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Cauchy problem for capillarity Van der Vaals model

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ABSTRACT. In this article, we consider the compressible Navier-Stokes equation with density dependent viscosity coefficients and a term of capillarity introduced formally by Van der Waals in [**8**]. This model includes at the same time the barotropic Navier-Stokes equations with variable viscosity coefficients, shallow-water system and the model introduced by Rohde in [**7**]. We first study the well-posedness of the model in critical regularity spaces with respect to the scaling of the associated equations. In a functional setting as close as possible to the physical energy spaces, we prove local in time strong solutions with general initial data.

1. Introduction

This paper is devoted to the Cauchy problem for the compressible Navier-Stokes equation with viscosity coefficients depending on the density and with a capillary term coming from the works of Van der Waals in [**8**]. This capillarity term modelize the behavior at the interfaces of a mixture liquid-vapor. More precisely Van der Waals assume that the thickness of the interfaces is null and introduce consequently a non-local capillarity term. Coquel, Rohde and theirs collaborators in [**2**], [**7**] have reactualized on a modern form the results of Van der Waals. In the sequel we will work in the infinite Euclidian space $\mathbf{R}^N$ with $N \geq 2$. Let ρ and u denote the density and the velocity of a compressible viscous fluid. As usual, ρ is a non-negative function and u is a vector valued function defined on $\mathbb{R}^N$. Then, the Navier-Stokes equation for compressible fluids endowed with internal capillarity studied in [**7**] reads:

$$(SW) \quad \begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)Du) - \nabla(\lambda(\rho)\mathrm{div} u) + \nabla P(\rho) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \kappa\rho\nabla D[\rho], \\ \rho_{/t=0} = \rho_0, \;\; \rho u_{/t=0} = \rho_0 u_0, \end{cases}$$

with $D[\rho] = \phi * \rho - \rho$ and where ϕ is chosen so that:

$$\phi \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi(x)dx = 1, \;\; \phi \;\text{ even, and }\; \phi \geq 0.$$

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Here $Du = \frac{1}{2}(\nabla u + ^t \nabla u)$ is the strain tensor, P the pressure is a suitably smooth function of the density ρ and μ, λ are the two Lamé viscosity coefficients. They depend in our case regularly on the density ρ and satisfy: $\mu > 0$ and $2\mu + N\lambda \geq 0$. Several physical models arise as a particular case of system (SW):

- when $\kappa = 0$ (SW) represents compressible Navier-Stokes model with variable viscosity coefficients. Moreover if $\mu(\rho) = \rho$, $\lambda(\rho) = 0$, $P(\rho) = \rho^2$, $N = 2$ then (SW) describes the system of shallow-water.
- when $\kappa \neq 0$ and μ, λ are constant, (SW) reduce to the model studied by Rohde in [**7**].

In the present article, we address the question of local-welposedness in critical functional framework for the scaling of the equations. More precisely we generalize here the result of Danchin in [**5**] by considering general viscosity coefficient and by including this nonlocal Korteweg capillarity term studies in the works of [**2**], [**7**] . Moreover we improve the results of [**7**] and [**5**] (Danchin obtain strong solution with following initial data $B^{\frac{N}{2}}_{2,1} \times (B^{\frac{N}{2}-1}_{2,1})^N$) by getting strong solution in finite time in general Besov space $B^{\frac{N}{p}}_{p,1} \times (B^{\frac{N}{p}-1}_{p,1})^N$ built on the space L^p with $1 \leq p \leq N$. To finish with, we will give a criterion of blow-up for these solutions where we need that ∇u is in $L^1(L^\infty)$. We can observe that our result is very close in dimension $N = 2$ of the energy initial data for the global weak solutions of Bresch and Desjardins in [**1**] (where it is assumed that $(\nabla \rho_0, \sqrt{\rho_0} u_0) \in L^2$), these solutions include the shallow-water system. To conclude, our result improves too the case of strong solution for the shallow-water system, where Wang and Xu in [**9**] obtain strong solution in finite time for $\rho_0 - 1$, $u_0 \in H^{2+s}$ with $s > 0$.
In the sequel we will work around a constant state $\bar{\rho} > 0$ (to simplify we assume from now that $\bar{\rho} = 1$), this motivates the following notation:

Definition 1.1. We will note in the sequel $q = \frac{\rho - \bar{\rho}}{\bar{\rho}}$ and $a = \frac{1}{\rho} - \frac{1}{\bar{\rho}}$.

We can now state our main results.

Theorem 1.2. *Let $p \in [1, N]$. Let $q_0 \in B^{\frac{N}{p}}_{p,1}$ and $u_0 \in B^{\frac{N}{p}-1}_{p,1}$. Under the assumptions that μ and $\mu + 2\lambda$ are strictly bounded away zero on $[\bar{\rho}(1 - 2\|q_0\|_{L^\infty}), \bar{\rho}(1 + 2\|q_0\|_{L^\infty})]$, there exists a time $T > 0$ such that then system (SW) has a unique solution (q, u) in $F^{\frac{N}{p}}$ with: $F^{\frac{N}{p}} = C(B^{\frac{N}{p}}_{p,1}) \times \big(L^1_T(B^{\frac{N}{p}+1}_{p,1}) \cap C_T(B^{\frac{N}{p}-1}_{p,1})\big)$.*

Theorem 1.3. *Let $p \in [1, N]$. Assume that (SW) has a solution $(q, u) \in C([0, T], B^{\frac{N}{p}}_{p,1} \times (B^{\frac{N}{p}-1}_{p,1})^N)$ on the time interval $[0, T)$ which satisfies the following conditions:*

- *the function q is in $L^\infty([0, T], B^{\frac{N}{p}}_{p,1})$ and ρ is bounded away from zero.*
- *we have $\int_0^T \|\nabla u\|_{L^\infty} dt < +\infty$.*

Then (q, u) may be continued beyond T.

