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Abstract

In this paper we are interested in the barotropic compressible Navier-Stokes system endowed with a non-local capillarity tensor depending on a small parameter ε such that it heuristically tends to the local Korteweg system. After giving some explanations about the capillarity (physical justification and purpose, motivations related to the theory of non-classical shocks (see [28])), we prove global well-posedness (in the whole space $\mathbb{R}^d$ with $d \geq 2$) for the non-local model, as well as the convergence, as ε goes to zero, to the solution of the local Korteweg system.

1 Introduction

1.1 Presentation of the models

This section is widely inspired by the works of F.Coquel, D.Diehl, C.Merkle, and C.Rohde and we refer to [32], [11] for an in-depth presentation of the capillary models.

The mathematical description of liquid-vapour phase interfaces has a long history and has been recently renewed in the 80's after the works of Dunn and Serrin (see [16]). The first investigations begin with the Young-Laplace theory which claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface.

The point is to describe the location (and the movement) of the interfaces and to understand whether an interface behaves as a discontinuity in the state space (sharp interface, SI) or whether the phase boundary corresponds to a more regular transition (diffuse interface, DI). The DI approach is often favored because it covers topological changes in the phase distribution (such as the separation of bubbles/drops), and because its mathematical and numerical study are much more simple. Indeed it only requires one set of equations to be solved in a single spatial domain (the density takes into account the different phases), instead of one system per phase coupled with free-boundary problems required by the SI model.

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1.1.1 Basic modelization setting

Let us consider a single-component fluid contained in an open set $\Omega \subset \mathbb{R}^d$ at constant temperature T^*. In the isothermal setting the density is denoted by $\rho : \Omega \to]0, b[$ $(b > 0)$. To be able to determine the two phases, we need to define the energy density function as:

$$W(\rho) = \rho f(\rho, T^*), \quad \rho \in]0, b[, \text{ and } f \in C^2(0, b). \tag{1.1}$$

W has to satisfy the following properties:

- $\exists\, \alpha_1 < \alpha_2 \in]0, b[$ such that $W'' > 0$ in $]0, \alpha_1[\cup]\alpha_2, b[$ and $W'' < 0$ in $]\alpha_1, \alpha_2[$.
- $\lim_{\rho \to 0} W(\rho) = \lim_{\rho \to b} W(\rho) = +\infty$.
- $W \geq 0$ in $]0, b[$.

We refer to [30] and [11] for a far more complete presentation. Since we consider the fluid as a single system, the phase state is determined by the density: the high variations of density define the location of the interfaces and we will say that the fluid at the position x is in the *vapour/elliptic (or spinodal)/liquid* phase if $\rho(x) \in]0, \alpha_1]/[\alpha_1, \alpha_2]/[\alpha_2, b[$. According to standard thermodynamical theory, we define the pressure as:

$$P : \rho \in]0, b[\quad \mapsto P(\rho) = \rho W'(\rho) - W(\rho) \in]0, +\infty[. \tag{1.2}$$

Thus the pressure is a *non-convex* and *non-monotone* function (because $P'(\rho) = \rho W''(\rho)$). A standard model for the pressure in two-phase mixtures is the well-known Van der Waals law given by:

$$P(\rho, T^*) = \frac{RT^* \rho}{b - \rho} - a\rho^2 \tag{1.3}$$

where R is the specific gas constant and a, b are positive constants (the parameter a controls the attractive forces between molecules of the fluid and b is related to the molecules size).

1.1.2 Variational Approach for sharp interfaces

One classically associates to this equilibrium problem for the liquid-vapor mixture (at constant temperature T^*, at least in an asymptotic configuration in time) a variational problem that consists in minimizing the free energy functional. For this, we introduce an admissibility set for the density which takes into account the fact that our interfaces have null thickness, more precisely we set:

$$A^0 = \{\rho \in L^1(\Omega)/W(\rho) \in L^1(\Omega), \int_\Omega \rho(x)dx = m\}.$$

Here we have prescribed the total mass by a constant $m > 0$. To find a static equilibrium, we need to minimize, when $\rho \in A^0$, the functional:

$$F^0[\rho] = \int_\Omega W(\rho(x))dx. \tag{1.4}$$

There are only two points $\beta_1 \in]0, \alpha_1[$ and $\beta_2 \in]\alpha_2, \beta[$, called the Maxwell states, where W is minimal. A *physically correct solution* is a minimizer that only takes the values

β_1 and β_2 and such that, away from the boundary, the length of the phase interface is minimal. For example bubbles ans drops are physically correct solutions. The problem is that if $\beta_1|\Omega| < m < \beta_2|\Omega|$ every piecewise constant function ρ taking its values in $\{\beta_1, \beta_2\}$, so that the total mass is m, is a minimizer. This allows drops and bubbles as well as an infinity of minimizers describing an arbitrary large number of phase changes (provided that the total mass is m) which clearly are physically wrong.

In order to identify the *good* solutions, one needs selection principles and Van der Waals, in the XIX-th century, seems to be the first to add a term related to the surface energy to select the physically correct solutions, modulo the introduction of a diffuse interface. This theory is widely accepted as a thermodynamically consistent model for equilibria.

Remark 1 *All the previous minimizers realize the phase transition as a jump between the vapour or liquid zones, without taking values in the spinodal zone (at least up to a set of zero Lebesgue measure). This is why we talk about sharp-interface model.*

1.1.3 The local diffuse interface approach

Van der Waals tried to overcome the lack of uniqueness for the SI approach by adding a term of capillarity (see [35]). His idea for selecting some relevant physical solution is to penalize the high variations of density (typically at the interfaces) by adding some derivative terms to the previous functional. More precisely, one considers a new admissibility set A^{local}, and we now search for a function ρ_ε which is a minimizer of the local Van der Waals functional:

$$F^\varepsilon_{local}[\rho_\varepsilon] = \int_\Omega (W(\rho_\varepsilon(x)) + \gamma\frac{\varepsilon^2}{2}|\nabla\rho_\varepsilon(x)|^2)dx, \quad \text{with} \quad A^{local} = H^1(\Omega) \cap A^0,$$

where $\gamma > 0$ is a capillarity coefficient and $\varepsilon > 0$ is a scaling parameter. Obviously, functions which exhibit jumps across a surface are not in $H^1(\Omega)$. By consequence, a possible two-phase minimizer ρ_ε cannot change its phase without taking values in the elliptic region, another way to express this phenomenon is to say that the constituent propagates continuously in the interfaces. This is the reason why the variational problem is called a diffuse interface approach (DI). Here the thickness of the interface is expected to be of size $O(\varepsilon)$.

This DI-approach may appear artificial at first sight but a result, obtained by Modica in [29], validates this method: if a sequence of minimizers ρ_ε converges when ε tends to zero, the limit is a physically relevant minimizer of the Sharp-Interface functional F^0.

1.1.4 The non-local diffuse interface approach

This alternative approach (also called global diffuse interface approach, introduced by Serrin et al in [16] and next by Coquel, et al. in [11], and Rohde in [31] and [32]) also consists in a modification of the sharp-interface functional F_0. This approach does not need to introduce high derivatives to penalize the high fluctuations of density. Let us consider the new following admissibility set:

$$A^{global} = A^0 \cap L^2(\Omega),$$

and let us choose a function $\phi \in L^1(\mathbb{R}^d)$ such that:

$$(|.|+|.|^2)\phi(.) \in L^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \phi(x)dx = 1, \quad \phi \text{ even, and } \phi \geq 0. \tag{1.5}$$

Such a function is called an interaction potential, and $\phi_\varepsilon(x) = \frac{1}{\varepsilon^d}\phi(\frac{x}{\varepsilon})$ is called scaled interaction potential. Then for $\gamma > 0$, we search for $\rho_\varepsilon \in A^{global}$ that minimizes the following modified functionnal called non-local Van der Waals functional:

$$F^{\varepsilon}_{global}[\rho_\varepsilon] = \int_\Omega \Big(W(\rho_\varepsilon(x)) + \frac{\gamma}{4}\int_\Omega \phi_\varepsilon(x-y)(\rho_\varepsilon(y)-\rho_\varepsilon(x))^2 dy\Big)dx.$$

Roughly speaking, the non-local term penalizes high fluctuations on an ε-scale if the interaction potential has most of its mass in a ball centered around zero and whose radius is of order ε. The set A^{global} contains functions with jumps but it can be proved that minimizers of F^{ε}_{global} take values also in the elliptic regions: the model belongs to the class of diffuse-interface models. The counterpart of the result of Modica, which also validates this approach, was proved by Alberti and Bellettini (see [1]).

Remark 2 *Let us consider F^{ε}_{global} with $\Omega = \mathbb{R}$ and assume that ρ is an analytic function (we refer to [32]). If we use an asymptotic expansion of the density into the non-local term in F^{ε}_{global} and perform the change of variable $r = \frac{x-y}{\varepsilon}$ we obtain:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(r)(\sum_{k\geq 1}\frac{1}{k!}(\varepsilon r)^k \frac{\partial^k}{\partial x^k}\rho(x))^2 dr dx \approx \int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon^2\phi(r)(r\frac{\partial}{\partial x}\rho(x))^2 dr dx.$$

Since $\int_{\mathbb{R}} \phi(r) r^2 dr < +\infty$ holds by definition of ϕ, the last expression is equal to the local penalty terms in F^{ε}_{local} (up to a multiplicative constant). In particular the scaling with respect to the small parameter ε fits.

1.1.5 Non classical shocks and the Euler problem

Another important approach using selection principles and capillarity concerns the discontinuous solutions of nonlinear hyperbolic systems of conservation laws.

Indeed the mathematical model of liquid-vapor flows should have special solutions that can be interpreted as dynamical phase transitions. In [32], the author shows that there are traveling-wave solutions for one-dimensional versions of the local and non-local Korteweg system that connect states in different phases (see below for these systems). This fact is one of the major arguments to accept the Korteweg systems (see below) as promising candidates to model the dynamics of liquid vapour flow in a reliable way.

An idea for selecting some relevant solutions of the Euler system with a Van der Waals pressure is to consider the solutions limit of solutions of (NSK) systems when the capillarity and the viscosity coefficients tend to 0. This condition is called the viscosity-capillarity criterium.

Remark 3 *In this case the Euler system is far from being a standard hyperbolic system:*

- *it is not hyperbolic (but elliptic) in $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}) \times \mathbb{R}$,*

- *the characteristic fields are not genuinely nonlinear in the hyperbolic part of the state space.*

Here the classical theory cannot be applied. There exists (see [17]) an infinity of weak ***entropy*** *solutions of the Riemann problem with initial states in different phases. It means that the entropy inequalities are not sufficiently discriminating when one characteristic field fails to be genuinely nonlinear.*

By a study of traveling waves (see [28] in the context of the Korteweg de Vries system, see also [7]) we can remark that generally the limit solutions violate the Oleinik conditions, more precisely the shocks are undercompressive (it means non classic in the sense of the Lax theory). We refer for more details to the book [28] of P. Lefloch.

An important research line (see [31], [32], [11]) is to model the capillarity tensor and to understand how fast the solutions converge to the Euler system when the capillarity and the viscosity coefficients tend to zero. If we assume the viscosity coefficient equal to ε with $\varepsilon \to 0$, then we have the three different regimes:

- $\kappa << \varepsilon^2$, the viscosity dominates.
- $\kappa \simeq \varepsilon^2$, intermediary regime.
- $\kappa >> \varepsilon^2$, the capillarity dominates.

A second important problem from a numerical point of view is to describe the link between the local and the non-local Korteweg system and more particularly how it plays on the choice of the thickness of the interfaces. We want here to address this question and give a speed of convergence between the local and non-local Korteweg system.

1.2 Results

1.2.1 Presentation of the results

The model to study the dynamics of a liquid-vapour mixture is based upon the compressible Navier-Stokes equations with a Van der Waals state law for ideal fluids. Endowed with such a state law, the system can describe the dynamics of multi-phases (liquid and vapour) but the effect of surface tension is still not taken into account. In this goal, Korteweg extended (in [27]) the Navier-Stokes equations by adding a capillarity tensor modeling the behavior at the interfaces. The additional term is directly related to the surface energy added to the free functional in order to obtain physically relevant minimizers. This local Navier-Stokes-Korteweg system belongs to the class of diffuse interface models and, to penalize the density fluctuations, it involves the density gradient and in fact it introduces, in the classical model, third-order derivatives of the density (and therefore numerical complications).

Alternatively, another way to penalize the high density variations consists in applying a zero order but non-local operator to the density gradient (see [30], [31]). Let us mention that it was the original idea of Van der Waals in [35] and that statistical mechanics tend to show that this non-local approach is in fact the correct one (see [32]). This model does not introduce additional orders of differentiation but a non-local integral operator which creates comparable numerical difficulties. We refer to the work of Benzoni on capillary fluids and phase transitions (see [4, 5] and [6]). For new developments we also refer to the work of M. Heida and J. Málek in [25].

More precisely, let ρ and u denote the density and the velocity of a compressible viscous fluid. As usual, ρ is a non-negative function and u is a vector-valued function defined on $\mathbb{R}^d$. In the sequel we will denote by $\mathcal{A}$ the following diffusion operator

$$\mathcal{A}u = \mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}\, u, \quad \text{with} \quad \mu > 0 \quad \text{and} \quad \nu = \lambda + 2\mu > 0.$$

The Navier-Stokes equation for compressible fluids endowed with internal capillarity reads:

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mathcal{A}u + \nabla(P(\rho)) = \kappa\rho\nabla D[\rho]. \end{cases}$$

In the local Korteweg system (NSK), the capillary term $D[\rho]$ is given by (see [16]):

$$D[\rho] = \Delta\rho,$$

and, in the non-local Korteweg system $(NSRW)$ (introduced in its modern form by C. Rohde in [31] and also [11], see Van der Waals [35] for the original works), if ϕ is an interaction potential which satisfies conditions (1.5), $D[\rho]$ is a non-local term:

$$D[\rho] = \phi * \rho - \rho.$$

Both of these systems have been studied in the context of existence of strong solutions in critical spaces for the scaling of the equations. For example, concerning the strong solutions, we refer to [15] (and [20] in the non isothermal case) for a study of (NSK) system, and to [19] for $(NSRW)$. Let us mention some new results in [23] of existence of global strong solutions with infinite energy for (NSK), in particular the initial density may be chosen discontinuous (which allows to consider initial data with null thickness interfaces), and a result of ill-posedness in the sense of the explosion of norm. In [22], are derived new blow-up conditions of Prodi-Serrin type for (NSK), which is new for a compressible system (one also obtains blow-up conditions involving a control on the vacuum of type $\frac{1}{\rho^\epsilon} \in L^\infty_T(L^1)$ for $\epsilon > 0$ arbitrary small).

If we compute the Fourier transform in the capillarity terms (simplified by the density), we obtain $(\widehat{\phi}(\xi)-1)\widehat{\rho}(\xi)$ in the non-local model, and $-|\xi|^2\widehat{\rho}(\xi)$ in the local model.

