# New entropy for Korteweg's system, existence of global weak solution and new blow-up criterion

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**Abstract**

This work is devoted to prove the existence of global weak solution for a general isothermal model of capillary fluids derived by J.E Dunn and J.Serrin (1985) (see [18]), which can be used as a phase transition model. More precisely we shall derive in a first part new entropy estimates for the density when we are dealing with specific capillarity coefficient $\kappa(\rho) = \frac{1}{\rho}$ (let us emphasize on the fact that this choice of capillarity exhibits particular regime flows in the case of the compressible Euler system with quantic pressure which corresponds here to the capillarity, see [2]). This allows us in particular to get enough compactness estimates in order to prove the stability of the global weak solution, the used method follows the works of A. Mellet and A. Vasseur (see [35]). Let us point out that the key of the proof is related to the introduction of a new *effective velocity* (which depends strongly on the structure of the viscosity and capillary coefficients).
In a second part, we shall give the main result of this paper which consists in new blow-up criterion of Prodi-Serrin type for the Korteweg system involving only a control on the vacuum. It is up our knowledge the first result of this type for a compressible fluid system.

## 1 Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by Van der Waals and Korteweg [45, 32] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [18, 30, 44]. The first investigations begin with the Young-Laplace theory which claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface. The main difficulty consists in describing the location and the movement of the interfaces.
Another major problem is to understand whether the interface behaves as a discontinuity in the state space (sharp interface SI) or whether the phase boundary corresponds to a more regular transition (diffuse interface, DI). The diffuse interface models have the advantage to consider only one set of equations in a single spatial domain (the density

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takes into account the different phases) which considerably simplifies the mathematical and numerical study (indeed in the case of sharp interfaces, we have to treat a problem with free boundary).
Another approach corresponds to determine equilibrium solutions which classically consists in the minimization of the free energy functional. Unfortunately this minimization problem has an infinity of solutions, and many of them are physically wrong. In order to overcome this difficulty, Van der Waals in the XIX-th century was the first to add a term of capillarity to select the physically correct solutions, modulo the introduction of a diffuse interface. This theory is widely accepted as a thermodynamically consistent model for equilibria. Alternatively, another way to penalize the high density variations consists in applying a zero order but non-local operator to the density gradient (we refer to [39], [40], [41]). We refer for a mathematical analysis on this system to [12, 14, 24, 27].
Let us now consider a fluid of density $\rho \geq 0$, velocity field $u \in \Omega$ (both are defined on a subset $\Omega$ with $\Omega = \mathbb{R}^N$ or the torus $\mathbb{T}^N$), we are now interested in the following compressible capillary fluid model, which can be derived from a Cahn-Hilliard free energy (see the pioneering work by J.- E. Dunn and J. Serrin in [18] and also in [1, 11, 21]). The conservation of mass and of momentum write:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu(\rho)\,\mathrm{Du}) + \nabla \mathrm{P}(\rho) = \mathrm{divK}, \end{cases} \tag{1.1}$$

where the Korteweg tensor reads as following:

$$\mathrm{div}K = \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa^{'}(\rho))|\nabla\rho|^2\big) - \mathrm{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big). \tag{1.2}$$

Here $\kappa$ is the capillary coefficient and is a regular function of the form $\kappa(\rho) = \kappa\rho^\alpha$ with $\alpha \in \mathbb{R}$. The term $\mathrm{div}K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor. $P(\rho) = a\rho^\gamma$ with $\gamma \geq 1$ is a general $\gamma$ law pressure term, $\mu(\rho) > 0$ is the viscosity coefficient and $Du = \frac{1}{2}(\nabla u +^t \nabla u)$ is the strain tensor.

**Remark 1** *In the sequel we are focising on the case of shallow-water viscosity coefficients, it means* $\mu(\rho) = 2\mu\rho$ *with* $\mu > 0$.