In the sequel, all the notations especially concerning the Besov spaces and the Chemin-Lerner spaces follow these of [**3**].

2. Proof of theorem 1.2

2.1. Estimates for parabolic system with variable coefficients. To avoid condition of smallness as in [**3**] on the initial density data, it is crucial to study very precisely the following parabolic system with variable coefficient which is obtained by linearizing the momentum equation:

$$\begin{cases} \partial_t u + v\cdot\nabla u + u\cdot\nabla w - b\big(\mathrm{div}(2\mu Du) + \nabla(\lambda \mathrm{div} u)\big) = f, \\ u_{/t=0} = u_0. \end{cases} \tag{2.1}$$

Above u is the unknown function. We assume that $u_0 \in B^s_{p,1}$ with $1 \le p \le N$ and $f \in L^1(0,T;B^s_{p,1})$, that v and w are time dependent vector-fields with coefficients in $L^1(0,T;B^{\frac{N}{p}+1}_{p,1})$, that b, μ and $2\mu+\lambda$ are bounded by below by positive constants $\underline{b}$, $\underline{\mu}$ and $2\mu+\lambda$ that $a = b-1$, $\mu' = \mu - \mu(1)$ and $\lambda' = \lambda - \lambda(1)$ belongs to $L^\infty(0,T;B^{\frac{N}{p}}_{p,1})$. We generalize now a result of [**5**] to the case of variable density and general Besov spaces.

Proposition 2.1. Let $\underline{\nu} = \underline{b}\min(\mu, \lambda+2\mu)$ and $\bar{\nu} = \mu + |\lambda+\mu|$. Assume that $s \in (-\frac{N}{p}, \frac{N}{p}-1]$. Let $m\in\mathbb{Z}$ be such that $b_m = 1 + S_m a$ and $a_{1,m} = a - S_m a$ satisfies for c small enough (depending only on N and on s):

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}^N} b_m(t,x) \ge \frac{\underline{b}}{2} \quad\text{and}\quad \|a - S_m a\|_{\widetilde{L}^\infty(0,T;B^{\frac{N}{p}}_{p,1})} \le c\frac{\underline{\nu}}{\bar{\nu}}. \tag{2.2}$$

We impose similar condition for μ_m, λ_m and $\mu'_{1,m} = \mu' - S_m\mu'$, $\lambda'_{1,m} = \lambda' - S_m\lambda'$. There exist two constants C and κ such that by setting:

$$V(t) = \int_0^t \|v\|_{B^{\frac{N}{p}+1}_{p,1}} d\tau,\ \ W(t) = \int_0^t \|w\|_{B^{\frac{N}{p}+1}_{p,1}} d\tau$$
$$\text{and}\quad Z_m(t) = 2^{2m}\bar{\nu}^2\underline{\nu}^{-1}\int_0^t (\|a\|^2_{B^{\frac{N}{p}}_{p,1}} + \|\mu'\|^2_{B^{\frac{N}{p}}_{p,1}} + \|\lambda'\|^2_{B^{\frac{N}{p}}_{p,1}}) d\tau,$$

we have for all $t\in[0,T]$,

$$\|u\|_{\widetilde{L}^\infty((0,T)\times B^s_{p,1})} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1((0,T)\times B^{s+2}_{p,1})} \le e^{C(V+W+Z_m)(t)}(\|u_0\|_{B^s_{p,1}} + \int_0^t e^{-C(V+W+Z_m)(\tau)}\|f(\tau)\|_{B^s_{p,1}} d\tau).$$

Proof. Let us first rewrite (2.1) as follows:

$$\partial_t u + v\cdot\nabla u + u\cdot\nabla w - b_m\big(\mathrm{div}\big(2\mu_m Du\big) + \nabla(\lambda_m \mathrm{div} u)\big) = f + E_m - u\cdot\nabla w, \tag{2.3}$$

Note that, because $-\frac{N}{p} < s \le \frac{N}{p} - 1$, the error term E_m and $u\cdot\nabla w$ may be estimated by:

$$\begin{aligned} \|E_m\|_{B^s_{p,1}} &\le (\|a_{1,m}\|_{B^{\frac{N}{p}}_{p,1}} + \|\mu'_{1,m}\|_{B^{\frac{N}{p}}_{p,1}} + \|\lambda'_{1,m}\|_{B^{\frac{N}{p}}_{p,1}})\|D^2 u\|_{B^s_{p,1}} \\ &\text{and}\quad \|u\cdot\nabla w\|_{B^s_{p,1}} \le \|\nabla w\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^s_{p,1}}. \end{aligned} \tag{2.4}$$