We want to see if, when $\widehat{\phi}(\xi)$ is formally "close" to $1-|\xi|^2$, we can expect the solutions of these models to be close. More precisely, the aim of this paper is to approximate the local Korteweg model (NSK) with a non-local model such as system $(NSRW)$. For that we will choose a specific function ϕ_ε in the capillarity tensor. The results are the following (we refer to section 1.4 for precise statements):

Theorem 1 *Given the initial density and velocity (ρ_0, u_0), the capillary coefficient κ/ε^2 and the scaled interaction potential $\widehat{\phi_\varepsilon}(\xi) = e^{-\varepsilon^2|\xi|^2}$, if ρ_0 is close enough to a constant state, and if u_0 is small enough then systems (NSK) and $(NSRW)$ have global solutions, and the solutions of $(NSRW)$ tend to those of (NSK) when ε goes to zero. The speed of convergence is of the form ε^α with $\alpha \in]0,1]$.*

1.2.2 The models

The non-local system we will consider in this paper is the following:

$$(NSRW_\varepsilon)\qquad \begin{cases} \partial_t\rho_\varepsilon + \mathrm{div}\,(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}\,(\rho_\varepsilon u \otimes u_\varepsilon) - \mathcal{A}u_\varepsilon + \nabla(P(\rho_\varepsilon)) = \rho_\varepsilon \dfrac{\kappa}{\varepsilon^2}\nabla(\phi_\varepsilon * \rho_\varepsilon - \rho_\varepsilon), \end{cases}$$

where we set:

$$\phi_\varepsilon = \frac{1}{\varepsilon^d}\phi(\frac{x}{\varepsilon}) \quad \text{with} \quad \phi(x) = \frac{1}{(2\pi)^d}e^{-\frac{|x|^2}{4}}$$

As the Fourier transform of ϕ is $\widehat{\phi}(\xi) = e^{-|\xi|^2}$ we have:

$$\widehat{\phi_\varepsilon}(\xi) = e^{-\varepsilon^2|\xi|^2},$$

and for a fixed ξ, when ε is small, $\dfrac{\widehat{\phi_\varepsilon}(\xi) - 1}{\varepsilon^2}$ is close to $-|\xi|^2$.

Remark 4 *We mention here that the choice of ϕ_ε and κ/ε^2 is in accordance with the physical relevant capillarity coefficient (see [32]).*

We will consider a density which is close to an equilibrium state $\overline{\rho}$ and we will introduce the change of function $\rho = \overline{\rho}(1+q)$. By simplicity we take $\overline{\rho} = 1$. The previous systems become:

$$(K)\qquad \begin{cases} \partial_t q + u.\nabla q + (1+q)\mathrm{div}\, u = 0, \\ \partial_t u + u.\nabla u - \mathcal{A}u + P'(1).\nabla q - \kappa\nabla\Delta q = K(q).\nabla q - I(q)\mathcal{A}u, \end{cases}$$

and

$$(RW_\varepsilon)\qquad \begin{cases} \partial_t q_\varepsilon + u_\varepsilon.\nabla q_\varepsilon + (1+q_\varepsilon)\mathrm{div}\, u_\varepsilon = 0, \\ \partial_t u_\varepsilon + u_\varepsilon.\nabla u_\varepsilon - \mathcal{A}u_\varepsilon + P'(1).\nabla q_\varepsilon - \dfrac{\kappa}{\varepsilon^2}\nabla(\phi_\varepsilon * q_\varepsilon - q_\varepsilon) \\ \qquad\qquad\qquad\qquad = K(q_\varepsilon).\nabla q_\varepsilon - I(q_\varepsilon)\mathcal{A}u_\varepsilon, \end{cases}$$

where K and I are real-valued functions defined on $\mathbb{R}$ given by:

$$K(q) = \left(P'(1) - \frac{P'(1+q)}{1+q}\right) \quad \text{and} \quad I(q) = \frac{q}{q+1}.$$

Remark 5 *In the sequel we will sometimes rewrite the term $K(q).\nabla q$ as $\nabla(G(q))$ where G is a primitive of K.*

Remark 6 *When $\overline{\rho} \neq 1$ the only changes are in the viscosity and capillarity coefficients, and in the expression of functions I and K.*

Let us now give results concerning the local and non-local Korteweg systems.

1.3 The Korteweg System

We refer to the appendix for a presentation of the Besov setting in which we will constantly work in this article.

Unlike the Navier-Stokes or $(NSRW)$ systems, here the velocity, as well as the density, is regularized for every frequency (in fact, like these systems, there is a frequency threshold in the Fourier modes, but in both low and high frequencies the density is parabolically regularized). In the sequel we choose $\overline{\rho}=1$ and we will denote $q_0=\rho_0-1$ and $q=\rho-1$ the density fluctuations.

Let us now recall some results from [15] about the strong solutions of the Korteweg system (for more simplicity, we do not mention here any exterior forcing term):

Theorem 2 *([15]) Assume that $P'(1)>0$, $\min(\mu,2\mu+\lambda)>0$, that the initial density fluctuation q_0 belongs to $\dot{B}_{2,1}^{\frac{d}{2}-1}\cap\dot{B}_{2,1}^{\frac{d}{2}}$, and that the initial velocity u_0 is in $(\dot{B}_{2,1}^{\frac{d}{2}-1})^d$. Then there exist constants $\eta_K>0$ and $C>0$ depending on κ, μ, ν, $P'(1)$ and d such that if:*

$$\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}\cap\dot{B}_{2,1}^{\frac{d}{2}}}+\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}\leq\eta_K$$

then system (K) has a unique global solution (ρ,u) such that the density fluctuation and the velocity satisfy:

$$\begin{cases} q\in\mathcal{C}(\mathbb{R}_+,\dot{B}_{2,1}^{\frac{d}{2}-1}\cap\dot{B}_{2,1}^{\frac{d}{2}})\cap L^1(\mathbb{R}_+,\dot{B}_{2,1}^{\frac{d}{2}+1}\cap\dot{B}_{2,1}^{\frac{d}{2}+2}),\\ u\in\mathcal{C}(\mathbb{R}_+,\dot{B}_{2,1}^{\frac{d}{2}-1})^d\cap L^1(\mathbb{R}_+,\dot{B}_{2,1}^{\frac{d}{2}+1})^d. \end{cases}$$

Moreover the norm of (q,u) in this space is estimated by the initial norm $C(\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}\cap\dot{B}_{2,1}^{\frac{d}{2}}}+\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}})$.

Further in this article R. Danchin and B. Desjardins provide a Fourier study of the linearized system and observe different behaviours whether the quantity $\nu^2-4\kappa$ is positive, negative of zero. In any case they obtain parabolic regularization.

1.4 Notations and main results

In this article we will often use hybrid Besov norms (see the appendix for properties):

Definition 1 *For $l_\varepsilon=[\frac{1}{2}\log_2(\frac{\gamma}{C_0\varepsilon^2})-1]$ and $s,t\in\mathbb{R}$, we define the following hybrid norms:*

$$\|q\|_{\dot{B}_\varepsilon^{s,t}}\overset{def}{=}\sum_{l\leq l_\varepsilon}2^{ls}\|\dot{\Delta}_l q\|_{L^2}+\sum_{l>l_\varepsilon}\frac{1}{\varepsilon^2}2^{lt}\|\dot{\Delta}_l q\|_{L^2}. \tag{1.6}$$

Remark 7 *We refer to (2.29) for the choice of l_ε. As $l_\varepsilon\simeq -C'\log\varepsilon$, the threshold l_ε between (parabolically regularized) low frequencies and (only dumped) high frequencies of the density goes to infinity as ε goes to zero.*

Definition 2 *The space E^s_ε is the set of functions (q,u) in*

$$\Big(\mathcal{C}_b(\mathbb{R}_+,\dot B^{s-1}_{2,1}\cap\dot B^{s}_{2,1})\cap L^1(\mathbb{R}_+,\dot B^{s+1,s}_{\varepsilon}\cap\dot B^{s+2,s}_{\varepsilon})\Big)\times\Big(\mathcal{C}_b(\mathbb{R}_+,\dot B^{s-1}_{2,1})\cap L^1(\mathbb{R}_+,\dot B^{s+1}_{2,1})\Big)^d$$

endowed with the norm

$$\|(q,u)\|_{E^s_\varepsilon}\overset{def}{=}\|u\|_{L^\infty\dot B^{s-1}_{2,1}}+\|q\|_{L^\infty\dot B^{s-1}_{2,1}}+\|q\|_{L^\infty\dot B^{s}_{2,1}}\\ +\|u\|_{L^1\dot B^{s+1}_{2,1}}+\|q\|_{L^1\dot B^{s+1,s}_{\varepsilon}}+\|q\|_{L^1\dot B^{s+2,s}_{\varepsilon}} \quad (1.7)$$

We first prove global well-posedness for system (RW_ε) by following similar ideas as in [19] and uniform estimates with respect to ε:

Theorem 3 *Let $\varepsilon>0$ and assume that $\min(\mu,2\mu+\lambda)>0$. There exist two positive constants η_R and C only depending on d, κ, μ, λ and $P'(1)$ such that if $q_0\in\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}$, $u_0\in\dot B^{\frac d2-1}_{2,1}$ and*

$$\|q_0\|_{\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}\le\eta_R$$

then system (RW_ε) has a unique global solution (ρ,u) with $(q,u)\in E^{\frac d2}_\varepsilon$ such that:

$$\|(q,u)\|_{E^{\frac d2}_\varepsilon}\le C(\|q_0\|_{\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}).$$

Remark 8 *Note that in the low frequency regime ($l\le l_\varepsilon$), the parabolic regularization for q is the same as for the Korteweg system, that is the low frequencies of q are in $\dot B^{\frac d2+1}_{2,1}\cap\dot B^{\frac d2+2}_{2,1}$.*

The main result in this article is the following: when the initial data are small enough (so that we have global solutions for (K) and (RW_ε)) the solution of (RW_ε) goes to the solution of (K) when ε goes to zero.

Theorem 4 *Assume that $\min(\mu,2\mu+\lambda)>0$, $P'(1)>0$ and that $q_0\in\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}$, $u_0\in\dot B^{\frac d2-1}_{2,1}$. There exists $0<\eta\le\min(\eta_K,\eta_R)$ such that if*

$$\|q_0\|_{\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}\le\eta,$$

then systems (K) and (RW_ε) both have global solutions and $\|(q_\varepsilon-q,u_\varepsilon-u)\|_{E^{\frac d2}_\varepsilon}$ tends to zero as ε goes to zero. Moreover, with the same notations as before, there exists a constant $C=C(\eta,\kappa,\overline{\rho},P'(1))>0$ such that for all $\alpha\in]0,1[$ (if $d=2$) or $\alpha\in]0,1]$ (if $d\ge3$), and for all $t\in\mathbb{R}_+$,

$$\|(q_\varepsilon-q,u_\varepsilon-u)\|_{E^{\frac d2-\alpha}_\varepsilon}=\|u_\varepsilon-u\|_{L^\infty_t\dot B^{\frac d2-\alpha-1}_{2,1}}+\|q_\varepsilon-q\|_{L^\infty_t\dot B^{\frac d2-\alpha-1}_{2,1}}+\|q_\varepsilon-q\|_{L^\infty_t\dot B^{\frac d2-\alpha}_{2,1}}\\ +\|u_\varepsilon-u\|_{L^1_t\dot B^{\frac d2-\alpha+1}_{2,1}}+\|q_\varepsilon-q\|_{L^1_t\dot B^{\frac d2-\alpha+1,\frac d2-\alpha}_{\varepsilon}}+\|q_\varepsilon-q\|_{L^1_t\dot B^{\frac d2-\alpha+2,\frac d2-\alpha}_{\varepsilon}}\le C\varepsilon^\alpha, \quad (1.8)$$

Remark 9 *The same results hold for any other function $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\forall \xi \in \mathbb{R}^d$, $\widehat{\phi}(\xi) = g(|\xi|^2)$ with:*

- *Function $g : \mathbb{R}_+ \to \mathbb{R}$ takes its values in $[0,1]$, with $g(0) = 1$,*
- *Function $h : x \mapsto \frac{1-g(x)}{x}$ is decreasing with $\lim_0 h = 1$ and $\lim_\infty h = 0$,*
- *Function $k : x \mapsto 1 - g(x)$ is increasing with $\lim_0 k = 0$ and $\lim_\infty k = 1$,*
- *For all $1 < \beta < 2$, there exists $C_\beta > 0$ such that for all $x \geq 0$,*

$$0 \leq \frac{g(x) - 1 + x}{x^\beta} \leq C_\beta.$$

Remark 10 *Another question is to get similar result for large data (bounded away from vacuum). Here the existence and estimates are local in time and the distinction between low and high frequencies is not needed.*

The structure of this article is the following: in the second section we will obtain a priori estimates on the linear system with convection terms. The third section is devoted to the existence and uniqueness of solutions for the non-local model, and in the last section we will obtain the convergence result of theorem 4. In the appendix, one will find the proofs of some estimates involving the special hybrid Besov norms introduced in this paper.

2 A priori estimates

In this section we focus on the following linear system ($\varepsilon > 0$ is fixed and for more simplicity we write (q, u) instead of $(q_\varepsilon, u_\varepsilon)$):

$$(LR_\varepsilon) \qquad \begin{cases} \partial_t q + v.\nabla q + \operatorname{div} u = F, \\ \partial_t u + v.\nabla u - \mathcal{A}u + p\nabla q - \dfrac{k}{\varepsilon^2}\nabla(\phi_\varepsilon * q - q) = G. \end{cases}$$

With

$$\mathcal{A}u = \mu\Delta u + (\lambda + \mu)\nabla \operatorname{div} u.$$

This section is devoted to the proof of the following a priori estimates (we refer to definition 4 in the appendix for $\widetilde{L}_t^p \dot{B}_{2,1}^s$-spaces):

Proposition 1 *Let $\varepsilon > 0$, $s \in \mathbb{R}$, $I = [0, T[$ or $[0, +\infty[$ and $v \in L^1(I, \dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L^2(I, \dot{B}_{2,1}^{\frac{d}{2}})$. Assume that (q, u) is a solution of System (LR_ε) defined on I. There exists a constant $C > 0$ depending on d, s, μ, ν, p, k, c_0 and C_0 such that for all $t \in I$,*

$$\begin{aligned} &\|u\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{s-1}} + \|q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{s-1}} + \|q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{s}} + \|u\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s+1}} + \|q\|_{\widetilde{L}_t^1 \dot{B}_{\varepsilon}^{s+1,s}} + \|q\|_{\widetilde{L}_t^1 \dot{B}_{\varepsilon}^{s+2,s}} \\ &\quad \leq C e^{C\int_0^t (\|\nabla v(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|v(\tau)\|^2_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau} \Big(\|u_0\|_{\dot{B}_{2,1}^{s-1}} + \|q_0\|_{\dot{B}_{2,1}^{s-1}} + \|q_0\|_{\dot{B}_{2,1}^{s}} \\ &\qquad + \|F\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s-1}} + \|F\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s}} + \|G\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s-1}} \Big). \end{aligned} \tag{2.9}$$

The proof of the proposition will be close to those in [12], [3], [13] or [19]: lead by the behaviour of the linearized system and using symmetrizers we get estimates on dyadic blocks and obtain the expected estimates. The difference here is that we need to carefully estimate penalized terms and we obtain, for the density, a frequency threshold depending on ε.

