Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D

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Abstract

We consider Navier-Stokes equations for compressible viscous fluids in one dimension. We prove the existence of global strong solution with large initial data for the shallow water system. The key ingredient of the proof relies to a new formulation of the compressible equations involving a new effective velocity v (see [8, 11, 10, 9]) such that the density verifies a parabolic equation. We estimate v in L^∞ norm which enables us to control the vacuum on the density.

1 Introduction

This paper is devoted to the existence of global strong solutions of the following Navier-Stokes equations for compressible isentropic flows:

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho)\partial_x u) + \partial_x P(\rho) = 0. \end{cases} \tag{1.1}$$

with possibly degenerate viscosity coefficient.
Throughout the paper, we will assume that the pressure $P(\rho)$ verifies a γ type law:

$$P(\rho) = \rho^\gamma,\ \gamma > 1. \tag{1.2}$$

Following the idea of [8, 11, 10, 9], setting $v = u + \partial_x\varphi(\rho)$ with $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$ we have (we refer to the appendix for more details on the computations):

$$\begin{cases} \partial_t\rho - p_x(\dfrac{\mu(\rho)}{\rho}\partial_x\rho) + \partial_x(\rho v) = 0, \\ \rho\partial_t v + \rho u\partial_x v + \partial_x P(\rho) = 0. \end{cases} \tag{1.3}$$

Let us mention that this change of unknown transform the system (1.1) as a parabolic equation on the density and a transport equation for the velocity (we will see in the sequel that v is not so far to verify a damped transport equation).It seems very surprising to observe that at the inverse of u which has a parabolic behavior, v has a hyperbolic

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behavior. In some sense in $1D$ the compressible Navier Stokes equations are a compressible Euler system with a viscous regularization on the density of the type $-p_x(\frac{\mu(\rho)}{\rho}\partial_x\rho)$ In the literature the authors are often interested in the constant viscosity case, however physically the viscosity of a gas depends on the temperature and on the density (in the isentropic case). Let us mention the case of the Chapman-Enskog viscosity law (see [3]) or the case of monoatomic gas ($\gamma = \frac{5}{3}$) where $\mu(\rho) = \rho^{\frac{1}{3}}$. More generally, $\mu(\rho)$ is expected to vanish as a power of the ρ on the vacuum. In this paper we are going to deal with degenerate viscosity coefficients which can be written under the form $\mu(\rho) = \mu\rho$ with corresponds to the shallow-water system.
Let us start by recalling some results on the existence of solutions for the one dimension case when the viscosity is constant positive. The existence of global weak solutions was first obtained by Kazhikhov and Shelukin [18] for smooth enough data close to the equilibrium (bounded away from zero). The case of discontinuous data (still bounded way from zero) was studied by Shelukin [28, 29, 30] and then by Serre [26, 27] and Hoff [12]. First results dealing with vacuum were also obtained by Shelukin [31]. In [14], Hoff extends the previous results by proving the existence of global weak solution with large discontinuous initial data having different limits at $x = \pm\infty$. In passing let us mention that the existence of global weak solution has been proved for the first time by P-L Lions in [20] and the result has been later refined by Feireisl et al ([7] and [6]).Concerning the uniqueness of the solution, Solonnikov in [33] obtained the existence of strong solution for smooth initial data in finite time. However, the regularity may blow up when the solution approach from the vacuum. A natural question is to understand if the vacuum may arise in finite time. Hoff and Smoller ([15]) show that any weak solution of the Navier-Stokes equation in one space dimension do not exhibit vacuum states, provided that no vacuum states are present initially.
Let us mention now some results on the 1D compressible Navier-Stokes equations when the viscosity coefficient depends on the density. This case has been studied in [21], [25], [35], [36], [16] and [19] in the framework of initial density admitting vacuum, more precisely the initial density is compactly supported and the authors are concerned by the evolution of the free boundary delimiting the vacuum. We are interested in the opposite situation, it means where there is no vacuum indeed in this case the velocity has a parabolic behavior which enables us to prove uniqueness results. This case has been studied by A. Mellet and A. Vasseur who proves in [24] the existence of global strong solution with large initial data when $\mu(\rho)$ verifies the following condition:

$$\begin{cases} \mu(\rho) \geq \nu\rho^{\alpha} \;\; \forall\rho \leq 1 \;\text{ for some } \alpha \in [0, \dfrac{1}{2}), \\ \mu(\rho) \geq \nu \;\; \forall\rho \geq 1. \end{cases}$$