Now applying Δ_q to equation (2.3) yields:

$$\begin{aligned} \frac{d}{dt}u_q + v\cdot\nabla u_q - \mu\mathrm{div}(b_m\nabla u_q) - (\lambda+\mu)\nabla(b_m \mathrm{div} u_q) &= f_q + E_{m,q} \\ &- \Delta_q(u\cdot\nabla w) + R_q + \widetilde{R}_q, \end{aligned} \tag{2.5}$$

where we denote by $u_q=\Delta_q u$ and R_q, $\widetilde{R}_q$ are classical commutators. Next multiplying both sides by $|u_q|^{p-2}u_q$, integrating by parts, using Hölder's inequalities the lemma A5 in [**3**]and the fact that $\mu\geq 0$ and $\lambda+2\mu\geq 0$, we get:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|u_q\|^p_{L^p}+\frac{\nu\underline{b}(p-1)}{p^2}2^{2q}\|u_q\|^p_{L^p}\leq \|u_q\|^{p-1}_{L^p}\big(\|f_q\|_{L^p}+\|E_{m,q}\|_{L^p}+\|\Delta_q(u\cdot\nabla w)\|_{L^p}\\
+\frac{1}{p}\|u_q\|_{L^p}\|\text{div}u\|_{L^\infty}+\|R_q\|_{L^p}+\|\widetilde{R}_q\|_{L^p}\big),\end{aligned}$$

which leads, after time integration to:

$$\begin{aligned}\|u_q\|_{L^p}+\frac{\nu\underline{b}(p-1)}{p}2^{2q}\int_0^t\|u_q\|_{L^p}d\tau\leq\|\Delta_q u_0\|_{L^p}+\int_0^t\big(\|f_q\|_{L^p}+\|E_{m,q}\|_{L^p}\big)d\tau\\
+\int_0^t\big(\|\Delta_q(u\cdot\nabla w)\|_{L^p}+\frac{1}{p}\|u_q\|_{L^p}\|\text{div}u\|_{L^\infty}+\|R_q\|_{L^p}+\|\widetilde{R}_q\|_{L^p}\big)d\tau,\end{aligned}$$

For commutators R_q and $\widetilde{R}_q$, we have the following estimates:

$$\|R_q\|_{L^p}\leq c_q 2^{-qs}\|v\|_{B^{\frac{N}{p}+1}_{p,1}}\|u\|_{B^s_{p,1}},$$

$$\|\widetilde{R}_q\|_{L^p}\leq c_q\bar{\nu}2^{-qs}(\|S_m a\|_{B^{\frac{N}{p}+1}_{p,1}}+\|S_m\mu^{'}\|_{B^{\frac{N}{p}+1}_{p,1}}+\|S_m\lambda^{'}\|_{B^{\frac{N}{p}+1}_{p,1}})\|Du\|_{B^s_{p,1}},$$

where $(c_q)_{q\in\mathbb{Z}}$ is a positive sequence such that $\sum_{q\in\mathbb{Z}}c_q=1$, and $\bar{\nu}=\mu+|\lambda+\mu|$. Note that, using Bernstein inequality, we have: $\|S_m a\|_{B^{\frac{N}{p}+1}_{p,1}}\leq 2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}$. Hence, using these latter estimates and multiplying by 2^{qs} and summing up on $q\in\mathbb{Z}$, we get for all $t\in[0,T]$:

$$\begin{aligned}\|u\|_{L^\infty_t(B^s_{p,1})}+\frac{\nu\underline{b}(p-1)}{p}\|u\|_{L^1_t(B^{s+2}_{p,1})}\leq\|u_0\|_{B^s_{p,1}}+\|f\|_{L^1_t(B^s_{p,1})}+C\int_0^t(\|v\|_{B^{\frac{N}{p}}_{p,1}}\\
+\|w\|_{B^{\frac{N}{p}+1}_{p,1}})\|u\|_{B^s_{p,1}}d\tau+C\bar{\nu}\int_0^t\big(\|a_{1,m}\|_{B^{\frac{N}{p}}_{p,1}}+\|\mu^{'}_{1,m}\|_{B^{\frac{N}{p}}_{p,1}}+\|\lambda^{'}_{1,m}\|_{B^{\frac{N}{p}}_{p,1}})\|u\|_{B^{s+2}_{p,1}}\\
+2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+1}_{p,1}}\big)d\tau,\end{aligned}$$

for a constant C depending only on N and s. Let $X(t)=\|u\|_{L^\infty_t(B^s_{p,1})}+\nu\underline{b}\|u\|_{L^1_t(B^{s+2}_{p,1})}$. Assuming that m has been chosen so large as to satisfy condition (2.2) and by interpolation, we have:

$$C\bar{\nu}\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+2}_{p,1}}\leq\kappa\underline{\nu}+\frac{C^2\bar{\nu}^2 2^{2m}}{4\kappa\underline{\nu}}\|a\|^2_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^s_{p,1}},$$

We conclude by using Grönwall lemma and this leads to the desired inequality. □

REMARK 2.2. The proof of the continuation criterion (theorem 1.3) relies on a better estimate which is available when $u=v=w$. In fact, by arguing as in the proof of the previous proposition and by using other commutator estimate, one can prove that under conditions (2.2), there exists constants C and κ such that:

$$\begin{aligned}\forall t\in[0,T],\ \|u\|_{L^\infty_t(B^s_{p,1})}+\kappa\underline{\nu}\|u\|_{L^1_t(B^{s+2}_{p,1})}\leq e^{C(U+Z_m)(t)}\big(\|u_0\|_{B^s_{p,1}}\\
+\int_0^t e^{-C(U+Z_m)(\tau)}\|f(\tau)\|_{B^s_{p,1}}d\tau\big)\quad\text{with}\quad U(t)=\int_0^t\|\nabla u\|_{L^\infty}d\tau.\end{aligned}$$

Proposition 2.1 fails in the limit case $s=-\frac{N}{p}$. One can however state the following result which will be the key to the proof of uniqueness.