We will localize in frenquency (there is no need here to separate the compressible and incompressible parts): using the Littlewood-Paley decomposition, for all $l \in \mathbb{Z}$ we define $q_l = \dot{\Delta}_l q$ and $u_l = \dot{\Delta}_l u$. We obtain the following system:

$$(LLR_\varepsilon) \qquad \begin{cases} \partial_t q_l + v.\nabla q_l + \operatorname{div} u_l = F_l + R_l, \\ \partial_t u_l + v.\nabla u_l - \mathcal{A} u_l + p\nabla q_l - \dfrac{k}{\varepsilon^2}\nabla(\phi_\varepsilon * q_l - q_l) = G_l + R'_l, \end{cases}$$

where $R_l = [v.\nabla, \dot{\Delta}_l]q$ and $R'_l = [v.\nabla, \dot{\Delta}_l]u$.

If $\alpha > 0$ is fixed and small enough (for example we can choose $\alpha = \min(\mu, \lambda+2\mu)/4$), we will introduce in the following:

$$h_l^2 = \|u_l\|_{L^2}^2 + p\|q_l\|_{L^2}^2 + \alpha\left(\nu\|\nabla q_l\|_{L^2}^2 + 2(u_l|\nabla q_l)_{L^2}\right) + \frac{k}{\varepsilon^2}(q_l|q_l - \phi_\varepsilon * q_l)_{L^2}.$$

2.1 Preliminary result

In order to prove the estimate we will show as a first step the following lemma:

Lemma 1 *Under the previous notations, if $\alpha > 0$ is small enough (for example $\alpha = \min(\mu, \lambda + 2\mu)/4$) there exist some $m > 0$ and $C > 0$ such that for all $l \in \mathbb{Z}$:*

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}h_l^2 + m\big(2^{2l}\|u_l\|_{L^2}^2 + \|\nabla q_l\|_{L^2}^2 + \frac{1}{\varepsilon^2}(\nabla q_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2}\big) \\ \leq C(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l^2 + C\big((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) \\ + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\big)h_l. \end{aligned} \tag{2.10}$$

Proof: If we compute the innerproduct in L^2 of the first equation from system (LLR_ε) by q_l, and the innerproduct of the second equation by u_l we obtain that for all $l \in \mathbb{Z}$:

$$\frac{1}{2}\frac{d}{dt}\|q_l\|_{L^2}^2 + (\operatorname{div} u_l|q_l)_{L^2} = -(v.\nabla q_l|q_l)_{L^2} + (F_l + R_l|q_l)_{L^2}, \tag{2.11}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_l\|_{L^2}^2 - (\mathcal{A}u_l|u_l)_{L^2} + p(\nabla q_l|u_l)_{L^2} - \frac{k}{\varepsilon^2}(\phi_\varepsilon * \nabla q_l - \nabla q_l|u_l)_{L^2} \\ = -(v.\nabla u_l|u_l)_{L^2} + (G_l + R'_l|u_l)_{L^2}. \end{aligned} \tag{2.12}$$

Using an integration by parts allows us to write that $(\operatorname{div} u_l|q_l)_{L^2} = -(\nabla q_l|u_l)_{L^2}$ and then to combine the previous estimates in order to eliminate these terms. We need to get rid of terms like this because they cannot be absorbed by the left-hand side and after the use of a Gronwall type estimate they introduce multiplicative constant terms e^{Ct} (which are problematic as we look for global time estimates). It is less easy to get rid of the term $\frac{k}{\varepsilon^2}(\phi_\varepsilon * \nabla q_l - \nabla q_l | u_l)_{L^2}$ and we will explain later how to do it.

Integrations by parts also provide that there exists a constant $C > 0$ such that:

$$\begin{cases} |(v.\nabla q_l | q_l)_{L^2}| \leq C \|\nabla v\|_{L^\infty} \|q_l\|_{L^2}^2, \\ |(v.\nabla u_l | u_l)_{L^2}| \leq C \|\nabla v\|_{L^\infty} \|u_l\|_{L^2}^2, \end{cases}$$

as well as the fact that

$$-(\mathcal{A}u_l | u_l)_{L^2} = \mu \|\nabla u_l\|_{L^2}^2 + (\lambda + \mu) \|\mathrm{div}\, u_l\|_{L^2}^2$$

which leads to

$$-(\mathcal{A}u_l | u_l)_{L^2} \geq \underline{\nu} \|\nabla u_l\|^2, \quad \text{where} \quad \nu = \lambda + 2\mu \quad \text{and} \quad \underline{\nu} = \min(\mu, \nu).$$

(if $\lambda + \mu \geq 0$ it is immediate, else we use that $\|\mathrm{div}\, u_l\|_{L^2}^2 \leq \|\nabla u_l\|^2$.) Combining these estimates we obtain that:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_l\|_{L^2}^2 + p\|q_l\|_{L^2}^2) + \underline{\nu} \|\nabla u_l\|_{L^2}^2 - \frac{k}{\varepsilon^2} (\phi_\varepsilon * \nabla q_l - \nabla q_l | u_l)_{L^2} \\ \leq C \|\nabla v\|_{L^\infty} (\|u_l\|_{L^2}^2 + p\|q_l\|_{L^2}^2) + p(\|F_l\|_{L^2} + \|R_l\|_{L^2}) \|q_l\|_{L^2} \\ + (\|G_l\|_{L^2} + \|R'_l\|_{L^2}) \|u_l\|_{L^2}. \end{aligned} \tag{2.13}$$

As we want to obtain L^1_t-regularization on the density, we need at least a term such as $\|q_l\|_{L^2}^2$ to appear in the left-hand side. If we compute the inner product of the equation on u_l by ∇q_l we get:

$$\begin{aligned} (\partial_t u_l | \nabla q_l)_{L^2} + (v.\nabla u_l | \nabla q_l)_{L^2} - (\mathcal{A}u_l | \nabla q_l)_{L^2} + p\|\nabla q_l\|_{L^2}^2 + \frac{k}{\varepsilon^2} (\nabla q_l - \phi_\varepsilon * \nabla q_l | \nabla q_l)_{L^2} \\ = (G_l + R'_l | \nabla q_l)_{L^2}. \end{aligned} \tag{2.14}$$

In order to get a full derivative, we need to estimate the term $(u_l | \partial_t \nabla q_l)$ so we will write the equation on ∇q_l and its innerproduct on u_l:

$$\partial_t \nabla q_l + \nabla(v.\nabla q_l) + \nabla \mathrm{div}\, u_l = \nabla F_l + \nabla R_l. \tag{2.15}$$

Before adding (2.14) to

$$(\partial_t \nabla q_l | u_l)_{L^2} + (\nabla(v.\nabla q_l) | u_l)_{L^2} + (\nabla \mathrm{div}\, u_l | u_l)_{L^2} = (\nabla F_l + \nabla R_l | u_l)_{L^2}, \tag{2.16}$$

we remark that a simple computation shows there exists a constant $C > 0$ such that:

$$|(v.\nabla u_l | \nabla q_l)_{L^2} + (\nabla(v.\nabla q_l) | u_l)_{L^2}| \leq C \|\nabla v\|_{L^\infty} \|\nabla q_l\|_{L^2} \|u_l\|_{L^2}.$$

Moreover, integrations by part also provide the fact that

$$(\mathcal{A}u_l | \nabla q_l)_{L^2} = \nu (u_l | \nabla \Delta q_l)_{L^2}.$$

Gathering these estimates, we obtain that:

$$\begin{aligned} \frac{d}{dt} (u_l | \nabla q_l)_{L^2} - \nu (u_l | \nabla \Delta q_l)_{L^2} - \|\mathrm{div}\, u_l\|_{L^2}^2 + p\|\nabla q_l\|_{L^2}^2 + \frac{k}{\varepsilon^2} (\nabla q_l - \phi_\varepsilon * \nabla q_l | \nabla q_l)_{L^2} \\ \leq C \|\nabla v\|_{L^\infty} \|\nabla q_l\|_{L^2} \|u_l\|_{L^2} + (\|G_l\|_{L^2} + \|R'_l\|_{L^2}) \|\nabla q_l\|_{L^2} + 2^l (\|F_l\|_{L^2} + \|R_l\|_{L^2}) \|u_l\|_{L^2}. \end{aligned} \tag{2.17}$$

Remark 11 *Note that* $\frac{k}{\varepsilon^2}(\nabla q_l - \phi_\varepsilon * \nabla q_l|\nabla q_l)_{L^2}$ *is nonnegative.*

In the previous equation, the term $(u_l|\nabla\Delta q_l)_{L^2}$ is dangerous because it introduces much more derivatives that we will be able to handle in the following, so we need to neutralize it. A simple way to do that is to compute the innerproduct of the equation of ∇q_l by ∇q_l. Using that:

$$(\nabla \mathrm{div}\, u_l|\nabla q_l)_{L^2} = (u_l|\nabla\Delta q_l)_{L^2},$$

and that

$$|(\nabla(v.\nabla q_l)|\nabla q_l)_{L^2}| \leq C\|\nabla v\|_{L^\infty}\|\nabla q_l\|^2_{L^2},$$

we get the following estimate:

$$\frac{1}{2}\frac{d}{dt}\|\nabla q_l\|^2_{L^2} + (u_l|\nabla\Delta q_l)_{L^2} \leq C\|\nabla v\|_{L^\infty}\|\nabla q_l\|^2_{L^2} + 2^l(\|F_l\|_{L^2} + \|R_l\|_{L^2})\|\nabla q_l\|_{L^2}, \quad (2.18)$$

so that, adding ν (2.18) to (2.17) will neutralize the previous troublesome term.

Remark 12 *Due to* $-\|\mathrm{div}\, u_l\|^2_{L^2}$ *appearing in (2.17) we have to be careful and look at (2.13)+*α *(*ν *(2.18)+(2.17)) with* $\alpha > 0$ *small enough (*$\alpha < \underline{\nu}$*) so that* $-\alpha\|\mathrm{div}\, u_l\|^2_{L^2}$ *can be absorbed by* $\underline{\nu}\|\nabla u_l\|^2_{L^2}$*.*

As announced we have to get rid of $\frac{k}{\varepsilon^2}(\phi_\varepsilon * \nabla q_l - \nabla q_l|u_l)_{L^2}$ which is introduced by estimate (2.13). A rough estimate will eventually provide some term in ε^{-2} which will ruin any asymptotic approach as ε goes to zero. The easiest way to take care of it is to consider the equation satisfied by the nonnegative term $\frac{k}{\varepsilon^2}(q_l|q_l - \phi_\varepsilon * q_l)_{L^2}$. Computing the innerproduct of the equation on q_l by $q_l - \phi_\varepsilon * q_l$ gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(q_l|q_l - \phi_\varepsilon * q_l)_{L^2} &- (u_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2}\\ &= -(v.\nabla q_l|q_l - \phi_\varepsilon * q_l)_{L^2} + (F_l + R_l|q_l - \phi_\varepsilon * q_l)_{L^2}. \quad (2.19)\end{aligned}$$

We need to be careful when estimating the right-hand side as this estimate will be multiplied by k/ε^2 and added to the others. Using Plancherel (up to a multiplicative constant) we introduce the following notation:

$$(q_l|q_l - \phi_\varepsilon * q_l)_{L^2} = \int_{\mathbb{R}^d} |\widehat{q_l}(\xi)|^2(1 - e^{-\varepsilon^2|\xi|^2})d\xi \overset{\text{def}}{=} \|\sqrt{1-\phi_\varepsilon} * q_l\|^2_{L^2}.$$

When estimating the right-hand side, we are forced to let the block $q_l - \phi_\varepsilon * q_l$ in one part, as our only possibility is to absorb some terms using remark 11. As the following function:

$$\begin{aligned} f: \quad &\mathbb{R}_+^* \to \mathbb{R}\\ &x \mapsto \frac{1-e^{-x}}{x}, \end{aligned} \quad (2.20)$$

is decreasing from 1 to 0, and as up to loosing some derivatives we want to get uniform estimates in ε, we can write that for all $\xi \in \mathbb{R}^d$:

$$1 - e^{-\varepsilon^2|\xi|^2} \leq \varepsilon^2|\xi|^2.$$

So, using Plancherel,

$$
\begin{aligned}
|(v.\nabla q_l|q_l-\phi_\varepsilon * q_l)_{L^2}| &= C|\int_{\mathbb{R}^d}\widehat{v.\nabla q_l}.\sqrt{1-e^{-\varepsilon^2|\xi|^2}}.\sqrt{1-e^{-\varepsilon^2|\xi|^2}}.\overline{\widehat{q_l}(\xi)}d\xi| \\
&\leq C\int_{\mathbb{R}^d}|\widehat{v.\nabla q_l}|.\varepsilon|\xi|.\sqrt{1-e^{-\varepsilon^2|\xi|^2}}.|\overline{\widehat{q_l}(\xi)}|d\xi. \quad (2.21)
\end{aligned}
$$

Thanks to the fact that we are localized in frequency, and using the Cauchy-Schwarz estimate, we obtain that

$$
\begin{aligned}
|(v.\nabla q_l|q_l-\phi_\varepsilon * q_l)_{L^2}| &\leq C\varepsilon 2^l\|v.\nabla q_l\|_{L^2}\|\sqrt{1-\phi_\varepsilon} * q_l\|_{L^2}^2 \\
&\leq C\varepsilon 2^l\|v\|_{L^\infty}\|\nabla q_l\|_{L^2}\sqrt{(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}}. \quad (2.22)
\end{aligned}
$$

Remark 13 *Using commutators would lead to get $\|\nabla v\|_{L^\infty}$ instead of $\|v\|_{L^\infty}$, but wouldn't improve the use of the estimates in the main result of the paper.*

Using the same arguments allows us to write:

$$
\begin{aligned}
|(F_l+R_l|q_l-\phi_\varepsilon * q_l)_{L^2}| &= C|\int_{\mathbb{R}^d}\widehat{F_l+R_l}.\sqrt{1-e^{-\varepsilon^2|\xi|^2}}.\sqrt{1-e^{-\varepsilon^2|\xi|^2}}.\overline{\widehat{q_l}(\xi)}d\xi| \\
&\leq C\varepsilon 2^l(\|F_l\|_{L^2}+\|R_l\|_{L^2})\sqrt{(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}}. \quad (2.23)
\end{aligned}
$$

Plugging (2.22) and (2.23) into (2.19), we obtain that:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}-(u_l|\nabla q_l-\phi_\varepsilon * \nabla q_l)_{L^2} &\leq C\varepsilon^2\|v\|_{L^\infty}\|\nabla q_l\|_{L^2}\sqrt{\frac{(\nabla q_l|\nabla q_l-\phi_\varepsilon * \nabla q_l)_{L^2}}{\varepsilon^2}} \\
&+ C\varepsilon^2 2^l(\|F_l\|_{L^2}+\|R_l\|_{L^2})\sqrt{\frac{(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}}{\varepsilon^2}}. \quad (2.24)
\end{aligned}
$$