In the sequel we shall study the more general system:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu\rho\,\nabla \mathrm{u}) - \mathrm{div}(\alpha\rho\,\nabla \mathrm{u}^{\mathrm{t}}) + \nabla(\mathrm{a}\rho^\gamma) = \mathrm{divK}, \end{cases} \tag{1.3}$$

$\mu$ and $\alpha$ are the two Lamé viscosity coefficients and satisfying:

$$\mu > 0 \;\text{ and }\; \mu \geq \alpha > 0.$$

In particular, it allows to write the diffusion tensor under the form $(\mu - \alpha)\mathrm{div}(\rho\nabla u) + \alpha\mathrm{div}(\rho Du)$ which implies the following energy inequality after multiplication of the momentum equation by $u$:

$$\begin{aligned}&\int_\Omega \big(\rho(t,x)|u(t,x)|^2+\frac{a}{\gamma-1}\rho^\gamma(t,x)+\kappa|\nabla\sqrt{\rho}|^2(t,x)\big)dx\\&\qquad\qquad+\int_0^t\int_\Omega \big((\mu-\alpha)\rho(t,x)|\nabla u|^2(t,x)+\alpha\rho(t,x)|Du|^2(t,x)\\&\qquad\qquad\qquad\leq C(\int_\Omega \big(\rho_0(x)|v_0(x)|^2+\frac{1}{\gamma-1}\rho_0^\gamma(x)+\kappa|\nabla\sqrt{\rho_0}|^2\big)dx).\end{aligned}\tag{1.4}$$

Now before recalling the main results on the existence of global weak solutions for compressible Navier Stokes equations and Korteweg system, we would like to point out also on an another aspect of the Korteweg system (1.3). Indeed this system is also used in a purely theoretical interest consisting in the selection of the physically relevant solutions of the Euler model by a vanishing capillarity-viscosity limit (especially when the system is not strictly hyperbolic, which is typically the case when the pressure is Van der Waals). Indeed in this last case at least when $N=1$ it is not possible to apply the classical theory of Lax for the Riemann problem (see [33]) and of Glimm (see [20]) with small $BV$ initial data in order to obtain the existence of global weak-entropy solution (we refer also to the work of Bianchini and Bressan see [6] for the uniqueness). Inthis spirit, we prove recently in [12] with F. Charve that the global strong solution of the Korteweg system in one dimension (we obtain also in this paper the existence of global strong solution in one dimension for Korteweg system inspired by DiPerna [17]) converges in the setting of a $\gamma$ law for the pressure ($P(\rho)=a\rho^\gamma$, $\gamma>1$) to weak entropy solution of the compressible Euler equations. In particular it justifies that the Korteweg system is suitable for selecting the physical solutions in the case where the Euler system is strictly hyperbolic. The problem remains however open for a Van der Waals pressure.

**Weak solutions for compressible Navier-Stokes and Korteweg system**

When the viscosity coefficients are constant and the pressure is $P(\rho)=a\rho^\gamma$, with $a>0$ and $\gamma>1$, Lions in [34] proved the global existence of weak solutions $(\rho,u)$ to the compressible Navier-Stokes system (which corresponds to the system (1.1) when $\kappa=0$ and $\mu(\rho)=\mu$ a constant) for $\gamma>\frac{N}{2}$ if $N\geq 4$, $\gamma\geq\frac{3N}{N+2}$ if $N=2,3$ and initial data $(\rho_0,m_0)$ such that:

$$\rho_0^\gamma\in L^1(\Omega)\ \text{ and }\ \frac{|m_0|^2}{\rho_0}\in L^1(\Omega).$$

Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties in $L^p_{loc}$ spaces for the density $\rho$ what is required in order to pass to the limit in the pressure term $P(\rho)=a\rho^\gamma$.

Let us mention that Feireisl in [19] generalized the result to $\gamma>\frac{N}{2}$ by obtaining renormalized solution without assuming that $\rho\in L^2_{loc}$, for this he introduces the concept of oscillation defect measure evaluating the loss of compactness.

In the context of the viscosity coefficients depending on the density the situation is radically different essentially because we loss the structure of *effective pressure* introduced in [34], in particular it is not clear how to get a gain of integrability on the density in order to deal with the pressure term. However let us mention some new results due to Mellet and Vasseur in [35] who prove the stability of the global weak solution when

$P(\rho) = a\rho^\gamma$ with $\gamma \geq 1$ by using new entropy estimates on the density due to Bresch and Desjardins (see [8]). We also refer to [9] in the case of a cold pressure. The main difficulty is coming from the degeneresence of the viscosity coefficient, indeed we lose the control of $\nabla u \in L^2((0,T) \times \mathbb{R}^N)$ what makes delicate the treatment of the term $\rho u \otimes u$ because the vacuum. In order to overcome this difficulty they obtained new entropy on the velocity which gives them a gain of integrability on the velocity.