PROPOSITION 2.3. Under condition (2.2), there exists two constants C and κ (with c, C, depending only on N, and κ universal) such that we have:

$$\|u\|_{L_t^\infty(B_{p,\infty}^{-\frac{N}{p}})}+\kappa\underline{\nu}\|u\|_{\widetilde{L}_t^1(B_{p,\infty}^{2-\frac{N}{p}})}\leq 2e^{C(V+W)(t)}(\|u_0\|_{B_{p,\infty}^{-\frac{N}{p}}}+\|f\|_{\widetilde{L}_t^1(B_{p,\infty}^{-\frac{N}{p}})}),$$

whenever $t\in[0,T]$ satisfies:

$$\bar{\nu}^2t\|a\|^2_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})}\leq c2^{-2m}\underline{\nu}. \tag{2.6}$$

2.2. The proof of existence for theorem 1.2. We smooth out the data as follows:

$$q_0^n=S_nq_0,\ \ u_0^n=S_nu_0\ \text{ and }\ f^n=S_nf.$$

Now according [**6**], one can solve (SW) with smooth initial data (q_0^n,u_0^n,f^n) on a time interval $[0,T_n]$. Let $\varepsilon>0$, we get solution checking:

$$q^n\in C([0,T_n],B_{p,1}^{\frac{N}{p}+\varepsilon})\ \ u^n\in C([0,T_n],B_{p,1}^{\frac{N}{p}-1+\varepsilon})\cap\widetilde{L}^1([0,T_n],B_{p,1}^{\frac{N}{p}+1+\varepsilon}). \tag{2.7}$$

2.2.1. *Uniform Estimates for* $(q^n,u^n)_{n\in\mathbb{N}}$. Let T_n be the lifespan of (q_n,u_n), that is the supremum of all $T>0$ such that (SW) with initial data (q_0^n,u_0^n) has a solution which satisfies (2.7). Let T be in $(0,T_n)$, we aim at getting uniform estimates in E_T for T small enough. For that, we need to introduce the solution u_L^n to the linear system:

$$\partial_tu_L^n-\mu(1)\Delta u_L^n-(\lambda+\mu)(1)\nabla\text{div}u_L^n=f^n,\ \ u_L^n(0)=u_0^n.$$

Now, the vector field $\tilde{u}^n=u^n-u_L^n$ satisfies the parabolic system:

$$\begin{aligned}&\partial_t\tilde{u}^n+u_L^n\cdot\nabla\tilde{u}^n+(1+a^n)\big(\text{div}(2\mu(1+q^n)D\tilde{u}^n)-\nabla(\lambda(1+q^n)\text{div}\tilde{u}^n)\big)=H^n,\\&\tilde{u}^n(0)=0.\end{aligned}$$

with (where we note $Au=(\mu(1)\Delta-(\lambda+\mu)(1)\nabla\text{div})u$):

$$H^n=a^nAu_L^n-u_L^n\cdot\nabla u_L^n-(1+a^n)\nabla P(1+q^n)+\phi*\nabla q^n-\nabla q^n$$

which has been studied in proposition 2.1. Define $m\in\mathbb{Z}$ by:

$$m=\inf\{p\in\mathbb{Z}/\ 2\bar{\nu}\sum_{l\geq p}2^{l\frac{N}{2}}\|\Delta_la_0\|_{L^2}\leq c\bar{\nu}\} \tag{2.8}$$

where c is small enough positive constant to be fixed hereafter. Let:

$$\bar{b}=1+\sup_{x\in\mathbb{R}^N}a_0(x),\ A_0=1+2\|a_0\|_{B_{p,1}^{\frac{N}{p}}},\ U_0=\|u_0\|_{B_{p,1}^{\frac{N}{p}}}+\|f\|_{L^1(B_{p,1}^{\frac{N}{p}-1})},$$

and $\widetilde{U}_0=2CU_0+4C\bar{\nu}A_0$ (where C stands for a large enough constant which will be determined when applying proposition 2.1). We assume that the following inequalities are fulfilled for some $\eta>0$ and $T>0$:

$$\|a^n-S_ma^n\|_{\widetilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}\leq c\underline{\nu}\bar{\nu}^{-1},\ \|a^n\|_{\widetilde{L}^\infty(B_{p,1}^{\frac{N}{p}})}\leq A_0,\ \|q^n\|_{\widetilde{L}^\infty(B_{p,1}^{\frac{N}{p}})}\leq A_0 \tag{H_1}$$

$$\frac{1}{2}\underline{b}\leq 1+a^n(t,x)\leq 2\bar{b}\ \text{ for all }\ (t,x)\in[0,T]\times\mathbb{R}^N, \tag{H_2}$$

$$\|u_L^n\|_{\widetilde{L}_T^1(B_{p,1}^{\frac{N}{p}+1})}\leq\eta\ \text{ and }\ \|\widetilde{u}^n\|_{\widetilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})}+\underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}_T^1(B_{p,1}^{\frac{N}{p}+1})}\leq\widetilde{U}_0\eta, \tag{H_3}$$