Now, when we compute (2.13)+α (ν (2.18)+(2.17))+$\frac{k}{\varepsilon^2}$(2.24), we will be able to neutralize or absorb any dangerous term. For that we will introduce as previously announced:

$$
h_l^2=\|u_l\|_{L^2}^2+p\|q_l\|_{L^2}^2+\alpha\left(\nu\|\nabla q_l\|_{L^2}^2+2(u_l|\nabla q_l)_{L^2}\right)+\frac{k}{\varepsilon^2}(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}.
$$

Using the Cauchy-Schwarz estimate, we can write that

$$
|(u_l|\nabla q_l)|_{L^2}\leq\|u_l\|_{L^2}.\|\nabla q_l\|_{L^2}\leq\frac{\nu}{4}\|\nabla q_l\|_{L^2}^2+\frac{1}{\nu}\|u_l\|_{L^2}^2,
$$

so

$$
\begin{aligned}
(1-\frac{2\alpha}{\nu})\|u_l\|_{L^2}^2+p\|q_l\|_{L^2}^2+\frac{\alpha\nu}{2}\|\nabla q_l\|_{L^2}^2+\frac{k}{\varepsilon^2}(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}\leq h_l^2 \\
\leq(1+\frac{2\alpha}{\nu})\|u_l\|_{L^2}^2+p\|q_l\|_{L^2}^2+\frac{3\alpha\nu}{2}\|\nabla q_l\|_{L^2}^2+\frac{k}{\varepsilon^2}(q_l|q_l-\phi_\varepsilon * q_l)_{L^2}. \quad (2.25)
\end{aligned}
$$

The last term is nonnegative and lesser than $\|\nabla q_l\|_{L^2}$. If we want h_l^2 to be equivalent to $\|u_l\|_{L^2}^2+\|q_l\|_{L^2}^2+\|\nabla q_l\|_{L^2}^2$ we need that $\alpha<\nu/2$. Remember that we already have a condition on α: for $-\alpha\|\operatorname{div} u_l\|_{L^2}^2$ to be absorbed by the left-hand side, we need $\alpha<\underline{\nu}$.

So we need $\alpha < \min(\mu, \frac{\nu}{2})$ and for example we will fix here once and for all $\alpha = \frac{\nu}{4}$. After computing (2.13)+α (ν (2.18)+(2.17))+$\frac{k}{\varepsilon^2}$(2.24) we get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}h_l^2 + (\underline{\nu} - \alpha)\|\nabla u_l\|_{L^2}^2 + \alpha p\|\nabla q_l\|_{L^2}^2 + \frac{\alpha k}{\varepsilon^2}(\nabla q_l - \phi_\varepsilon * \nabla q_l|\nabla q_l)_{L^2}\\ \leq C\|\nabla v\|_{L^\infty}h_l^2 + C\left((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R_l'\|_{L^2}\right)h_l\\ + Ck2^l(\|F_l\|_{L^2} + \|R_l\|_{L^2})\sqrt{\frac{(q_l|q_l - \phi_\varepsilon * q_l)_{L^2}}{\varepsilon^2}}\\ + Ck\|v\|_{L^\infty}\|\nabla q_l\|_{L^2}\sqrt{\frac{(\nabla q_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2}}{\varepsilon^2}}. \quad (2.26)\end{aligned}$$

Thanks to the definition of h_l, we have:

$$k2^l(\|F_l\|_{L^2} + \|R_l\|_{L^2})\sqrt{\frac{(q_l|q_l - \phi_\varepsilon * q_l)_{L^2}}{\varepsilon^2}} \leq \sqrt{k}2^l(\|F_l\|_{L^2} + \|R_l\|_{L^2})h_l,$$

and using the classical estimate $ab \leq (a^2+b^2)/2$, we obtain for the last term:

$$\begin{aligned}Ck\|v\|_{L^\infty}\|\nabla q_l\|_{L^2}\sqrt{\frac{(\nabla q_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2}}{\varepsilon^2}}\\ \leq \frac{\alpha k}{2\varepsilon^2}(\nabla q_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2} + \frac{kC^2}{2\alpha}\|v\|_{L^\infty}^2\|\nabla q_l\|_{L^2}^2. \quad (2.27)\end{aligned}$$

So, if we denote by $m = \min(\underline{\nu} - \alpha, \alpha p, \frac{\alpha k}{2}) > 0$ we have proved there exists a constant $C > 0$ such that for all $l \in \mathbb{Z}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}h_l^2 + m\left(\|\nabla u_l\|_{L^2}^2 + \|\nabla q_l\|_{L^2}^2 + \frac{1}{\varepsilon^2}(\nabla q_l - \phi_\varepsilon * \nabla q_l|\nabla q_l)_{L^2}\right)\\ \leq C(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l^2 + C\left((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R_l'\|_{L^2}\right)h_l,\\ (2.28)\end{aligned}$$

which ends the proof of lemma 1. ■

2.2 Proof of the proposition

According to the previous estimate, we observe that each term of h_l except ∇q_l is parabolically regularized. The key point is that thanks to $\frac{1}{\varepsilon^2}(q_l|q_l - \phi_\varepsilon * q_l)_{L^2}$ (that appears in h_l) we will be able to get regularization on ∇q_l for some low frequencies. This leads to the hybrid Besov spaces defined in the introduction (and the threshold between the two regimes is at $-\log\varepsilon$).

-The low frequency case: More precisely, the frequency threshold is simply given by the study of the decreasing function f introduced in (2.20). As $f(z)$ goes to 1 when z goes to zero, there exists some $\gamma > 0$ such that for all $0 \leq z \leq \gamma$, $f(z) \geq \frac{1}{2}$. If we denote by $0 < c_0 < C_0$ the radii of the annulus $\mathcal{C}$ used in the dyadic decomposition, let the threshold l_ε be the greatest integer such that $\varepsilon^2 2^{2l_\varepsilon} C_0 \leq \gamma$ (that is $l_\varepsilon = [\frac{1}{2}\log_2(\frac{\gamma}{C_0\varepsilon^2}) - 1]$). Then for all frequency $l \leq l_\varepsilon$ and all $\xi \in 2^l\mathcal{C}$, we have:

$$(\varepsilon|\xi|)^2 \leq \varepsilon^2 C_0^2 2^{2l_\varepsilon} \leq \gamma \quad \text{and} \quad 1 - e^{-\varepsilon^2|\xi|^2} \geq \frac{\varepsilon^2|\xi|^2}{2}, \tag{2.29}$$

and then for all $l \leq l_\varepsilon$,

$$(\nabla q_l - \phi_\varepsilon * \nabla q_l|\nabla q_l)_{L^2} = C\int_{2^l\mathcal{C}} (1 - e^{-\varepsilon^2|\xi|^2})|\widehat{\nabla q_l}(\xi)|^2 d\xi \geq C\frac{\varepsilon^2 c_0^2 2^{2l}}{2}\|\nabla q_l\|_{L^2}^2.$$

So plugging this into the estimate given by lemma 1 gives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}h_l^2 + m\left(\|\nabla u_l\|_{L^2}^2 + \|\nabla q_l\|_{L^2}^2 + \frac{2^{2l}}{2\varepsilon^2}(q_l - \phi_\varepsilon * q_l|q_l)_{L^2} + C\frac{c_0^2 2^{2l}}{2}\|\nabla q_l\|_{L^2}^2\right)\\ \leq C(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l^2 + C\left((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\right)h_l,\end{aligned} \tag{2.30}$$

and we have obtained that for all $l \leq l_\varepsilon$:

$$\begin{aligned}&\frac{d}{dt}h_l + \frac{m}{2}2^{2l}h_l\\ &\leq C(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l + C\left((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\right),\end{aligned} \tag{2.31}$$

that is for all $l \leq l_\varepsilon$ and all t:

$$\begin{aligned}&h_l(t) + m2^{2l}\int_0^t h_l(\tau)d\tau\\ &\leq h_l(0) + C\int_0^t (\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l + C\left((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\right)d\tau.\end{aligned} \tag{2.32}$$

Remark 14 *Notice that as $l_\varepsilon \sim \log_2(\frac{\sqrt{\gamma}}{2C_0\varepsilon})$, and goes to infinity when ε goes to zero. The parabolic regularization eventually reaches every frequency.*

-The high frequency case: Remember that l_ε is characterized by:

$$\begin{cases}\varepsilon 2^{l_\varepsilon} C_0 \leq \sqrt{\gamma},\\ \varepsilon 2^{l_\varepsilon+1} C_0 > \sqrt{\gamma}\end{cases},$$

So, for all $l \geq l_\varepsilon + 1$ and all $\xi \in 2^l\mathcal{C}$, we have

$$\varepsilon^2|\xi|^2 \geq \varepsilon^2 2^{2l}c_0^2 \geq \varepsilon^2 2^{2(l_\varepsilon+1)}C_0^2(\frac{c_0}{C_0})^2 > \gamma(\frac{c_0}{C_0})^2. \tag{2.33}$$

Let us go back to the estimate given by lemma 1. We can rewrite it into:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}h_l^2 + m2^{2l}\left(\|u_l\|_{L^2}^2 + \|q_l\|_{L^2}^2\right) + \frac{2^{2l}}{2\varepsilon^2}(q_l|q_l - \phi_\varepsilon * q_l)_{L^2} + \frac{1}{2\varepsilon^2}(\nabla q_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2}\\ \leq C(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l^2 + C\left((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\right)h_l.\end{aligned} \tag{2.34}$$

Using Plancherel, (2.33) together with the fact that function $x \mapsto 1-e^{-x}$ is increasing, we get that:

$$(\nabla q_l|\nabla q_l - \phi_\varepsilon * \nabla q_l)_{L^2} = C\int_{2^l\mathcal{C}} (1-e^{-\varepsilon^2|\xi|^2})|\widehat{\nabla q_l}(\xi)|^2 d\xi \geq (1-e^{-\gamma(\frac{c_0}{C_0})^2})\|\nabla q_l\|_{L^2}^2.$$

As $2^{2l} \geq \frac{\gamma}{\varepsilon^2 C_0^2}$, (and greater than 1 if ε is small enough) if we denote by $m' = m.\min(\frac{\gamma}{C_0^2}, 1-e^{-\gamma(\frac{c_0}{C_0})^2}) > 0$, our estimate becomes: for all $l \geq l_\varepsilon + 1$,

$$\begin{aligned}&\frac{d}{dt}h_l + \frac{m'}{\varepsilon^2}h_l\\ &\leq C(\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l + C\Big((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R_l'\|_{L^2}\Big), \quad (2.35)\end{aligned}$$

that is for all $l \geq l_\varepsilon + 1$ and all t:

$$\begin{aligned}&h_l(t) + \frac{m'}{\varepsilon^2}\int_0^t h_l(\tau)d\tau\\ &\leq h_l(0) + C\int_0^t (\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l + C\Big((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R_l'\|_{L^2}\Big) d\tau.\end{aligned} \quad (2.36)$$

We need to find a way to get parabolic regularization for u_l. Classically (we refer for example to [12], [3] or [19]) we will use the new information given by the previous estimate in the energy estimate for u_l:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_l\|_{L^2}^2 + \underline{\nu}2^{2l}\|u_l\|_{L^2}^2\\ &\leq C\|\nabla v\|_{L^\infty}\|u_l\|_{L^2}^2 + (\|G_l\|_{L^2} + \|R_l'\|_{L^2})\|u_l\|_{L^2} + p\|\nabla q_l\|_{L^2}.\|u_l\|_{L^2} + \frac{k}{\varepsilon^2}\|\nabla q_l - \phi_\varepsilon * \nabla q_l\|_{L^2}.\|u_l\|_{L^2},\end{aligned} \quad (2.37)$$

and then, simplifying and integrating in time:

$$\begin{aligned}\|u_l(t)\|_{L^2} + \underline{\nu}2^{2l}\int_0^t \|u_l(\tau)\|_{L^2}d\tau &\leq \|u_l(0)\|_{L^2}\\ &+ C\int_0^t \Big(\|\nabla v\|_{L^\infty}\|u_l\|_{L^2} + (\|G_l\|_{L^2} + \|R_l'\|_{L^2} + (p+\frac{k}{\varepsilon^2})\|\nabla q_l\|_{L^2}\Big) d\tau. \quad (2.38)\end{aligned}$$

Thanks to (2.25), estimate (2.36) implies

$$\frac{m'}{\varepsilon^2}\int_0^t \|\nabla q_l(\tau)\|_{L^2}d\tau \leq C'\frac{m'}{\varepsilon^2}\int_0^t h_l(\tau)d\tau,$$

so that we can estimate the last term of the left-hand side of the previous estimate and then there exists a constant such that for all $l \geq l_\varepsilon + 1$ and all t:

$$\begin{aligned}&h_l(t) + \underline{\nu}2^{2l}\int_0^t \|u_l(\tau)\|_{L^2}d\tau + \frac{m'}{\varepsilon^2}\int_0^t h_l(\tau)d\tau\\ &\leq h_l(0) + C\int_0^t (\|\nabla v\|_{L^\infty} + \|v\|_{L^\infty}^2)h_l + C\Big((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R_l'\|_{L^2}\Big) d\tau.\end{aligned} \quad (2.39)$$

Remember that thanks to (2.25):

$$h_l \sim \|u_l\|_{L^2} + \|q_l\|_{L^2} + \|\nabla q_l\|_{L^2}.$$

Let us collect what we have obtained for the low and high frequencies: if we use the Gronwall lemma to (2.32) and (2.39) we obtain that for all t, for all $l \leq l_\varepsilon$:

$$\begin{aligned} \|u_l(t)\|_{L^2} + \|q_l(t)\|_{L^2} + \|\nabla q_l(t)\|_{L^2} + m 2^{2l} \int_0^t (\|u_l(\tau)\|_{L^2} + \|q_l(\tau)\|_{L^2} + \|\nabla q_l(\tau)\|_{L^2})\, d\tau \\ \leq e^{C\int_0^t (\|\nabla v(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2) d\tau} \Big(\|u_l(0)\|_{L^2} + \|q_l(0)\|_{L^2} + \|\nabla q_l(0)\|_{L^2} \\ + C\int_0^t \big((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\big) d\tau \Big), \end{aligned} \tag{2.40}$$

and for $l \geq l_\varepsilon + 1$:

$$\begin{aligned} \|u_l(t)\|_{L^2} + \|q_l(t)\|_{L^2} + \|\nabla q_l(t)\|_{L^2} + \int_0^t \Big(\underline{\nu} 2^{2l} \|u_l(\tau)\|_{L^2} + \frac{m'}{\varepsilon^2}\big(\|q_l(\tau)\|_{L^2} + \|\nabla q_l(\tau)\|_{L^2}\big)\Big)\, d\tau \\ \leq e^{C\int_0^t (\|\nabla v(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2) d\tau} \Big(\|u_l(0)\|_{L^2} + \|q_l(0)\|_{L^2} + \|\nabla q_l(0)\|_{L^2} \\ + C\int_0^t \big((1+2^l)(\|F_l\|_{L^2} + \|R_l\|_{L^2}) + \|G_l\|_{L^2} + \|R'_l\|_{L^2}\big) d\tau \Big), \end{aligned} \tag{2.41}$$