The key tool of the proof consist in controlling the vacuum by using an entropy derived by D. Bresch and B. Desjardins in [2] in the multi d case (In the one dimensional case, a similar inequality was introduced earlier by V. A. Vaigant [34] for flows with constant viscosity, see also V. V. Shelukin [32]).
. Indeed it allows to A. Mellet and A. Vasseur to control in norm $L^\infty(L^2)$ the quantity $\partial_x\rho^{\alpha-\frac{1}{2}}$, by Sobolev embedding they can estimate the L^∞ norm on $\frac{1}{\rho}$. Similar arguments allows to control the L^∞ norm of the density ρ. It is then sufficient to the authors for showing that the solution of V. Solonnikov in [33] can be extended for any time T (or in

other word that there is no blow up of the regularity in finite time). We would like also to mention that A. Mellet and A. Vasseur prove the stability of the global weak solution for compressible Navier-Stokes equation with viscosity coefficient verifying the relation used in [2].
The goal of this paper consists in extending the result of [24] to the case where $\mu(\rho) = \mu\rho^\alpha$ with $\alpha > 0$ and in particular $\alpha \geq \frac{1}{2}$. The idea consists in studying the system (1.3) and to prove that v remains in $L^\infty_t(L^\infty)$ for any $t > 0$. To do this we shall verify that v verifies a damped transport equation with a remainder term bounded in $L^2(L^\infty)$. We can then use the maximum principle on the first equation of (1.3) in order to control ρ and $\frac{1}{\rho}$ in L^∞. We can then prove that the solution of V. Solonnikov in [33] does not blow up for any time $t > 0$ which is sufficient to show the existence of global strong solution.

In the next section we state our main result. Section 3 deal with the proof of the theorem 2.1 and we postpone a appendix in order to explain the equivalence between the system (1.1) and the system (1.3).

2 Main result

Following D. Hoff in [14], we work with positive initial data having (possible different) positive limits at $x = \pm$. We fix constant positive density $\rho_+ > 0$ and $\rho_- > 0$ and a smooth monotone function $\bar{\rho}(x)$ such that:

$$\bar{\rho}(x) = \rho_\pm \text{ when } x_\pm \geq 1, \;\; \bar{\rho}(x) > 0 \;\forall x \in \mathbb{R}. \tag{2.4}$$

In the sequel in order to simplify, we will deal with density of the form $\mu(\rho) = \mu\rho^\alpha$ with $\alpha > 0$ and in addition we assume that it exists $C > 0$ such that:

$$\mu(\rho) \leq C + CP(\rho) \;\; \forall \rho \geq 0. \tag{2.5}$$

Our main theorem is the following:

Theorem 2.1 *Assume that $\mu(\rho) = \mu\rho$ and the initial data ρ_0 and u_0 satisfy:*

$$\begin{aligned} &0 < \alpha_0 \leq \rho_0(x) \leq \beta_0 < +\infty, \\ &\rho_0 - \bar{\rho} \in H^1(\mathbb{R}), \\ &u_0 - \bar{u} \in H^1(\mathbb{R}), \\ &\partial_x \rho_0 \in L^\infty, \end{aligned} \tag{2.6}$$

for some constant α_0 and β_0. Assume that $p(\rho) = a\rho^\gamma$ with $\gamma \geq 2$. Then there exists a global strong solution (ρ, u) of system (1.3) on $\mathbb{R}^+ \times \mathbb{R}$ such that for every $T > 0$:

$$\begin{aligned} &\rho - \bar{\rho} \in L^\infty(0, T; H^1(\mathbb{R})), \\ &u - \bar{u} \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^2(0, T, H^2(\mathbb{R})), \\ &v \in L^\infty_T(L^\infty(\mathbb{R})). \end{aligned}$$

Moreover for every $T > 0$, there exists constant $\alpha(T)$ and $\beta(T)$ such that

$$0 < \alpha(T) \leq \rho(t, x) \leq \beta(T) < +\infty \;\; \forall (t, x) \in (0, T) \times \mathbb{R}.$$

Finally if $\mu(\rho) \geq \mu > 0$ for all $\rho \geq 0$, if μ is uniformly Lipschitz then this solution is unique in the class of weak solutions satisfying the usual entropy inequality (3.10).

Remark 1 *This result extend the work [24] to the classical shallow water system.*

When the viscosity coefficient $\mu(\rho)$ satisfies:

$$\mu(\rho) \geq \nu > 0 \ \ \forall \rho \geq 0, \tag{2.7}$$

the existence of strong solution with large initial data in finite time is classical. An extension to the case of degenerate viscosity coefficient follows the same ideas than in [33]. More precisely we have:

Proposition 2.1 *([33]). Let (ρ_0, u_0) satisfy (2.6) and assume that μ satisfies $\mu(\rho) = \mu\rho^\alpha$ with $\alpha \geq 0$ then there exists T_0 depending on $\alpha_0, \beta_0, \|\rho_0 - \bar{\rho}\|_{H^1}$ and $\|u_0 - \bar{u}\|_{H^1}$ such that (1.1) has a unique solution (ρ, u) on $(0, T_0)$ satisfying:*

$$\begin{aligned} &\rho - \bar{\rho} \in L^\infty(0, T_1, H^1(\mathbb{R})),\ \partial_t \rho \in L^2((0, T_1) \times \mathbb{R}), \\ &u - \bar{u} \in L^2(0, T_1, H^2(\mathbb{R})),\ \partial_t u \in L^2((0, T_1) \times \mathbb{R}) \end{aligned}$$

for all $T_1 < T_0$.
Moreover, there exist some $\alpha(T) > 0$ and $\beta(T) < +\infty$ such that $\alpha(t) \leq \rho(, x) \leq \beta(t)$ for all $t \in (0, T_0)$.

Remark 2 *In order to prove the theorem 2.1, it will be sufficient to show that we can control $\alpha(T)$, $\beta(T)$, $\|\rho(T) - \bar{\rho}\|_{H^1}$ and $\|u(T) - \bar{u}\|_{H^1}$ for any $T > 0$. In other words we are interested in proving that these quantities do not blow up when T goes to T_0 (in particular that $\alpha(T)$ does not go to 0 and the other quantities to $+\infty$). It will imply then that $T_0 = +\infty$.*

3 Proof of theorem 2.1

We are going to follow the proof of Mellet and Vasseur in [24] except in the way to control the vacuum $\frac{1}{\rho}$ in L^∞. To do this we are going to observe that v is in L^∞ which ensures that no vacuum arises by using the maximum principle to the first equation of system (1.3).

3.1 Entropy inequalities

Let us start by recalling some classical entropy inequalities (we refer to [24] for more details). Let us define the well-known relative entropy, for any functions $U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$ and $\widetilde{U}\begin{pmatrix} \widetilde{\rho} \\ \widetilde{\rho u} \end{pmatrix}$ we set:

$$\begin{aligned} \mathcal{H}(U/\widetilde{U}) &= \mathcal{H}(U) - \mathcal{H}(\widetilde{U}) - D\mathcal{H}(\widetilde{U})(U - \widetilde{U}), \\ &= \rho(u - \bar{u})^2 + p(\rho/\widetilde{\rho}), \end{aligned}$$

where $p(\rho/\widetilde{\rho})$ is the relative entropy associated to $\frac{1}{\gamma-1}\rho^\gamma$:

$$p(\rho/\widetilde{\rho}) = \frac{1}{\gamma-1}\rho^\gamma - \frac{1}{\gamma-1}\bar{\rho}^\gamma - \frac{\gamma}{\gamma-1}\bar{\rho}^{\gamma-1}(\rho-\bar{\rho}).$$

Let us mention that since p is convex, the function $p(\rho/\widetilde{\rho})$ remains positive for every ρ and $p(\rho/\widetilde{\rho}) = 0$ if and only if $\rho = \bar{\rho}$.
Mellet and Vasseur in [24] have obtained the following entropy inequalities.

Lemma 1 *Let (ρ, u) be a solution of system (1.1) satisfying the entropy inequality:*

$$\partial_t \mathcal{H}(U) + \partial_x[F(U) - \mu(\rho)u\partial_x u] + \mu(\rho)|\partial_x u|^2 \leq 0; \tag{3.8}$$

assume that the initial data (ρ_0, u_0) satisfies:

$$\int_{\mathbb{R}} \mathcal{H}(U_0/\bar{U})dx = \int_{\mathbb{R}} [\rho_0 \frac{(u_0-\bar{u})^2}{2} + p(\rho_0/\bar{\rho})]dx < +\infty, \tag{3.9}$$

then for every $T > 0$, there exists a positive constant $C(T)$ such that:

$$\sup_{[0,T]} \int_{\mathbb{R}} [\rho \frac{(u-\bar{u})^2}{2} + p(\rho/\bar{\rho})]dx + \int_0^T \int_{\mathbb{R}} \mu(\rho)(\partial_x u)^2 dx\, dt \leq C(T). \tag{3.10}$$

The constant $C(T)$ depends only on $T > 0$, $\bar{U}$, the initial value U_0, γ and on the constant C appearing in (2.5).

Remark 3 *Let us point out that when both $\bar{\rho}$ and ρ_0 are bounded above and below away from zero, we can prove that:*

$$p(\rho_0/\bar{\rho}) \leq C(\rho_0 - \bar{\rho})^2.$$

In particular (3.9)is verified under the assumptions of theorem 2.1.