To be more precisely $\mu^n-\mu(1)$ and $\lambda^n-\lambda(1)$ have to check the same assumption than (H_1) it is left to the reader. We know that there exists a small time $\widetilde{T}^n$ with $0<\widetilde{T}^n<T^n$ such that those conditions are verified. Remark that since: $1+S_m a^n=1+a^n+(S_m a^n-a^n)$, assumptions (H_1) and (H_2) combined with the embedding $B^{\frac{N}{p}}_{p,1}\hookrightarrow L^\infty$ insure that:

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^N}(1+S_m a^n)(t,x)\geq\frac{1}{4}\underline{b}. \tag{2.9}$$

provided c has been chosen small enough. We are going to prove that under suitable assumptions on T and η (to be specified below) condition (H_1) to (H_3) are satisfied on $[0,T]$ with strict inequalities. Since all those conditions depend continuously on the time variable and are strictly satisfied initially, a basic boobstrap argument insures that (H_1) to (H_3) are indeed satisfied for T with $0<\widetilde{T}^n<T$ and T independent of n. First we shall assume that η satisfies:

$$C(1+\underline{\nu}^{-1}\widetilde{U}_0)\eta\leq\log 2 \tag{2.10}$$

so that denoting $\widetilde{U}^n(t)=\int_0^t\|\widetilde{u}^n\|_{B^{\frac{N}{p}+1}_{p,1}}d\tau$ and $U^n_L(t)=\int_0^t\|u^n_L\|_{B^{\frac{N}{p}+1}_{p,1}}d\tau$, we have, according to (H_3):

$$e^{C(U^n_L+\widetilde{U}^n)(T)}<2\ \text{ and }\ e^{C(U^n_L+\widetilde{U}^n)(T)}-1\leq\frac{C}{\log 2}(U^n_L+\widetilde{U}^n)(T)\leq 1. \tag{2.11}$$

In order to bound a^n in $\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})$, we use paraproduct and classical result on transport equation (see [**5**]):

$$\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}<1+2\|a_0\|_{B^{\frac{N}{p}}_{p,1}}=A_0. \tag{2.12}$$

We proceed similarly to bound $\|q^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}$. Now by applying results on transport equation which yields for all $m\in\mathbb{Z}$, we get:

$$\sum_{l\geq m}2^{l\frac{N}{p}}\|\Delta_l a^n\|_{L^\infty_T(L^p)}\leq\sum_{l\geq m}2^{l\frac{N}{p}}\|\Delta_l a_0\|_{L^p}+(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})\big(e^{C(U^n_L+\widetilde{U}^n)(T)}-1\big).$$

Using (2.10) and (H_3), we thus have:

$$\|a^n-S_m a^n\|_{L^\infty_T(B^{\frac{N}{p}}_{p,1})}\leq\sum_{l\geq m}2^{l\frac{N}{p}}\|\Delta_l a_0\|_{L^p}+\frac{C}{\log 2}(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})(1+\underline{\nu}^{-1}\widetilde{L}_0)\eta.$$

Hence (H_1) is strictly satisfied provided that η further satisfies:

$$\frac{C}{\log 2}(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})(1+\underline{\nu}^{-1}\widetilde{L}_0)\eta<\frac{c\underline{\nu}}{2\bar{\nu}}. \tag{2.13}$$

Next, applying classical estimates on heat equation yields:

$$\|u^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1}_{p,1})}\leq U_0, \tag{2.14}$$

$$\kappa\nu\|u^n_L\|_{L^1_T(B^{\frac{N}{p}+1}_{p,1})}\leq\sum_{l\in\mathbb{Z}}2^{l(\frac{N}{p}-1)}(1-e^{-\kappa\nu 2^{2l}T})(\|\Delta_l u_0\|_{L^p}+\|\Delta_l f\|_{L^1(\mathbb{R}^+,L^p)}). \tag{2.15}$$

Hence taking T such that:

$$\sum_{l\in\mathbb{Z}}2^{l(\frac{N}{p}-1)}(1-e^{-\kappa\nu 2^{2l}T})(\|\Delta_l u_0\|_{L^p}+\|\Delta_l f\|_{L^1(\mathbb{R}^+,L^p)})<\kappa\eta\nu, \tag{2.16}$$

insures a strictly inequality for the first estimate of (H_4). Now we have to choose:

$$T<\frac{2^{-2m}\underline{\nu}}{C\bar{\nu}^2A_0^2}. \tag{2.17}$$

Since (H_1), (2.17) and (2.9) are satisfied, proposition 2.1 may be applied, we get :

$$\begin{aligned}&\|\widetilde{u}^n\|_{\widetilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})}+\underline{\nu}\|\widetilde{u}^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})}\\&\leq Ce^{C(U_L^n+\widetilde{U}^n)(T)}\int_0^T\big(\|a^nAu_L^n\|_{B_{p,1}^{\frac{N}{p}-1}}+\|u_L^n\cdot\nabla u_L^n\|_{B_{p,1}^{\frac{N}{p}-1}}+\|\nabla q^n\|_{B_{p,1}^{\frac{N}{p}-1}}\big)dt.\end{aligned}$$