If we use the definition of the hybrid norm introduced in the first section of this paper, after multiplying these estimates by 2^{s-1} and a summation over $l \in \mathbb{Z}$, we obtain that there exists a constant $C > 0$ such that:

$$\begin{aligned} \|u\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{s-1}} + \|q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{s-1}} + \|q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{s}} + \|u\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s+1}} + \|q\|_{\widetilde{L}_t^1 \dot{B}_{\varepsilon}^{s+1,s}} + \|q\|_{\widetilde{L}_t^1 \dot{B}_{\varepsilon}^{s+2,s}} \\ \leq C e^{C\int_0^t (\|\nabla v(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2) d\tau} \Big(\|u_l(0)\|_{\dot{B}_{2,1}^{s-1}} + \|q_l(0)\|_{\dot{B}_{2,1}^{s-1}} + \|\nabla q_l(0)\|_{\dot{B}_{2,1}^{s}} \\ + \|F\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s-1}} + \|F\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s}} + \|G\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{s-1}} + \int_0^t \big(\sum_{l\in\mathbb{Z}} (2^{l(s-1)} + 2^{ls}) \|R_l\|_{L^2} + 2^{l(s-1)} \|R'_l\|_{L^2} \big) d\tau \Big). \end{aligned} \tag{2.42}$$

Recall that $R_l = [v.\nabla, \dot{\Delta}_l]q$ and $R'_l = [v.\nabla, \dot{\Delta}_l]u$. We refer for example to [3] (lemma 2.96 and estimate 10.10) for the following well-known results:

Proposition 2 *There exists a constant $C > 0$ such that for all $v \in \dot{B}_{2,1}^{\frac{d}{2}+1}$ and $g \in \dot{B}_{2,1}^s$, if $r_l = [v.\nabla, \dot{\Delta}_l]g$ there exists a nonnegative sequence $c \in l^1(\mathbb{Z})$, with $\sum_{l\in\mathbb{Z}} c_l = 1$, such that for all $l \in \mathbb{Z}$:*

$$\|r_l\|_{L^2} \leq C c_l 2^{-ls} \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|g\|_{\dot{B}_{2,1}^s}.$$

Then there exist three nonnegative sequences c, c', and $c'' \in l^1(\mathbb{Z})$ (with $\sum_{l\in\mathbb{Z}} c_l^{(','')} = 1$)

such that for all $l \in \mathbb{Z}$:

$$\begin{cases} \|R_l\|_{L^2} \leq C c_l 2^{-ls} \|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|q\|_{\dot{B}^{s}_{2,1}}, \\ \|R_l\|_{L^2} \leq C c_l' 2^{-l(s-1)} \|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|q\|_{\dot{B}^{s-1}_{2,1}}, \\ \|R_l'\|_{L^2} \leq C c_l'' 2^{-l(s-1)} \|\nabla v\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|u\|_{\dot{B}^{s-1}_{2,1}}. \end{cases}$$

This allows to write:

$$\int_0^t \big(\sum_{l\in\mathbb{Z}} (2^{l(s-1)} + 2^{ls}) \|R_l\|_{L^2} + 2^{l(s-1)} \|R_l'\|_{L^2} \big) d\tau \\ \leq C \int_0^t \|\nabla v(\tau)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} (\|q\|_{\dot{B}^{s-1}_{2,1}} + \|q\|_{\dot{B}^{s}_{2,1}} + \|u\|_{\dot{B}^{s-1}_{2,1}}) d\tau. \quad (2.43)$$

Plugging this into estimate (2.42), and using once more the Gronwall lemma finally gives:

$$\|u\|_{\widetilde{L}^\infty_t \dot{B}^{s-1}_{2,1}} + \|q\|_{\widetilde{L}^\infty_t \dot{B}^{s-1}_{2,1}} + \|q\|_{\widetilde{L}^\infty_t \dot{B}^{s}_{2,1}} + \|u\|_{\widetilde{L}^1_t \dot{B}^{s+1}_{2,1}} + \|q\|_{\widetilde{L}^1_t \dot{B}^{s+1,s}_{\varepsilon}} + \|q\|_{\widetilde{L}^1_t \dot{B}^{s+2,s}_{\varepsilon}} \\ \leq C e^{C\int_0^t (\|\nabla v(\tau)\|_{L^\infty} + \|v(\tau)\|_{L^\infty}^2 + \|\nabla v(\tau)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}) d\tau} \Big(\|u_0\|_{\dot{B}^{s-1}_{2,1}} \\ + \|q_0\|_{\dot{B}^{s-1}_{2,1}} + \|\nabla q_0\|_{\dot{B}^{s-1}_{2,1}} + \|F\|_{\widetilde{L}^1_t \dot{B}^{s-1}_{2,1}} + \|F\|_{\widetilde{L}^1_t \dot{B}^{s}_{2,1}} + \|G\|_{\widetilde{L}^1_t \dot{B}^{s-1}_{2,1}} \Big). \quad (2.44)$$

which, thanks to the injection $\dot{B}^{\frac{d}{2}}_{2,1} \hookrightarrow L^\infty$, ends the proof of the proposition. ■

3 Proof of theorem 3

3.1 Existence

The proof of the global well-posedness of system (RW_ε) is classical and follows the lines of the proof for the compressible Navier-Stokes system (Friedrichs approximation and uniform estimates, we refer to [15], [12], [3] (section 10.2.3) or [19]). The only difference here is that we use, when proving the existence, the estimate given by Proposition 1, which ensures uniform estimates with respect to ε (and takes care of the additional penalized capillary term). We obtain the existence part of theorem 3 under the assumption of small initial data.

3.2 Uniqueness

Once again, as ε is fixed (the negative powers of ε won't cause any trouble), the computations are the same as for the compressible Navier-Stokes system. As in [3] or [19], we successively use the transport estimate for the density fluctuation ([3] theorem 3.14) and the transport-diffusion estimate (2.9) for the velocity ([3] Proposition 10.3). We need to separate the cases $d \geq 3$ and $d = 2$ (the case $d = 2$ is more difficult because of endpoints for the remainder estimates in the Littlewood-Paley paradecomposition) and we obtain the following result:

Theorem 5 *Let $d \geq 2$ and (q_i, u_i) $(i \in \{1,2\})$ two solutions of (RW_ε) with the same initial data on the same interval $[0, T^*[$ and both belonging to $E_\varepsilon^{\frac{d}{2}}(T^*)$. There exists $m > 0$ such that if, for $i \in \{1,2\}$*

$$\|q_i\|_{\widetilde{L}^\infty_{T^*}(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}})} \leq m$$

then $(q_1, u_1) = (q_2, u_2)$ on $[0, T^*[$.

4 Proof of Theorem 4

In this section we will show that the solution of (RW_ε) goes to the solution of (K) and give estimates of the speed of convergence as ε goes to zero. For that we will once more use Proposition 1.

As said in the introduction, if the initial data satisfy

$$\|q_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}}} + \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \eta \leq \min(\eta_K, \eta_R),$$

then systems (K) and (RW_ε) both have global solutions $(q_\varepsilon, u_\varepsilon)$ and (q, u), and with the same notations as before, there exists a constant $C = C(\eta, \kappa) > 0$ such that for all $t \in \mathbb{R}$ we have,

$$\|u_\varepsilon\|_{\widetilde{L}^\infty_t \dot{B}_{2,1}^{\frac{d}{2}-1}} + \|q_\varepsilon\|_{\widetilde{L}^\infty_t \dot{B}_{2,1}^{\frac{d}{2}-1}} + \|q_\varepsilon\|_{\widetilde{L}^\infty_t \dot{B}_{2,1}^{\frac{d}{2}}} + \|u_\varepsilon\|_{L^1_t \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q_\varepsilon\|_{L^1_t \dot{B}_\varepsilon^{\frac{d}{2}+1, \frac{d}{2}}} + \|q_\varepsilon\|_{L^1_t \dot{B}_\varepsilon^{\frac{d}{2}+2, \frac{d}{2}}} \leq 2C\eta, \quad (4.45)$$

and

$$\|u\|_{\widetilde{L}^\infty_t \dot{B}_{2,1}^{\frac{d}{2}-1}} + \|q\|_{\widetilde{L}^\infty_t \dot{B}_{2,1}^{\frac{d}{2}-1}} + \|q\|_{\widetilde{L}^\infty_t \dot{B}_{2,1}^{\frac{d}{2}}} + \|u\|_{L^1_t \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q\|_{L^1_t \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q\|_{L^1_t \dot{B}_{2,1}^{\frac{d}{2}+2}} \leq 2C\eta. \quad (4.46)$$

Up to an additional forcing term, let us rewrite system (K) with a capillary term as in system (RW_ε) (section 1.2.2):

$$(K) \quad \begin{cases} \partial_t q + u.\nabla q + (1+q)\operatorname{div} u = 0, \\ \partial_t u + u.\nabla u - \mathcal{A}u + P'(1).\nabla q - \dfrac{\kappa}{\varepsilon^2}\nabla(\phi_\varepsilon * q - q) = K(q).\nabla q - I(q)\mathcal{A}u + R_\varepsilon, \end{cases}$$

where the rest $R_\varepsilon \overset{def}{=} -\frac{\kappa}{\varepsilon^2}\nabla(\phi_\varepsilon * q - q - \varepsilon^2 \Delta q)$ and we recall that K and I are the following real-valued functions defined on $\mathbb{R}_+$ (whose value is 0 when $q = 0$):

$$K(q) = \left(P'(1) - \frac{P'(1+q)}{1+q}\right) \quad \text{and} \quad I(q) = \frac{q}{q+1}.$$

In order to estimate the new term R_ε we use the following lemma (which can be proved by a simple function study):

Lemma 2 *For all $\beta \in]1,2[$ there exists a constant $C_\beta > 0$ such that for all $x \geq 0$ we have:*

$$0 \leq \frac{e^{-x} - 1 + x}{x^\beta} \leq C_\beta$$

so that we immediately obtain:

Corollary 1 *For all $s \in \mathbb{R}$ and $\beta \in]1,2[$, if $q \in \dot{B}_{2,1}^{s+1+2\beta}$ then $R_\varepsilon \in \dot{B}_{2,1}^s$ and there exists a constant $C_\beta > 0$ such that we have*

$$\|R_\varepsilon\|_{\dot{B}_{2,1}^s} \leq \kappa C_\beta \varepsilon^{2(\beta-1)} \|q\|_{\dot{B}_{2,1}^{s+1+2\beta}}.$$

Remark 15 *In the following we will use this estimate for $\beta \in]1,2[$ and s such that $s + 2\beta \in [\frac{d}{2}, \frac{d}{2}+1]$. More precisely, we will take $s = \frac{d}{2} - \alpha - 1$ (with $\alpha \in]0,1]$), and the previous condition turns into $2\beta \in [1+\alpha, 2+\alpha]$.*

Let us now write the system satisfied by the difference $(\delta q, \delta u) = (q_\varepsilon - q, u_\varepsilon - u)$:

$$\begin{cases} \partial_t \delta q + u_\varepsilon.\nabla \delta q + \operatorname{div} \delta u = \delta F, \\ \partial_t \delta u + u_\varepsilon.\nabla \delta u - \mathcal{A}\delta u + P'(1).\nabla \delta q - \dfrac{\kappa}{\varepsilon^2}\nabla(\phi_\varepsilon * \delta q - \delta q) = \delta G - R_\varepsilon, \end{cases} \tag{4.47}$$

where

$$\begin{cases} \delta F \overset{def}{=} \sum_{i=1}^3 \delta F_i \\ \delta G \overset{def}{=} \sum_{i=1}^5 \delta G_i \end{cases} \quad \text{with} \quad \begin{cases} \delta F_1 = -\delta u.\nabla q \\ \delta F_2 = -\delta q.\operatorname{div} u_\varepsilon \\ \delta F_3 = -q.\operatorname{div} \delta u \end{cases} \quad \text{and} \quad \begin{cases} \delta G_1 = -\delta u.\nabla u \\ \delta G_2 = (K(q_\varepsilon) - K(q)).\nabla q_\varepsilon \\ \delta G_3 = K(q).\nabla \delta q \\ \delta G_4 = (I(q_\varepsilon) - I(q))\mathcal{A}u_\varepsilon \\ \delta G_5 = -I(q)\mathcal{A}\delta u. \end{cases}$$

We will estimate $(\delta q, \delta u)$ in less regular spaces. Let us show that for $\alpha < \min(1, d-1)$ (α will be precised later):

$$\begin{cases} \delta q \in \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} \cap \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha} \cap L_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+1, \frac{d}{2}-\alpha} \cap L_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+2, \frac{d}{2}-\alpha} \\ \delta u \in \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} \cap L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}. \end{cases}$$

Like previously, from estimates (4.46) and (4.45) we obtain (using Proposition 7 with $s = \frac{d}{2}$ for q_ε, and interpolation for q) that q_ε and q are uniformly bounded (with respect to t and ε) in $\widetilde{L}_t^2 \dot{B}_{2,1}^{\frac{d}{2}}$ and that for all $p \in [1, \infty]$, u_ε and u are uniformly bounded in $\widetilde{L}_t^p \dot{B}_{2,1}^{\frac{d}{2}+\frac{2}{p}-1}$.

1. Classically we use $\partial_t q_\varepsilon = -u_\varepsilon.\nabla q_\varepsilon - \operatorname{div} u_\varepsilon - q_\varepsilon \operatorname{div} u_\varepsilon$, and estimate the right-hand side:
   - Taking $p = \frac{2}{1-\alpha}$ we get $\operatorname{div} u_\varepsilon \in \widetilde{L}_t^{\frac{2}{1-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$.

- As $u_\varepsilon \in \widetilde{L}_t^{\frac{2}{1-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha}$ from (5.60) we get that

$$\begin{cases} \|q_\varepsilon \mathrm{div}\, u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|q_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\mathrm{div}\, u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \\ \|u_\varepsilon \nabla q_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|q_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}}. \end{cases}$$

So that $\partial_t q_\varepsilon$ is bounded (uniformly with respect to ε and t) in $L_t^{\frac{2}{1-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$ and then

$$q_\varepsilon \in \mathcal{C}_t^{\frac{1+\alpha}{2}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} \quad \text{and } q_\varepsilon - q_\varepsilon(0) = q_\varepsilon - q_0 \in L_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$$

uniformly in ε but not in t. As the same is true for q, we get for all $t \geq 0$

$$\delta q \in \mathcal{C}_t \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} \cap L_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}.$$

By interpolation, as $\delta q \in L_t^\infty \dot{B}_{2,1}^{\frac{d}{2}}$, we also have $\delta q \in \mathcal{C}_t \dot{B}_{2,1}^{\frac{d}{2}-\alpha} \cap L_t^\infty \dot{B}_{2,1}^{\frac{d}{2}}$ for all $t \geq 0$ and $\delta q \in L_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+1,\frac{d}{2}-\alpha} \cap L_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+2,\frac{d}{2}-\alpha}$ (at this stage we cannot get uniform bounds in t nor ε, and this is not a problem for fixed t and ε).