Lemma 2 *Assume that $\mu(\rho)$ is a C^2 function, and let (ρ, u) be a solution of system (1.1) such that:*

$$u - \bar{u} \in L^2((0,T); H^2(\mathbb{R})),\ \rho - \bar{\rho} \in L^\infty((0,T); H^1(\mathbb{R})),\ 0 < m \leq \rho \leq M. \tag{3.11}$$

Then there exists $C(T)$ such that the following inequality holds:

$$\begin{aligned} &\sup_{[0,T]} \int [\frac{1}{2}\rho|(u-\bar{u}) + \partial_x(\varphi(\rho))|^2 + p(\rho/\bar{\rho})|dx \\ &\qquad\qquad + \int_0^T \int_{\mathbb{R}} \partial_x(\varphi(\rho))\partial_x(\rho^\gamma)dxdt \leq C(T), \end{aligned} \tag{3.12}$$

and φ verifying $\varphi'(\rho) = \frac{\mu(\rho)}{\rho^2}$. The constant $C(T)$ depends only on $T > 0$, $(\bar{\rho}, \bar{u})$, the initial daat U_0, γ and on the constant C on (2.5).

3.2 Proof of theorem 2.1

The proposition 2.1 shows the existence of global strong solution (ρ, u) for the system (1.1) on a finite time interval $(0, T_0)$ with $T_0 > 0$. We are interested in proving that $T_0 = +\infty$. To do this we are going to follow the idea of the remark 2 which will ensure that we have no blow up of the solution (ρ, u).

3.2.1 A priori estimates

Since the initial datum (ρ_0, u_0) satisfies (2.6), we have:

$$\int \rho_0 (u_0 - \bar{u})^2 dx < +\infty \text{ and } \int \rho_0 |\partial_x(\varphi(\rho_0))|^2 dx < +\infty.$$

In addition (ρ, u) verifies (3.11), using lemmas 1 and 2 it yields for any $T < T_0$ (L^γ_2 is the Orlicz space, see [20]):

$$\begin{aligned}
&\|\sqrt{\rho}(u - \bar{u})\|_{L^\infty((0,T),L^2)} \leq C(T), \\
&\|\rho\|_{L^\infty((0,T),L^1_{loc} \cap L^\gamma_l oc)} \leq C(T), \\
&\|\rho - \bar{\rho}\|_{L^\infty((0,T),L^\gamma_2)} \leq C(T), \\
&\|\sqrt{\mu(\rho)}\partial_x u\|_{L^2((0,T),L^2)} \leq C(T),
\end{aligned} \tag{3.13}$$

and:

$$\begin{aligned}
&\|\sqrt{\rho}\partial_x \varphi(\rho)\|_{L^\infty((0,T),L^2)} \leq C(T), \\
&\|\partial_x \rho^\gamma \cdot \partial_x \varphi(\rho)\|_{L^2((0,T),L^2)} \leq C(T).
\end{aligned} \tag{3.14}$$

3.2.2 Uniform bounds for the density

The first proposition shows that the density is bounded by below and by above.

Proposition 3.2 *For every $T > 0$ there exists constants $\alpha(T)$ and $\beta(T)$ such that for all $T < T_0$:*

$$\alpha(T) \leq \rho(t, x) \leq \beta(T) \;\; \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Proof: Since (ρ, u) is a regular solution which verifies the system (1.1) on $(0, T_0)$, it is also solution of the system (1.3) on $(0, T_0)$ with

$$\begin{cases}
\partial_t \rho - \partial_x(\dfrac{\mu(\rho)}{\rho}\partial_x \rho) + \partial_x(\rho v) = 0, \\
\rho \partial_t v + \rho u \partial_x v + \partial_x P(\rho) = 0.
\end{cases}$$

We are interested in proving that v remains in $L^\infty_t(L^\infty)$. In order to do this since $P(\rho) = \rho^\gamma$ with $\gamma > 1$ and $v = u + \partial_x \varphi(\rho)$ we have:

$$\rho \partial_t v + \rho u \partial_x v + \frac{\gamma \rho^{\gamma+1}}{\mu(\rho)} v = \frac{\gamma \rho^{\gamma+1}}{\mu(\rho)} u.$$

It gives:

$$\partial_t v + u\partial_x v + \frac{\gamma\rho^\gamma}{\mu(\rho)} v = \frac{\gamma\rho^\gamma}{\mu(\rho)} u.$$

In particular we deduce that: We have then using the characteristics method and defining the flow by:

$$\begin{cases} \partial_t \chi(t,x) = u(t,\chi(t,x)), \\ \chi(0,x) = x. \end{cases}$$

It comes:

$$\partial_t[v(t,\chi(t,x))] + \frac{\gamma\rho^\gamma}{\mu(\rho)}(t,\chi(t,x))v(t,\chi(t,x)) = \frac{\gamma\rho^\gamma}{\mu(\rho)}(t,\chi(t,x))u(t,\chi(t,x)). \tag{3.15}$$