By taking advantage of the paraproduct, we end up with:

$$\begin{aligned}&\|\widetilde{u}^n\|_{\widetilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})}+\underline{\nu}\|\widetilde{u}^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})}\leq Ce^{C(U_L^n+\widetilde{U}^n)(T)}\\&\times\big(C\|u_L^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})}(\bar{\nu}\|a^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}-1})}+\|u_L^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}-1})})+C_gT\|q^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}})}\big)dt.\end{aligned}$$

with $C>0$. Now, using assumptions (H_1), (H_3), and inserting (2.11) we obtain:

$$\|\widetilde{u}^n\|_{\widetilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})}+\|\widetilde{u}^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})}\leq 2C(\bar{\nu}A_0+U_0)\eta+2C_gTA_0,$$

hence (H_3) is satisfied with a strict inequality provided:

$$C_gT<C\bar{\nu}\eta. \tag{2.18}$$

In the goal to check whether (H_2) is satisfied, we use the fact that:

$$a^n-a_0=S_m(a^n-a_0)+(Id-S_m)(a^n-a_0)+\sum_{l>n}\Delta_la_0,$$

whence, using $B_{p,1}^{\frac{N}{p}}\hookrightarrow L^\infty$ and assuming (with no loss of generality) that $n\geq m$,

$$\begin{aligned}\|a^n-a_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\leq C\big(&\|S_m(a^n-a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})}+\|a^n-S_ma^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})}\\&+2\sum_{l\geq m}2^{l\frac{N}{p}}\|\Delta_la_0\|_{L^p}\big).\end{aligned}$$

One can, in view of the previous computations, assume that:

$$C\big(\|a^n-S_ma^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})}+2\sum_{l\geq m}2^{l\frac{N}{p}}\|\Delta_la_0\|_{L^p}\big)\leq\frac{b}{4}.$$

As for the term $\|S_m(a^n-a_0)\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})}$, it may be bounded:

$$\begin{aligned}\|S_m(a^n-a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})}\leq(1+\|a_0\|_{B_{p,1}^{\frac{N}{p}}})(e^{C(\widetilde{U}^n+U_L^n)(T)}-1)+C2^{2m}\sqrt{T}\|a_0\|_{B_{p,1}^{\frac{N}{p}}}\\\times\|u^n\|_{L_T^2(B_{p,1}^{\frac{N}{p}})}.\end{aligned}$$

Note that under assumptions (H_5), (H_6), (2.10) and (2.13), the first term in the right-hand side may be bounded by $\frac{b}{8}$. Hence using interpolation, (2.14) and the assumptions (2.10) and (2.13), we end up with:

$$\|S_m(a^n-a_0)\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})}\leq\frac{b}{8}+C2^m\sqrt{T}\|a_0\|_{B_{2,1}^{\frac{N}{2}}}\sqrt{\eta(U_0+\widetilde{U}_0\eta)(1+\underline{\nu}^{-1}\widetilde{U}_0}.$$

Assuming in addition that T satisfies:

$$C2^m\sqrt{T}\|a_0\|_{B^{\frac{N}{2}}_{2,1}}\sqrt{\eta(U_0+\widetilde{U}_0\eta)(1+\underline{\nu}^{-1}\widetilde{U}_0}<\frac{b}{8}, \tag{2.19}$$

and using the assumption $\underline{b}\leq 1+a_0\leq\bar{b}$ yields (H_2) with a strict inequality. One can now conclude that if $T<T^n$ has been chosen so that conditions (2.16), (2.18), (2.17) and (2.19) are satisfied (with η verifying (2.10) and (2.13), and m defined in (2.8) and $n\geq m$ then (a^n,q^n,u^n) satisfies (H_1) to $(H_3$ and is bounded independently of n on $[0,T]$. We still have to state that T^n may be bounded by below by the supremum $\bar{T}$ of all times T such that (2.16), (2.18), (2.17) and (2.19) are satisfied. This is actually a consequence of the uniform bounds we have just obtained, and of continuation criterion of theorem 1.2. We finally obtain $T^n\geq\bar{T}$.

2.2.2. *Existence of solutions.* The existence of a solution stems from compactness properties for the sequence $(q^n,u^n)_{n\in\mathbb{N}}$ by using some results of type Ascoli.

LEMMA 2.4. *The sequence $(\partial_t\tilde{q}^n,\partial_t\tilde{u}^n)_{n\in\mathbb{N}}$ is uniformly bounded for some $\alpha>1$ in:*

$$L^2(0,T;B^{\frac{N}{p}-1}_{p,1})\times(L^\alpha(0,T;B^{\frac{N}{p}-2}_{p,1}))^N.$$