Remark 16 *As we had to estimate $\|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}}$, this requires $\frac{d}{2}-\alpha \in [\frac{d}{2}-1, \frac{d}{2}+1]$ that is $\alpha \in [-1,1]$.*

2. Similarly we use

$$\partial_t u_\varepsilon = -u_\varepsilon.\nabla u_\varepsilon + \mathcal{A}u_\varepsilon - P'(\overline{\rho}).\nabla q_\varepsilon + \frac{\kappa\overline{\rho}}{\varepsilon^2}\nabla(\phi_\varepsilon * q_\varepsilon - q_\varepsilon) + K(q_\varepsilon).\nabla q_\varepsilon - I(q_\varepsilon)\mathcal{A}u_\varepsilon,$$

- Taking $p = \frac{2}{2-\alpha}$ we get $\mathcal{A}u_\varepsilon \in \widetilde{L}_t^{\frac{2}{2-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$.

Remark 17 *As we estimate $\|\mathcal{A}u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}}$, this requires $\frac{d}{2}-\alpha+1 \in [\frac{d}{2}-1, \frac{d}{2}+1]$ that is $\alpha \in [0,2]$.*

- As $\nabla q_\varepsilon \in \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-2}$ we have $\nabla q_\varepsilon \in \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$.
- Thanks to (5.60) we prove that

$$\|u_\varepsilon.\nabla u_\varepsilon\|_{\widetilde{L}_t^{\frac{2}{2-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|u_\varepsilon\|_{\widetilde{L}_t^2 \dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u_\varepsilon\|_{\widetilde{L}_t^{\frac{2}{1-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}}$$

$$\|I(q_\varepsilon)\mathcal{A}u_\varepsilon\|_{\widetilde{L}_t^{\frac{2}{2-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C_\eta\|q_\varepsilon\|_{\widetilde{L}_t^2 \dot{B}_{2,1}^{\frac{d}{2}}} \|\mathcal{A}u_\varepsilon\|_{\widetilde{L}_t^{\frac{2}{2-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}}$$

$$\|K(q_\varepsilon)\nabla q_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|K(q_\varepsilon)\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}} \|\nabla q_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$$

  We obtain

$$\|K(q_\varepsilon)\nabla q_\varepsilon\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|K(q_\varepsilon)\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha}} \|\nabla q_\varepsilon\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-1}}.$$

Remark 18 *As we used Proposition 4 we need* $\frac{d}{2} - \alpha > 0$.

So $\partial_t u_\varepsilon$ is bounded in the space $\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} + \widetilde{L}_t^{\frac{2}{2-\alpha}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$ (uniformly in t but not in ε) and then

$$u_\varepsilon \in \mathcal{C}_t^{\frac{\alpha}{2}} \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} \quad \text{and } u_\varepsilon - u_\varepsilon(0) = u_\varepsilon - u_0 \in L_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}$$

With the same argument we used for δq, we obtain that $\delta u \in \widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1} \cap L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}$.

We can now use Proposition 1 with $s = \frac{d}{2} - \alpha$ (if $\alpha < \min(1, d-1)$, the following computations are true for the cases $d = 2$ and $d \geq 3$), we obtain:

$$\begin{aligned}
&\|\delta u\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} + \|\delta q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} + \|\delta q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha}} + \|\delta u\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}} \\
&\quad + \|\delta q\|_{\widetilde{L}_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+1, \frac{d}{2}-\alpha}} + \|\delta q\|_{\widetilde{L}_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+2, \frac{d}{2}-\alpha}} \leq C e^{C \int_0^t (\|\nabla u_\varepsilon(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|u_\varepsilon(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2) d\tau} \\
&\qquad \times \Big(\|\delta F\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} + \|\delta F\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha}} + \|\delta G\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} + \|R_\varepsilon\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \Big). \quad (4.48)
\end{aligned}$$

Thanks to (5.60) we estimate the external force terms:

- $\|\delta F_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} = \|\delta u.\nabla q\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.\|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}}$,
- $\|\delta F_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}} \leq C \left(\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}.\|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}} + \|q\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.\|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}} \right)$,
- $\|\delta F_2\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} = \|\delta q.\operatorname{div} u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|\delta q\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}$,
- $\|\delta F_2\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}} \leq C\|\delta q\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}} \|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}$,
- $\|\delta F_3\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} = \|q\operatorname{div} \delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}}$,
- $\|\delta F_3\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha}} \leq C\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}}$.

Recall that from Corollary 1 (see also Remark 15),

$$\|R_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq \kappa C_\beta \varepsilon^{2(\beta-1)} \|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+2\beta-\alpha}}.$$

And concerning the other forcing terms in the equation on δu:

- $\|\delta G_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} = \|\delta u.\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq C\|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}$,

- $\|\delta G_4\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} = \|\left(I(q_\varepsilon)-I(q)\right)\mathcal{A}u_\varepsilon\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} \leq C\|I(q_\varepsilon)-I(q)\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}}\|u_\varepsilon\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}$,

  and using Proposition 4, we can estimate the first term:

  $$\|I(q_\varepsilon)-I(q)\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}} \leq C(\|q_\varepsilon\|_{L^\infty},\|q\|_{L^\infty})\left(|I'(0)|+\|q_\varepsilon\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)\|q_\varepsilon-q\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}},$$

  so there exists C_η (bounded with respect to η) so that

  $$\|\delta G_4\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} \leq C_\eta\|\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}}\|u_\varepsilon\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}}.$$

- $\|\delta G_5\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} = \|I(q)\mathcal{A}\delta u\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} \leq C\|I(q)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\delta u\|_{\dot{B}^{\frac{d}{2}-\alpha+1}_{2,1}}$,

  and similarly, as $I(0)=0$, $\|I(q)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \leq C_0(\|q\|_{L^\infty})\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}}$ so that

  $$\|\delta G_5\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} \leq C_\eta\|q\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\|\delta u\|_{\dot{B}^{\frac{d}{2}-\alpha+1}_{2,1}}.$$

The last two terms require more attention: as we want to get global in time estimates, we cannot afford to loose any power of t so we need to use any available L^1_t-type estimates of q and q_ε.

- If $\alpha < d-1$ then we can use Proposition 8 with $s=\frac{d}{2}-\alpha$ and $t=\frac{d}{2}-1$ (and here $s+t>0$):

  $$\begin{aligned}\|\delta G_2\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} &= \|\left(K(q_\varepsilon)-K(q)\right).\nabla q_\varepsilon\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}}\\ &\leq C\left(\|K(q_\varepsilon)-K(q)\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}}+\|K(q_\varepsilon)-K(q)\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}}\right)\|\nabla q_\varepsilon\|_{\dot{B}^{\frac{d}{2},\frac{d}{2}-1}_{\varepsilon}}. \quad (4.49)\end{aligned}$$

  Moreover as $\alpha\in]0,d-1[$, we have $\frac{d}{2}-\alpha, \frac{d}{2}-\alpha-1\in]-\frac{d}{2},\frac{d}{2}]$, and we can use Proposition 4 with $K'(0)=P'(1)\neq 0$,

  $$\|K(q_\varepsilon)-K(q)\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}}+\|K(q_\varepsilon)-K(q)\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}} \leq C_\eta\left(\|\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}}+\|\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}}\right)$$

  and we obtain:

  $$\|\delta G_2\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} \leq C_\eta\left(\|\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}}+\|\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha}_{2,1}}\right)\|q_\varepsilon\|_{\dot{B}^{\frac{d}{2}+1,\frac{d}{2}}_{\varepsilon}}.$$

- Similarly, Proposition 8 with $s=\frac{d}{2}$ and $t=\frac{d}{2}-\alpha-1$ ($s+t>0$) implies:

  $$\begin{aligned}\|\delta G_3\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}} &= \|K(q).\nabla\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha-1}_{2,1}}\\ &\leq C\left(\|K(q)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|K(q)\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\right)\|\nabla\delta q\|_{\dot{B}^{\frac{d}{2}-\alpha,\frac{d}{2}-\alpha-1}_{\varepsilon}}. \quad (4.50)\end{aligned}$$

One more time, as $K(0)=0$, Proposition 4 allows to write:

$$\|\delta G_3\|_{\dot B^{\frac d2-\alpha-1}_{2,1}} \le C_\eta \left(\|q\|_{\dot B^{\frac d2-1}_{2,1}} + \|q\|_{\dot B^{\frac d2}_{2,1}}\right) \|\delta q\|_{\dot B_\varepsilon^{\frac d2-\alpha+1,\frac d2-\alpha}}.$$

Remark 19 *As we used Proposition 4 with $s=\frac d2-1$, the previous estimate is valid only when $d\ge 3$.*

In the case $d=2$, as $q\in \widetilde L^\infty_t \dot B^{\frac d2-1}_{2,1}\cap \widetilde L^1_t \dot B^{\frac d2+1}_{2,1}$, thanks to interpolation, we obtain that $q\in \widetilde L^2_t \dot B^{\frac d2}_{2,1}$.

Similarly, but instead of using interpolation we use Proposition 7, as $\delta q\in \widetilde L^\infty_t(\dot B^{\frac d2-\alpha-1}_{2,1}\cap \dot B^{\frac d2-\alpha}_{2,1})\cap \widetilde L^1_t \dot B_\varepsilon^{\frac d2-\alpha+1,\frac d2-\alpha}$, we obtain $\delta q\in \widetilde L^2_t \dot B^{\frac d2-\alpha}_{2,1}$. So

$$\|\delta G_3\|_{\dot B^{\frac d2-\alpha-1}_{2,1}} \le C\|K(q)\|_{\dot B^{\frac d2}_{2,1}} \|\nabla \delta q\|_{\dot B^{\frac d2-\alpha-1}_{2,1}}.$$

As $\|K(q)\|_{\dot B^{\frac d2}_{2,1}} \le C(\|q\|_{L^\infty})\|q\|_{\dot B^{\frac d2}_{2,1}} \le C_\eta \|q\|_{\dot B^{\frac d2}_{2,1}} \le C\|q\|_{\dot B^{\frac d2}_{2,1}}$, we obtain

$$\begin{aligned}
\|\delta G_3\|_{\dot B^{\frac d2-\alpha-1}_{2,1}} &\le C\|q\|^{\frac12}_{\dot B^{\frac d2-1}_{2,1}} \|q\|^{\frac12}_{\dot B^{\frac d2+1}_{2,1}} \left(\|\delta q\|_{\dot B^{\frac d2-\alpha}_{2,1}} + \|\delta q\|_{\dot B^{\frac d2-\alpha-1}_{2,1}}\right)^{\frac12} \|\delta q\|^{\frac12}_{\dot B_\varepsilon^{\frac d2-\alpha+1,\frac d2-\alpha}} \\
&\le \frac{C^2}{2}\|q\|_{\dot B^{\frac d2-1}_{2,1}} \|\delta q\|_{\dot B_\varepsilon^{\frac d2-\alpha+1,\frac d2-\alpha}} + \frac12 \|q\|_{\dot B^{\frac d2+1}_{2,1}} \left(\|\delta q\|_{\dot B^{\frac d2-\alpha}_{2,1}} + \|\delta q\|_{\dot B^{\frac d2-\alpha-1}_{2,1}}\right). \quad (4.51)
\end{aligned}$$

Remark 20 *As we used Proposition 4 we need the sum of indices s and t to be positive so $\alpha<d-1$.*

If we define

$$\begin{aligned}
h(t) \stackrel{def}{=} \|\delta u\|_{\widetilde L^\infty_t \dot B^{\frac d2-\alpha-1}_{2,1}} + \|\delta q\|_{\widetilde L^\infty_t \dot B^{\frac d2-\alpha-1}_{2,1}} + \|\delta q\|_{\widetilde L^\infty_t \dot B^{\frac d2-\alpha}_{2,1}} + \|\delta u\|_{\widetilde L^1_t \dot B^{\frac d2-\alpha+1}_{2,1}} \\
+ \|\delta q\|_{\widetilde L^1_t \dot B_\varepsilon^{\frac d2-\alpha+1,\frac d2-\alpha}} + \|\delta q\|_{\widetilde L^1_t \dot B_\varepsilon^{\frac d2-\alpha+2,\frac d2-\alpha}} \quad (4.52)
\end{aligned}$$

then collecting the previous estimates on the external terms in (4.48), there exists $C_\eta>0$ (bounded when $\eta\in[0,1]$) such that we have for all $t\ge 0$ (and in any case $d=2$ or $d\ge 3$),

$$\begin{aligned}
h(t) \le C_\eta e^{2C\eta+4C^2\eta^2}\Big(\int_0^t h(\tau)\big(\|q\|_{\dot B^{\frac d2+1}_{2,1}} + \|q\|_{\dot B^{\frac d2+2}_{2,1}} + \|u_\varepsilon\|_{\dot B^{\frac d2+1}_{2,1}} \\
+ \|u\|_{\dot B^{\frac d2+1}_{2,1}} + \|q_\varepsilon\|_{\dot B_\varepsilon^{\frac d2+1,\frac d2}}\big)d\tau + \kappa C_\beta \varepsilon^{2(\beta-1)}\|q\|_{\widetilde L^1_t \dot B^{\frac d2+2\beta-\alpha}_{2,1}} \\
+ \big(\|q\|_{\widetilde L^\infty_t \dot B^{\frac d2-1}_{2,1}} + \|q\|_{\widetilde L^\infty_t \dot B^{\frac d2}_{2,1}}\big).\big(\|\delta q\|_{\widetilde L^1_t \dot B_\varepsilon^{\frac d2-\alpha+1,\frac d2-\alpha}} + \|\delta u\|_{\widetilde L^1_t \dot B^{\frac d2+1-\alpha}_{2,1}}\big)\Big). \quad (4.53)
\end{aligned}$$

Thanks to (4.45) and (4.46), this estimate turns into:

$$h(t) \leq C_\eta \Bigg(\int_0^t h(\tau) \Big(\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q_\varepsilon\|_{\dot{B}_\varepsilon^{\frac{d}{2}+1,\frac{d}{2}}} \Big) d\tau \\ + \kappa C_\beta \varepsilon^{2(\beta-1)} \|q\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}+2\beta-\alpha}} + 4\eta h(t) \Bigg). \quad (4.54)$$

The term $\|q\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}+2\beta-\alpha}}$ can be estimated by interpolation (thanks again to (4.46)) if $1 \leq 2\beta - \alpha \leq 2$ and in this case it is less than $2C.\eta$. Let us collect all the conditions on α (see remarks 16 to 20) and β:

$$\begin{cases} \alpha \in]0,1], & \alpha < \frac{d}{2}, \quad \alpha < d-1, \\ \beta \in]1,2[, & \frac{1}{2} + \frac{\alpha}{2} \leq \beta \leq 1 + \frac{\alpha}{2}. \end{cases}$$

