Global stability of weak solutions for a multilayer Saint-Venant model with interactions between the layers

Bernard Di Martino, Boris Haspot, Yohan Penel

Global stability of weak solutions for a multilayer Saint-Venant model with interactions between the layers

Bernard Di Martino[*‡] Boris Haspot[§] Yohan Penel[†‡]

December 2, 2016

Abstract

In this paper we investigate the existence of global weak solutions for the multilayer model introduced by Audusse *et al.* [3] which is related to incompressible free surface flows. We prove the global stability of weak solutions over the torus. We observe that this model admits the so called BD-entropy and a gain of integrability on the velocity in the spirit of the work of Mellet and Vasseur [27]. It allows us to obtain enough compactness estimates in order to show the stability of global weak solutions.

1 Introduction

The issue of modelling and simulating free-surface flows is extensively addressed in the literature. It is of major interest for a large amount of engineering applications such as the design of harbours, the protection of coasts, the production of energy or the prevention of natural hazards. Depending on the wavelengths of hydrodynamic processes at stake, several models of reduced complexity have been designed.

A renowned simplified model implemented in many industrial codes is the system of viscous Shallow Water (SW) equations [16, 25] which consists of a hyperbolic 1^{st} order partial differential equation (PDE) modelling the conservation of volume and of a 2^{nd} order PDE for the momentum. The SW equations are dedicated to a specific regime of water flows, namely when dispersion effects can be neglected and for water heights small compared to the characteristic longitudinal length of the domain. For such flows, the SW equations turn out to provide reliable numerical results.

From SW equations to the Navier-Stokes (NS) equations, there exists in the literature a hierarchy of models of increasing complexity including Boussinesq type models [17, 24, 31–33] with higher order derivatives to account for dispersion effects (necessary for modelling shoaling) or non-hydrostatic models [9, 11] with a larger amount of unknowns (like the hydrodynamic pressure). This process aims at widening the range of applications of hydrodynamic models.

For the specific regime addressed by the SW equations, another technique consists in splitting the flow into horizontal layers similar to a discretisation procedure along the vertical axis in order to improve the accuracy of the results. In this framework, the SW equations correspond to a coarse vertical mesh with a single layer. As a consequence, this multilayer approach is still relevant for non shallow flows.

First, such models have been introduced with 2 or 3 layers for immiscible multifluid flows [10, 28, 29]. They were then extended to an arbitrary number of layers without [2, 4] or with [3, 14, 15, 23] mass transfer between layers. A major consequence is the noticeable increase of the number of unknowns related to the number of layers. In the inviscid case, open questions like the hyperbolicity of the model still hold (let us mention that recently Aguillon *et al.* in [1] proved the well-posedness of the Riemann problem for a two layer model). In the present work, we focus on the viscous case. We

[*]Université de Corse, SPE, UMR CNRS 6134, Campus Grimaldi, BP 52, 20250 Corte, France.
[†]ANGE project-team (Inria, Cerema, UPMC, CNRS), 2 rue Simone Iff, CS 42112, 75589 Paris, France.
[‡]Sorbonne Université, UMPC Univ. Paris 06, LJLL, UMR CNRS 7958, F-75005 Paris, France.
[§]Université Paris Dauphine, PSL Research University, Ceremade, UMR CNRS 7534, Place du Maréchal De Lattre De Tassigny 75775 Paris cedex 16 (France), haspot@ceremade.dauphine.fr

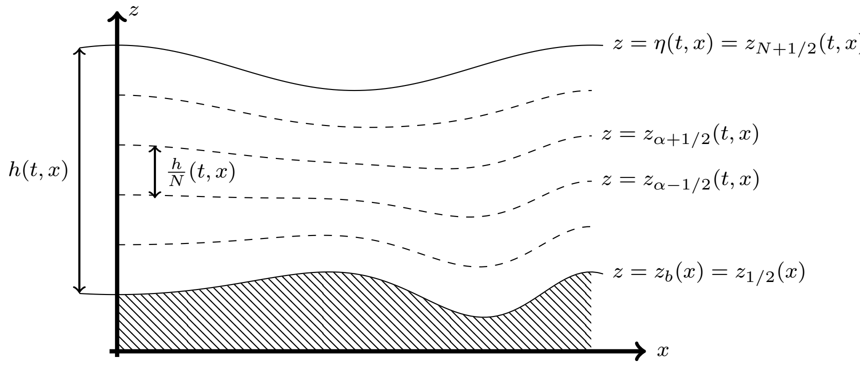

Figure 1: Multilayer approach

are interested in proving the stability of the global weak solutions (in a future work, we will consider the construction of global approximate solution which will imply the existence of global weak solutions).

In the sequel of this section, the equations are detailed (§ 1.1) and a review of classical techniques to obtain global existence of weak solutions is presented (§ 1.2). The main result is stated in § 1.3 (Th. 1.1).

1.1 The multilayer Saint-Venant model

We consider in this paper a multilayer description of a geophysical flow with a free surface and a varying topography. N is the number of layers which might correspond to physical discontinuities but in the present approach layers are predetermined elements of the discretisation.

Horizontal layers ℓ_α are separated by given surfaces $z = z_{\alpha+1/2}(t, \boldsymbol{x})$ where $\boldsymbol{x} \in \mathbb{T}^d$, with $d \in \{1, 2, 3\}$.[1] See Figure 1 for notations. Without loss of generality, we assume that all layers have the same thickness $h_\alpha = z_{\alpha+\frac{1}{2}} - z_{\alpha-\frac{1}{2}} = h/N$, so that we have

$$z_{\alpha+\frac{1}{2}}(t,\boldsymbol{x}) = z_b(\boldsymbol{x}) + \frac{\alpha}{N} h(t,\boldsymbol{x}).$$

The multilayer approach amounts to approximating the velocity field by a layer-wise constant function through a Galerkin discretisation procedure. More precisely, u_α denotes an approximation of the average velocity over the layer ℓ_α

$$u_\alpha(t,\boldsymbol{x}) \approx \frac{N}{h(t,\boldsymbol{x})} \int_{z_{\alpha-\frac{1}{2}}(t,\boldsymbol{x})}^{z_{\alpha+\frac{1}{2}}(t,\boldsymbol{x})} u(t,\boldsymbol{x},z)\,\mathrm{d}z$$

where u satisfies the Navier-Stokes equations. Let us introduce the notations

$$\mathbf{u} = (u_1, \ldots, u_N) \in \mathbb{R}^N, \quad \text{and} \quad \bar{u} = \frac{1}{N}\sum_{\alpha=1}^{N} u_\alpha.$$

The multilayer Saint-Venant model proposed by Audusse *et al.* [3] is obtained by integrating the hydrostatic Navier-Stokes equations over each layer, which reads with $\alpha \in \{1, \ldots, N\}$:

$$\begin{cases} \partial_t h + \operatorname{div}(h\bar{u}) = 0, \\ \partial_t(hu_\alpha) + \operatorname{div}(hu_\alpha \otimes u_\alpha) + \dfrac{g}{2}\nabla h^2 = -gh\nabla z_b + N\left(u_{\alpha+\frac{1}{2}} G_{\alpha+\frac{1}{2}} - u_{\alpha-\frac{1}{2}} G_{\alpha-\frac{1}{2}}\right) \\ \qquad\qquad + \operatorname{div}\big(4\nu h D(u_\alpha)\big) + \kappa(u_{\alpha+1} - u_\alpha) - \kappa(u_\alpha - u_{\alpha-1}), \\ \big(h(0,\cdot), u_1(0,\cdot), \ldots, u_N(0,\cdot)\big) = (h_0, u_{1,0}, \ldots, u_{N,0}), \end{cases} \tag{1.1}$$

with $u_0 = u_1$ and $u_{N+1} = u_N$. The model has for initial data $(h_0, u_{\alpha,0})$ with $\alpha \in \{1, \ldots, N\}$. The term $z_b(\boldsymbol{x})$ denotes the bottom topography assumed sufficiently smooth and stationary. In the sequel we assume that $z_b \in W^{1,\infty}(\mathbb{T}^d)$.

[1] The model considered here results from an averaging process over the vertical axis applied to the Navier-Stokes equations. Hence it has a smaller dimension: $d = 1$ corresponds to the 2D Navier-Stokes equations and $d = 2$ to the 3D equations.

The term $D(u)=\frac{1}{2}(\nabla u+\nabla^T u)$ corresponds to the stress tensor. $G_{\alpha+\frac{1}{2}}$ represents the mass flux through the interface $z=z_{\alpha+\frac{1}{2}}$ from $\ell_{\alpha+1}$ to ℓ_α. It is defined by

$$G_{\alpha+\frac{1}{2}}=\frac{1}{N^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\operatorname{div}\big(h(u_j-u_i)\big). \tag{1.2}$$

Let us consider standard kinematic boundary conditions at the surface and at the bottom which read

$$G_{\frac{1}{2}}=G_{N+\frac{1}{2}}=0.$$

Finally, we define the value of the interface velocity $u_{\alpha+\frac{1}{2}}$ by means of an upwind formula

$$u_{\alpha+\frac{1}{2}}=\begin{cases}u_\alpha, & \text{if } G_{\alpha+\frac{1}{2}}\le 0,\\ u_{\alpha+1}, & \text{otherwise.}\end{cases} \tag{1.3}$$

Let us observe that using the definition (1.3), we have:

$$G_{\alpha+\frac{1}{2}}u_{\alpha+\frac{1}{2}}=\frac{1}{2}G_{\alpha+\frac{1}{2}}(u_\alpha+u_{\alpha+1})-\frac{1}{2}|G_{\alpha+\frac{1}{2}}|(u_\alpha-u_{\alpha+1}). \tag{1.4}$$

Remark 1. *In [14], the authors consider the following choice:*

$$u_{\alpha+\frac{1}{2}}=\frac{1}{2}(u_\alpha+u_{\alpha+1}).$$

From (1.4)*, we notice that our choice for $u_{\alpha+\frac{1}{2}}$ gives the same value for $G_{\alpha+\frac{1}{2}}u_{\alpha+\frac{1}{2}}$ as in [14] up to an additional term $-\frac{1}{2}|G_{\alpha+\frac{1}{2}}|(u_\alpha-u_{\alpha+1})$.*

Remark 2. *Let us notice that the definition 1.2 implies the following partial mass law*

$$\partial_t h+\operatorname{div}(hu_\alpha)=N(G_{\alpha+\frac{1}{2}}-G_{\alpha-\frac{1}{2}}). \tag{1.5}$$

Remark 3. *In [3], the friction term at the leading order is derived from a finite difference method with respect to the vertical variable: $N^2\nu\frac{u_{\alpha+1}-u_\alpha}{h}$. Unfortunately we are not able to deal with this term in our proof of stability, mainly because we have not enough compactness information in order to treat this term. Indeed it would require some control on $\frac{1}{h}$ in suitable Lebesgue spaces.*

In case of immiscible multilayer fluids, some other modelling approaches of the friction term are proposed. For example in [36], the friction term is defined by

$$\operatorname{fric}(v_1,v_2)=-\xi B(h_1,h_2)(v_1-v_2),\qquad B(h_1,h_2)=\frac{h_1h_2}{\frac{\rho_1}{\rho_2}h_1+\frac{\rho_2}{\rho_1}h_2}$$

where ρ_α (resp. v_α) is the density (resp. velocity) of each layer and ξ a positive constant [30]. Consequently, if densities and height are the same, the viscous term is proportional to h. In this paper we use a more conventional friction term of the form $-\kappa(v_2-v_1)$ where κ is a positive constant.

1.2 Main results of existence of global solutions for the compressible Navier-Stokes equations

A large amount of papers in the literature has been devoted to the study of existence of global weak solutions for the compressible Navier-Stokes system. The first result of existence is due to P.-L. Lions in [22] where the mono-layer counterpart of (1.1) with constant viscosity coefficients is considered for a gamma law $P(\rho)=\rho^\gamma$ (with P the pressure and ρ the density which replaces the height h) with $\gamma\ge\frac{9}{5}$ in dimension 3 and $\gamma\ge\frac{3}{2}$ in dimension 2. One of the main ingredient in order to assess the compactness of the global approximate solutions is the so-called theory of renormalised solutions and the introduction of the effective pressure. In [13], Feireisl *et al.* extend the latter result to the case $\gamma>\frac{d}{2}$ by introducing the notion of defect measures. Let us mention that recently Bresch and Jabin [8] generalised these results to the case of anisotropic viscous tensors. The case of the compressible Navier-Stokes equations with degenerate viscosity coefficients is completely different in terms of analysis. Indeed one of the main issues is due to the fact that there is no equivalent to the so-called "effective pressure" (or in other words we cannot invert the viscous stress tensor). Recently several authors obtained significant progress on the existence of global weak solutions with degenerate viscosity coefficients. Bresch and Desjardins [6] introduced a new entropy (the so-called BD entropy) which gives new estimates on the gradient of the density provided that the viscosity coefficients μ and λ verify the following algebraic relation

$$\lambda(\rho) = 2\mu'(\rho) - 2\mu(\rho). \tag{1.6}$$

Let us mention that a particular choice of viscosity coefficients $\lambda(\rho) = 0$ and $\mu(\rho) = \mu\rho$ satisfying (1.6) leads to the so-called *viscous shallow water system* which corresponds to our problem in the mono layer framework. At least heuristically Bresch and Desjardins observed that the quantity $\frac{\mu'(\rho)}{\sqrt{\rho}}\nabla\rho$ is conserved in $L^\infty(0,T;L^2(\mathbb{R}^d))$ norm for any $T > 0$. This allows to prove the existence of global weak solutions [6] with either a drag friction or a cold pressure term (a pressure that is singular at the vacuum). The addition of a friction term allows to get a gain of integrability on the velocity which provides enough compactness estimates in order to deal with the stability of the term $\rho u \otimes u$. Indeed compared with the constant viscosity case there is no control on the gradient of the velocity ∇u in $L^2(\mathbb{R}^+, L^2(\mathbb{R}^d))$ and it is not possible to apply classical Sobolev embeddings to deal with the term $\rho u \otimes u$. This is related to the fact that the viscosity coefficients are degenerate (see the relation (1.6)). The same remark holds when a cold pressure is added. We refer also to [7, 37] for more developments on the existence of global weak solutions with a cold pressure or with a drag friction.

The problem of stability of global weak solutions for classical γ law (when $1 < \gamma < +\infty$ for $d = 2$ and $1 < \gamma < 3$ for $d = 3$) has been solved by Mellet and Vasseur [27]. To do this they introduced a new energy estimate allowing a gain of integrability on the velocity. However the problem of existence of global weak solutions remains open. Indeed it remains to prove the existence of global approximate solutions of the system verifying uniformly energy estimates, BD entropy and the gain of integrability *à la Mellet-Vasseur* which is tricky. However recently for the particular case of the *shallow water system*, the proof has been completed simultaneously and independently by Vasseur and Yu [34, 35] and Li and Xin [21] using different methods.

Concerning the existence of global strong solutions with large initial data for degenerate viscosity coefficients, the problem remains completely open in dimensions greater than 1. We can however mention some results in the case $d = 1$. For viscosity coefficients of the form $\mu(\rho) = \rho^\alpha$ with $0 < \alpha < \frac{1}{2}$, the BD entropy allows to bound the density from below. It allowed Mellet and Vasseur [26] to prove the existence of global strong solutions for initial density far away from vacuum. Indeed the BD entropy gives a bound on $\partial_x(\rho^{\alpha-\frac{1}{2}})$ in $L^\infty(0,T;L^2(\mathbb{R}))$ for all $T > 0$ and a control on ρ^{-1} in $L^\infty(0,T;L^\infty(\mathbb{R}))$ from Sobolev embeddings. Next it is classical to propagate any regularity on the density and the velocity in order to prove the uniqueness. This result has been recently extended by the second author in [18] to the case of general degenerate viscosity coefficients $\alpha \geq \frac{1}{2}$ and in particular the *shallow water system* ($\alpha = 1$) which corresponds to System (1.1) for $N = 1$. The main idea was to rewrite the system by introducing a suitable effective velocity v and apply a maximum principle.

Let us also recall some results on multilayer systems. To our knowledge most existence results concern immiscible fluid flows. In other words it means that there is no mass flux between each layer at the interface, in particular $G_{\alpha+\frac{1}{2}} = 0$ for any α. In [12, 28] the authors obtained existence results of weak solutions for the bilayer case with a viscous term of the form $\nu\Delta u_\alpha$. In [36], it is proven stability of global weak solutions for viscous terms like in (1.1) with surface tensions and with test functions depending on the density itself. When mass transfer is involved, let us mention the work from Fernández Nieto *et al.* [14] who construct numerical solutions of finite element type satisfying the classical energy inequality.

In our paper, we prove the stability of the global weak solution of System (1.1). The main difficulty comes from the terms describing the transfer of flux between the layers which are not taken into account in the immiscible case. In particular it makes the analysis more difficult when we wish to prove the BD entropy and the gain of integrability *à la Mellet-Vasseur* which ensures the stability of the convection term. These two estimates are the cornerstone of the proof of stability of global weak solutions following the argument developed in [27]. However the lack of compactness for the mass flux terms prevents from recovering the expected limit. This is due to the fact that we can not prove the convergence almost everywhere of the terms $G^n_{\alpha+\frac{1}{2}}$.