In the case $\kappa > 0$, we can observe that via the energy inequality (1.4), the density $\sqrt{\rho}$ belongs in $L^\infty(0,\infty,\dot{H}^1(\mathbb{R}^N))$. Hence, in contrast to the non capillary case one can easily pass to the limit in the pressure term. However let us emphasize on a new obstruction which consists in dealing with the quadratic terms in gradient of the density appearing in the capillary tensor (see (1.2)), recently Bresch, Desjardins and Lin in [10] got some stability result for the global weak solutions of the Korteweg model with some specific viscosity coefficients and capillarity coefficient $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ and $\kappa(\rho) = \kappa$ a constant. However the global weak solutions of D. Bresch, B. Desjardins and C-K. Lin require some specific test functions which depend on the solution itself (in other words they obtain the stability of global weak solution for the Korteweg system where the momentum equation is multiplied by $\rho$). In [31], Jüngel obtains by using an effective velocity $v$ the existence of global weak solution when $\kappa(\rho) = \frac{1}{\rho}$ modulo that as in [10] the test functions depends on the density $\rho$.

In [26], we improve this result by showing the existence of global weak solution with small initial data in the energy space for specific choices on the capillary coefficients and with general viscosity coefficient. Comparing with the results of [10], we get global weak solutions with general test function $\varphi \in C^0_\infty(\mathbb{R}^N)$ not depending on the density $\rho$. In fact we have extracted of the structure of capillarity term a new energy inequality using fractionnary derivative which allows a gain of derivative on the density $\rho$.

In the present paper we are interested in proving the stability of the global weak solution for large initial data and without any condition on the test functions as in [10].

### Global strong solutions and Blow-up criterion

Concerning the existence of global strong solution with small initial data, we would like to mention the works of Danchin and Desjardins [16] in the framework of critical initial data for the scaling of the equation. This last result has been recently improved in [22] by working with more general initial data space which are the same for the velocity $u_0$ than Cannone-Meyer-Planchon (see [15]) for incompressible Navier-Stokes equations. Let us also point out a result of global strong solution with large initial data on the rotational part when we add friction term for the Korteweg system (see [29]).

In a second part we are presenting the main result of this paper where we exhibit new blow-up criterion for the Korteweg system (1.3). Generally for the compressible system the classical blow-up criterion follow the Beale-Kato-Majda criterion discovered in the context of incompressible Euler equation (see [4]) which consists roughly speaking in controlling the Lipschitz norm on the velocity $\int_0^T \|\nabla u(s)\|_{L^\infty} ds$. Indeed it appears crucial to control the velocity in a Lipschitz norm in order to estimate the density via the mass equation. It is one of the main difference with incompressible Navier-Stokes equation, how to deal with the density, that is why in particular it seems very tricky to generalize the famous Prodi-Serrin criterion (see [38, 43]) to compressible Navier-Stokes equations.

However we would like to mention in the context of the compressible Navier-Stokes equations with constant viscosity coefficient a recent result involving only a control in a $L^p_T(L^q(\Omega))$ norm on the density (see [28]). We would like to emphasize on the importance of the structure of the viscosity coefficient, indeed in the case of the compressible Navier Stokes equations when the viscosity coefficients are constant, we can exhibit a structure of effective velocity (see [34, 28]) which plays a crucial role in the sense that this effective velocity is regular in some sense. When the viscosity coefficients depend on the density, the picture is less clear (in particular the problem of global weak solution remains open). In our case we are going to prove that we can obtain new blow-up criterion involving only a control on the vacuum, or more precisely a control on $\frac{1}{\rho}$ in $L^\infty_T(L^p)$ for suitable $p$ with $T>0$. Similarly we also get a criterion of Prodi-Serrin for the Korteweg system for an *effective velocity* $v$ (in particular the strong solutions blow up in time $T$ if $\|v\|_{L^p_T(L^q)}=+\infty$ with $\frac{1}{p}+\frac{N}{2q}=\frac{1}{2}$).

## 1.1 Derivation of the models

We are going to prove that we can derive new entropy estimates when we choose specific regime for the coefficients of viscosity and of capillarity. Indeed in the sequel we will consider the following physical coefficients:

$$\mu(\rho)=\mu\rho \quad \text{and} \quad \kappa(\rho)=\frac{\kappa}{\rho},$$

with $\mu,\kappa>0$.