We deduce then that we have to solve the Cauchy equation with $f(t) = v(t,\chi(t,x)$:

$$\begin{cases} \partial_t f(t) + \frac{\gamma\rho^\gamma}{\mu(\rho)}(t,\chi(t,x))f(t) = \frac{\gamma\rho^\gamma}{\mu(\rho)}(t,\chi(t,x))u(t,\chi(t,x)), \\ f(0) = v_0(x) \end{cases} \tag{3.16}$$

It gives:

$$f(t) = v_0(x) + \gamma\int_0^t \frac{\rho^\gamma}{\mu(\rho)}(s,\chi(s,x))u(s,\chi(s,x))e^{-\gamma\int_s^t \frac{\rho^\gamma}{\mu(\rho)}(s',\chi(s',x))ds'}ds.$$

We have in particular:

$$v(t,x) = v_0(\chi(-t,x)) + \gamma\int_0^t \frac{\rho^\gamma}{\mu(\rho)}(s,\chi(s-t,x))u(s,\chi(s-t,x))e^{-\gamma\int_s^t \frac{\rho^\gamma}{\mu(\rho)}(s',\chi(s'-t,x))ds'}ds. \tag{3.17}$$

Since $\mu(\rho) = \mu\rho$ we know via the estimate (3.14) that $\partial_x\sqrt{\rho}$ belongs in $L^\infty_T(L^2)$. In addition we know that $\rho - \bar{\rho}$ is in $L^\infty_T(L^2_\gamma)$ using (3.13) which is embedded in $L^\infty_T(L^2)$ since $\gamma \geq 2$. Since $(\sqrt{\rho} - \sqrt{\bar{\rho}}) = \frac{\rho-\bar{\rho}}{\sqrt{\rho}+\sqrt{\bar{\rho}}}$ we deduce that $(\sqrt{\rho} - \sqrt{\bar{\rho}})$ is in $L^\infty_T(L^2)$. In other word we have prove that $\sqrt{\rho} - \sqrt{\bar{\rho}}$ is in $L^\infty_t(H^1)$. By Sobolev embedding it implies that ρ is in $L^\infty_T(L^\infty)$ with:

$$\|\rho\|_{L^\infty_T(L^\infty)} \leq \beta(T). \tag{3.18}$$

Multiplying the momentum equation of (1.1) by $u|u|^\epsilon$ with $\epsilon > 0$ small enough as in [23] and integrating by parts we prove that $\rho^{\frac{1}{2+\epsilon}}u$ belongs to $L^\infty_{loc}(L^{2+\epsilon}(\mathbb{R}))$. We recall now that since:

$$\partial_x(\rho u) = \sqrt{\rho}\sqrt{\rho}\partial_x u + 2\rho^{\frac{1}{2+\epsilon}}\rho^{\frac{1}{2}-\frac{1}{2+\epsilon}}u\partial_x\sqrt{\rho},$$

it implies via the estimate (3.14) and (3.18) that $\partial_x(\rho u)$ belongs in $L^2_T(L^2(\mathbb{R})) + L^\infty_T(L^p(\mathbb{R}))$ which is embedded in $L^2_T(L^2(\mathbb{R}) + L^p(\mathbb{R}))$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{2+\epsilon}$. By the Riesz Thorin theorem it implies that the Fourier transform $F(\partial_x(\rho u))$ is in $L^2_T(L^2(\mathbb{R}) + L^{p'}(\mathbb{R}))$ with $\frac{1}{p} + \frac{1}{p'} = 1$. It implies in particular that for $|\xi| \geq 1$ and Hölder inequality $\mathcal{F}(\rho u)$ is in $L^2_T(L^1(\mathbb{R}))$. And since $\rho u = \sqrt{\rho}\sqrt{\rho}u$ is in $L^2_T(L^2)$ we deduce by Plancherel theorem that $\mathcal{F}(\rho u)$ is in

$L^2_T(L^1(\mathbb{R}))$ for $|\xi|\leq 1$. It implies that $\mathcal{F}(\rho u)$ is in $L^2_T(L^1(\mathbb{R}))$. We deduce that ρu is in $L^2_T(L^\infty(\mathbb{R}))$. Finally we have from (3.17) when $\gamma\geq 2$ and since $\mu(\rho)=\mu\rho$:

$$
\begin{aligned}
|v(t,x)| &\leq \|v_0\|_{L^\infty}+\frac{\gamma}{\mu}\int_0^t \|\rho^{\gamma-2}(s,\cdot)\|_{L^\infty}\|\rho u(s,\cdot)\|_{L^\infty} e^{\frac{\gamma}{\mu}(t-s)\|\rho\|^{\gamma-1}_{L^\infty}}ds,\\
&\leq \|v_0\|_{L^\infty}+\frac{\gamma}{\mu}\|\rho^{\gamma-2}\|_{L^\infty_t(L^\infty)}\|\rho u\|_{L^2_t(L^\infty)} e^{\frac{\gamma}{\mu}t\|\rho\|^{\gamma-1}_{L^\infty}}\sqrt{t}.
\end{aligned}
$$