In a future work, we will prove the existence of global weak solution. It remains essentially to construct global approximate solutions verifying uniformly all the entropy inequalities (in order to do this, we follow the method developed in [34, 35].

1.3 Main results

Before stating the result (Th. 1.1), we define the notion of weak solutions which differs depending on the space dimension d.

Definition 1. *Let $d = 1$. $(h, u_1, \dots, u_N)$ is said to be a global weak solution of* (1.1) *supplemented with initial conditions*

$$h(0, \cdot) = h_0, \;\; (hu_\alpha)(0, \cdot) = m_{\alpha,0}, \tag{1.7}$$

such that for any $\alpha \in \{1, \dots, N\}$:

$$\begin{gathered} h_0 \in L^1(\mathbb{T}^1), \;\; \sqrt{h_0}\partial_x \log h_0 \in L^2(\mathbb{T}^1), \;\; h_0 \geq 0, \\ \sqrt{h_0}|u_{\alpha,0}| \in L^2(\mathbb{T}^1), \;\; u_{\alpha,0} \in L^\infty(\mathbb{T}^1), \\ \sqrt{h_0}|u_{\alpha,0}|\sqrt{\log(1 + |u_{\alpha,0}|^2)} \in L^2(\mathbb{T}^1), \end{gathered} \tag{1.8}$$

if the following smoothness assumptions are satisfied for any $\alpha \in \{1, \dots, N\}$:

- $h \in L^\infty\left(0, T; L^1(\mathbb{T}^1)\right), \; \partial_x\sqrt{h} \in L^\infty\left(0, T; L^2(\mathbb{T}^1)\right), \; \sqrt{h}u_\alpha \in L^\infty\left(0, T; L^2(\mathbb{T}^1)\right),$
- $\sqrt{h}\partial_x u_\alpha \in L^2\left((0, T) \times \mathbb{T}^1\right), \; \sqrt{h}|u_\alpha|\sqrt{\log(1 + |u_\alpha|^2)} \in L^\infty\left(0, T; L^2(\mathbb{T}^1)\right),$

with $h \geq 0$ satisfying in the sense of distributions over $[0, T] \times \mathbb{T}^1$ for any $\alpha \in \{1, \dots, N\}$:

$$\begin{cases} \partial_t h + \partial_x(hu_\alpha) = N(G_{\alpha+\frac{1}{2}} - G_{\alpha-\frac{1}{2}}), \\ h(0, \cdot) = h_0, \end{cases}$$

and if the following equality holds for all smooth test functions $\varphi(t, x)$ with compact support such that $\varphi(T, \cdot) = 0$, we have:

$$\begin{aligned} &\int_{\mathbb{T}^1} m_{\alpha,0}\varphi(0, \cdot)\,\mathrm{d}x + \int_0^T \int_{\mathbb{T}^1} \left[\sqrt{h}\left(\sqrt{h}u_\alpha\right)\partial_t\varphi + \sqrt{h}u_\alpha \times \sqrt{h}u_\alpha \times \partial_x\varphi + \frac{g}{2}h^2\partial_x\varphi\right]\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{T}^1} \left[N\left(\frac{G_{\alpha+\frac{1}{2}}}{2}(u_\alpha + u_{\alpha+1}) - \frac{M_{\alpha+\frac{1}{2}}}{2}\mathbb{1}_{\{h>0\}}(u_\alpha - u_{\alpha+1}) - \frac{1}{2}G_{\alpha-\frac{1}{2}}(u_{\alpha-1} + u_\alpha)\right.\right. \\ &\quad \left.\left. + \frac{M_{\alpha-\frac{1}{2}}}{2}\mathbb{1}_{\{h>0\}}(u_{\alpha-1} - u_\alpha)\right) - gh\partial_x z_b + \kappa(u_{\alpha+1} - 2u_\alpha + u_{\alpha-1})\right]\varphi\,\mathrm{d}x\mathrm{d}t - \langle 4\nu h\, D(u_\alpha), \partial_x\varphi\rangle = 0, \end{aligned} \tag{1.9}$$

where $M_{\alpha+\frac{1}{2}}$ is the weak limit in $L^2((0, T) \times \mathbb{T}^1)$ of $\frac{G^n_{\alpha+\frac{1}{2}}}{h^n}$. Moreover, we give sense to the diffusion term and the flux term:

$$\langle 4\nu h\, D(u_\alpha), \partial_x\varphi\rangle = -\int_0^T \int_{\mathbb{T}^1} \sqrt{h}\partial_x u_\alpha\big(\sqrt{h}\partial_{xx}\varphi + 2\partial_x\varphi\,\partial_x\sqrt{h}\big)\,\mathrm{d}x\mathrm{d}t \tag{1.10}$$

and $G_{\alpha+\frac{1}{2}}$ defined by (1.2).

Definition 2. *Let $d \in \{2, 3\}$. $(h, u_1, \dots, u_N)$ is said to be a global weak solution of* (1.1) *supplemented with initial conditions*

$$h(0, \cdot) = h_0, \;\; (hu_\alpha)(0, \cdot) = m_{\alpha,0}, \tag{1.11}$$

such that for any $\alpha \in \{1, \dots, N\}$:

$$\begin{gathered} h_0 \in L^1(\mathbb{T}^d), \;\; \sqrt{h_0}\nabla \log h_0 \in L^2(\mathbb{T}^d), \;\; h_0 \geq 0, \\ \sqrt{h_0}|u_{\alpha,0}| \in L^2(\mathbb{T}^d), \\ \sqrt{h_0}|u_{\alpha,0}|\sqrt{\log(1 + |u_{\alpha,0}|^2)} \in L^2(\mathbb{T}^d), \end{gathered} \tag{1.12}$$

if the following smoothness assumptions are satisfied for any $\alpha \in \{1, \dots, N\}$:

- $h \in L^\infty\left(0,T;L^1(\mathbb{T}^d)\right)$, $\nabla\sqrt{h} \in L^\infty\left(0,T;L^2(\mathbb{T}^d)\right)$, $\sqrt{h}u_\alpha \in L^\infty\left(0,T;L^2(\mathbb{T}^d)\right)$,
- $\sqrt{h}\nabla u_\alpha \in L^2\left((0,T)\times\mathbb{T}^d\right)$, $\sqrt{h}|u_\alpha|\sqrt{\log(1+|u_\alpha|^2)} \in L^\infty\left(0,T;L^2(\mathbb{T}^d)\right)$,

with $h \geq 0$ satisfying in the sense of distributions over $[0,T]\times\mathbb{T}^d$ for any $\alpha \in \{1,\dots,N\}$:

$$\begin{cases}\partial_t h + \operatorname{div}(hu_\alpha) = N(G_{\alpha+\frac{1}{2}} - G_{\alpha-\frac{1}{2}}),\\ h(0,\cdot) = h_0,\end{cases}$$

and if the following equality holds for all smooth test functions $\varphi(t,\boldsymbol{x})$ with compact support such that $\varphi(T,\cdot)=0$, we have:

$$\begin{aligned}\int_{\mathbb{T}^d} m_{\alpha,0}\cdot\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x} + \int_0^T\int_{\mathbb{T}^d}\left[\sqrt{h}\left(\sqrt{h}u_\alpha\right)\partial_t\varphi + \sqrt{h}u_\alpha\otimes\sqrt{h}u_\alpha : \nabla\varphi + \frac{g}{2}h^2\operatorname{div}\varphi\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\ + \int_0^T\int_{\mathbb{T}^d}\Bigg[N\Bigg(\frac{G_{\alpha+\frac{1}{2}}}{2}(u_\alpha+u_{\alpha+1}) - \frac{M_{\alpha+\frac{1}{2}}}{2}\sqrt{h}(u_\alpha-u_{\alpha+1}) - \frac{G_{\alpha-\frac{1}{2}}}{2}(u_{\alpha-1}+u_\alpha) + \frac{M_{\alpha-\frac{1}{2}}}{2}\sqrt{h}(u_{\alpha-1}-u_\alpha)\Bigg)\\ - gh\nabla z_b + \kappa(u_{\alpha+1}-2u_\alpha+u_{\alpha-1})\Bigg]\cdot\varphi\,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \langle 4\nu h\,D(u_\alpha),\nabla\varphi\rangle = 0,\end{aligned} \quad (1.13)$$

where $M_{\alpha+\frac{1}{2}}$ is the weak limit in $L^2((0,T)\times\mathbb{T}^d)$ of $\frac{G^n_{\alpha+\frac{1}{2}}}{h^n}$. Moreover, we give sense to the diffusion term and the flux term:

$$\langle 4\nu h\,D(u_\alpha),\nabla\varphi\rangle = -\sum_{i,j}\int_0^T\int_{\mathbb{T}^d}\sqrt{h}u_{\alpha,i}\left(\sqrt{h}\partial_{jj}\varphi_i + 2\partial_j\varphi_i\,\partial_j\sqrt{h}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}t \quad (1.14)$$

and $G_{\alpha+\frac{1}{2}}$ defined by (1.2).

(1.15)

Remark 4. *In the previous definitions, the sequences $(h^n)_{n\in\mathbb{N}}$ and $(G^n_{\alpha+\frac{1}{2}})_{n\in\mathbb{N}}$ are related to the sequence $(h^n,u^n)_{n\in\mathbb{N}}$ of global weak solutions defined in Theorem 1.1.*

Let us state now our main result about global weak solutions for the multilayer system (1.1).

Theorem 1.1. *Given $1\leq d\leq 3$ and $(h_0,m_{1,0},\dots,m_{N,0})$ initial data verifying the assumption (1.8) and (1.12). Let us assume that there exists a sequence of global weak solutions $(h^n,u_1^n,\dots,u_N^n)_{n\in\mathbb{N}}$ for System (1.1) such that the energy inequalities (2.1), (2.5), (2.9) and (2.21) are uniformly verified. In particular the corresponding initial data are chosen such that:*

$$h_0^n > 0,\ h_0^n \xrightarrow[n\to+\infty]{} h_0 \ \text{ in }\ L^1(\mathbb{T}^d),\ h_0^n u_{0,\alpha}^n \xrightarrow[n\to+\infty]{} h_0 u_{0,\alpha} \quad (1.16)$$

and satisfy the following bounds (where $C>0$ is independent from n):

$$\int_{\mathbb{T}^d}\left(\sum_{\alpha=1}^N h_0^n\frac{|u_0^n|^2}{2} + (h_0^n)^2\right)\mathrm{d}\boldsymbol{x} < C,\quad \int_{\mathbb{T}^d}\left|\nabla\sqrt{h_0^n}\right|^2\mathrm{d}\boldsymbol{x} < C, \quad (1.17)$$

and when $d\geq 2$:

$$\int_{\mathbb{T}^d}\sum_{\alpha=1}^N h_0^n\frac{1+|u_0^n|^2}{2}\log\left(1+|u_{0,\alpha}^n|^2\right)\mathrm{d}\boldsymbol{x} < C \quad (1.18)$$

or when $d=1$:

$$\forall\,\alpha\in\{1,\dots,N\},\ \|u_{0,\alpha}^n\|_{L^\infty(\mathbb{T}^1)} < C. \quad (1.19)$$

In addition we assume that h^n is a continuous function on $\mathbb{R}^+\times\mathbb{T}^d$ such that for any $(t,x)\in\mathbb{R}^+\times\mathbb{T}^d$, we have:

$$h^n(t,x) > 0.$$

Then up to a subsequence, $(h^n,\sqrt{h^n}u_1^n,\dots,\sqrt{h^n}u_N^n)$ converges strongly to a global weak solution $(h,\sqrt{h}u_1,\dots,\sqrt{h}u_N)$ of System (1.1) in the sense of Definition 1 or 2. More precisely, h^n converges strongly in $C((0,T);L^{\frac{3}{2}}(\mathbb{T}^d))$, $\sqrt{h^n}u_\alpha^n$ converges strongly in $L^2((0,T);L^2(\mathbb{T}^d))$ and the momentum $m_\alpha^n = h^n u_\alpha^n$ converges strongly in $L^1((0,T);L^1(\mathbb{T}^d))$ for any $T>0$. When $d=1$, h is continuous on $\mathbb{R}^+\times\mathbb{T}^1$.

Remark 5. *In the paper of Audusse* et al. *[3] the authors claim that the modelling is physically relevant in terms of thermodynamics since they exhibit a classical energy. In our result in order to prove the stability of global weak solutions we need in addition to show the BD entropy which is another hint of the physical interest of the model.*

Remark 6. *In a future work, we shall prove the existence of such a sequence* $(h^n, u^n)_{n\in\mathbb{N}}$ *of global regular approximate solutions verifying uniformly* (2.1)*,* (2.5)*,* (2.9) *and* (2.21)*.*

Remark 7. *Let us emphasize that in Theorem 1.1, it seems difficult to deal with the mass transfer flux, essentially because we are not able to prove that* $|G^n_{\alpha+\frac{1}{2}}|(u^n_{\alpha+1} - u^n_\alpha)$ *converges in the sense of distributions to* $|G_{\alpha+\frac{1}{2}}|(u_{\alpha+1} - u_\alpha)$*. Indeed it is not clear to prove the convergence almost everywhere of* $G^n_{\alpha+\frac{1}{2}}$*.*

In [14], with the choice $u_{\alpha+\frac{1}{2}} = \frac{1}{2}(u_\alpha + u_{\alpha+1})$*, the additional term* $|G^n_{\alpha+\frac{1}{2}}|(u^n_{\alpha+1} - u^n_\alpha)$ *does not appear. Then it is easy to deal with the mass transfer term. For this specific choice for* $u_{\alpha+\frac{1}{2}}$*, we are also able to prove the BD entropy but it seems tricky to obtain a gain of integrability à la Mellet Vasseur. For this reason we have not enough compactness information to treat the convection term.*

We would like to mention that we could obtain global weak solutions for the system proposed in [14] if we consider friction terms of the form $h|u_\alpha|^{1+\epsilon}u_\alpha$ *with* $\epsilon > 0$ *in each layer. Indeed in this case the friction terms ensure directly a gain of integrability on the velocity.*

The paper unfolds as follows. In Section 2, we give new estimates for System (1.1) involving the BD entropy and some gain of integrability on the velocity u_α. In Section 3, we show the stability of global weak solutions following the arguments developed in [27]. We postpone an appendix which details some computations and prove the BD entropy for the choice $u_{\alpha+\frac{1}{2}} = \frac{1}{2}(u_\alpha + u_{\alpha+1})$ used in [14].

2 A priori energy estimates

In this section, we are interested in proving at least heuristically different energy estimates: the classical energy of the system, the BD entropy (see [5]) which is less obvious and an equivalent of the Mellet-Vasseur estimate from [27].

2.1 Classical energy

Proposition 2.1. *Let* $(h, u_1, \ldots, u_N)$ *be a classical solution of System* (1.1)*. Then, the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} E\,\mathrm{d}\boldsymbol{x} + \sum_{\alpha=1}^N \int_{\mathbb{T}^d} 4\nu h|D(u_\alpha)|^2\,\mathrm{d}\boldsymbol{x} + N\sum_{\alpha=1}^N \int_{\mathbb{T}^d} \kappa|u_{\alpha+1} - u_\alpha|^2\,\mathrm{d}\boldsymbol{x} + \frac{N}{2}\sum_{\alpha=1}^N \int_{\mathbb{T}^d} |u_{\alpha+1} - u_\alpha|^2 |G_{\alpha+\frac{1}{2}}|\,\mathrm{d}\boldsymbol{x} = 0 \quad (2.1)$$

with

$$E = \frac{1}{2}\left(Ngh^2 + \sum_{\alpha=1}^N h|u_\alpha|^2\right) + Nghz_b. \quad (2.2)$$

Hereafter, $A : B = \sum_{i,j} A_{ij}B_{ij}$ denotes the scalar product upon matrices and $|A|^2 = A : A$.