This obviously implies that $d \geq 2$, and if α is fixed in $]0,1[$ (when $d=2$) or in $]0,1]$ (when $d \geq 3$), and if we choose $\beta = 1 + \frac{\alpha}{2}$ all the conditions are satisfied. Moreover if $\eta > 0$ is such that $4\eta.C_\eta \leq \frac{1}{2}$ then the last term in 4.54 can be absorbed by the left-hand side, and tanks to the Gronwall lemma, for all $t \geq 0$, we have

$$h(t) \leq C_{\eta,\alpha} \kappa \varepsilon^\alpha e^{\int_0^t \left(\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q_\varepsilon\|_{\dot{B}_\varepsilon^{\frac{d}{2}+1,\frac{d}{2}}} \right) d\tau} \quad (4.55)$$

And finally, thanks to (4.46) and (4.45), we obtain that for all $t \in \mathbb{R}$,

$$\|\delta u\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} + \|\delta q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} + \|\delta q\|_{\widetilde{L}_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha}} + \|\delta u\|_{\widetilde{L}_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha+1}} \\ + \|\delta q\|_{\widetilde{L}_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+1,\frac{d}{2}-\alpha}} + \|\delta q\|_{\widetilde{L}_t^1 \dot{B}_\varepsilon^{\frac{d}{2}-\alpha+2,\frac{d}{2}-\alpha}} \leq C_{\eta,\alpha} \kappa \varepsilon^\alpha. \quad (4.56)$$

which ends the proof of the theorem. ■

Remark 21 *If we had not estimated this way the terms δG_2 and δG_3, we would have to write that $\delta G_2 + \delta G_3 = \nabla(L(q_\varepsilon) - L(q))$ and then due to the regularity of δq, we could only obtain that $\|\delta G_2 + \delta G_2\|_{L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-\alpha-1}} \leq tC_\eta \|\delta q\|_{L_t^\infty \dot{B}_{2,1}^{\frac{d}{2}-\alpha}}$, which prevents any global in time estimate.*

Remark 22 *Estimate (4.56) is stronger than the estimate stated in the theorem.*

Remark 23 *The case $\alpha = 0$ is interesting. Indeed, estimate (4.55) turns into:*

$$h(t) \leq C_\eta \kappa \|R_\varepsilon\|_{L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-1}} e^{\int_0^t \left(\|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|q_\varepsilon\|_{\dot{B}_\varepsilon^{\frac{d}{2}+1,\frac{d}{2}}} \right) d\tau}$$

We easily check that $\|R_\varepsilon\|_{L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \kappa \sum_{j \in \mathbb{Z}} 2^{j\frac{d}{2}} \frac{e^{-\varepsilon^2 C^2 2^{2j}} - 1 + \varepsilon^2 C^2 2^{2j}}{\varepsilon^2} \|\dot{\Delta}_j q\|_{L_t^1 L^2}$. As $q \in L_t^1 \dot{B}_{2,1}^{\frac{d}{2}+2}$ we can prove, thanks to the Lebesgue dominated convergence theorem for series, that $\|R_\varepsilon\|_{L_t^1 \dot{B}_{2,1}^{\frac{d}{2}-1}}$ goes to zero as ε goes to zero. And then $h(t)$ also goes to zero, but the estimate does not provide any information on the speed of convergence.

5 Appendix

The first part is devoted to a quick presentation of the Littlewood-Paley theory, and the second to specific properties for hybrid Besov norms used in this paper.

5.1 Besov spaces

5.1.1 Littlewood-Paley theory

As usual, the Fourier transform of u with respect to the space variable will be denoted by $\mathcal{F}(u)$ or $\widehat{u}$. In this section we will state classical definitions and properties concerning the homogeneous dyadic decomposition with respect to the Fourier variable. We will recall some classical results and we refer to [3] (Chapter 2) for proofs (and more general properties).

To build the Littlewood-Paley decomposition, we need to fix a smooth radial function χ supported in (for example) the ball $B(0, \frac{4}{3})$, equal to 1 in a neighborhood of $B(0, \frac{3}{4})$ and such that $r \mapsto \chi(r.e_r)$ is nonincreasing over $\mathbb{R}_+$. So that if we define $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$, then φ is compactly supported in the annulus $\{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and we have that,

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1. \tag{5.57}$$

Then we can define the *dyadic blocks* $(\dot{\Delta}_l)_{l \in \mathbb{Z}}$ by $\dot{\Delta}_l := \varphi(2^{-l}D)$ (that is $\widehat{\dot{\Delta}_l u} = \varphi(2^{-l}\xi)\widehat{u}(\xi)$) so that, formally, we have

$$u = \sum_l \dot{\Delta}_l u \tag{5.58}$$

As (5.57) is satisfied for $\xi \neq 0$, the previous formal equality holds true for tempered distributions *modulo polynomials.* A way to avoid working modulo polynomials is to consider the set $\mathcal{S}'_h$ of tempered distributions u such that

$$\lim_{l \to -\infty} \|\dot{S}_l u\|_{L^\infty} = 0,$$

where $\dot{S}_l$ stands for the low frequency cut-off defined by $\dot{S}_l := \chi(2^{-l}D)$. If $u \in \mathcal{S}'_h$, (5.58) is true and we can write that $\dot{S}_l u = \sum_{k \leq l-1} \dot{\Delta}_q u$. We can now define the homogeneous Besov spaces used in this article:

Definition 3 *For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we set*

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_l 2^{rls} \|\Delta_l u\|^r_{L^p}\right)^{\frac{1}{r}} \quad \text{if} \quad r < \infty \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,\infty}} := \sup_l 2^{ls} \|\Delta_l u\|_{L^p}.$$

We then define the space $\dot{B}^s_{p,r}$ as the subset of distributions $u \in \mathcal{S}'_h$ such that $\|u\|_{\dot{B}^s_{p,r}}$ is finite.

Once more, we refer to [3] (chapter 2) for properties of the inhomogeneous and homogeneous Besov spaces. Among these properties, let us mention:

- for any $p \in [1, \infty]$ we have the following chain of continuous embeddings:

$$\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty};$$

- if $p < \infty$ then $B^{\frac{d}{p}}_{p,1}$ is an algebra continuously embedded in the set of continuous functions decaying to 0 at infinity;

- for any smooth homogeneous of degree m function F on $\mathbb{R}^d \setminus \{0\}$ the operator $F(D)$ maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-m}_{p,r}$. This implies that the gradient operator maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-1}_{p,r}$.

We refer to [3] (lemma 2.1) for the Bernstein lemma (describing how derivatives act on spectrally localized functions), that entails the following embedding result:

Proposition 3 *For all $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in the space $\dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*

In this paper, we shall mainly work with functions or distributions depending on both the time variable t and the space variable x. We shall denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X. For $p \in [1, \infty]$, the notation $L^p(I; X)$ stands for the set of measurable functions on I with values in X such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$.

In the case where $I = [0, T]$, the space $L^p([0, T]; X)$ (resp. $\mathcal{C}([0, T]; X)$) will also be denoted by $L^p_T X$ (resp. $\mathcal{C}_T X$). Finally, if $I = \mathbb{R}^+$ we shall alternately use the notation $L^p X$.

The Littlewood-Paley decomposition enables us to work with spectrally localized (hence smooth) functions rather than with rough objects. We naturally obtain bounds for each dyadic block in spaces of type $L^\rho_T L^p$. Going from those type of bounds to estimates in $L^\rho_T \dot{B}^s_{p,r}$ requires to perform a summation in $\ell^r(\mathbb{Z})$. When doing so however, we *do not* bound the $L^\rho_T \dot{B}^s_{p,r}$ norm for the time integration has been performed *before* the ℓ^r summation. This leads to the following notation (after J.-Y. Chemin and N. Lerner in [10]):

Definition 4 *For $T > 0$, $s \in \mathbb{R}$ and $1 \leq r, \rho \leq \infty$, we set*

$$\|u\|_{\widetilde{L}^\rho_T \dot{B}^s_{p,r}} := \big\| 2^{js} \|\dot{\Delta}_q u\|_{L^\rho_T L^p} \big\|_{\ell^r(\mathbb{Z})}.$$

One can then define the space $\widetilde{L}^\rho_T \dot{B}^s_{p,r}$ as the set of tempered distributions u over $(0, T) \times \mathbb{R}^d$ such that $\lim_{q \to -\infty} \dot{S}_q u = 0$ in $L^\rho([0, T]; L^\infty(\mathbb{R}^d))$ and $\|u\|_{\widetilde{L}^\rho_T \dot{B}^s_{p,r}} < \infty$. The letter T is omitted for functions defined over $\mathbb{R}^+$. The spaces $\widetilde{L}^\rho_T \dot{B}^s_{p,r}$ may be compared with the spaces $L^\rho_T \dot{B}^s_{p,r}$ through the Minkowski inequality: we have

$$\|u\|_{\widetilde{L}^\rho_T \dot{B}^s_{p,r}} \leq \|u\|_{L^\rho_T \dot{B}^s_{p,r}} \quad \text{if} \quad r \geq \rho \quad \text{and} \quad \|u\|_{\widetilde{L}^\rho_T \dot{B}^s_{p,r}} \geq \|u\|_{L^\rho_T \dot{B}^s_{p,r}} \quad \text{if} \quad r \leq \rho.$$

All the properties of continuity for the product and composition which are true in Besov spaces remain true in the above spaces. The time exponent just behaves according to Hölder's inequality.

Let us now recall a few nonlinear estimates in Besov spaces. Formally, any product of two distributions u and v may be decomposed into

$$uv = T_u v + T_v u + R(u,v), \text{ where} \tag{5.59}$$

$$T_u v := \sum_l \dot{S}_{l-1} u \dot{\Delta}_l v, \quad T_v u := \sum_l \dot{S}_{l-1} v \dot{\Delta}_l u \text{ and } R(u,v) := \sum_l \sum_{|l'-l|\leq 1} \dot{\Delta}_l u \, \dot{\Delta}_{l'} v.$$

The above operator T is called "paraproduct" whereas R is called "remainder". The decomposition (5.59) has been introduced by J.-M. Bony in [8].

In this article we will frequently use the following estimates (we refer to [3] section 2.6, [12], [19] for general statements, more properties of continuity for the paraproduct and remainder operators, sometimes adapted to $\widetilde{L}_T^\rho \dot{B}_{p,r}^s$ spaces): under the same assumptions there exists a constant $C > 0$ such that:

$$\|\dot{T}_u v\|_{\dot{B}_{2,1}^s} \leq C\|u\|_{L^\infty} \|v\|_{\dot{B}_{2,1}^s} \leq C\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|v\|_{\dot{B}_{2,1}^s},$$

$$\|\dot{T}_u v\|_{\dot{B}_{2,1}^{s+t}} \leq C\|u\|_{\dot{B}_{\infty,\infty}^t} \|v\|_{\dot{B}_{2,1}^s} \leq C\|u\|_{\dot{B}_{2,1}^{t+\frac{d}{2}}} \|v\|_{\dot{B}_{2,1}^s} \quad (t<0),$$

$$\|\dot{R}(u,v)\|_{\dot{B}_{2,1}^{s_1+s_2}} \leq C\|u\|_{\dot{B}_{\infty,\infty}^{s_1}} \|v\|_{\dot{B}_{2,1}^{s_2}} \leq C\|u\|_{\dot{B}_{2,1}^{s_1+\frac{d}{2}}} \|v\|_{\dot{B}_{2,1}^{s_2}} \quad (s_1+s_2>0),$$

$$\|\dot{R}(u,v)\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{d}{2}}} \leq C\|u\|_{\dot{B}_{2,1}^{s_1}} \|v\|_{\dot{B}_{2,\infty}^{s_2}} \leq C\|u\|_{\dot{B}_{2,1}^{s_1}} \|v\|_{\dot{B}_{2,1}^{s_2}} \quad (s_1+s_2>0). \tag{5.60}$$

Let us now turn to the composition estimates. We refer for example to [3] (Theorem 2.59, corollary 2.63)):

Proposition 4 *1. Let $s>0$, $u \in \dot{B}_{2,1}^s \cap L^\infty$ and $F \in W_{loc}^{[s]+2,\infty}(\mathbb{R}^d)$ such that $F(0)=0$. Then $F(u) \in \dot{B}_{2,1}^s$ and there exists a function of one variable C_0 only depending on s, d and F such that*

$$\|F(u)\|_{\dot{B}_{2,1}^s} \leq C_0(\|u\|_{L^\infty}) \|u\|_{\dot{B}_{2,1}^s}.$$

2. If u and $v \in \dot{B}_{2,1}^{\frac{d}{2}}$ and if $v-u \in \dot{B}_{2,1}^s$ for $s \in]-\frac{d}{2}, \frac{d}{2}]$ and $G \in W_{loc}^{[s]+3,\infty}(\mathbb{R}^d)$, then $G(v)-G(u)$ belongs to $\dot{B}_{2,1}^s$ and there exists a function of two variables C only depending on s, d and G such that

$$\|G(v)-G(u)\|_{\dot{B}_{2,1}^s} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) \left(|G'(0)| + \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \|v-u\|_{\dot{B}_{2,1}^s}.$$

5.1.2 Hybrid Besov spaces

As explained, in the compressible Navier-Stokes system, the density fluctuation has two distinct behaviours in some low and high frequencies, separated by a frequency threshold. This leads to the definition of the hybrid Besov spaces. Let us begin with the spaces that are introduced by R. Danchin in [12] or [3] (we will use these spaces to prove estimates with the Hybrid norms introduced in (1.6)):

Definition 5 *For $\alpha > 0$, $r \in [0, \infty]$ and $s \in \mathbb{R}$ we denote*

$$\|u\|_{\widetilde{B}_\alpha^{s,r}} \stackrel{def}{=} \sum_{l \in \mathbb{Z}} 2^{ls} \max(\alpha, 2^{-l})^{1-\frac{2}{r}} \|\dot{\Delta}_l u\|_{L^2}$$

In the present paper, we will only use these norms with $r \in \{1, \infty\}$:

$$\|u\|_{\widetilde{B}_\alpha^{s,\infty}} = \sum_{l \le \log_2(\frac{1}{\alpha})} 2^{l(s-1)} \|\dot{\Delta}_l u\|_{L^2} + \sum_{l > \log_2(\frac{1}{\alpha})} \alpha 2^{ls} \|\dot{\Delta}_l u\|_{L^2}, \text{ and}$$

$$\|u\|_{\widetilde{B}_\alpha^{s,1}} = \sum_{l \le \log_2(\frac{1}{\alpha})} 2^{l(s+1)} \|\dot{\Delta}_l u\|_{L^2} + \sum_{l > \log_2(\frac{1}{\alpha})} \frac{1}{\alpha} 2^{ls} \|\dot{\Delta}_l u\|_{L^2},$$

Remark 24 *As stated in [3] we have the equivalence*

$$\frac{1}{2} \left(\|u\|_{\dot{B}_{2,1}^{s-1}} + \alpha \|u\|_{\dot{B}_{2,1}^{s}} \right) \le \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \le \|u\|_{\dot{B}_{2,1}^{s-1}} + \alpha \|u\|_{\dot{B}_{2,1}^{s}}.$$

We refer to definition 1.6 for the precise expression of the hybrid norms used in this article. Let us just mention that this particuliar hybrid norm is accurate for our problem, but it is also related to the hybrid norms introduced by R. Danchin:

Remark 25 *If ε is small enough, we have $\|u\|_{\widetilde{B}_1^{s,1}} \le \|u\|_{\widetilde{B}_\varepsilon^{s,1}} \le \|u\|_{\dot{B}_\varepsilon^{s+1,s}}$*

5.2 Results on hybrid norms

In this section we state and prove some estimates involving the hybrid Besov spaces introduced in the beginning of the paper. These estimates are very close to the ones proved in [12].