In particular we have proved that v belongs in $L^\infty_{loc}(L^\infty)(\mathbb{R}))$. By using the mass equation we deduce using the maximum principle (see [17]) that $\frac{1}{\rho}$ is in $L^\infty_{loc}(L^\infty(\mathbb{R}))$. ■

Proposition 3.3 *There exists a constant $C(T)$ such that:*

$$\|\rho(t,x)-\bar{\rho}(x)\|_{L^\infty((0,T),H^1(\mathbb{R}))}\leq C(T).$$

Proof: Proposition 3.3 yields from the fact that $\partial_x\rho=2\sqrt{\rho}\partial_x\rho$ which is in $L^\infty_T(L^2)$ using the previous proposition and (3.14). By (3.13) we know that $\rho-\bar{\rho}$ is in $L^\infty_T(L^2)$. It achieves the proof. ■ And the result follows.

3.2.3 Uniform bounds for the velocity

We follow the argument of [24] that we recall for completeness.

Proposition 3.4 *There exists a constant $C(T)$ such that:*

$$\|u-\bar{u}\|_{L^2((0,T),H^2(\mathbb{R}))}\leq C(T),$$

and:

$$\|\partial_t u\|_{L^2((0,T),L^2(\mathbb{R}))}\leq C(T).$$

In particular $u-\bar{u}\in C((0,T),H^1(\mathbb{R}))$.

Proof: First, we observethat $u-\bar{u}$ is bounded in $L^2((0,T),H^1(\mathbb{R}))$. Indeed since we have $\rho\geq\alpha>0$ using (3.10) it implies that $\partial_x u$ is bounded in $L^2((0,T)\times\mathbb{R})$ and $u-\bar{u}$ is bounded in $L^\infty((0,T),L^2(\mathbb{R}))$. Therefore $u-\bar{u}$ is bounded in $L^2((0,T),H^1(\mathbb{R}))$.
We deduce via the mass equation that $\partial_t\rho$ is bounded in $L^2((0,T)\times\mathbb{R})$. Since $\rho-\bar{\rho}$ is bounded in $L^\infty((0,T),H^1(\mathbb{R}))$, it yields (see [1] for example) that ρ belongs to $C^{s_0}((0,T)\times\mathbb{R})$ for some s_0 in $(0,1)$.
Next let us rewrite the momentum equation as follows:

$$\partial_t u-\partial_x(\frac{\mu(\rho)}{\rho}\partial_x u)=-\gamma\rho^{\gamma-2}\partial_x\rho+\bar{u}\partial_x u+\big(\partial_x\varphi(\rho)-(u-\bar{u})\big)\partial_x u. \tag{3.19}$$

In order to obtain new estimates on u, we are going to control the right hand side of (3.19). The first term $\rho^{\gamma-2}\partial_x\rho$ is bounded in $L^\infty((0,T),L^2(\mathbb{R}))$ (thanks to proposition

3.3 and 3.2). The second term is bounded in $L^2((0,T)\times\mathbb{R})$ since $\bar{u}$ is in L^∞. For the last part following [24], we write (using Hölder inequality and interpolation inequality):

$$
\begin{aligned}
&\|\big(\partial_x\varphi(\rho)-(u-\bar{u})\big)\partial_x u\|_{L^2((0,T),L^{\frac{4}{3}}(\mathbb{R}))}\\
&\leq \|\partial_x\varphi(\rho)-(u-\bar{u})\|_{L^\infty((0,T),L^2(\mathbb{R}))}\|\partial_x u\|_{L^2((0,T),L^4(\mathbb{R}))}\\
&\leq \|\partial_x\varphi(\rho)-(u-\bar{u})\|_{L^\infty((0,T),L^2(\mathbb{R}))}\|\partial_x u\|^{\frac{2}{3}}_{L^2((0,T),L^2(\mathbb{R}))}\|\partial_x u\|^{\frac{1}{3}}_{L^2((0,T),W^{1,\frac{4}{3}}(\mathbb{R}))}\\
&\leq C\|\partial_x u\|^{\frac{1}{3}}_{L^2((0,T),W^{1,\frac{4}{3}}(\mathbb{R}))}
\end{aligned}
$$