Proof. We follow here the arguments of [3]. The main difficulty concerns the coupling between the different equalities through the flux terms. Simplifications arise only after summing the equations. Multiplying the momentum equations (1.1) by u_α and summing over α, we obtain:

1. $\displaystyle\sum_{\alpha=1}^N \int_{\mathbb{T}^d} [\partial_t(hu_\alpha) + \operatorname{div}(hu_\alpha \otimes u_\alpha)] \cdot u_\alpha\,\mathrm{d}\boldsymbol{x} - N\sum_{\alpha=1}^N \int_{\mathbb{T}^d} \left(u_{\alpha+\frac{1}{2}}G_{\alpha+\frac{1}{2}} - u_{\alpha-\frac{1}{2}}G_{\alpha-\frac{1}{2}}\right) \cdot u_\alpha\,\mathrm{d}\boldsymbol{x}$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} h|u_\alpha|^2\,\mathrm{d}\boldsymbol{x} + \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|u_\alpha|^2(G_{\alpha+\frac12}-G_{\alpha-\frac12})\,\mathrm{d}\boldsymbol{x}$$
$$- N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\left(u_{\alpha+\frac12}G_{\alpha+\frac12}-u_{\alpha-\frac12}G_{\alpha-\frac12}\right)\cdot u_\alpha\,\mathrm{d}\boldsymbol{x};$$

2. $\displaystyle \frac12\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} hu_\alpha\cdot\nabla h^2\,\mathrm{d}\boldsymbol{x} = N\frac{g}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}h^2\,\mathrm{d}\boldsymbol{x};$

3. $\displaystyle \sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} u_\alpha\cdot\mathrm{div}\,(4\nu hD(u_\alpha))\,\mathrm{d}\boldsymbol{x} = -\int_{\mathbb{T}^d}4\nu h\sum_{\alpha=1}^{N}|D(u_\alpha)|^2\,\mathrm{d}\boldsymbol{x};$

4. $\displaystyle \sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\left[(u_{\alpha+1}-u_\alpha)\cdot u_\alpha-(u_\alpha-u_{\alpha-1})\cdot u_\alpha\right]\,\mathrm{d}\boldsymbol{x} = -\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|u_{\alpha+1}-u_\alpha|^2\,\mathrm{d}\boldsymbol{x};$

5. $\displaystyle \sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} hu_\alpha\cdot\nabla z_b\,\mathrm{d}\boldsymbol{x} = N\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}z_b h\,\mathrm{d}\boldsymbol{x}.$

Let us observe now that:

$$\frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}G_{\alpha+\frac12}\left(|u_\alpha|^2-|u_{\alpha+1}|^2\right)\,\mathrm{d}\boldsymbol{x} - N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\left(u_{\alpha+\frac12}G_{\alpha+\frac12}-u_{\alpha-\frac12}G_{\alpha-\frac12}\right)\cdot u_\alpha\,\mathrm{d}\boldsymbol{x}$$
$$= \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|G_{\alpha+\frac12}|\,|u_{\alpha+1}-u_\alpha|^2\,\mathrm{d}\boldsymbol{x}. \quad (2.3)$$

Combining the different previous estimates and (2.3), we obtain the energy estimate (2.1). □

2.2 BD entropy

Unfortunately this last energy inequality is not sufficient in order to prove the existence of global weak solutions. Indeed we need additional compactness information to deal with the pressure term and the convection terms. As in [5], we would like to prove that a BD entropy estimate is satisfied. Let us introduce as in [18, 19] the effective velocity $v_\alpha = u_\alpha + 4\nu\nabla\log h$. Then system (1.1) can be written

$$\begin{cases} \partial_t h + \mathrm{div}(h\bar{u}) = 0, \\ h\left[\partial_t v_\alpha + (u_\alpha\cdot\nabla)v_\alpha\right] - 2\nu\,\mathrm{div}\,(h\ \mathrm{curl}\,v_\alpha) + \frac{g}{2}\nabla h^2 = -gh\nabla z_b \\ \qquad + N\left(G_{\alpha+\frac12}(u_{\alpha+\frac12}-u_\alpha) - G_{\alpha-\frac12}(u_{\alpha-\frac12}-u_\alpha)\right) + 4\nu Nh\nabla\left(\frac{G_{\alpha+\frac12}-G_{\alpha-\frac12}}{h}\right) \\ \qquad\qquad + \kappa(u_{\alpha+1}-u_\alpha) - \kappa(u_\alpha-u_{\alpha-1}), \end{cases} \quad (2.4)$$

with $\mathrm{curl}\,v_\alpha = (\nabla v_\alpha - \nabla^T v_\alpha)$ the vorticity.

Proposition 2.2 (BD entropy). *If we assume that $(h, u_1, \ldots, u_N)$ is a smooth solution of System (1.1), then*

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}h\sum_{\alpha=1}^{N}|v_\alpha|^2\,\mathrm{d}\boldsymbol{x} + \frac{Ng}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}h^2\,\mathrm{d}\boldsymbol{x} + Ng\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}z_b h\,\mathrm{d}\boldsymbol{x} + 2\nu\int_{\mathbb{T}^d}h|\,\mathrm{curl}\,v_\alpha|^2\,\mathrm{d}\boldsymbol{x}$$
$$+ 4N\nu g\int_{\mathbb{T}^d}|\nabla h|^2\,\mathrm{d}\boldsymbol{x} + 4N\nu g\int_{\mathbb{T}^d}\nabla z_b\cdot\nabla h\,\mathrm{d}\boldsymbol{x} + \kappa\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|v_{\alpha+1}-v_\alpha|^2\,\mathrm{d}\boldsymbol{x}$$
$$+ \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|G_{\alpha+\frac12}|\,|v_{\alpha+1}-v_\alpha|^2\,\mathrm{d}\boldsymbol{x} + \sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^d}\frac{1}{hN^2}\sum_{j=\alpha+1}^{N}\left(\mathrm{div}(h(u_\alpha-u_j))\right)^2\,\mathrm{d}\boldsymbol{x} = 0. \quad (2.5)$$

Remark 8. *From this estimate we deduce two new pieces of information which are essential to obtain convergence results. Firstly, $\sqrt{h}v_\alpha$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$. We deduce that $\sqrt{h}\nabla\log h = 2\nabla\sqrt{h}$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$. This is the crucial point ensured by the BD entropy. On the other hand, thanks to (1.2), the last term of this estimate also gives a bound for $G_{\alpha+\frac{1}{2}}/\sqrt{h}$ in $L^2(0,T;L^2(\mathbb{T}^d))$ that enables to give sense to the term $u_{\alpha+\frac{1}{2}}G_{\alpha+\frac{1}{2}}$.*

Remark 9. *We mention that we can also obtain energy and BD entropy with the choice for $u_{\alpha+\frac{1}{2}}$ used in [14].*

Proof. Multiplying the momentum equations of (2.4) by v_α, integrated over $\mathbb{T}^d$ and summing over α we get:

1. $\displaystyle\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} h\,(\partial_t v_\alpha + (u_\alpha\cdot\nabla)v_\alpha)\cdot v_\alpha\,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\left[\frac{\mathrm{d}}{\mathrm{d}t}h|v_\alpha|^2 + NG_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2-|v_\alpha|^2)\right]\,\mathrm{d}\boldsymbol{x};$
2. $\displaystyle\sum_{\alpha=1}^{N}\frac{g}{2}\int_{\mathbb{T}^d} v_\alpha\cdot\nabla h^2\,\mathrm{d}\boldsymbol{x} = \frac{Ng}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h^2\,\mathrm{d}\boldsymbol{x} + 4N\nu g\int_{\mathbb{T}^d}|\nabla h|^2\,\mathrm{d}\boldsymbol{x};$
3. $\displaystyle\sum_{\alpha=1}^{N} g\int_{\mathbb{T}^d} h\nabla z_b\cdot v_\alpha\,\mathrm{d}\boldsymbol{x} = Ng\int_{\mathbb{T}^d} z_b h\,\mathrm{d}\boldsymbol{x} + 4gN\nu\int_{\mathbb{T}^d}\nabla z_b\cdot\nabla h\mathrm{d}\boldsymbol{x};$
4. $\displaystyle\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} v_\alpha\cdot 2\nu\,\mathrm{div}(h\ \mathrm{curl}\,v_\alpha))\,\mathrm{d}\boldsymbol{x} = -\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} 2\nu h|\,\mathrm{curl}\,v_\alpha|^2\,\mathrm{d}\boldsymbol{x};$
5. Since for all $\alpha\in\{1,\dots,N\}$ we have $v_\alpha - v_{\alpha-1} = u_\alpha - u_{\alpha-1}$ we deduce that:
$$\kappa\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}(u_{\alpha+1}-u_\alpha)\cdot v_\alpha - (u_\alpha - u_{\alpha-1})\cdot v_\alpha\,\mathrm{d}\boldsymbol{x} = -\kappa\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|v_{\alpha+1}-v_\alpha|^2\,\mathrm{d}\boldsymbol{x}.$$

Combining the previous estimate we have:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h\sum_{\alpha=1}^{N}|v_\alpha|^2\,\mathrm{d}\boldsymbol{x} + \frac{Ng}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h^2\,\mathrm{d}\boldsymbol{x} + Ng\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} z_b h\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^d} 2\nu h|\,\mathrm{curl}\,v_\alpha|^2\,\mathrm{d}\boldsymbol{x}\\
&\qquad + 4N\nu g\int_{\mathbb{T}^d}|\nabla h|^2\,\mathrm{d}\boldsymbol{x} + 4N\nu g\int_{\mathbb{T}^d}\nabla z_b\cdot\nabla h\,\mathrm{d}\boldsymbol{x} + \kappa\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|v_{\alpha+1}-v_\alpha|^2\,\mathrm{d}\boldsymbol{x}\\
&\quad + \frac{1}{2}\sum_{\alpha=1}^{N} NG_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2-|v_\alpha|^2)\,\mathrm{d}\boldsymbol{x} - N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\Big(G_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}}-u_\alpha) - G_{\alpha-\frac{1}{2}}(u_{\alpha-\frac{1}{2}}-u_\alpha)\Big)\cdot v_\alpha\mathrm{d}\boldsymbol{x}\\
&\qquad\qquad - 4\nu N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} h\nabla\left(\frac{G_{\alpha+\frac{1}{2}}-G_{\alpha-\frac{1}{2}}}{h}\right)\cdot v_\alpha\mathrm{d}\boldsymbol{x} = 0. \qquad (2.6)
\end{aligned}$$

Next we have due to $G_{\frac{1}{2}} = G_{N+\frac{1}{2}} = 0$:

$$
\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{N} G_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2 - |v_\alpha|^2) - \sum_{\alpha=1}^{N}\Big(G_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}} - u_\alpha) - G_{\alpha-\frac{1}{2}}(u_{\alpha-\frac{1}{2}} - u_\alpha)\Big)\cdot v_\alpha \\
&\quad = \frac{1}{2}\sum_{\alpha=1}^{N} G_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2 - |v_\alpha|^2) - \sum_{\alpha=1}^{N}(G_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}} - u_\alpha)\cdot v_\alpha - \sum_{\alpha=0}^{N-1} G_{\alpha+\frac{1}{2}}(u_{\alpha+1} - u_{\alpha+\frac{1}{2}})\cdot v_{\alpha+1} \\
&\quad = \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[u_{\alpha+\frac{1}{2}}\cdot(v_\alpha - v_{\alpha+1}) + (u_{\alpha+1}\cdot v_{\alpha+1} - u_\alpha\cdot v_\alpha) - \frac{1}{2}(|v_{\alpha+1}|^2 - |v_\alpha|^2)\right] \\
&\quad = \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[u_{\alpha+\frac{1}{2}}\cdot(u_\alpha - u_{\alpha+1}) + |u_{\alpha+1}|^2 - |u_\alpha|^2 + 4\nu(u_{\alpha+1} - u_\alpha)\cdot\nabla\log h - \frac{1}{2}(|v_{\alpha+1}|^2 - |v_\alpha|^2)\right]; \\
&\quad = \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[u_{\alpha+\frac{1}{2}}\cdot(u_\alpha - u_{\alpha+1}) + \frac{1}{2}(|u_{\alpha+1}|^2 - |u_\alpha|^2)\right], \\
&\quad = \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}(u_\alpha - u_{\alpha+1})\cdot\left(u_{\alpha+\frac{1}{2}} - \frac{1}{2}(u_{\alpha+1} + u_\alpha)\right),
\end{aligned}
$$

since

$$
\begin{aligned}
|v_{\alpha+1}|^2 - |v_\alpha|^2 &= (u_{\alpha+1} - u_\alpha)\cdot(u_{\alpha+1} + u_\alpha + 8\nu\nabla\log h) \\
&= |u_{\alpha+1}|^2 - |u_\alpha|^2 + 8\nu(u_{\alpha+1} - u_\alpha)\cdot\nabla\log h.
\end{aligned}
$$

Then, by the definition 1.3 of $u_{\alpha+\frac{1}{2}}$ we have:

- If $G_{\alpha+\frac{1}{2}} > 0$ then $u_{\alpha+\frac{1}{2}} = u_{\alpha+1}$ and:
$$\sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}(u_\alpha - u_{\alpha+1})\cdot\left(u_{\alpha+\frac{1}{2}} - \frac{1}{2}(u_{\alpha+1} + u_\alpha)\right) = \frac{1}{2}\sum_{\alpha=1}^{N-1}|G_{\alpha+\frac{1}{2}}|(u_{\alpha+1} - u_\alpha)^2;$$
- If $G_{\alpha+\frac{1}{2}} \le 0$ then $u_{\alpha+\frac{1}{2}} = u_\alpha$ and:
$$\sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}(u_\alpha - u_{\alpha+1})\cdot\left(u_{\alpha+\frac{1}{2}} - \frac{1}{2}(u_{\alpha+1} + u_\alpha)\right) = \frac{1}{2}\sum_{\alpha=1}^{N-1}|G_{\alpha+\frac{1}{2}}|(u_{\alpha+1} - u_\alpha)^2.$$

Finally we have proved that:

$$
\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{N} G_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2 - |v_\alpha|^2) - \sum_{\alpha=1}^{N}\Big(G_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}} - u_\alpha) - G_{\alpha-\frac{1}{2}}(u_{\alpha-\frac{1}{2}} - u_\alpha)\Big)\cdot v_\alpha \\
&\qquad = \sum_{\alpha=1}^{N}\frac{|G_{\alpha+\frac{1}{2}}|}{2}|u_{\alpha+1} - u_\alpha|^2.
\end{aligned}
\tag{2.7}
$$

From the relation (1.2) and by integration by parts, we have

$$
\begin{aligned}
\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} hv_\alpha\cdot\nabla\left(\frac{G_{\alpha+\frac{1}{2}} - G_{\alpha-\frac{1}{2}}}{h}\right)\,\mathrm{d}\boldsymbol{x} &= \sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^d} h(v_\alpha - v_{\alpha+1})\cdot\nabla\left(\frac{G_{\alpha+\frac{1}{2}}}{h}\right)\,\mathrm{d}\boldsymbol{x} \\
&= -\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^d}\frac{G_{\alpha+\frac{1}{2}}}{h}\operatorname{div}\left(h(u_\alpha - u_{\alpha+1})\right)\,\mathrm{d}\boldsymbol{x} \\
&= -\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^d}\frac{1}{hN^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\operatorname{div}(h(u_j - u_i))\operatorname{div}(h(u_\alpha - u_{\alpha+1}))\,\mathrm{d}\boldsymbol{x} \\
&= -\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^d}\frac{1}{hN^2}\sum_{j=\alpha+1}^{N}\left[\operatorname{div}(h(u_\alpha - u_j))\right]^2\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{2.8}
$$

A detailed proof of the latter relation is given in 4.1. Combining (2.6), (2.7) and (2.8) implies the estimate (2.5). □

2.3 Mellet-Vasseur logarithmic estimate

In order to deal with the convection term $hu_\alpha \otimes u_\alpha$ which is only bounded $L^\infty(0,T;L^1(\mathbb{T}^d))$, it is important to get a gain of integrability on the velocity as in [27]. Let us mention that in [5], in order to overcome this difficulty the authors need to work with a friction term. We have the following result.

Proposition 2.3 (MV inequality)**.** *If we assume that $(h, u_1, \dots, u_N)$ is a smooth solution of System* (1.1)*, then*

$$\sum_{\alpha=1}^{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\left[h\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right)\right]\,\mathrm{d}\boldsymbol{x} + 3\nu\int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right]|D(u_\alpha)|^2\,\mathrm{d}\boldsymbol{x}\right)$$
$$\leq \sum_{\alpha=1}^{N}\left(C\left(\int_{\mathbb{T}^d} h|\nabla u_\alpha|^2\,\mathrm{d}\boldsymbol{x}\right) + C\left(\int_{\mathbb{T}^d} h^{\frac{6-\delta}{2-\delta}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{2-\delta}{2}} \times \left(\int_{\mathbb{T}^d} h\left[2+\log\left(1+|u_\alpha|^2\right)\right]^{\frac{2}{\delta}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{\delta}{2}}\right.$$
$$\left. + g\int_{\mathbb{T}^d} h\frac{1+|u_\alpha|^2}{2}\left[1+\log(1+|u_\alpha|^2)\right]|\nabla z_b|\,\mathrm{d}\boldsymbol{x}\right) \qquad (2.9)$$

for any $\delta \in (0,2)$ and for some constant $C \geq 0$.