**Remark 1** *Let us give some explanations on this choice of capillarity $\kappa(\rho)=\frac{\kappa}{\rho}$, indeed this regime flows exhibits particular phenomena in the case of the compressible Korteweg Euler system (which is called quantum compressible Euler system when $\kappa(\rho)=\frac{\kappa}{\rho}$). Indeed, at least heuristically, the system is equivalent via the Madelung transform to the Gross-Pitaevskii equations which are globally well-posed for large initial data in dimension $N=1,2,3$ (we refer to [5]). One of the main difficulty to pass from Gross-Pitaevskii to Quantic Euler consists in dealing with the vacuum. This is one of the reasons why the mathematical community is interested in building solitons for this type of problem (one of the main other reasons corresponds to give a negative answer to the problem of scattering and after to study the stability of the soliton). Finally we would also like to mention very interesting results of global weak solutions for the compressible quantic Euler equation with a regime $\kappa(\rho)=\frac{1}{\rho}$ due to Antonelli and Marcati (see [2])*

Let us do some computation in order to express in a friendly way the capillary tensor, more precisely we obtain that (see the appendix for more details):

$$\mathrm{div}K=\kappa\mathrm{div}(\rho\nabla\nabla\ln\rho)=\kappa\mathrm{div}(\rho D(\nabla\ln\rho)).$$

It means in particular that the capillary term has the form of the viscosity tensor. It is then natural to introduce the new unknown $v=u+\frac{\kappa}{\mu}\nabla\ln\rho$ (let us point out that Jüngel in [31] has used the same type of effective velocity). We now want to rewrite system (1.3) in terms of the variables $(\rho,v)$.When $\alpha=\frac{\kappa}{\mu}$ we have the following system:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho v)-\dfrac{\kappa}{\mu}\Delta\rho=0,\\ \rho\partial_t v+\rho u\cdot\nabla v-\mathrm{div}(\mu\rho\,\nabla \mathrm{v})+\nabla \mathrm{P}(\rho)=0,\end{cases} \tag{1.5}$$

When $\alpha = 0$ and $\kappa = \mu^2$, we obtain the following simplified model:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) - \dfrac{\kappa}{\mu} \Delta \rho = 0, \\ \rho \partial_t v + \rho u \cdot \nabla v - \mathrm{div}(\mu \rho \, \nabla \mathrm{v}) + \nabla \mathrm{P}(\rho) = 0, \end{cases} \tag{1.6}$$

**Remark 2** *Let us give some few words on the choice $\kappa = \mu^2$. It gives a specific structure of effective velocity, but is also reasonable on a physic point of view (see [41] for more details). Finally we recall that in [14], we prove in the one dimension case the convergence of the global strong solution of the Korteweg system to an entropy weak solution when $\kappa = \mu^2 = \epsilon^2$. This algebraic relation between $\kappa$ and $\mu^2$ corresponds to an intermediary regime, indeed an important research line (see [41]]) is to model the capillarity tensor and to understand how fast the solutions converges to the Euler system when the capillarity and the viscosity coefficients tends to zero. We want point out here that it exists three different regimes, more precisely if we assume the viscosity coefficient equal to $\epsilon$ with $\epsilon \to 0$. Then we have the three different regimes:*

1. *$\kappa << \epsilon^2$, the viscosity dominates so the parabolic effects is primordial.*
2. *$\kappa = \epsilon^2$, intermediary regime.*
3. *$\kappa >> \epsilon^2$, the capillarity dominates so the dispersive effects are predominant.*

For more details on the computation, we refer to the appendix. Let us mention that system () is also equivalent to the following system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) - \dfrac{\kappa}{\mu} \Delta \rho = 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho u \otimes v) - \mathrm{div}(\mu \rho \, \nabla \mathrm{v}) + \nabla \mathrm{P}(\rho) = 0, \end{cases} \tag{1.7}$$

Our goal is now to prove new entropy inequalities for these two systems and to prove the stability of global weak solution for these two systems. To do this we are inspired by works of Mellet and A. Vasseur in [35].