(here we have used (3.12) and proposition 3.2). Using regularity results for parabolic equation of the form (3.19) (note that that the diffusion coefficient is in $C^{s_0}((0,T)\times\mathbb{R})$ give:

$$\|\partial_x u\|_{L^2((0,T),W^{1,\frac{4}{3}}(\mathbb{R}))}\leq C\|\partial_x u\|^{\frac{1}{3}}_{L^2((0,T),W^{1,\frac{4}{3}}(\mathbb{R}))}+C,$$

and by bootstrap:

$$\|\partial_x u\|_{L^2((0,T),W^{1,\frac{4}{3}}(\mathbb{R}))}\leq C.$$

Sobolev inequalities implies that $\partial_x u$ is bounded in $L^2((0,T),L^\infty(\mathbb{R}))$. Finally, we have shown that the right hand side in (3.19) is bounded in $L^2((0,T),L^2(\mathbb{R}))$, and classical regularity results for parabolic equation ensure that $u-\bar{u}$ is bounded in $L^2((0,T),H^2(\mathbb{R}))$ and $\partial_t u$ is bounded in $L^2((0,T),L^2(\mathbb{R}))$. This concludes the proof. ■ We have prove the existence of global solution since now using proposition 2.1 we deduce that $T_0=+\infty$. The uniqueness in this class of solution is a direct consequence of proposition 2.1 where the uniqueness is obtained.

4 Appendix

Let us explain why the system (1.1) is equivalent to the system (1.3) via the change of unknown $v=u+\partial_x\varphi(\rho)$.

Proposition 4.5 *We can formally rewrite the system (1.1) as follows with $v=u+\partial_x\varphi(\rho)$ ($\varphi'(\rho)=\frac{\mu(\rho)}{\rho^2}$):*

$$
\begin{cases}
\partial_t\rho-\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\rho)+\partial_x(\rho v)=0,\\
\rho\partial_t v+\rho u\partial_x v+\partial_x P(\rho)=0,\\
(\rho,u)_{/t=0}=(\rho_0,u_0).
\end{cases}
\tag{4.20}
$$

Proof: As observed in [8, 11, 10, 9], we are interested in rewriting the system (1.1) in terms of the following unknown $v=u+\nabla\varphi(\rho)$ where $\varphi'(\rho)=\frac{\mu(\rho)}{\rho^2}$. We have then by using the mass equation:

$$\partial_t\rho+\partial_x(\rho v)-\partial_x(\rho\varphi'(\rho)\partial_x\rho)=0.$$

and it gives:

$$\partial_t\rho+\partial_x(\rho v)-\partial_x(\frac{\mu(\rho)}{\rho}\partial_x\rho)=0.$$

Next we have via the transport equation:

$$\rho\partial_t[\partial_x(\varphi(\rho))] + \rho\partial_x\big(\frac{\mu(\rho)}{\rho^2}\partial_x(\rho u)\big) = 0. \tag{4.21}$$

Next we have:

$$\begin{aligned}\rho\partial_x\big(\frac{\mu(\rho)}{\rho^2}\partial_x(\rho u)\big) &= \rho\partial_x(\frac{\mu(\rho)}{\rho}\partial_x u + u\partial_x\varphi(\rho)),\\ &= \mu(\rho)\partial_{xx}u + \rho\partial_x(\frac{\mu(\rho)}{\rho})\partial_x u + \rho\partial_x(u\partial_x\varphi(\rho)),\\ &= \mu(\rho)\partial_{xx}u + \partial_x\mu(\rho)\,\partial_x u - \frac{\mu(\rho)}{\rho}\partial_x\rho\,\partial_x u + \rho\partial_x(u\partial_x\varphi(\rho)),\\ &= \partial_x(\mu(\rho)\partial_x u) + \rho u\partial_{xx}\varphi(\rho).\end{aligned}$$

And we obtain adding the equality (4.21) to the momentum equation:

$$\rho\partial_t v + \rho u\partial_x v + \partial_x P(\rho) = 0.$$

It concludes the proof. ∎

References

[1] H. Amann. Compact embeddings of vector-valued Sobolev and Besov spaces. *Glas. Mat. Ser. III*, 35(55) (1): 161-177, 2000. Dedicated to the memory of Branko Najman.

[2] D. Bresch and B. Desjardins, Existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, *Journal de Mathématiques Pures et Appliqués*, Volume 87, Issue 1, January 2007, Pages 57-90.

[3] S. Chapman and T.G Cowling. The mathematical theory of non-uniform gases. An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases. *Third edition, prepared in co-operation with D. Burnett. Cambridge University Press, London, 1970.*

[4] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2): 167-179; 1979.

[5] R. J. DiPerna and P-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math?,* 98(3): 511-547, 1989.

[6] E. Feireisl. On the motion of a viscous compressible, and heat conducting fluid. *Indiana Univ. Math. J.*, 53(6): 1705-1738, 2004.

[7] E. Feireisl, A. Novotný and H. Petzeltòva. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4): 358-392, 2001.