Remark 10. *Let us mention that it seems difficult to obtain a similar result with the choice for $u_{\alpha+\frac{1}{2}}$ used in [14].*

Proof. Let us first rewrite equations (1.1) under the non-conservative form

$$\begin{cases} \partial_t h + \mathrm{div}(h\bar{u}) = 0, & (2.10a)\\ h\left[\partial_t u_\alpha + (u_\alpha\cdot\nabla)u_\alpha\right] + \frac{g}{2}\nabla h^2 = -gh\nabla z_b + NG_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}} - u_\alpha) & \\ \qquad + NG_{\alpha-\frac{1}{2}}(u_\alpha - u_{\alpha-\frac{1}{2}}) + \mathrm{div}\left(4\nu h D(u_\alpha)\right) + \kappa(u_{\alpha+1}-u_\alpha) - \kappa(u_\alpha - u_{\alpha-1}). & (2.10b)\end{cases}$$

Let us set

$$\Phi(x) = \frac{1+x^2}{2}\log(1+x^2) \text{ and } \phi(x) = \Phi'(x) = x\left[1+\log(1+x^2)\right]. \qquad (2.11)$$

Let us notice that

$$\partial\left(\frac{1+|u|^2}{2}\log\left(1+|u|^2\right)\right) = (u\cdot\partial u)\left[1+\log\left(1+|u|^2\right)\right].$$

We multiply equation (2.10b) by $u_\alpha\left[1+\log\left(1+|u_\alpha|^2\right)\right]$ and we integrate over $\mathbb{T}^d$. Hence each term becomes

1. $\displaystyle\int_{\mathbb{T}^d} hu_\alpha\cdot\partial_t u_\alpha\left[1+\log(1+|u_\alpha|^2)\right]\mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^d}\left[\partial_t\big(h\Phi(|u_\alpha|)\big) - \Phi(|u_\alpha|)\partial_t h\right]\mathrm{d}\boldsymbol{x}.$
2. Using (1.5), we obtain by integration by parts:
$$\int_{\mathbb{T}^d} hu_\alpha\cdot(u_\alpha\cdot\nabla)u_\alpha\left[1+\log(1+|u_\alpha|^2)\right]\mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^d} hu_\alpha\cdot\nabla\Phi(|u_\alpha|)\,\mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{T}^d}\Phi(|u_\alpha|)\left[\partial_t h - N(G_{\alpha+1/2} - G_{\alpha-1/2})\right]\mathrm{d}\boldsymbol{x}.$$
3. For the pressure term we apply the same approach as in [27]
$$\left|\int_{\mathbb{T}^d}\left[1+\log(1+|u_\alpha|^2)\right]u_\alpha\cdot\nabla h^2\mathrm{d}\boldsymbol{x}\right|$$
$$\leq \left|\sum_{i,j}\int_{\mathbb{T}^d} h^2\frac{2u_{\alpha_i}u_{\alpha_k}}{1+|u_\alpha|^2}\partial_i u_{\alpha_k}\mathrm{d}\boldsymbol{x}\right| + \left|\int_{\mathbb{T}^d} h^2\left[1+\log\left(1+|u_\alpha|^2\right)\right](\mathrm{div}\,u_\alpha)\right|\mathrm{d}\boldsymbol{x}$$
$$\leq 2\left(\int_{\mathbb{T}^d} h|\nabla u_\alpha|^2\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}\left(\int_{\mathbb{T}^d} h^3\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}} + \left|\int_{\mathbb{T}^d} h^2\left[1+\log\left(1+|u_\alpha|^2\right)\right](\mathrm{div}\,u_\alpha)\mathrm{d}\boldsymbol{x}\right|.$$

Since

$$(\mathrm{div}\, u)^2 = \sum_i \sum_j \partial_i u_i \partial_j u_j \le \sum_i \sum_j \frac{1}{2}\left((\partial_i u_i)^2 + (\partial_j u_j)^2\right) \le d\,|D(u)|^2$$

where d is the dimension of the space, we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d} h^2 \left[1+\log\left(1+|u_\alpha|^2\right)\right](\mathrm{div}\, u_\alpha)\mathrm{d}\boldsymbol{x}\right| \\
&\quad \le \left(\int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right](\mathrm{div}\, u_\alpha)^2 \mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} h^3 \left[1+\log\left(1+|u_\alpha|^2\right)\right]\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}} \\
&\quad \le \nu \int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right]\, |D(u_\alpha)|^2 \mathrm{d}\boldsymbol{x} + C_\nu \int_{\mathbb{T}^d} h^3\left[1+\log\left(1+|u_\alpha|^2\right)\right]\mathrm{d}\boldsymbol{x}
\end{aligned}$$

according to the Young's inequality with $C_\nu = \frac{d}{4\nu}$. It follows that for C'_ν large enough:

$$\begin{aligned}
&\left|\int_{\mathbb{T}^d} \left[1+\log(1+|u_\alpha|^2)\right] u_\alpha \cdot \nabla h^2 \mathrm{d}\boldsymbol{x}\right| \\
&\quad \le \nu\left(\int_{\mathbb{T}^d} h|\nabla u_\alpha|^2 \mathrm{d}\boldsymbol{x}\right) + \nu \int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right]\, |D(u_\alpha)|^2 \mathrm{d}\boldsymbol{x} \\
&\qquad + C'_\nu \int_{\mathbb{T}^d} h^3 \left[1+\log\left(1+|u_\alpha|^2\right)\right]\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Finally, for any $\delta \in (0,2)$, the last term is bounded by means of the Hölder's inequality

$$\int_{\mathbb{T}^d} h^3\left[1+\log\left(1+|u_\alpha|^2\right)\right]\mathrm{d}\boldsymbol{x} \le \left(\int_{\mathbb{T}^d} h^{\frac{6-\delta}{2-\delta}} \mathrm{d}\boldsymbol{x}\right)^{\frac{2-\delta}{2}} \times \left(\int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right]^{\frac{2}{\delta}} \mathrm{d}\boldsymbol{x}\right)^{\frac{\delta}{2}}.$$

4. Likewise we have

$$\int_{\mathbb{T}^d} h u_\alpha \cdot \nabla z_b \left[1+\log(1+|u_\alpha|^2)\right]\mathrm{d}\boldsymbol{x} \le \int_{\mathbb{T}^d} h\frac{1+|u_\alpha|^2}{2}\left[1+\log(1+|u_\alpha|^2)\right]\, |\nabla z_b| \mathrm{d}\boldsymbol{x}.$$

Since z_b is assumed bounded in $W^{1,\infty}$ this term can be treated by the Grönwall's lemma.

5. For the viscous terms we have

$$\begin{aligned}
&\int_{\mathbb{T}^d} u_\alpha \cdot \mathrm{div}\left(4\nu h D(u_\alpha)\right)\left[1+\log(1+|u_\alpha|^2)\right]\mathrm{d}\boldsymbol{x} \\
&\qquad = -4\nu \int_{\mathbb{T}^d} h\left[1+\log(1+|u_\alpha|^2)\right]|D(u_\alpha)|^2 \mathrm{d}\boldsymbol{x} - \sum_{i,j} 8\nu \int_{\mathbb{T}^d} h \frac{u_{\alpha_i} u_\alpha \cdot \partial_j u_\alpha}{1+|u_\alpha|^2} D_{ij}(u_\alpha)\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

and we have for $C_\alpha > 0$ large enough:

$$\begin{aligned}
&\int_{\mathbb{T}^d} u_\alpha \cdot \mathrm{div}\left(4\nu h D(u_\alpha)\right)\left[1+\log(1+|u_\alpha|^2)\right]\mathrm{d}\boldsymbol{x} + 4\nu \int_{\mathbb{T}^d} h\left[1+\log(1+|u_\alpha|^2)\right]|D(u_\alpha)|^2 \mathrm{d}\boldsymbol{x} \\
&\qquad \le C_\alpha \int_{\mathbb{T}^d} h|\nabla u_\alpha|^2 \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

6. For the friction terms (since by definition $u_0 = u_1$ and $u_N = u_{N+1}$) we have:

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \int_{\mathbb{T}^d} \left[(u_{\alpha+1}-u_\alpha)\cdot u_\alpha - (u_\alpha - u_{\alpha-1})\cdot u_\alpha\right]\left[1+\log(1+|u_\alpha|^2)\right]\mathrm{d}\boldsymbol{x} \\
&\qquad = -\sum_{\alpha=1}^{N-1} \int_{\mathbb{T}^d} (u_\alpha - u_{\alpha+1})\cdot \Big[u_\alpha\left[1+\log(1+|u_\alpha|^2)\right] - u_{\alpha+1}\left[1+\log(1+|u_{\alpha+1}|^2)\right]\Big]\mathrm{d}\boldsymbol{x} \le 0
\end{aligned}$$

since the function ϕ defined by (2.11) is increasing.

Combining all the previous estimates, we have:

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d}\left[h\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right)\right]\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{T}^d}3\nu h\left[1+\log\left(1+|u_\alpha|^2\right)\right]|D(u_\alpha)|^2\,\mathrm{d}\boldsymbol{x}\bigg)\\
&\qquad+\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\bigg([G_{\alpha+\frac12}(u_{\alpha+\frac12}-u_\alpha)+G_{\alpha-\frac12}(u_\alpha-u_{\alpha-\frac12})]\cdot u_\alpha\left[1+\log(1+|u_\alpha|^2)\right]\\
&\qquad\qquad+\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right)(G_{\alpha+1/2}-G_{\alpha-1/2})\bigg)\mathrm{d}\boldsymbol{x}\\
&\le\sum_{\alpha=1}^{N}\bigg(C\left(\int_{\mathbb{T}^d}h|\nabla u_\alpha|^2\,\mathrm{d}\boldsymbol{x}\right)+C\left(\int_{\mathbb{T}^d}h^{\frac{6-\delta}{2-\delta}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{2-\delta}{2}}\times\left(\int_{\mathbb{T}^d}h\left[2+\log\left(1+|u_\alpha|^2\right)\right]^{\frac{2}{\delta}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{\delta}{2}}\\
&\qquad+g\int_{\mathbb{T}^d}h\frac{1+|u_\alpha|^2}{2}\left[1+\log(1+|u_\alpha|^2)\right]|\nabla z_b|\,\mathrm{d}\boldsymbol{x}\bigg);
\end{aligned}$$

We have now since $G_{\frac12}=G_{N+\frac12}=0$:

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}[G_{\alpha+\frac12}(u_{\alpha+\frac12}-u_\alpha)+G_{\alpha-\frac12}(u_\alpha-u_{\alpha-\frac12})]\cdot u_\alpha\left[1+\log(1+|u_\alpha|^2)\right]\\
&\qquad+\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right)(G_{\alpha+\frac12}-G_{\alpha-\frac12})\mathrm{d}\boldsymbol{x}\\
&=\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}G_{\alpha+\frac12}(u_{\alpha+\frac12}-u_\alpha)\cdot u_\alpha\left[1+\log(1+|u_\alpha|^2)\right]\,\mathrm{d}\boldsymbol{x}\\
&\quad+\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}G_{\alpha-\frac12}(u_\alpha-u_{\alpha-\frac12})]\cdot u_\alpha\left[1+\log(1+|u_\alpha|^2)\right]\,\mathrm{d}\boldsymbol{x}\\
&\quad\quad+\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right)(G_{\alpha+\frac12}-G_{\alpha-\frac12})\mathrm{d}\boldsymbol{x}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha=1}^{N} \int_{\mathbb{T}^d} G_{\alpha+\frac12}(u_{\alpha+\frac12} - u_\alpha)\cdot u_\alpha \left[1+\log(1+|u_\alpha|^2)\right] \,\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{\alpha=0}^{N-1} \int_{\mathbb{T}^d} G_{\alpha+\frac12}(u_{\alpha+1} - u_{\alpha+\frac12})]\cdot u_{\alpha+1} \left[1+\log(1+|u_{\alpha+1}|^2)\right] \,\mathrm{d}\boldsymbol{x} \\
&\qquad + \sum_{\alpha=1}^{N} \int_{\mathbb{T}^d} \frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right) G_{\alpha+\frac12}\mathrm{d}\boldsymbol{x} \\
&\qquad - \sum_{\alpha=0}^{N-1} \int_{\mathbb{T}^d} \frac{1+|u_{\alpha+1}|^2}{2}\log\left(1+|u_{\alpha+1}|^2\right) G_{\alpha+\frac12}\mathrm{d}\boldsymbol{x} \\
&= \sum_{\alpha=1}^{N-1} \int_{\mathbb{T}^d} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\le 0\}}(u_{\alpha+1}-u_\alpha)\cdot u_{\alpha+1}(1+\log(1+|u_{\alpha+1}|^2)) \\
&\qquad + G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\ge 0\}}(u_{\alpha+1}-u_\alpha)\cdot u_\alpha\left[1+\log(1+|u_\alpha|^2)\right] \\
&\qquad + G_{\alpha+\frac12}\left(\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right) - \frac{1+|u_{\alpha+1}|^2}{2}\log(1+|u_{\alpha+1}|^2)\right)\mathrm{d}\boldsymbol{x} \\
&= -\sum_{\alpha=1}^{N-1} \int_{\mathbb{T}^d} |G_{\alpha+\frac12}|\mathbb{1}_{\{G_{\alpha+\frac12}\le 0\}}(u_{\alpha+1}-u_\alpha)\cdot u_{\alpha+1}(1+\log(1+|u_{\alpha+1}|^2)) \\
&\qquad + |G_{\alpha+\frac12}|\mathbb{1}_{\{G_{\alpha+\frac12}\ge 0\}}(u_\alpha-u_{\alpha+1})\cdot u_\alpha\left[1+\log(1+|u_\alpha|^2)\right] \\
&\qquad + |G_{\alpha+\frac12}|\mathbb{1}_{\{G_{\alpha+\frac12}\le 0\}}\left(\frac{1+|u_\alpha|^2}{2}\log\left(1+|u_\alpha|^2\right) - \frac{1+|u_{\alpha+1}|^2}{2}\log(1+|u_{\alpha+1}|^2)\right)\mathrm{d}\boldsymbol{x} \\
&\qquad + |G_{\alpha+\frac12}|\mathbb{1}_{\{G_{\alpha+\frac12}\ge 0\}}\left(\frac{1+|u_{\alpha+1}|^2}{2}\log\left(1+|u_{\alpha+1}|^2\right) - \frac{1+|u_\alpha|^2}{2}\log(1+|u_\alpha|^2)\right)\mathrm{d}\boldsymbol{x} \\
&= -\sum_{\alpha=1}^{N-1} \int_{\mathbb{T}^d} |G_{\alpha+\frac12}|\left[\mathbb{1}_{\{G_{\alpha+\frac12}\le 0\}}\Psi(u_\alpha,u_{\alpha+1}) + \mathbb{1}_{\{G_{\alpha+\frac12}\ge 0\}}\Psi(u_{\alpha+1},u_\alpha)\right]\mathrm{d}\boldsymbol{x} \le 0.
\end{aligned}
$$

Indeed, the function $\Psi : (x,y) \mapsto y(y-x)\left[1+\log(1+y^2)\right]+\Phi(x)-\Phi(y)$ satisfies $\Psi(y,y)=0$ and $\partial_x\Psi(x,y)=\phi(x)-\phi(y)$. As the function ϕ defined by (2.11) is increasing, it shows that $\Psi(x,y)\ge 0$. Finally, we obtain the estimate

$$
\begin{aligned}
&\sum_{\alpha=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h\Phi(|u_\alpha|)\mathrm{d}\boldsymbol{x} + 3\nu\int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right]|D(u_\alpha)|^2\mathrm{d}\boldsymbol{x}) \\
&\quad \le \sum_{\alpha=1}^{N} C''_\nu\left(\int_{\mathbb{T}^d} h|\nabla u_\alpha|^2\mathrm{d}\boldsymbol{x}\right) + C'_\nu\left(\int_{\mathbb{T}^d} h^{\frac{6-\delta}{2-\delta}}\mathrm{d}\boldsymbol{x}\right)^{\frac{2-\delta}{2}} \times \left(\int_{\mathbb{T}^d} h\left[1+\log\left(1+|u_\alpha|^2\right)\right]^{\frac{2}{\delta}}\mathrm{d}\boldsymbol{x}\right)^{\frac{\delta}{2}} \\
&\qquad\qquad + g\int_{\mathbb{T}^d} h\frac{1+u_\alpha^2}{2}\left[1+\log(1+|u_\alpha|^2)\right]|\nabla z_b|\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Remark 11. *Let us notice that Ψ satisfies the following estimate*

$$
\frac{(y-x)^2}{2}\left[1+\log\bigl(1+\min(x,y)^2\bigr)\right] \le \Psi(x,y) \le \frac{(y-x)^2}{2}\left[3+\log\bigl(1+\max(x,y)^2\bigr)\right].
$$

Indeed given (2.11) *we check that*

$$
\Psi(x,y) = (y-x)\int_0^1 \left[\Phi'(y)-\Phi'\bigl(y+s(x-y)\bigr)\right]\,\mathrm{d}s = (y-x)^2\int_0^1\int_0^1 s\Phi''\bigl(y+s(1-t)(x-y)\bigr)\,\mathrm{d}s\mathrm{d}t
$$

with $\Phi''(z) = 1+\frac{2z^2}{1+z^2}+\log(1+z^2)$. *We obtain the bound above inserting* $\min(x,y)\le y+s(1-t)(x-y)\le\max(x,y)$ *for all* $(t,s)\in[0,1]^2$ *into the integral.*

We observe that the function $\Psi(u_\alpha, u_{\alpha+1})$ mesaures the distance between u_α and $u_{\alpha+1}$ in the Mellet-Vasseur inequality (2.9). In particular, this provides a better estimate for $u_{\alpha+1}-u_\alpha$ in $L^2_T(L^2)$ according to the classical energy inequality. Indeed we get here a factor $\log\big(1+\min(u_\alpha, u_{\alpha+1})^2\big)$.

2.4 Mellet-Vasseur gain of integrability on the velocity when $d = 1$

We are interested now in proving a gain of integrability on the velocity u_α when $d = 1$. Our goal is to prove that u_α belongs to $L^\infty([0,T]\times\mathbb{T}^1)$.