PROOF. The notation u.b will stand for uniformly bounded. We start with show that $\partial_t\tilde{q}^n$ is u.b in $L^2(0,T;B^{\frac{N}{p}-1}_{p,1})$. Since u^n is u.b in $L^2_T(B^{\frac{N}{p}}_{p,1})$ and ∇q^n is u.b in $L^\infty_T(B^{\frac{N}{p}-1}_{p,1})$, then $u^n\cdot\nabla q^n$ is u.b in $L^2_T(B^{\frac{N}{p}-1}_{p,1})$. Similar arguments enable us to conclude for the term $(1+q^n)\mathrm{div}u^n$ which is u.b in $L^2_T(B^{\frac{N}{p}}_{p,1})$. Let us now study $\partial_t\tilde{u}^{n+1}$. Since u^n is u.b in $L^\infty(B^{\frac{N}{p}-1}_{p,1})$ and ∇u^n is u.b in $L^2(B^{\frac{N}{p}-1}_{p,1})$, so $u^n\cdot\nabla u^n$ is u.b in $L^2(B^{\frac{N}{p}-2}_{p,1})$ thus in $L^2(B^{\frac{N}{p}-2}_{p,1})$. The other terms follow the same estimates and are left to the reader. □

Now, let us turn to the proof of the existence of a solution by using some Ascoli results and the properties of compactness showed in the lemma 2.4. According lemma 2.4, $(q^n,u^n)_{n\in\mathbb{N}}$ is u.b in: $C^{\frac{1}{2}}([0,T];B^{\frac{N}{p}-1}_{p,1})\times(C^{1-\frac{1}{\alpha}}([0,T];B^{\frac{N}{p}-2}_{p,1}))^N$, thus is uniformly equicontinuous in $C(([0,T];B^{\frac{N}{p}-1}_{p,1})\times(B^{\frac{N}{p}-2}_{p,1})^N)$. On the other hand we have the following result of compactness, for any $\phi\in C^\infty_0(\mathbb{R}^N)$, $s\in\mathbb{R}$, $\delta>0$ the application $u\to\phi u$ is compact from $B^s_{p,1}$ to $B^{s-\delta}_{p,1}$. Applying Ascoli's theorem, we infer that up to an extraction $(q^n,u^n)_{n\in\mathbb{N}}$ converges for the distributions to a limit $(\bar{q},\bar{u})$ which belongs to: $C^{\frac{1}{2}}([0,T];B^{\frac{N}{p}-1}_{p,1})\times(C^{1-\frac{1}{\alpha}}([0,T];B^{\frac{N}{p}-2}_{p,1}))^N$. Using again uniform estimates and proceeding as, we gather that (q,u) solves (SW) and belongs to $F^{\frac{N}{p}}$.

2.3. Proof of the uniqueness for theorem 1.2. We are interested here in the most complicated case when $p=N$, the other cases can be deduced by embedding. Let (q_1,u_1), (q_2,u_2) belong to $F^{\frac{N}{p}}$ with the same initial data. We set $(\delta q,\delta u)=(q_2-q_1,u_2-u_1)$. We can then write the system (SW) as follows:

$$\begin{cases}\partial_t\delta q+u_2\cdot\nabla\delta q=-\delta u\cdot\nabla q_1-\delta q\mathrm{div}u_2-(1+q_1)\mathrm{div}\delta u,\\ \partial_t\delta u+u^2\cdot\nabla\delta u+\delta u\cdot\nabla u^1-(1+a^1)\big(\mathrm{div}(2\mu(\rho_1)D\delta u)+\nabla(\lambda(\rho_1)\nabla\delta u)\big)\\ =\kappa(\phi*\nabla\delta q-\nabla\delta q)-\nabla\big(P(\rho_1)-P(\rho_2)\big)+A(\delta q,u_2),\end{cases}$$

with $A(\delta q, u_2)$ a rest term depending essentially of δq. Fix an integer m such that:

$$1+\inf_{(t,x)\in[0,T]\times\mathbb{R}^N} S_m a^1 \geq \frac{b}{2} \quad \text{and} \quad \|1-S_m a^1\|_{\widetilde{L}^\infty(B^1_{N,1})} \leq c\frac{\underline{\nu}}{\bar{\nu}}, \tag{2.20}$$

we have the same properties for $\mu-\mu(1)$, $\lambda-\lambda(1)$ and we define T_1 as the supremum of all positive time such that:

$$t\leq T \quad \text{and} \quad t\bar{\nu}^2\|a^1\|^2_{\widetilde{L}^\infty(B^1_{N,1})} \leq c2^{-2m}\underline{\nu}. \tag{2.21}$$

Remark that by classical properties on transport equation a^1 belongs to $\widetilde{C}_T(B^1_{N,1})$ so that the above two assumptions are satisfied if m has been chosen large enough. For bounding δq in $L^\infty_T(B^0_{N,\infty})$, we apply estimates on transport equation. We get $\forall t\in[0,T]$:

$$\|\delta q(t)\|_{B^0_{N,\infty}} \leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}\|-\delta u\cdot\nabla q_1-\delta q\,\mathrm{div}u_2-(1+q_1)\mathrm{div}\delta u\|_{B^0_{N,\infty}}d\tau,$$

hence using that the product of two functions maps $B^0_{N,\infty}\times B^1_{N,1}$ in $B^0_{N,\infty}$, and applying Gronwall lemma,

$$\|\delta q(t)\|_{B^0_{N,\infty}} \leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}(1+\|q^1\|_{B^1_{N,1}})\|\delta u\|_{B^1_{N,1}}d\tau. \tag{2.22}$$