[8] B. Haspot, From the solution of compressible Navier-Stokes equation to solutions of porous media equation, rate of convergence. to appear in *Comptes rendus à l'Académie des Sciences.*

[9] B. Haspot, Hyperbolic Problems: Theory, Numerics, Applications Proceedings of the 14th International Conference on Hyperbolic Problems held in Padova, June 25-29, 2012, p. 667-674, 2014.

[10] B. Haspot, Porous media, Fast diffusion equations and the existence of global weak solution for the quasi-solution of compressible Navier-Stokes equations. *Preprint* 2013 hal-00770248.

[11] B. Haspot, New formulation of the compressible Navier-Stokes equations and parabolicity of the density. *Preprint* 2014 Hal.

[12] D. Hoff. Global existence for 1D compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Amer. Math. Doc.*, 303(1): 169-181; 1987.

[13] D. Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. *J. Differential Equations*, 120(1): 215-254, 1995.

[14] D. Hoff. Global solutions of the equations of one-dimensional, compressible flow with large data and forces, and with differing end states. *Z. Angew. Math. Phys.*, 49(5): 774-785, 1998.

[15] D. Hoff and J. Smoller. Non-formation of vacuum states for compressible Navier-Stokes equations. *Comm. Math. Phys.*, 216(2): 255-276, 2001.

[16] S. Jiang, Z. Xin and P. Zhang. Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density dependent viscosity. *Methods and applications of analysis* Vol. 12, No. 3, pp. 239252, September 2005.

[17] O. A Ladyzhenskaya, V. A Solonnikov and N. N Uraltseva, Linear and quasilinear equations of parabolic type, *AMS translations, Providence*, 1968.

[18] A. V. Kazhikhov and V. V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *Prikl. Mat. Mech.*, 41(2): 282-291, 1977.

[19] H-L. Li, J. Li and Z. Xin. Vanishing of Vacuum States and Blow-up Phenomena of the Compressible Navier-Stokes Equations. *Commun. Math. Phys.* 281, 401444 (2008).

[20] P-L. Lions. Mathematical topics in fluid mechanics. *Vol. 2, Volume 10 of Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press Oxford University Press, New-York, 1998. Compressible models, Oxford science publications.

[21] T-P Liu, Z. Xin and T. Yang, Vacuum states for compressible flow.*Discrete Contin. Dynam. Systems*, 4(1): 1-32, 1998.

[22] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. *Proc. Japan Acas. Ser. A Math. Sci.*, 55(9): 337-342,1979.

[23] A. Mellet and A. Vasseur, On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations* 32 (2007), no. 1-3, 431–452.

[24] A. Mellet and A. Vasseur, Existence and Uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.* 39 (2007/08), no. 4, 1344–1365.

[25] M. Okada, S. Matusu-Necasova and T. Makino. Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity. *Ann. Univ. Ferrara Sez.* VII (N.S); 48:1-20, 2002.

[26] D. Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(13): 639-642, 1986.

[27] D. Serre. Sur l'équation monodimensionnelle d'un fluide visqueux, compressible et conducteur de chaleur. *C. R. Acad. Sci. Paris Sér. I Math.*, 303 (14): 703-706, 1986.

[28] V. V. Shelukin. Motion with a contact discontinuity in a viscous heat conducting gas. *Dinamika Sploshn. Sredy*, (57): 131-152, 1982.

[29] V. V. Shelukin. Evolution of a contact discontinuity in the barotropic flow of a viscous gas. *Prikl. Mat. Mekh.* 47(5): 870-872, 1983.

[30] V. V. Shelukin. On the structure of generalized solutions of the one-dimensional equations of a polytropic viscous gas. *Prikl. Mat. Mekh.*, 48(6): 912-920; 1984.

[31] V. V. Shelukin. Boundary value problems for equations of a barotropic viscous gas with nonnegative initial density. *Dinamika Sploshn. Sredy*, (74): 108-125, 162-163, 1986.

[32] V. V. Shelukin. A shear flow problem for the compressible Navier-Stokes equations. *Internat. J. Non-linear Mech.*, 33(2): 247-257, 1998.

[33] V. A. Solonnikov. The solvability of the initial-boundary value problem for the equations of motion of a viscous compressible fluid. *Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (LOMI), 56: 128-142, 197, 1976. Investigations on linear operators and theory of functions, VI.

[34] V. A Vaigant. Nonhomogeneous boundary value problems for equations of a viscous heat-conducting gas. *Dinamika Sploshn. Sredy,* (97): 3-21, 212, 1990.

[35] T. Yang, Z-a Y and C. Zhu, Compressible Navier-Stokes equations with density-dependent viscosity and vacuum. *Comm. Partial Differential Equations*, 26(5-6): 965-981, 2001.

[36] T. Yang and C. Zhu, Compressible Navier-Stokes equations with degenerate viscosity coefficients and vacuum. *Comm. Math. Phys.*, 230(2): 329-363, 2002.