Proposition 2.4 (MV inequality)**.** *If we assume that $(h, u_1, \ldots, u_N)$ is a smooth solution of System* (1.1)*, then we have for any $p \geq 0$ and any $t \in [0,T]$ with $T > 0$:*

$$\begin{aligned}\Big(\sum_{\alpha=1}^{N}\frac{1}{p+2}\int_{\mathbb{T}^1} h|u_\alpha|^{p+2}(t,x)\mathrm{d}x\Big)^{\frac{1}{p+2}} \leq C_T\bigg(\Big(\sum_{\alpha=1}^{N}\frac{1}{p+2}\int_{\mathbb{T}^1} h_0|u_{0,\alpha}|^{p+2}\mathrm{d}x\Big)^{\frac{1}{p+2}} + T(1+\|h\|^4_{L^\infty_T(L^\infty)}) \\ + \|h\|_{L^\infty_T(L^\infty)}T\|\nabla z_b\|_{L^\infty_T(L^\infty)}\bigg).\end{aligned} \tag{2.12}$$

Proof. For $p \geq 0$ we have

1. $\displaystyle\int_{\mathbb{T}^1} h\partial_t u_\alpha u_\alpha |u_\alpha|^p \mathrm{d}x = \frac{1}{p+2}\int_{\mathbb{T}^1} h\partial_t |u_\alpha|^{p+2}\mathrm{d}x.$
2. $\displaystyle\int_{\mathbb{T}^1} hu_\alpha\partial_x u_\alpha u_\alpha|u_\alpha|^p\mathrm{d}x = \frac{1}{p+2}\int_{\mathbb{T}^1}|u_\alpha|^{p+2}\partial_t h\mathrm{d}x - \frac{N}{p+2}\int_{\mathbb{T}^1}|u_\alpha|^{p+2}(G_{\alpha+1/2}-G_{\alpha-1/2})\mathrm{d}x.$
3. $\displaystyle\int_{\mathbb{T}^1}\partial_x\left(4\nu h\partial_x u_\alpha\right)u_\alpha|u_\alpha|^p\mathrm{d}x = -(p+1)\int_{\mathbb{T}^1}4\nu h(\partial_x u_\alpha)^2|u_\alpha|^p\mathrm{d}x$
4. Using Young's inequality, we have

$$\begin{aligned}\int_{\mathbb{T}^1}\sum_{\alpha=1}^{N}\left[(u_{\alpha+1}-u_\alpha)-(u_\alpha-u_{\alpha-1})\right]u_\alpha|u_\alpha|^p\mathrm{d}x \leqslant \sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1}\frac{1}{p+2}|u_{\alpha+1}|^{p+2}\mathrm{d}x + \int_{\mathbb{T}^1}\frac{p+1}{p+2}|u_\alpha|^{p+2}]\mathrm{d}x \\ - \int_{\mathbb{T}^1}|u_\alpha|^{p+2}]\mathrm{d}x - \sum_{\alpha=2}^{N}\int_{\mathbb{T}^1}\frac{1}{p+2}|u_\alpha|^{p+2}\mathrm{d}x - \int_{\mathbb{T}^1}\frac{p+1}{p+2}|u_{\alpha-1}|^{p+2}]\mathrm{d}x - \int_{\mathbb{T}^1}|u_\alpha|^{p+2}]\mathrm{d}x \leqslant 0.\end{aligned}$$

5. All terms including $G_{\alpha+1/2}$ including last term of the right hand of the first item give (passing all terme at the left of the equality),

$$\begin{aligned}&-N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^1}[G_{\alpha+\frac12}(u_{\alpha+\frac12}-u_\alpha)+G_{\alpha-\frac12}(u_\alpha-u_{\alpha-\frac12})]u_\alpha|u_\alpha|^{p+1}\mathrm{d}x - \frac{N}{p+2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^1}|u_\alpha|^{p+2}(G_{\alpha+\frac12}-G_{\alpha-\frac12})\mathrm{d}x \\ &= -N\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1}G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}(u_{\alpha+1}-u_\alpha)u_\alpha|u_\alpha|^{p+1}\mathrm{d}x \\ &- N\sum_{\alpha=2}^{N}\int_{\mathbb{T}^1}G_{\alpha-\frac12}\mathbb{1}_{\{G_{\alpha-\frac12}\leqslant 0\}}(u_\alpha-u_{\alpha-1})u_\alpha|u_\alpha|^{p+1}\mathrm{d}x - N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^1}|u_\alpha|^{p+2}(G_{\alpha+\frac12}-G_{\alpha-\frac12})\mathrm{d}x \\ &= N\frac{p+1}{p+2}\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1}G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}|u_\alpha|^{p+2}\mathrm{d}x - N\frac{p+1}{p+2}\sum_{\alpha=2}^{N}\int_{\mathbb{T}^1}G_{\alpha-\frac12}\mathbb{1}_{\{G_{\alpha-\frac12}\leqslant 0\}}|u_\alpha|^{p+2}\mathrm{d}x \\ &- \frac{N}{p+2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^1}|u_\alpha|^{p+2}G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\leqslant 0\}}\mathrm{d}x + \frac{N}{p+2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^1}|u_\alpha|^{p+2}G_{\alpha-\frac12}\mathbb{1}_{\{G_{\alpha-\frac12}\geqslant 0\}}\mathrm{d}x\end{aligned}$$

$-N\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}u_{\alpha+1}u_\alpha|u_\alpha|^{p+1}\mathrm{d}x+N\sum_{\alpha=2}^{N}\int_{\mathbb{T}^1} G_{\alpha-\frac12}\mathbb{1}_{\{G_{\alpha-\frac12}\leqslant 0\}}u_{\alpha-1}u_\alpha|u_\alpha|^{p+1}\mathrm{d}x$ Using an usual Young's inequality we obtain

$$
\begin{aligned}
&N\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}u_{\alpha+1}u_\alpha|u_\alpha|^{p+1}\mathrm{d}x\\
&\qquad\leqslant \frac{p+1}{p+2}N\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}|u_\alpha|^{p+2}\mathrm{d}x+\frac{N}{p+2}\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}|u_{\alpha+1}|^{p+2}\mathrm{d}x \qquad (2.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&-N\sum_{\alpha=2}^{N}\int_{\mathbb{T}^1} G_{\alpha-\frac12}\mathbb{1}_{\{G_{\alpha-\frac12}\leqslant 0\}}u_{\alpha-1}u_\alpha|u_\alpha|^{p+1}\mathrm{d}x\\
&\qquad\leqslant -N\frac{p+1}{p+2}\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}|u_\alpha|^{p+2}\mathrm{d}x-\frac{N}{p+2}\sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^1} G_{\alpha+\frac12}\mathbb{1}_{\{G_{\alpha+\frac12}\geqslant 0\}}|u_{\alpha+1}|^{p+2}\mathrm{d}x. \qquad (2.14)
\end{aligned}
$$

We deduce

$$
\begin{aligned}
&\sum_{\alpha=1}^{N}\frac{1}{p+2}\int_{\mathbb{T}^1}\partial_t(h|u_\alpha|^{p+2})\mathrm{d}x+4(p+1)\int_{\mathbb{T}^1}\nu h(\partial_x u_\alpha)^2|u_\alpha|^p\mathrm{d}x\\
&\qquad\leqslant \sum_{\alpha=1}^{N}-\frac{g}{2}\int_{\mathbb{T}^1}\partial_x h^2\cdot u_\alpha|u_\alpha|^p\mathrm{d}x-g\int_{\mathbb{T}^1}h\partial_x z_b u_\alpha|u_\alpha|^p\mathrm{d}x. \qquad (2.15)
\end{aligned}
$$

We are going now to follow some ideas developed in [20] in order to get a gain of integrability on u for any $p\geq 0$. By integration by parts we have since (h,u_α) are regular:

$$
-\frac{g}{2}\int_0^t\int_{\mathbb{T}^1}\partial_x h^2 u_\alpha|u_\alpha|^p dsdx=\frac{g}{2}\int_0^t\int_{\mathbb{T}^1}\partial_x(|u_\alpha|^p u_\alpha)h^2 dsdx.
$$

Using Young inequality we get for $\epsilon>0$:

$$
|\frac{g}{2}\int_0^t\int_{\mathbb{T}^1}h^2|u_\alpha|^p\partial_x u_\alpha\mathrm{d}x\mathrm{d}s|\lesssim\left(\frac{\epsilon}{2}\int_0^t\int_{\mathbb{T}^1}h|u_\alpha|^p|\partial_x u_\alpha|^2\mathrm{d}x\mathrm{d}s+\frac{g^2}{8\epsilon}\int_0^t\int_{\mathbb{T}^1}h^3|u_\alpha|^p\mathrm{d}x\mathrm{d}s\right). \qquad (2.16)
$$

Plugging (2.16) in (2.15) with $\epsilon=1$ and since $\partial_x(|u_\alpha|^p u_\alpha)=(p+1)|u_\alpha|^p\partial_x u_\alpha$ we have using again Young inequality:

$$
\begin{aligned}
&\sum_{\alpha=1}^{N}\frac{1}{p+2}\int_{\mathbb{T}^1}h(t,x)|u_\alpha(t,x)|^{p+2}\mathrm{d}x+(p+1)\big(4\nu-\frac12\big)\int_0^t\int_{\mathbb{T}^1}h(\partial_x u_\alpha)^2|u_\alpha|^p\mathrm{d}x\mathrm{d}s\\
&\leqslant\sum_{\alpha=1}^{N}\frac{1}{p+2}\int_{\mathbb{T}^1}h_0(x)|u_{0,\alpha}(x)|^{p+2}\mathrm{d}x+\sum_{\alpha=1}^{N}\frac{g^2(p+1)}{8}\int_0^t\int_{\mathbb{T}^1}h^3|u_\alpha|^p\mathrm{d}x\mathrm{d}s-g\int_0^t\int_{\mathbb{T}^1}h\partial_x z_b u_\alpha|u_\alpha|^p\mathrm{d}x\mathrm{d}s\\
&\leqslant\sum_{\alpha=1}^{N}\frac{1}{p+2}\int_{\mathbb{T}^1}h_0(x)|u_{0,\alpha}(x)|^{p+2}\mathrm{d}x+\sum_{\alpha=1}^{N}\frac{g^2(p+1)}{8}\frac{1}{(p+2)\epsilon}\int_0^t\int_{\mathbb{T}^1}h|u_\alpha|^{p+2}\mathrm{d}x\mathrm{d}s dxds \qquad (2.17)\\
&+\sum_{\alpha=1}^{N}\Big(\frac{g^2\epsilon(p+1)^2}{8}\frac{1}{(p+2)}\int_0^t\int_{\mathbb{T}^1}h^{\frac{2(p+3)}{p+2}}(s,x)\mathrm{d}x\mathrm{d}s+\frac{p+1}{(p+2)\epsilon'}g\int_0^t\int_{\mathbb{T}^1}h|u_\alpha|^{p+2}(s,x)\mathrm{d}x\mathrm{d}s\\
&+\frac{\epsilon' g}{p+2}\int_0^t\int_{\mathbb{T}^1}h(\partial_x z_b)^{p+2}(s,x)dxds\Big).
\end{aligned}
$$

We deduce that:

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{\mathbb{T}^1} h(t,x)|u_\alpha(t,x)|^{p+2}\mathrm{d}x + (p+1)(p+2)\big(4\nu-\frac{1}{2}\big)\int_0^t\int_{\mathbb{T}^1} h(\partial_x u_\alpha)^2|u_\alpha|^p\mathrm{d}x\mathrm{d}s \\
&\leqslant \sum_{\alpha=1}^{N}\int_{\mathbb{T}^1} h_0(x)|u_{0,\alpha}(x)|^{p+2}\mathrm{d}x + \sum_{\alpha=1}^{N}(p+1)\big(\frac{g^2}{8}\frac{1}{\epsilon}+\frac{1}{\epsilon'}\big)\int_0^t\int_{\mathbb{T}^1} h|u_\alpha|^{p+2}\mathrm{d}x\mathrm{d}s dxds \\
&+\sum_{\alpha=1}^{N}\frac{g^2\epsilon(p+1)^2}{8}\int_0^t\int_{\mathbb{T}^1} h^{\frac{2(p+3)}{p+2}}(s,x)\mathrm{d}x\mathrm{d}s + \epsilon' g\int_0^t\int_{\mathbb{T}^1} h(\partial_x z_b)^{p+2}(s,x)dxds.
\end{aligned} \tag{2.18}$$

Using Grönwall lemma, we have for $C_1>0, C_2>0$ and $C_3>0$ independent on p and for any $t\in[0,T]$:

$$\begin{aligned}
\sum_{\alpha=1}^{N}\|h^{\frac{1}{p+2}}u_\alpha(t)\|_{L^{p+2}}^{p+2} \le C_2 e^{C_1T(p+1)}\big(&\sum_{\alpha=1}^{N}\|h^{\frac{1}{p+2}}u_\alpha(t)\|_{L^{p+2}}^{p+2} + T(p+1)^2(1+\|h\|_{L^\infty_T(L^\infty)}^4) \\
&+ C_3\|h\|_{L^\infty_T(L^\infty)}T\|\nabla z_b\|_{L^\infty_T(L^{p+2})}^{p+2}\big).
\end{aligned} \tag{2.19}$$

Next we have that for any $p\ge 0$, there exists $C_1'>0$, $C_2'>0$, $C_3'>0$ independent on p:

$$\begin{aligned}
\big(\sum_{\alpha=1}^{N}\|h^{\frac{1}{p+2}}u_\alpha(t)\|_{L^{p+2}}^{p+2}\big)^{\frac{1}{p+2}} \le C_2' e^{C_1'T}\big(&\big(\sum_{\alpha=1}^{N}\|h^{\frac{1}{p+2}}u_\alpha(t)\|_{L^{p+2}}^{p+2}\big)^{\frac{1}{p+2}} + T(1+\|h\|_{L^\infty_T(L^\infty)}^4) \\
&+ C_3'\|h\|_{L^\infty_T(L^\infty)}T\|\nabla z_b\|_{L^\infty_T(L^\infty)}\big).
\end{aligned} \tag{2.20}$$

It ends up the proof. □

We deduce in particular the following proposition.

Proposition 2.5. *Under the same assumption as in Proposition 2.4, let us consider $(h^n,u^n)_{n\in\mathbb{N}}$ a regularising sequence of System (1.1) such that h^n verifies for any $t>0$:*

$$h^n(t,x)\ge C_{n,t}>0 \ \ \forall x\in\mathbb{T}^1.$$

We have then for any $\alpha\in\{1,\dots,N\}$ and for all $n\in\mathbb{N}$:

$$\|u_\alpha^n\|_{L^\infty_T(L^\infty(\mathbb{T}^1))}\le C_T \ \ \forall T>0. \tag{2.21}$$

Proof: We observe that $\forall\epsilon>0$ sufficiently small, we have for any $p\ge 2$ and $t\in(0,T)$:

$$\begin{aligned}
\|(h^n)^{\frac{1}{p}}u_\alpha^n(t)\|_{L^p} &\ge \big(\int_{\{x,\ |u_\alpha^n(t,x)|\ge\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon\}} h^n(t,x)|u_\alpha^n|^p(t,x)dx\big)^{\frac{1}{p}} \\
&\ge (\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon)\big(\int_{\{x,\ |u_\alpha^n(t,x)|\ge\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon\}} h^n(t,x)dx\big)^{\frac{1}{p}} \\
&\ge (\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon)C_{n,T}^{\frac{1}{p}}\left|\{x,\ |u_\alpha^n(t,x)|\ge\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon\}\right|^{\frac{1}{p}}
\end{aligned} \tag{2.22}$$

Since we have $C_{n,T}>0$ and $\left|\{x,\ |u_\alpha^n(t,x)|\ge\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon\}\right|>0$, we can pass to the limit when p goes to $+\infty$ in (2.22). It implies that for any $\epsilon>0$, we get using (2.12):

$$\|u_\alpha^n(t,\cdot)\|_{L^\infty}-\epsilon\le C_T.$$

It concludes the proof of the proposition.

□

3 Stability of global weak solutions

Let us assume that we have proved the existence of a sequence of global approximate weak solutions $(h^n, u^n)_{n\in\mathbb{Z}}$ verifying uniformly all estimates of section 2. In addition we assume that:

$$h^n(t,x) > 0 \text{ almost everywhere on } (0,+\infty)\times\mathbb{T}^d. \tag{3.1}$$

In other word there exists C independent on $n\in\mathbb{N}$ such that for every α and all $T>0$ we have:

$$\left\|\sqrt{h^n}u^n_\alpha\right\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C, \tag{3.2a}$$

$$\|h^n\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C, \tag{3.2b}$$

$$\left\|\sqrt{h^n}\nabla u^n_\alpha\right\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C, \tag{3.2c}$$

$$\|\nabla h^n\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C, \tag{3.2d}$$

$$\left\|\nabla\sqrt{h^n}\right\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C. \tag{3.2e}$$

The initial data satisfy the following conditions:

$$\begin{gathered}
h^n_0 \text{ is bounded in } L^2(\mathbb{T}^d),\ h^n_0 \ge 0 \text{ a.e. in } \mathbb{T}^d,\\
h^n_0|u^n_{\alpha,0}|^2 = |m^n_{\alpha,0}|^2/h^n_0 \text{ is bounded in } L^1(\mathbb{T}^d),\\
\nabla\sqrt{h^n_0} \text{ is bounded in } L^2(\mathbb{T}^d),\\
\int_{\mathbb{T}^d} h^n_0\frac{1+|u^n_{\alpha,0}|^2}{2}\log(1+|u^n_{\alpha,0}|^2)\mathrm{d}\boldsymbol{x} \le C.
\end{gathered} \tag{3.3}$$

The proof of the stability of the sequence $(h^n, u^n)_{n\in\mathbb{N}}$ follows the same arguments as in [27]. We adapt them to our case.

