text{loc}} t^{-1/3} B^s_{q,2}$ or not, we may simply set $E = \{\varphi \in L^\infty H^s \cap L^2 B^s_{p,2} : \sup_{t\geq 0} t^{1/3}\|\varphi\|_{B^s_{q,2}} < \infty\}$ and repeat the fixed point argument in $L^\infty H^s \cap L^2 B^s_{p,2} \cap E$. $\qquad\square$

Corollary 7.0. *For* $s > N/2 - 2/3$, $Z_0 \in H^s \cap B^s_{a',2}$, $1/a = 1/2 - 1/(3N)$, *there is* $\varepsilon_0 > 0$ *such that*

$$\forall \varepsilon \leq \varepsilon_0, \ \|Z_0\|_{B^s_{a',2} \cap H^s} \leq \varepsilon \Rightarrow \sup_{t\geq 0} t^{1/3}\|Z(t)\|_{B^{s-1/3}_{q,2}} \lesssim 2\varepsilon.$$

Proof. The embedding $B^s_{a,2} \hookrightarrow B^{s-1/3}_{q,2}$ and the dispersion estimate give:

$$\|e^{-itH} Z_0\|_{B^{s-1/3}_{q,2}} \lesssim \|e^{-itH} Z_0\|_{B^s_{a,2}} \lesssim \frac{1}{t^{N(1/2-1/a)}} \|Z_0\|_{B^s_{a',2}} = \frac{\|Z_0\|_{H^{s,a'}}}{t^{1/3}},$$

so that we can conclude by applying proposition 1. $\qquad\square$

Proposition 7.12. *For* $s > N/2 - 1/6$, $1/a = 1/2 - 1/(3N)$, *if* $U^{-1}V\varphi_0 \in H^s \cap B^s_{a',2}$ *there is* ε_0 *such that for any* $\varepsilon \leq \varepsilon_0$

$$\|U^{-1}V\varphi_0\|_{H^s \cap B^s_{a',2}} \leq \varepsilon_0 \ \text{and} \ \|e^{-it\Delta}\varphi_0\|_{L^\infty([0,1]\times\mathbb{R}^N)} \leq 1/4 \Rightarrow \|\varphi\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^N)} \leq 1/2.$$

Proof. If $\|U^{-1}V\varphi_0\|_{H^s \cap B^s_{a',2}} << 1$, from (7.254) corollary 7.0 applies and for $t \geq 1$, $\|Z(t)\|_{B^{s-1/3}_{q,2}} << 1$. Using then (7.255) we get

$$\forall t \geq 1, \ \|\varphi(t)\|_{L^\infty} \lesssim \|\varphi\|_{B^{s-1/3}_{q,2}} \lesssim \|Z\|_{B^{s-1/3}_{q,2}} << 1.$$

The bound on $\|\varphi\|_{L^\infty([0,1]\times\mathbb{R}^N)}$ is then obtained thanks to theorem 6.1 :

$$\begin{aligned}
\|\varphi\|_{L^\infty([0,1]\times\mathbb{R}^N)} &\leq \|e^{-it\Delta}\varphi_0\|_{L^\infty([0,1]\times\mathbb{R}^N)} + C\|\varphi_0\|_{H^s} + C\left\| \int_0^t e^{i(t-s)\Delta} F(\varphi) ds \right\|_{H^{s+1/2}} \\
&\leq \frac{1}{4} + C\varepsilon_0.
\end{aligned}$$

$\qquad\square$

Remark 29. The only reason why we must work with $s > N/2 - 1/6$ is to control the Duhamel term for $t \leq 1$. Smallness for $t \geq 1$ only requires $s > N/2 - 1/3$ thus the assumption $U^{-1}V\varphi_0 \in H^s \cap H^{s,a'}$ is probably a bit too strong.

Proof of the theorem 2.4

The existence of global weak solution follows the same line than for $N = 3$ and the uniqueness too.

References

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