Next, using proposition 2.3 combined with paraproduct theory, we get for all $t\in[0,T_1]$:

$$\begin{aligned}\|\delta u\|_{\widetilde{L}^1_T(B^1_{N,\infty})} \leq Ce^{C(U^1+U^2)(t)}\int_0^t(1+\|q^1\|_{B^1_{N,1}}+\|q^2\|_{B^1_{N,1}}\\ +\|u^2\|_{B^1_{N,1}})\|\delta a\|_{B^0_{N,\infty}}d\tau.\end{aligned} \tag{2.23}$$

In order to control the term $\|\delta u\|_{B^1_{N,1}}$ which appears in the right-hand side of (2.22), we make use of the following logarithmic interpolation inequality whose proof may be found in [**4**], page 120:

$$\|\delta u\|_{L^1_t(B^1_{N,\infty})} \leq \|\delta u\|_{\widetilde{L}^1_t(B^1_{N,\infty})}\log\big(e+\frac{\|\delta u\|_{\widetilde{L}^1_t(B^0_{N,\infty})}+\|\delta u\|_{\widetilde{L}^1_t(B^2_{N,\infty})}}{\|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})}}\big). \tag{2.24}$$

Because u^1 and u^2 belong to $\widetilde{L}^\infty_T(B^0_{N,1})\cap L^1_T(B^2_{N,1})$, the numerator in the right-hand side may be bounded by some constant C_T depending only on T and on the norms of u^1 and u^2. Therefore inserting (2.22) in (2.23) and taking advantage of (2.24), we get for all $t\in[0,T_1]$ with:

$$\|\delta u\|_{\widetilde{L}^1_T(B^1_{N,\infty})} \leq C(1+\|a^1\|_{\widetilde{L}^\infty_T(B^1_{N,1})})$$

$$\int_0^t(1+\|q^1\|_{B^1_{N,1}}+\|q^2\|_{B^1_{N,1}}+\|u^2\|_{B^1_{N,1}})\|\delta u\|_{\widetilde{L}^1_\tau(B^1_{N,\infty})}\log\big(e+C_T\|\delta u\|^{-1}_{\widetilde{L}^1_\tau(B^1_{N,\infty})}\big)d\tau.$$

Since the function $t\to\|q^1(t)\|_{B^1_{N,1}}+\|q^2(t)\|_{B^1_{N,1}}+\|u^2(t)\|_{B^2_{N,1}}$ is integrable on $[0,T]$, and:

$$\int_0^1\frac{dr}{r\log(e+C_Tr^{-1})}=+\infty$$

Osgood lemma yields $\|\delta u\|_{\widetilde{L}^1_T(B^1_{N,1})}=0$. The definition of m depends only on T and that (2.20) is satisfied on $[0,T]$. Hence, the above arguments may be repeated until the whole interval $[0,T]$ is exhausted. This yields uniqueness on $[0,T]$.

3. Continuation criterion

In this section, we prove theorem 1.3. So we assume that we are given a solution (q,u) to (SW) which belongs to $F_{T'}^{\frac{N}{p}}$ for all $T^{'}<T$ and such that conditions of theorem 1.3 are satisfied. Fix an integer m such that conditions (2.2) is fullfiled. Hence, taking advantage of remark 2.2 and using results of composition, we get for some constant C and all $t\in[0,T)$,

$$\|u\|_{\widetilde{L}_t^{\infty}(B_{p,1}^{\frac{N}{p}-1})}+\kappa\underline{\nu}\|u\|_{\widetilde{L}_t^{1}(B_{p,1}^{\frac{N}{p}+1})}\\ \leq e^{C\int_0^t\big(\|\nabla u\|_{L^\infty}+2^{2m}\underline{\nu}^{-1}\bar{\nu}^2\|q\|^2_{B_{p,1}^{\frac{N}{p}}}\big)d\tau}\big(\|u_0\|_{B_{p,1}^{\frac{N}{p}-1}}+\|f\|_{\widetilde{L}_t^{1}(B_{p,1}^{\frac{N}{p}-1})}+C\int_0^t\|q\|^2_{B_{p,1}^{\frac{N}{p}}})d\tau\big).$$

This yields a bound on $\|u\|_{\widetilde{L}_t^{\infty}(B_{p,1}^{\frac{N}{p}-1})}$ and on $\|u\|_{\widetilde{L}_t^{1}(B_{p,1}^{\frac{N}{p}+1})}$ depending only on the data and on m, $\underline{\nu}$, $\bar{\nu}$, $\|q\|_{\widetilde{L}_t^{1}(B_{p,1}^{\frac{N}{p}})}$ and $\|\nabla u\|_{L^1_T(L^\infty)}$. Of course due to $\|q\|_{L_T^{\infty}(B_{p,1}^{\frac{N}{p}})}$, we also have $\|q\|_{\widetilde{L}_T^{\infty}(B_{p,1}^{\frac{N}{p}})}$. By replacing $\|\Delta_q q_0\|_{L^p}$ and $\|\Delta_q u_0\|_{L^p}$ by $\|\Delta_q q\|_{L_T^\infty(L^p)}$ and $\|\Delta_q u\|_{L_T^\infty(L^p)}$ in the definition (2.8) of m and in the lower bounds (2.16), (2.17) and (2.19) that we have obtained for the existence time, we obtain an $\varepsilon>0$ such that (SW) with data $q(T-\varepsilon)$ and $u(T-\varepsilon)$ has a solution on $[0,3\varepsilon]$. Since the solution (q,u) is unique on $[0,T)$, this provides a continuation of (q,u) beyond T.

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