Proposition 5 *Let $s \in \mathbb{R}$, $\alpha > 0$. For all $q \in \widetilde{B}_\alpha^{s,\infty} \cap \widetilde{B}_\alpha^{s,1}$, we have*

$$\|q\|_{\dot{B}_{2,1}^{s}}^2 \le \|q\|_{\widetilde{B}_\alpha^{s,\infty}} \|q\|_{\widetilde{B}_\alpha^{s,1}}$$

Proof : This proof is classical (see [12] appendix). We have:

$$\begin{aligned}
\|q\|_{\dot{B}_{2,1}^{s}}^2 &= \sum_{l \in \mathbb{Z}} 2^{ls} \|\dot{\Delta}_l q\|_{L^2} \\
&= \sum_{l \in \mathbb{Z}} (2^{ls} \max(\alpha, 2^{-l}) \|\dot{\Delta}_l q\|_{L^2})^{\frac{1}{2}} (\frac{2^{ls}}{\max(\alpha, 2^{-l})} \|\dot{\Delta}_l q\|_{L^2})^{\frac{1}{2}} \\
&\le \left(\sum_{l \in \mathbb{Z}} 2^{ls} \max(\alpha, 2^{-l}) \|\dot{\Delta}_l q\|_{L^2} \right)^{\frac{1}{2}} \left(\sum_{l \in \mathbb{Z}} \frac{2^{ls}}{\max(\alpha, 2^{-l})} \|\dot{\Delta}_l q\|_{L^2} \right)^{\frac{1}{2}} \\
&\le \|q\|_{\widetilde{B}_\alpha^{s,\infty}} \|q\|_{\widetilde{B}_\alpha^{s,1}}. \blacksquare \qquad (5.61)
\end{aligned}$$

Proposition 6 *Let $s \le d/2$, $t \le d/2 - 1$ such that $s + t > 0$. There exists $C > 0$ such that for all $(u, v) \in \widetilde{B}_\alpha^{s,\infty} \times \widetilde{B}_\alpha^{t,1}$,*

$$\|uv\|_{\dot{B}_{2,1}^{s+t-\frac{d}{2}}} \le C \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \|v\|_{\widetilde{B}_\alpha^{t,1}}.$$

Proof : Using the Bony decomposition (see 5.59), we have $uv = T_u v + T_v u + R(u,v)$ and we will separately estimate these terms in $\dot{B}^{s+t-\frac{d}{2}}_{2,1}$ thanks to the following lemma (see [12] Proposition 5.3.)

Lemma 3 *Let $\alpha > 0$, $a, b \in \mathbb{R}$. Then we have*

$$\frac{\max(\alpha, 2^{-a})}{\max(\alpha, 2^{-b})} \leq \begin{cases} 1 & \text{if } \; a \geq b, \\ 2^{b-a} & \text{if } \; a \leq b. \end{cases}$$

1- Let us begin with $T_u v = \sum_{q\in\mathbb{Z}} S_{q-1}u.\Delta_q v$. As for all $q \in \mathbb{Z}$, $S_{q-1}u.\Delta_q v$ has its frequencies localized in $2^q\mathcal{C}'$ (where $\mathcal{C}'$ is a ring) we will bound the norm by estimating $2^{q(s+t-\frac{d}{2})}\|S_{q-1}u.\Delta_q v\|_{L^2}$. Using the Bernstein lemma,

$$\begin{aligned} 2^{q(s+t-\frac{d}{2})}\|S_{q-1}u.\Delta_q v\|_{L^2} &\leq 2^{q(s+t-\frac{d}{2})}\left(\sum_{q'\leq q-2}\|\Delta_{q'}u\|_{L^\infty}\right).\|\Delta_q v\|_{L^2} \\ &\leq 2^{q(s+t-\frac{d}{2})}\left(\sum_{q'\leq q-2}2^{q'\frac{d}{2}}\|\Delta_{q'}u\|_{L^2}\right).\|\Delta_q v\|_{L^2} \quad (5.62)\end{aligned}$$

As $(u,v) \in \widetilde{B}^{s,\infty}_\alpha \times \widetilde{B}^{t,1}_\alpha$, there exist two nonnegative sequences $c, c' \in l^1(\mathbb{Z})$ such that $\|c\|_{l^1} \leq 1$, $\|c'\|_{l^1} \leq 1$ and for all $j \in \mathbb{Z}$,

$$\|\Delta_j u\|_{L^2} \leq \frac{2^{-js}c_j}{\max(\alpha, 2^{-j})}\|u\|_{\widetilde{B}^{s,\infty}_\alpha} \quad \text{and} \quad \|\Delta_j v\|_{L^2} \leq 2^{-jt}c'_j \max(\alpha, 2^{-j})\|v\|_{\widetilde{B}^{s,1}_\alpha}, \quad (5.63)$$

so we can write that

$$2^{q(s+t-\frac{d}{2})}\|S_{q-1}u.\Delta_q v\|_{L^2} \leq \sum_{q'\leq q-2} 2^{(q-q')(s-\frac{d}{2})}c_{q'}c'_q\frac{\max(\alpha, 2^{-q})}{\max(\alpha, 2^{-q'})}\|u\|_{\widetilde{B}^{s,\infty}_\alpha}\|v\|_{\widetilde{B}^{s,1}_\alpha}.$$

Thanks to the previous lemma, as $q' \leq q-2$ we obtain

$$2^{q(s+t-\frac{d}{2})}\|S_{q-1}u.\Delta_q v\|_{L^2} \leq \left(\sum_{q'\leq q-2} 2^{(q-q')(s-\frac{d}{2})}c_{q'}\right)c'_q\|u\|_{\widetilde{B}^{s,\infty}_\alpha}\|v\|_{\widetilde{B}^{s,1}_\alpha}.$$

If we denote

$$a_q = \begin{cases} 2^{q(s-\frac{d}{2})} & \text{if } q \geq 2 \\ 0 & \text{if } q < 2, \end{cases}$$

then $\left(\sum_{q'\leq q-2} 2^{(q-q')(s-\frac{d}{2})}c_{q'}\right) = (a*c)_q$, and as $c, c' \in l^1(\mathbb{Z})$, $a \in l^\infty(\mathbb{Z})$ if and only if $s \leq \frac{d}{2}$ then $(a*c).c' \in l^1(\mathbb{Z})$ when $s \leq \frac{d}{2}$ and

$$\|T_u v\|_{\dot{B}^{s+t-\frac{d}{2}}_{2,1}} \leq C\|u\|_{\widetilde{B}^{s,\infty}_\alpha}\|v\|_{\widetilde{B}^{s,1}_\alpha}.$$

2- Let us now turn to $T_v u$. The same arguments as previously give that for all $q \in \mathbb{Z}$,

$$2^{q(s+t-\frac{d}{2})}\|S_{q-1}v.\Delta_q u\|_{L^2} \leq \sum_{q' \leq q-2} 2^{(q-q')(t-\frac{d}{2})} c_{q'} c'_q \frac{\max(\alpha, 2^{-q'})}{\max(\alpha, 2^{-q})} \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \|v\|_{\widetilde{B}_\alpha^{s,1}}.$$

This time, as $q' \leq q-2$ thanks to the previous lemma, $\frac{\max(\alpha,2^{-q'})}{\max(\alpha,2^{-q})} \leq 2^{q-q'}$ and

$$2^{q(s+t-\frac{d}{2})}\|S_{q-1}u.\Delta_q v\|_{L^2} \leq \left(\sum_{q' \leq q-2} 2^{(q-q')(t-\frac{d}{2}+1)} c_{q'}\right) c'_q \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \|v\|_{\widetilde{B}_\alpha^{s,1}}.$$

Similarly, if we denote

$$b_q = \begin{cases} 2^{q(t-\frac{d}{2}+1)} & \text{if } q \geq 2 \\ 0 & \text{if } q < 2, \end{cases}$$

Then if $t \leq \frac{d}{2} - 1$ we get that $\|(b * c).c'\|_{l^1(\mathbb{Z})} \leq \|b\|_{l^\infty(\mathbb{Z})} \|c\|_{l^1(\mathbb{Z})} \|c'\|_{l^1(\mathbb{Z})}$ and

$$\|T_v u\|_{\dot{B}_{2,1}^{s+t-\frac{d}{2}}} \leq C \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \|v\|_{\widetilde{B}_\alpha^{s,1}}.$$

3- Finally, let us look at the remainder $R(u,v) = \Delta_q u(\Delta_{q-1}v + \Delta_q v + \Delta_{q+1}v)$. Here when $q \in \mathbb{Z}$ the frequencies are only in the ball $2^q\mathcal{B}$ and we estimate, for all j, $2^{j(s+t-\frac{d}{2})}\|\Delta_j R(u,v)\|_{L^2}$. If we perform the same computations as before we will end with the condition $s+t > \frac{d}{2}$. We can get $s+t > 0$ if we first estimate the L^2-norm by the L^1-norm (using the spectral localization) and then use the Hölder estimate. Thanks to the Bernstein lemma:

$$\begin{aligned} 2^{j(s+t-\frac{d}{2})}\|\Delta_j R(u,v)\|_{L^2} &\leq 2^{j(s+t-\frac{d}{2})} 2^{j\frac{d}{2}} \|\Delta_j R(u,v)\|_{L^1} \\ &\leq 2^{j(s+t)} \sum_{q \geq j-N_0} \|\Delta_q u\|_{L^2} (\|\Delta_{q-1}v\|_{L^2} + \|\Delta_q v\|_{L^2} + \|\Delta_{q+1}v\|_{L^2}). \end{aligned} \quad (5.64)$$

As $(u,v) \in \widetilde{B}_\alpha^{s,\infty} \times \widetilde{B}_\alpha^{t,1}$, we can use (5.63) and get:

$$\begin{aligned} 2^{j(s+t-\frac{d}{2})}\|\Delta_j R(u,v)\|_{L^2} &\leq 2^{j(s+t)} \sum_{q \geq j-N_0} \frac{2^{-qs} c_q}{\max(\alpha, 2^{-q})} \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \times 2^{-qt} \|v\|_{\widetilde{B}_\alpha^{s,1}} \\ &\times \left(2^t c'_{q-1} \max(\alpha, 2^{-(q-1)}) + c'_q \max(\alpha, 2^{-q}) + 2^{-t} c'_{q+1} \max(\alpha, 2^{-(q+1)})\right). \end{aligned} \quad (5.65)$$

Thanks again to lemma 3 and (5.63), we obtain:

$$\begin{aligned} 2^{j(s+t-\frac{d}{2})}&\|\Delta_j R(u,v)\|_{L^2} \\ &\leq (1 + 2^t + 2^{-t}) \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \|v\|_{\widetilde{B}_\alpha^{s,1}} \sum_{q \geq j-N_0} 2^{(j-q)(s+t)} c_q (c'_{q-1} + c'_q + c'_{q+1}) \\ &\leq (1 + 2^t + 2^{-t}) \|u\|_{\widetilde{B}_\alpha^{s,\infty}} \|v\|_{\widetilde{B}_\alpha^{s,1}} \sum_{q \geq j-N_0} 2^{(j-q)(s+t)} c_q. \end{aligned} \quad (5.66)$$

Once again, if we denote

$$d_q = \begin{cases} 2^{j(s+t)} & \text{if } j \leq N_0 \\ 0 & \text{else,} \end{cases}$$

As we want $(d * c) \in l^1(\mathbb{Z})$ we now need that $d \in l^1(\mathbb{Z})$ ($d \in l^\infty(\mathbb{Z})$ is not enough as we only could estimate $c'_{q-1} + c'_q + c'_{q+1} \leq 1$ in L^∞ instead of L^1) which is true if and only if $s + t > 0$. So when $s + t > 0$ we finally get:

$$\|R(u,v)\|_{\dot{B}_{2,1}^{s+t-\frac{d}{2}}} \leq C_{s,t}\|u\|_{\widetilde{B}_\alpha^{s,\infty}}\|v\|_{\widetilde{B}_\alpha^{s,1}}.$$

which ends the proof of the proposition. ■

Remark 26 *For all* $q \in \dot{B}_{2,1}^{s-1} \cap \dot{B}_{2,1}^s = \widetilde{B}_1^{s,\infty}$ *we have*

$$\|q\|_{\widetilde{B}_1^{s,\infty}} \leq \|q\|_{\dot{B}_{2,1}^{s-1}} + \|q\|_{\dot{B}_{2,1}^s}$$

and thanks to remark 25, when $\varepsilon > 0$ *is small enough, for all* $q \in \dot{B}_\varepsilon^{s+1,s}$*, we have*

$$\|q\|_{\widetilde{B}_1^{s,1}} \leq \|q\|_{\dot{B}_\varepsilon^{s+1,s}},$$

so we can use the hybrid norms introduced in (1.6) and we will in fact use the following results:

Proposition 7 *Let* $s \in \mathbb{R}$*. There exists a constant* $C > 0$ *such that for all* $\varepsilon > 0$*, and all* $q \in \dot{B}_{2,1}^{s-1} \cap \dot{B}_{2,1}^s \cap \dot{B}_\varepsilon^{s+1,s}$*, we have*

$$\|q\|_{\dot{B}_{2,1}^s}^2 \leq C(\|q\|_{\dot{B}_{2,1}^{s-1}} + \|q\|_{\dot{B}_{2,1}^s})\|q\|_{\dot{B}_\varepsilon^{s+1,s}}$$

Proposition 8 *Let* $s \leq d/2$*,* $t \leq d/2 - 1$ *such that* $s + t > 0$*. There exists* $C > 0$ *such that for all* $\varepsilon > 0$*, and all* $(u,v) \in (\dot{B}_{2,1}^{s-1} \cap \dot{B}_{2,1}^s) \times \dot{B}_\varepsilon^{t+1,t}$*,*

$$\|uv\|_{\dot{B}_{2,1}^{s+t-\frac{d}{2}}} \leq C(\|u\|_{\dot{B}_{2,1}^{s-1}} + \|u\|_{\dot{B}_{2,1}^s})\|v\|_{\dot{B}_\varepsilon^{t+1,t}}.$$

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