Step 1: Convergence of $\sqrt{h^n}$

Lemma 3.1. *We have that for any $T>0$:*

$$\begin{gathered}
\sqrt{h^n} \textit{ is bounded unifomly in } L^\infty\left(0,T;H^1(\mathbb{T}^d)\right),\\
\partial_t\sqrt{h^n} \textit{ is bounded uniformly in } L^2\left(0,T;H^{-1}(\mathbb{T}^d)\right).
\end{gathered}$$

As a consequence, up to a subsequence, $\left(\sqrt{h^n}\right)$ converges a.e. and strongly in $L^2\left(0,T;L^2(\mathbb{T}^d)\right)$. We write

$$\sqrt{h^n}\longrightarrow\sqrt{h} \textit{ a.e. and } L^2((0,T)\times\mathbb{T}^d) \textit{ strong.}$$

Moreover, (h^n) converges strongly to h in $C\left([0,T];L^{\frac{3}{2}}(\mathbb{T}^d)\right)$.

Proof. $\sqrt{h^n}$ is uniformly bounded in $L^\infty((0,T),H^1(\mathbb{T}^d))$ due to (3.2b) and (3.2e). The estimate on $\partial_t\sqrt{h^n}$ can be deduced from the mass equation since

$$\partial_t\sqrt{h^n} = \frac{1}{2}\sqrt{h^n}\,\mathrm{div}\,\bar{u}^n - \mathrm{div}\left(\sqrt{h^n}\bar{u}^n\right). \tag{3.4}$$

The first term in the right hand side is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$ and the second term is bounded in $L^\infty(0,T;H^{-1}(\mathbb{T}^d))$, so it implies that $\partial_t\sqrt{h^n}$ is uniformly bounded in $L^2((0,T),H^{-1}(\mathbb{T}^d))$. Aubin Lions Lemma gives directly the strong convergence in $L^2((0,T)\times\mathbb{T}^d)$.

To prove the convergence in $C([0,T];L^{\frac{3}{2}}(\mathbb{T}^d))$ we first deduce by Sobolev embedding that $\sqrt{h^n}$ is bounded in $L^\infty(0,T;L^6(\mathbb{T}^d))$. We deduce that

$$h^n\bar{u}^n = \frac{\sqrt{h^n}}{N}\sum_{\alpha=1}^N\left(\sqrt{h^n}u^n_\alpha\right) \text{ is bounded in } L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^d)).$$

The continuity equation thus yields $\partial_t h^n$ bounded in $L^\infty(0,T;W^{-1,\frac{3}{2}}(\mathbb{T}^d))$ and since $\nabla h^n = 2\sqrt{h^n}\nabla\sqrt{h^n}$ we have h^n bounded in $L^\infty((0,T),W^{1,\frac{3}{2}}(\mathbb{T}^d))$. From Aubin Lions theorem we deduce that h^n converges to h up a subsequence in $C([0,T];L^{\frac{3}{2}}(\mathbb{T}^d))$. □

Step 2: Convergence of the pressure

Lemma 3.2. *The pressure $(h^n)^2$ is bounded in $L^r((0,T)\times\mathbb{T}^d)$ for all $r\in[1,2[$. In particular, $(h^n)^2$ converge to h^2 strongly in $L^r((0,T)\times\mathbb{T}^d)$ for every $T>0$ and $1\le r<\frac{5}{3}$.*

Proof. From inequalities (3.2b) and (3.2d) we deduce that $h^n\in L^2(0,T;H^1(\mathbb{T}^d))$. We deduce $h^n\in L^2(0,T;L^6(\mathbb{T}^d))$. In addition h^n is bounded in $L^\infty((0,T),L^2(\mathbb{T}^d))$. By interpolation h^n is bounded in $L^{\frac{10}{3}}((0,T)\times\mathbb{T}^d)$. We conclude recalling that $(h^n)^2$ converges almost everywhere to h^2 and is uniformly bounded in $L^{\frac{5}{3}}((0,T)\times\mathbb{T}^d)$, it implies then the strong convergence of $(h^n)^2$ to h^2 in $L^r((0,T)\times\mathbb{T}^d)$ for $1\le r<\frac{5}{3}$. □

Step 3: Bound for $\sqrt{h^n}u_\alpha^n$

Lemma 3.3. *$h^n|u_\alpha^n|^2\log(1+|u_\alpha^n|^2)$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$.*

Proof. We use the inequality given by the Mellet-Vasseur approach and it implies that :

$$\sum_{\alpha=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h^n\frac{1+|u_\alpha^n|^2}{2}\log(1+|u_\alpha^n|^2)\mathrm{d}\boldsymbol{x}+\int_{\mathbb{T}^d}3\nu h^n(1+\log(1+|u_\alpha^n|^2))|D(u_\alpha^n)|^2\mathrm{d}\boldsymbol{x}$$
$$\le C+C\sum_{\alpha=1}^N\left(\int_{\mathbb{T}^d}(h^n)^{\frac{6-\delta}{2-\delta}}\mathrm{d}\boldsymbol{x}\right)^{\frac{2-\delta}{2}}\times\left(\int_{\mathbb{T}^d}[2+\log(1+|u_\alpha^n|^2)]^{\frac{2}{\delta}}h^n\mathrm{d}\boldsymbol{x}\right)^{\frac{\delta}{2}}$$
$$+g\sum_{\alpha=1}^N\int_{\mathbb{T}^d}h^n\frac{1+(u_\alpha^n)^2}{2}(1+\log(1+|u_\alpha^n|^2))\nabla z_b\mathrm{d}\boldsymbol{x}$$

for any $\delta\in(0,2)$. Since h^n is uniformly bounded in $L^{\frac{10}{3}}((0,T)\times\mathbb{T}^d)$ we deduce that for δ small enough:

$$C\left(\int_{\mathbb{T}^d}(h^n)^{\frac{6-\delta}{2-\delta}}\mathrm{d}\boldsymbol{x}\right)^{\frac{2-\delta}{2}}\times\left(\int_{\mathbb{T}^d}[2+\log(1+|u_\alpha^n|^2)]^{\frac{2}{\delta}}h^n\mathrm{d}\boldsymbol{x}\right)^{\frac{\delta}{2}}\le C.$$

Since z_b is bounded in $W^{1,\infty}(\mathbb{T}^d)$, applying Grönwall's lemma we deduce that

$$\sum_{\alpha=1}^N\int_{\mathbb{T}^d}h^n\frac{1+|u_\alpha^n|^2}{2}\log(1+|u_\alpha^n|^2)(t)\mathrm{d}\boldsymbol{x}\le C_T\qquad\forall t\in(0,T). \tag{3.5}$$

□

Step 4: Convergence of the momentum

Lemma 3.4. *Up to a subsequence, the momentum $m_\alpha^n=u_\alpha^n h^n$ converges strongly in $L^2(0,T;L^p(\mathbb{T}^d))$ to some $m_\alpha(t,\boldsymbol{x})$ for all $p\in[1,2)$. In particular*

$$h^n u_\alpha^n \xrightarrow[n\to+\infty]{} m_\alpha \text{ almost everywhere in } \mathbb{T}^d\times(0,T)$$

Proof. We have $h^n u_\alpha^n=\sqrt{h^n}\sqrt{h^n}u_\alpha^n$. Since $\sqrt{h^n}$ is bounded in $L^\infty(0,T,L^6(\mathbb{T}^d))$ we deduce that $h^n u_\alpha^n$ is bounded in $L^\infty(0,T;L^{\frac{3}{2}}(\mathbb{T}^d))$. Next, since $\nabla(h^n u_\alpha^n)=\sqrt{h^n}\sqrt{h^n}\nabla u_\alpha^n+2\sqrt{h^n}u_\alpha^n\nabla\sqrt{h^n}$, we obtain that $\nabla(h^n u_\alpha^n)$ is bounded in

$L^2((0,T),L^1(\mathbb{T}^d))$.
In particular, we have

$$h^n u_\alpha^n \text{ is bounded in } L^2(0,T;W^{1,1}(\mathbb{T}^d)).$$

Let us bound now $\partial_t(h^n u_\alpha^n)$ in order to apply the Aubin-Lions lemma. Let us consider the momentum equation (1.1-b,c,d). We have then the uniform bounded estimates using in particular the fact that $h^n u_\alpha^n$ is bounded in $L^2(0,T;W^{1,1}(\mathbb{T}^d))$:

$$\begin{aligned}
&\operatorname{div}(h^n u_\alpha^n \otimes u_\alpha^n) = \operatorname{div}(\sqrt{h}^n u_\alpha^n \otimes \sqrt{h} u_\alpha^n) \in L^\infty(0,T;W^{-1,1}(\mathbb{T}^d))\\
&\operatorname{div}(hDu_\alpha^n) = \operatorname{div}\big(D(h^n u_\alpha^n) - \frac{1}{2}(u_\alpha^n \otimes \nabla h^n +^t (u_\alpha^n \otimes \nabla h^n))\big) \in L^2(0,T;W^{-1,1}(\mathbb{T}^d))\\
&\nabla h^2 \in L^\infty(0,T;W^{-1,1}(\mathbb{T}^d)).
\end{aligned}$$

Now using the remark 8, we have:

$$u^n_{\alpha+\frac{1}{2}} G^n_{\alpha+\frac{1}{2}} = \frac{G^n_{\alpha+\frac{1}{2}}}{\sqrt{h^n}} \sqrt{h^n} u^n_{\alpha+\frac{1}{2}} \in L^2(0,T;L^1(\mathbb{T}^d)).$$

In addition we know that $u_\alpha^n - u_{\alpha+1}^n$ is uniformly bounded in $L^2((0,T)\times\mathbb{T}^d)$ and $h^n \nabla z_b$ is bounded in $L^\infty((0,T),L^2(\mathbb{T}^d))$. In conclusion $\partial_t(h^n u_\alpha^n)$ is uniformly bounded in $L^2(0,T;W^{-1,1}(\mathbb{T}^d))$ and using the Aubin Lions lemma we deduce that $h^n u_\alpha^n$ converges strongly in $L^2((0,T),W^{s,1}(\mathbb{T}^d))$ for $-1 \le s < 1$. By Sobolev embedding we obtain what we wish. □

Note that we can define $u_\alpha(t,\boldsymbol{x}) = m_\alpha(t,\boldsymbol{x})/h(t,\boldsymbol{x})$ in $E = \{(t,\boldsymbol{x}); h(t,\boldsymbol{x}) > 0\}$ but $u_\alpha(t,\boldsymbol{x})$ is not uniquely defined in the vacuum set E^c. In order to define properly u_α on $\{h=0\}$, we have to study the weak limit of the terms $u_{\alpha+1}^n - u_\alpha^n$.

Step 5: Convergence of $u_{\alpha+1}^n - u_\alpha^n$

We know via the energy estimate (2.1) that $(u_{\alpha+1}^n - u_\alpha^n)$ is uniformly bounded in $L^2((0,T)\times\mathbb{T}^d)$ then $(u_{\alpha+1}^n - u_\alpha^n)$ converges weakly in $L^2_T(L^2)$ to m_α^1 up to a subsequence. Now when $d \ge 1$, we have since $\mathbb{1}_{\{h=0\}} \in L^\infty_T(L^\infty)$

$$(u_{\alpha+1}^n - u_\alpha^n)\mathbb{1}_{\{h=0\}} \xrightarrow[n\to+\infty]{} m_\alpha^{\mathbb{1}}\mathbb{1}_{\{h=0\}} \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^d).$$

Since u_α^n converges almost everywhere to u_α on $\{h>0\}$. We have then:

$$(u_{\alpha+1}^n - u_\alpha^n)\mathbb{1}_{\{h>0\}} \xrightarrow[n\to+\infty]{} (u_{\alpha+1} - u_\alpha)\mathbb{1}_{\{h>0\}} = m_\alpha^1\mathbb{1}_{\{h>0\}} \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^d).$$

We assume now that $u_1 = 0$ on $\{h=0\}$. It defines all the values of u_α on $\{h=0\}$ since u_1 is defined. Indeed we have $u_1 = 0$ on $\{h=0\}$ and by iteration $u_2 = m_1^1$ on $\{h=0\}$ and so one. When $d=1$, we know that u_α is uniformly bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ then up to a subsequence since h is continuous:

$$u_\alpha^n\mathbb{1}_{\{h=0\}} \xrightarrow[n\to+\infty]{} u_\alpha\mathbb{1}_{\{h=0\}} \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^1).$$

Step 6: Convergence of $\sqrt{h^n}u_\alpha^n$

Lemma 3.5. *The quantity $\sqrt{h^n}u_\alpha^n$ converges strongly in $L^2((0,T)\times\mathbb{T}^d)$ to $m_\alpha/\sqrt{h}$ (defined to be zero when $h=0$).*

In particular, we have $m_\alpha(t,\boldsymbol{x}) = 0$. a.e. on E^c and there exists a function $u_\alpha(t,\boldsymbol{x})$ such that $m_\alpha(t,\boldsymbol{x}) = h(t,\boldsymbol{x})u_\alpha(t,\boldsymbol{x})$ and

$$\sqrt{h^n}u_\alpha^n \longrightarrow \sqrt{h}u_\alpha \text{ strongly in } L^2((0,T)\times\mathbb{T}^d) \tag{3.6}$$

Proof. Since $m_\alpha^n/\sqrt{h^n}$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$, Fatou's lemma yields for almost every $t \in (0,T)$

$$\int \liminf_{n\to+\infty} \frac{(m_\alpha^n)^2}{h^n}(t)\mathrm{d}\boldsymbol{x} \le \liminf_{n\to+\infty} \int \frac{(m_\alpha^n)^2}{h^n}(t)\mathrm{d}\boldsymbol{x} < \infty$$

In particular, we have $m_\alpha(t,\boldsymbol{x}) = 0$ *a.e.* in $\{h(t,\boldsymbol{x}) = 0\}$. So, if we define the limit velocity $u_\alpha(t,\boldsymbol{x})$ by setting $u_\alpha(t,\boldsymbol{x}) = m_\alpha(t,\boldsymbol{x})/h(t,\boldsymbol{x})$ when $h(t,\boldsymbol{x}) \neq 0$ and $u_\alpha(t,\boldsymbol{x}) = 0$ when $h(t,\boldsymbol{x}) = 0$, we have

$$m_\alpha(t,\boldsymbol{x}) = h(t,\boldsymbol{x})u_\alpha(t,\boldsymbol{x})$$

and

$$\int_{\mathbb{T}^d} \frac{m_\alpha^2}{h}\mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^d} h|u_\alpha|^2\mathrm{d}\boldsymbol{x} < \infty$$

Moreover, Fatou's lemma yields that for almost every $t \in (0,T)$

$$\begin{aligned}
&\int_{\mathbb{T}^d} h|u_\alpha|^2\log(1+|u_\alpha|^2)(t)\mathrm{d}\boldsymbol{x} = \int_{\{h>0\}} h|u_\alpha|^2\log(1+|u_\alpha|^2)(t)\mathrm{d}\boldsymbol{x} \\
&= \int_{\{h>0\}} \liminf_{n\to+\infty} h^n|u_\alpha^n|^2\log(1+|u_\alpha^n|^2)(t)\mathrm{d}\boldsymbol{x} \leq \liminf_{n\to+\infty}\int_{\mathbb{T}^d} h^n|u_\alpha^n|^2\log(1+|u_\alpha^n|^2)(t)\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Let us point out that since $u_\alpha^n = \frac{m_\alpha^n}{h^n}$ has a limit on $\{h>0\}$ which is u_α and in addition $\frac{m_\alpha^n}{h^n}$ is well defined because $h^n > 0$ almost everywhere. We deduce that $h|u_\alpha|^2\log(1+|u_\alpha|^2)$ is in $L^\infty(0,T;L^1(\mathbb{T}^d))$.
Next, since m_α^n and h^n converge almost everywhere, it is readily seen that in $\{h(t,\boldsymbol{x}) \neq 0\}$, $\sqrt{h^n}u_\alpha^n = m_\alpha^n/\sqrt{h^n}$ converges almost everywhere to $\sqrt{h}u_\alpha = m_\alpha/\sqrt{h}$ (we observe that $m_\alpha^n/\sqrt{h^n}$ has a sense since $h^n > 0$ almost everywhere). Moreover, we have

$$\sqrt{h^n}u_\alpha^n\mathbb{1}_{\{|u_\alpha^n|\leq M\}} \xrightarrow[n\to+\infty]{} \sqrt{h}u_\alpha\mathbb{1}_{\{|u_\alpha|\leq M\}} \text{ almost everywhere.} \tag{3.7}$$

As a matter of fact, the convergence holds almost everywhere in $\{h(t,\boldsymbol{x}) \neq 0\}$, and in $\{h(t,\boldsymbol{x}) = 0\}$, we have $\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\leq M} \leq M\sqrt{h^n} \to 0$. To conclude to proof of lemma, for $M > 0$, there exists $C > 0$ such that:

$$\begin{aligned}
\int_0^T\int_{\mathbb{T}^d} |\sqrt{h^n}u_\alpha^n - \sqrt{h}u_\alpha|^2\mathrm{d}\boldsymbol{x} \leq & C\int_0^T\int_{\mathbb{T}^d} |\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\leq M} - \sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\leq M}|^2\mathrm{d}\boldsymbol{x} \\
&+ C\int_0^T\int_{\mathbb{T}^d} |\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\geq M}|^2\mathrm{d}\boldsymbol{x} \\
&+ C\int_0^T\int_{\mathbb{T}^d} |\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\geq M}|^2\mathrm{d}\boldsymbol{x}
\end{aligned}$$

We observe that:

$$\int_0^T\int_{\mathbb{T}^d} |\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\geq M}|^2\mathrm{d}\boldsymbol{x} \leq \frac{1}{\log(1+M^2)}\int_0^T\int_{\mathbb{T}^d} h^n|u_\alpha^n|^2\log(1+|u_\alpha^n|^2)\mathrm{d}\boldsymbol{x}$$

and

$$\int_0^T\int_{\mathbb{T}^d} |\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\geq M}|^2\mathrm{d}\boldsymbol{x} \leq \frac{1}{\log(1+M^2)}\int_{\mathbb{T}^d} h|u_\alpha|^2\log(1+|u_\alpha|^2)\mathrm{d}\boldsymbol{x}$$

We have now:

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}|\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M}-\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\le M}|^2\mathrm{d}\boldsymbol{x}\\
&\le C\Big(\int_0^T\int_{\mathbb{T}^d}|\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}\le M}-\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\le M,\sqrt{h}\le M}|^2\mathrm{d}\boldsymbol{x}\\
&\qquad\qquad+\int_0^T\int_{\mathbb{T}^d}|\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}>M}-\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\le M,\sqrt{h}>M}|^2\mathrm{d}\boldsymbol{x}\Big)\\
&\le C\Big(\int_0^T\int_{\mathbb{T}^d}|\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}\le M}-\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\le M,\sqrt{h}\le M}|^2\mathrm{d}\boldsymbol{x}\\
&+\int_0^T\int_{\mathbb{T}^d}|(\sqrt{h^n}-\sqrt{h})u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}>M}|\mathrm{d}\boldsymbol{x}+\int_0^T\int_{\mathbb{T}^d}|\sqrt{h}(u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}>M}-u_\alpha\mathbb{1}_{|u_\alpha|\le M,\sqrt{h}>M})|^2\mathrm{d}\boldsymbol{x}\Big)\\
&\le C\Big(\int_0^T\int_{\mathbb{T}^d}|\sqrt{h^n}u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}\le M}-\sqrt{h}u_\alpha\mathbb{1}_{|u_\alpha|\le M,\sqrt{h}\le M}|^2\mathrm{d}\boldsymbol{x}\\
&+\int_0^T\int_{\mathbb{T}^d}|(\sqrt{h^n}-\sqrt{h})u_\alpha^n\mathbb{1}_{|u_\alpha^n|\le M,\sqrt{h^n}>M}|\mathrm{d}\boldsymbol{x}+\int_0^T\int_{\mathbb{T}^d}|\sqrt{h}\mathbb{1}_{\{\sqrt{h}>M\}}(u_\alpha^n\mathbb{1}_{\{|u_\alpha^n|\le M\}}-u_\alpha\mathbb{1}_{\{|u_\alpha|\le M\}})|^2\mathrm{d}\boldsymbol{x}\Big)\\
&+\int_0^T\int_{\mathbb{T}^d}\sqrt{h}|u_\alpha^n\mathbb{1}_{\{|u_\alpha^n|\le M\}}|\,|\mathbb{1}_{\{|\sqrt{h}>M\}}-\mathbb{1}_{\{|\sqrt{h^n}>M\}}|\mathrm{d}\boldsymbol{x}
\end{aligned}$$

The first term on the right hand side converges to 0 when M goes to $+\infty$ by dominated convergence. The second term converges to 0 when n goes to $+\infty$ since $\sqrt{h^n}$ converges strongly to $\sqrt{h}$ in $L^2((0,T)\times\mathbb{T}^d)$. The third and fourth term converge to 0 when M goes to $+\infty$ when we apply the Tchebytchev lemma. We deduce that:

$$\limsup_{n\to+\infty}\int_{\mathbb{T}^d}|\sqrt{h^n}u_\alpha^n-\sqrt{h}u_\alpha|^2\mathrm{d}\boldsymbol{x}=0.$$

□

Step 7: Convergence of the diffusion terms

Lemma 3.6. *We have*

$$\operatorname{div}(h_n\nabla u_\alpha^n)\to\operatorname{div}(h\nabla u)\ \textit{in}\ \mathcal{D}'((0,T)\times\mathbb{T}^d)\tag{3.8}$$

$$\operatorname{div}(h_n\nabla^T u_\alpha^n)\to\operatorname{div}(h\nabla^T u)\ \textit{in}\ \mathcal{D}'((0,T)\times\mathbb{T}^d).\tag{3.9}$$

Proof. Let $\phi(t,x)$ a test function, then

$$\begin{aligned}
\int_0^T\int_{\mathbb{T}^d}\operatorname{div}(h^n\nabla u_\alpha^n)\phi\,\mathrm{d}\boldsymbol{x}&=-\int_0^T\int_{\mathbb{T}^d}h^n\nabla u_\alpha^n:\nabla\phi\,\mathrm{d}\boldsymbol{x}\\
&=\int_0^T\int_{\mathbb{T}^d}(\nabla h^n\cdot\nabla\phi)\cdot u_\alpha^n\,\mathrm{d}\boldsymbol{x}+\int_0^T\int_{\mathbb{T}^d}h^n u_\alpha^n\cdot\Delta\phi\mathrm{d}\boldsymbol{x}
\end{aligned}$$

Thanks to lemma 3.4, $h^n u_{\alpha,n}$ converges strongly in $L^2(0,T;L^p(\mathbb{T}^d))$ with $1\le p<2$. This is enough to prove the convergence of the second term. For the first term, we have $\nabla h^n.u_\alpha^n=2\nabla\sqrt{h^n}\cdot\sqrt{h^n}u_\alpha^n$, we know that $\sqrt{h^n}u_\alpha^n$ converges strongly in $L^2((0,T)\times\mathbb{T}^d)$ and $\nabla\sqrt{h^n}$ converges weakly in $L^2((0,T)\times\mathbb{T}^d)$ then $\nabla h^n.u_\alpha^n$ converges in the sense of distributions to $\nabla h\cdot u_\alpha$.

Step 7: Convergence of $G^n_{\alpha+\frac12}u^n_{\alpha+\frac12}$ Let us recall that we have:

$$G^n_{\alpha+\frac12}=\frac{1}{N^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\operatorname{div}\big(h^n(u_j^n-u_i^n)\big).$$

Since we know that $\sqrt{h^n}u_j^n$ converges strongly to $\sqrt{h}u_j$ in $L^2_{t,x}$ and $\sqrt{h^n}$ converges strongly to $\sqrt{h}$ we deduce that $G^n_{\alpha+\frac{1}{2}}$ converges in the sense of distributions to $G_{\alpha+\frac{1}{2}}$ with:

$$G_{\alpha+\frac{1}{2}} = \frac{1}{N^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\operatorname{div}\big(h(u_j-u_i)\big).$$

We recall that we have:

$$G^n_{\alpha+\frac{1}{2}}u^n_{\alpha+\frac{1}{2}} = \frac{1}{2}G^n_{\alpha+\frac{1}{2}}(u^n_\alpha+u^n_{\alpha+1}) - \frac{1}{2}|G^n_{\alpha+\frac{1}{2}}|(u^n_\alpha-u^n_{\alpha+1}). \tag{3.10}$$

Let us consider the first term on the right hand side $G^n_{\alpha+\frac{1}{2}}(u^n_\alpha+u^n_{\alpha+1})$. We are going to show that $G^n_{\alpha+\frac{1}{2}}(u^n_\alpha+u^n_{\alpha+1})$ converges in the sense of the distribution to $G_{\alpha+\frac{1}{2}}(u_\alpha+u_{\alpha+1})$. Let us take φ a C^∞ function with compact support in $(0,T)\times\mathbb{T}^d$, we have then:

$$\int_0^T\int_{\mathbb{T}^d} G^n_{\alpha+\frac{1}{2}}u^n_\alpha\varphi dxdt = \frac{1}{N^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\int_0^T\int_{\mathbb{T}^d}\operatorname{div}\big(h^n(u^n_j-u^n_i)\big)u^n_\alpha\varphi dxdt$$
$$= -\frac{1}{N^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\int_0^T\int_{\mathbb{T}^d}\big(\sqrt{h^n}(u^n_j-u^n_i)\cdot\sqrt{h^n}\nabla u^n_\alpha\varphi - \sqrt{h^n}(u^n_j-u^n_i)\cdot\nabla\varphi\sqrt{h^n}u^n_\alpha\big)dxdt$$

Using the fact that $\sqrt{h^n}u^n_\alpha$ converges strongly to $\sqrt{h}u_\alpha$ in $L^\infty_T(L^2)$, that $\sqrt{h^n}\nabla u^n$ converges weakly up to a subsequence in $L^2_T(L^2)$ to $\sqrt{h}\nabla u$ (indeed we have this convergence also in the sense of the distribution), we deduce that:

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d} G^n_{\alpha+\frac{1}{2}}u^n_\alpha\,\varphi dxdt\\ &\xrightarrow[n\to+\infty]{} -\frac{1}{N^2}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\int_0^T\int_{\mathbb{T}^d}\big(\sqrt{h}(u_j-u_i)\cdot\sqrt{h}\nabla u_\alpha\varphi - \sqrt{h}(u_j-u_i)\cdot\nabla\varphi\sqrt{h}u_\alpha\big)dxdt\\ &\xrightarrow[n\to+\infty]{} \int_0^T\int_{\mathbb{T}^d} G_{\alpha+\frac{1}{2}}u_\alpha\varphi dxdt.\end{aligned} \tag{3.11}$$

We proceed similarly for the term $G^n_{\alpha+\frac{1}{2}}u^n_{\alpha+1}$. Let us writte now the second term on the right hand side of (3.10) as follows:

$$|G^n_{\alpha+\frac{1}{2}}|(u^n_\alpha-u^n_{\alpha+1}) = \mathbb{1}_{\{h=0\}}|G^n_{\alpha+\frac{1}{2}}|(u^n_\alpha-u^n_{\alpha+1}) + \mathbb{1}_{\{h>0\}}|G^n_{\alpha+\frac{1}{2}}|(u^n_\alpha-u^n_{\alpha+1}). \tag{3.12}$$

We know that $|G^n_{\alpha+\frac{1}{2}}| = |\frac{G^n_{\alpha+\frac{1}{2}}}{\sqrt{h^n}}\sqrt{h^n}|$ is uniformly bounded in $L^2_T(L^{\frac{3}{2}})$. We are going to prove now that $\mathbb{1}_{\{h=0\}}||G^n_{\alpha+\frac{1}{2}}|$ converges strongly to 0 in $L^1_T(L^1)$. We have then:

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|G^n_{\alpha+\frac{1}{2}}|dxdt \le \int_0^T\int_{\mathbb{T}^d}\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\mathbb{1}_{\{h=0\}}|\operatorname{div}\big(h^n(u^n_j-u^n_i)\big)|dxdt\\ &\le \sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\int_0^T\int_{\mathbb{T}^d}\big(\mathbb{1}_{\{h=0\}}\sqrt{h^n}|\sqrt{h^n}\operatorname{div}(u^n_j-u^n_i)| + 2\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}|u^n_j-u^n_i|\big)dxdt\end{aligned} \tag{3.13}$$

Since $|\sqrt{h^n}\operatorname{div}(u^n_j-u^n_i)|$ is uniformly bounded in $L^2_T(L^2)$ and $\sqrt{h^n}\mathbb{1}_{\{h=0\}}$ converges strongly to 0 in $L^p_T(L^{6-\epsilon})$ for any $p\ge 2$ and $\epsilon>0$, we obtain using Hölder inequality that:

$$\sum_{j=1}^{\alpha}\sum_{i=\alpha+1}^{N}\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}\sqrt{h^n}|\sqrt{h^n}\operatorname{div}(u^n_j-u^n_i)|dxdt \xrightarrow[n\to+\infty]{} 0.$$

Let us estimate now the second term on the right hand side of (3.13), we have then:

$$\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}|u^n_j-u^n_i|dxdt \le \int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}(|u^n_j|+|u^n_i|)dxdt$$

Let us consider simply the term in u_j^n, we have then:

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}|u_i^n|dxdt \leq \int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}|u_i^n|\mathbb{1}_{\{|u_i^n|\leq M\}}dxdt\\
&+\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}|u_i^n|\mathbb{1}_{\{|u_i^n|>M\}}dxdt\\
&\leq M\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}dxdt+\frac{1}{(\log(1+M^2))^{\frac{1}{2}}}\int_0^T\int_{\mathbb{T}^d}\mathbb{1}_{\{h=0\}}|\nabla\sqrt{h^n}|\sqrt{h^n}|u_i^n|(\log(1+|u_i^n|^2)^{\frac{1}{2}}dxdt
\end{aligned}$$

The first term on the right hand side goes to 0 when n goes to $+\infty$ since $\sqrt{h^n}\mathbb{1}_{\{h=0\}}$ converges strongly to 0 in $L_T^p(L^{6-\epsilon})$ and $\nabla\sqrt{h^n}$ is uniformly bounded in $L_T^\infty(L^2)$. The second term goes also to 0 when M goes to $+\infty$ and because the integral is uniformly bounded using the Mellet Vasseur inequality.
It proves that:

$$|G^n_{\alpha+\frac{1}{2}}|\mathbb{1}_{\{h=0\}}\xrightarrow[n\to+\infty]{}\ \text{ in }\ L^1((0,T)\times\mathbb{T}^d). \tag{3.14}$$

Let us deal now with the dimension $d=1$. We have using (3.14) and the fact that $|G^n_{\alpha+\frac{1}{2}}|$ is uniformly bounded in $L^2((0,T)\times\mathbb{T}^1)$ that $|G^n_{\alpha+\frac{1}{2}}|\mathbb{1}_{\{h=0\}}$ converges strongly in $L^{2-\epsilon}((0,T)\times\mathbb{T}^1)$ for any $\epsilon>0$. Now since we know that $u_\alpha^n-u_{\alpha+1}^n$ (see proposition 2.5) is uniformly bounded in $L_T^\infty(L^\infty)$, we deduce that:

$$|G^n_{\alpha+\frac{1}{2}}|\mathbb{1}_{\{h=0\}}(u_\alpha^n-u_{\alpha+1}^n)\xrightarrow[n\to+\infty]{}0\ \text{ in }\ \mathcal{D}'((0,T)\times\mathbb{T}^1).$$

Let us now consider the term $|G^n_{\alpha+\frac{1}{2}}|\mathbb{1}_{\{h>0\}}(u_\alpha^n-u_{\alpha+1}^n)$. On the set $\{h>0\}$, we know that $u_\alpha^n-u_{\alpha+1}^n$ converges almost everywhere to $u_\alpha-u_{\alpha+1}$ and that this term is uniformly bounded in $L_T^\infty(L^\infty)$. We deduce in particular that $\mathbb{1}_{\{h>0\}}(u_\alpha^n-u_{\alpha+1}^n)$ converges strongly in $L^p((0,T)\times\mathbb{T}^1)$ for any $p\geq 2$ to $\mathbb{1}_{\{h>0\}}(u_\alpha-u_{\alpha+1})$. Moreover $|G^n_{\alpha+\frac{1}{2}}|$ is uniformly bounded in $L^2((0,T)\times\mathbb{T}^1)$ then up to a subsequence, it converges weakly in $L^2((0,T)\times\mathbb{T}^1)$ to a function $M_{\alpha+\frac{1}{2}}\in L^2((0,T)\times\mathbb{T}^1)$. We deduce then that:

$$|G^n_{\alpha+\frac{1}{2}}|\mathbb{1}_{\{h>0\}}(u_\alpha^n-u_{\alpha+1}^n)\xrightarrow[n\to+\infty]{}\mathbb{1}_{\{h>0\}}M_{\alpha+\frac{1}{2}}(u_\alpha-u_{\alpha+1})\ \text{ in }\ \mathcal{D}'((0,T)\times\mathbb{T}^1). \tag{3.15}$$

In conclusion we have proved that:

$$G^n_{\alpha+\frac{1}{2}}u^n_{\alpha+\frac{1}{2}}\xrightarrow[n\to+\infty]{}\frac{1}{2}G_{\alpha+\frac{1}{2}}(u_\alpha+u_{\alpha+1})-\frac{1}{2}M_{\alpha+\frac{1}{2}}\mathbb{1}_{\{h>0\}}(u_\alpha-u_{\alpha+1})\ \text{ in }\ \mathcal{D}'((0,T)\times\mathbb{T}^1). \tag{3.16}$$

Let us consider now the case $d\geq 2$, we have seen that:

$$\frac{1}{2}G^n_{\alpha+\frac{1}{2}}(u_\alpha^n+u_{\alpha+1}^n)\xrightarrow[n\to+\infty]{}\frac{1}{2}G_{\alpha+\frac{1}{2}}(u_\alpha+u_{\alpha+1})\ \text{ in }\ \mathcal{D}'((0,T)\times\mathbb{T}^d). \tag{3.17}$$

In addition we know that $\frac{|G^n_{\alpha+\frac{1}{2}}|}{\sqrt{h^n}}$ is uniformly bounded in $L_T^2(L^2)$, it implies that up to a subsequence it converges to $M_{\alpha+\frac{1}{2}}$ in $L_T^2(L^2)$. In addition we know that $\sqrt{h^n}(u_\alpha^n-u_{\alpha+1}^n)$ converges strongly in $L_T^2(L^2)$ to $G_{\alpha+\frac{1}{2}}(u_\alpha-u_{\alpha+1})$. We have then:

$$\frac{1}{2}|G^n_{\alpha+\frac{1}{2}}|(u_\alpha^n+u_{\alpha+1}^n)\xrightarrow[n\to+\infty]{}\frac{1}{2}M_{\alpha+\frac{1}{2}}\sqrt{h}(u_\alpha+u_{\alpha+1})\ \text{ in }\ \mathcal{D}'((0,T)\times\mathbb{T}^d). \tag{3.18}$$

Finally we have prove that:

$$G^n_{\alpha+\frac{1}{2}}u^n_{\alpha+\frac{1}{2}}\xrightarrow[n\to+\infty]{}\frac{1}{2}G_{\alpha+\frac{1}{2}}(u_\alpha+u_{\alpha+1})-\frac{1}{2}M_{\alpha+\frac{1}{2}}\sqrt{h}(u_\alpha-u_{\alpha+1})\ \text{ in }\ \mathcal{D}'((0,T)\times\mathbb{T}^d). \tag{3.19}$$

□

4 Appendix

4.1 A useful identity for the proof of Proposition 2.2

We aim at proving the identity

$$\forall\, n \geq 2,\ \sum_{i=1}^{n-1}\sum_{j=1}^{i}\sum_{k=i+1}^{n}(u_j-u_k)(u_i-u_{i+1}) = \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(u_i-u_k)^2.$$

This is equivalent to showing

$$E_n := \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\left[\sum_{j=1}^{i}(u_j-u_k)(u_i-u_{i+1}) - (u_i-u_k)^2\right] = 0. \tag{4.1}$$

Let us first notice that ($k \geq i$) due to a telescoping procedure

$$(u_i-u_k)^2 = (u_i-u_k)\sum_{j=i}^{k-1}(u_j-u_{j+1}). \tag{4.2}$$

Inserting (4.2) into (4.1) and switching series twice, we get

$$\begin{aligned}
E_n &= \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\left[\sum_{j=1}^{i}(u_j-u_k)(u_i-u_{i+1}) - (u_i-u_k)\sum_{j=i}^{k-1}(u_j-u_{j+1})\right]\\
&= \sum_{k=2}^{n}\sum_{i=1}^{k-1}\left[\sum_{j=1}^{i}(u_j-u_k)(u_i-u_{i+1}) - (u_i-u_k)\sum_{j=i}^{k-1}(u_j-u_{j+1})\right]\\
&= \sum_{k=2}^{n}\left[\sum_{i=1}^{k-1}\sum_{j=1}^{i}(u_j-u_k)(u_i-u_{i+1}) - \sum_{i=1}^{k-1}(u_i-u_k)\sum_{j=i}^{k-1}(u_j-u_{j+1})\right]\\
&= \sum_{k=2}^{n}\underbrace{\left[\sum_{j=1}^{k-1}\sum_{i=j}^{k-1}(u_j-u_k)(u_i-u_{i+1}) - \sum_{i=1}^{k-1}\sum_{j=i}^{k-1}(u_i-u_k)(u_j-u_{j+1})\right]}_{=0}
\end{aligned}$$

which ends the proof.

4.2 Energy and BD entropy when $u_{\alpha+\frac{1}{2}} = \frac{1}{2}(u_\alpha + u_{\alpha+1})$ [14]

We recall that $G_{\frac{1}{2}} = G_{N+\frac{1}{2}} = 0$. With this definition of $u_{\alpha+\frac{1}{2}}$, the energy due to the term in $G_{\alpha+\frac{1}{2}}$ computed in section 2 gives

$$\begin{aligned}
&\frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} G_{\alpha+\frac{1}{2}}\left(|u_\alpha|^2 - |u_{\alpha+1}|^2\right)\,\mathrm{d}\boldsymbol{x} - N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\left(u_{\alpha+\frac{1}{2}}G_{\alpha+\frac{1}{2}} - u_{\alpha-\frac{1}{2}}G_{\alpha-\frac{1}{2}}\right)\cdot u_\alpha\,\mathrm{d}\boldsymbol{x}\\
&\quad = \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} G_{\alpha+\frac{1}{2}}\left(|u_\alpha|^2 + |u_{\alpha+1}|^2\right)\,\mathrm{d}\boldsymbol{x} - \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\left((u_{\alpha+1}+u_\alpha)G_{\alpha+\frac{1}{2}} - (u_{\alpha-1}-u_\alpha)G_{\alpha-\frac{1}{2}}\right)\cdot u_\alpha\,\mathrm{d}\boldsymbol{x}\\
&\quad = \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} G_{\alpha+\frac{1}{2}}\left(|u_\alpha|^2 - |u_{\alpha+1}|^2\right)\,\mathrm{d}\boldsymbol{x} - \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} u_{\alpha+1}\cdot u_\alpha G_{\alpha+\frac{1}{2}} + |u_\alpha|^2 G_{\alpha+\frac{1}{2}} - u_{\alpha-1}\cdot u_\alpha G_{\alpha-\frac{1}{2}} - |u_\alpha|^2 G_{\alpha-\frac{1}{2}}\,\mathrm{d}\boldsymbol{x}\\
&\quad = \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} G_{\alpha+\frac{1}{2}}\left(|u_\alpha|^2 - |u_{\alpha+1}|^2\right)\,\mathrm{d}\boldsymbol{x} - \frac{N}{2}\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} u_{\alpha+1}\cdot u_\alpha G_{\alpha+\frac{1}{2}} + |u_\alpha|^2 G_{\alpha+\frac{1}{2}}\\
&\quad + \frac{N}{2}\sum_{\alpha=0}^{N-1} u_\alpha\cdot u_{\alpha+1}G_{\alpha+\frac{1}{2}} + |u_{\alpha+1}|^2 G_{\alpha+\frac{1}{2}}\,\mathrm{d}\boldsymbol{x} = 0
\end{aligned} \tag{4.3}$$

and the energy is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} E\,\mathrm{d}\boldsymbol{x} + \sum_{\alpha=1}^{N}\int_{\mathbb{T}^d} 4\nu h|D(u_\alpha)|^2\,\mathrm{d}\boldsymbol{x} + N\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}\kappa(u_{\alpha+1}-u_\alpha)^2\,\mathrm{d}\boldsymbol{x} = 0 \tag{4.4}$$

with

$$E = \frac{1}{2}\left(Ngh^2 + \sum_{\alpha=1}^{N} hu_\alpha^2\right) + Ngz_b h. \tag{4.5}$$

For the BD-entropy, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha=1}^{N} G_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2 - |v_\alpha|^2) - \sum_{\alpha=1}^{N}\Big(G_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}} - u_\alpha) - G_{\alpha-\frac{1}{2}}(u_{\alpha-\frac{1}{2}} - u_\alpha)\Big)\cdot v_\alpha \\
&= \frac{1}{2}\sum_{\alpha=1}^{N} G_{\alpha+\frac{1}{2}}(|v_{\alpha+1}|^2 - |v_\alpha|^2) - \sum_{\alpha=1}^{N}(G_{\alpha+\frac{1}{2}}(u_{\alpha+\frac{1}{2}} - u_\alpha)\cdot v_\alpha + \sum_{\alpha=0}^{N-1} G_{\alpha+\frac{1}{2}}(u_{\alpha+1} - u_{\alpha+\frac{1}{2}})\cdot v_{\alpha+1} \\
&= \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[u_{\alpha+\frac{1}{2}}\cdot(v_\alpha - v_{\alpha+1}) + (u_{\alpha+1}\cdot v_{\alpha+1} - u_\alpha\cdot v_\alpha) - \frac{1}{2}(|v_{\alpha+1}|^2 - |v_\alpha|^2)\right] \\
&= \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[u_{\alpha+\frac{1}{2}}\cdot(u_\alpha - u_{\alpha+1}) + |u_{\alpha+1}|^2 - |u_\alpha|^2 + 4\nu(u_{\alpha+1} - u_\alpha)\cdot\nabla\log h - \frac{1}{2}(|v_{\alpha+1}|^2 - |v_\alpha|^2)\right]; \\
&= \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[u_{\alpha+\frac{1}{2}}\cdot(u_\alpha - u_{\alpha+1}) + \frac{1}{2}(|u_{\alpha+1}|^2 - |u_\alpha|^2)\right], \\
&= \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[(u_\alpha - u_{\alpha+1})(u_{\alpha+\frac{1}{2}} - \frac{1}{2}u_{\alpha+1} - \frac{1}{2}u_\alpha)\right], \\
&= \sum_{\alpha=1}^{N-1} -G_{\alpha+\frac{1}{2}}\left[(u_\alpha - u_{\alpha+1})(u_{\alpha+\frac{1}{2}} - \frac{1}{2}(u_{\alpha+1} + u_\alpha)\right] = 0.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h\sum_{\alpha=1}^{N}|v_\alpha|^2\,\mathrm{d}\boldsymbol{x} + \frac{Ng}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h^2\,\mathrm{d}\boldsymbol{x} + Ng\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} z_b h\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{T}^d} 2\nu h|\operatorname{curl} v_\alpha|^2\,\mathrm{d}\boldsymbol{x} \\
&\quad + 4N\nu g\int_{\mathbb{T}^d}|\nabla h|^2\,\mathrm{d}\boldsymbol{x} + 4N\nu g\int_{\mathbb{T}^d}\nabla z_b\cdot\nabla h\,\mathrm{d}\boldsymbol{x} + \kappa\sum_{\alpha=1}^{N}\int_{\mathbb{T}^d}|v_{\alpha+1} - v_\alpha|^2\,\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{\alpha=1}^{N-1}\int_{\mathbb{T}^d}\frac{1}{hN^2}\sum_{j=\alpha+1}^{N}\big(\operatorname{div}(h(u_\alpha - u_j))\big)^2\,\mathrm{d}\boldsymbol{x} = 0.
\end{aligned} \tag{4.6}$$

Acknowledgements

The work presented in this paper was supported in part by CNRS-INSU, TelluS-INSMI-MI program, project CORSURF. It was realized during the secondment of the first and second author in the Ange Inria team. The second author has been partially funded by the ANR project INFAMIE ANR-15-CE40-0011.

References

[1] Audusse E Godlewski E Aguillon, N and M. Parisot. Hyperbolicity of the layer wise discretized hydrostatic Euler equations: The bilayer case. Preprint, 2016.

[2] E. Audusse. A multilayer Saint-Venant model: derivation and numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 5(2):189–214, 2005.

[3] E. Audusse, M.-O. Bristeau, B. Perthame, and J. Sainte-Marie. A multilayer Saint-Venant system with mass exchanges for shallow water flows. Derivation and numerical validation. *Math. Model. Numer. Anal.*, 45(1):169–200, 2011.

[4] F. Bouchut and V. Zeitlin. A robust well-balanced scheme for multi-layer shallow water equations. *Discrete Contin. Dyn. Syst. Ser. B*, 13(4):739–758, 2010.

[5] D. Bresch and B. Desjardins. Existence of weak solutions for 2d viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1-2):211–223, 2003.

[6] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models. *J. Math. Pures Appl.*, 86:362–368, 2006.

[7] D. Bresch, B. Desjardins, and E. Zatorska. Two-velocity hydrodynamics in fluid mechanics: Part II existence of global κ-entropy solutions to compressible Navier-Stokes systems with degenerate viscosities. *J. Math. Pures Appl.*, 104(4):801–836, 2015.

[8] D. Bresch and E. Jabin. Global Existence of Weak Solutions for Compresssible Navier–Stokes Equations: Thermodynamically unstable pressure and anisotropic viscous stress tensor. arXiv:1507.04629.

[9] M.-O. Bristeau and J. Sainte-Marie. Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems. *Discrete Contin. Dyn. Syst. Ser. B*, 10(4), 2008.

[10] M. Castro, J. Macías, and C. Parés. A Q-scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system. *ESAIM: Math. Model. Numer. Anal.*, 35(1):107–127, 2001.

[11] V. Casulli. A semi-implicit finite difference method for non-hydrostatic, free-surface flows. *Int. J. Numer. Methods Fluids*, 30(4):425–440, 1999.

[12] B. Di Martino, M. Peybernes, and P. Orenga. On a bi-layer shallow water problem with the rigid-lid hypothesis. *Math. Models Methods Appl. Sci.*, 15:843–869, 2005.

[13] E. Feireisl and A. Novotny. *Singular Limits in Thermodynamics of Viscous Fluids.* Adv. Math. Fluid Mech. Birkhaüser, 2009.

[14] E.D. Fernández-Nieto, E.H. Koné, and T. Chacón-Rebollo. A multilayer method for the hydrostatic Navier-Stokes equations: a particular weak solution. *J. Sci. Comput.*, 60(2):408–437, 2014.

[15] E.D. Fernández-Nieto, Morales De Luna T. Koné, E.H., and R. Bürger. A multilayer shallow water system for polydisperse sedimentation. *J. Comput. Phys.*, 238(1):281–314, 2013.

[16] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. *Discrete Contin. Dyn. Syst. Ser. B*, 1(1):89–102, 2001.

[17] A.E. Green and P.M. Naghdi. A derivation of equations for wave propagation in water of variable depth. *J. Fluid Mech.*, 78(02):237–246, 1976.

[18] B. Haspot. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D. hal-01082319, arXiv:1411.5503.

[19] B. Haspot. New formulation of the compressible Navier-Stokes equations and parabolicity of the density. hal-01081580.

[20] B. Haspot. Existence of global strong solution for the Korteweg system with quantum pressure when $N \geq 2$. *Math. Ann.*, 2016.

[21] J. Li and Z. Xin. Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities. arXiv:1504.06826, 2015.

[22] P.L. Lions. *Mathematical topics in fluid mechanics. Vol. 2: compressible models.* Oxford Science Publications, 1998.

[23] P.J. Lynett and P.L.-F. Liu. Linear analysis of the multi-layer model. *Coast. Eng.*, 51(5):439–454, 2004.

[24] P.A. Madsen, R. Murray, and O.R. Sørensen. A new form of the Boussinesq equations with improved linear dispersion characteristics. *Coastal Eng.*, 15(4):371–388, 1991.

[25] F. Marche. Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *Eur. J. Mech. B Fluids*, 26(1):49–63, 2007.

[26] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.*, 39(4):1344–1365, 2007.

[27] A. Mellet and A. Vasseur. On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 32(1-3):431–452, 2007.

[28] M.L. Muñoz-Ruiz, F.J. Chatelon, and P. Orenga. On a bi-layer shallow-water problem. *Nonlinear Anal. Real World Appl.*, 4(1):139–171, 2003.

[29] G. Narbona-Reina, J.D.D. Zabsonré, E.D. Fernández-Nieto, and D. Bresch. Derivation of a bilayer model for shallow water equations with viscosity. numerical validation. *Comput. Model. Eng. Sci.*, 43(1):27–72, 2009.

[30] G. Narbona-Reina, J.D.D. Zabsonré, E.D. Fernández-Nieto, and D. Bresch. Derivation of a bilayer model for Shallow Water equations with viscosity. Numerical validation. *Comput. Model. Eng. Sci.*, 43(1):27–71, 2009.

[31] O. Nwogu. Alternative form of Boussinesq equations for nearshore wave propagation. *J. Waterway, Port, Coastal, Ocean Eng.*, 119(6):618–638, 1993.

[32] D.H. Peregrine. Long waves on a beach. *J. Fluid Mech.*, 27(04):815–827, 1967.

[33] F. Serre. Contribution à l'étude des écoulements permanents et variables dans les canaux. *La Houille Blanche*, (6):830–872, 1953.

[34] A. Vasseur and C. Yu. Global weak solution for 3D degenerate compressible Navier-Stokes equations. arXiv:1501.06803, 2015.

[35] A. Vasseur and C. Yu. Global weak solutions to compressible quantum Navier-Stokes equations with damping. arXiv:1503.06894, 2015.

[36] J.D.D. Zabsonré and G. Narbona-Reina. Existence of a global weak solution for a 2D viscous bi-layer Shallow Water model. *Nonlinear Anal. Real World Appl.*, 10:2971–2984, 2009.

[37] E. Zatorska. On the flow of chemically reacting gaseous mixture. *J. Diff. Equ.*, 253:3471–3500, 2012.