# 2 Notations and main result

## 2.1 Existence of global weak solution for Korteweg system

We say that $(\rho, v)$ is a weak solution of (1.7) on $[0, T] \times \Omega$, which the following initial conditions

$$\rho_{/t=0} = \rho_0 \geq 0, \;\; \rho u_{/t=0} = m_0. \tag{2.8}$$

with:

$$\begin{aligned} &\rho_0 \in L^\gamma(\Omega) \cap L^1(\Omega), \; \sqrt{\rho_0} \nabla \ln \rho_0 \in L^2(\Omega), \; \rho_0 \geq 0, \\ &\sqrt{\rho_0} v_0 \in L^2(\Omega), \; \rho_0^{\frac{1}{2+\delta}} v_0 \in L^{2+\delta}(\Omega) \quad \text{for somme small } \delta. \end{aligned} \tag{2.9}$$

if

- $\rho \in L^\infty_T(L^1(\Omega) \cap L^\gamma(\Omega))$, $\sqrt{\rho} \in L^\infty_T(L^2(\Omega))$,
- $\sqrt{\rho} v \in L^\infty_T(L^2(\Omega))$,

- $\sqrt{\rho}\,\nabla v \in L^2((0,T)\times\Omega)$,

with $\rho \geq 0$ and $(\rho, \sqrt{\rho}v)$ satisfying in $\mathcal{D}'([0,T]\times\mathbb{R}^N)$:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\sqrt{\rho}\sqrt{\rho}v) - \dfrac{\kappa}{\mu}\Delta\rho = 0, \\ \rho(0,x) = \rho_0(x). \end{cases}$$

and if the following equality holds for all $\varphi(t,x)$ smooth test function with compact support such that $\varphi(T,\cdot) = 0$:

$$\begin{aligned} \int_\Omega (\rho v)_0 \cdot \varphi(0,\cdot)dx + \int_0^T \int_\Omega \sqrt{\rho}(\sqrt{\rho}v)\partial_t\varphi + \sqrt{\rho}u \otimes \sqrt{\rho}v : \nabla\varphi dx \\ + \int_0^T \int_\Omega \rho^\gamma \mathrm{div}\varphi - < \mu\rho\,\nabla v, \nabla\varphi >= 0, \end{aligned} \tag{2.10}$$

where we give sense to the diffusion terms by rewriting him according to $\sqrt{\rho}$ and $\sqrt{\rho}v$:

$$< \rho\,\nabla v, \nabla\varphi >= -\int \sqrt{\rho}(\sqrt{\rho}v_j)\partial_{ii}\varphi_j dx\,dt - \int 2\sqrt{\rho}v_j\partial_i\sqrt{\rho}\partial_i\varphi_j dx\,dt$$

Similarly we have exactly the same type of definition for system (1.1).

$$< \rho\nabla v, \nabla\varphi >= -\int \sqrt{\rho}(\sqrt{\rho}v_j)\partial_{ii}\varphi_j dx\,dt - \int 2\sqrt{\rho}v_j\partial_i\sqrt{\rho}\partial_i\varphi_j dx\,dt.$$

**Main results**

We obtain in this paper the existence of global weak solutions (more exactly the stability of global weak solutions) for systems (1.5) and (1.7). For system (1.7) we obtain the following theorem.

**Theorem 2.1** *Let $1 < \gamma < p$ with $p = +\infty$ if $N = 2$ and $p = 3$ if $N = 3$. Assume that we have a sequence $(\rho_n, v_n)$ with $v_n = u_n + \frac{\kappa}{\mu}\nabla\log\rho_n$ of weak solutions of system (1.7) satisfying entropy inequalities (3.23) and (3.24) with initial data:*

$$(\rho_n)_{/t=0} = \rho_0^n(x) \quad \textit{and} \quad (\rho_n v_n)_{/t=0} = \rho_0^n v_0^n(x)$$

*where $\rho_0^n$ and $v_0^n$ such that:*

$$\rho_0^n \geq 0, \;\; \rho_0^n \to \rho_0 \;\; \textit{in}\, L^1(\Omega), \;\; \rho_0^n v_0^n \to \rho_0 v_0 \;\; \textit{in}\, L^1(\Omega), \tag{2.11}$$

*and satisfy the following bounds (with $C$ constant independent on $n$):*

$$\int_\Omega (\rho_0^n \frac{|v_0^n|^2}{2} + \frac{a(\rho_0^n)^\gamma}{\gamma-1}) < C, \;\; \int_\Omega \frac{1}{\rho_0^n}|\nabla\ln\rho_0^n|^2 dx < C, \tag{2.12}$$

*and:*

$$\int_\Omega \rho_0^n \frac{|v_0^n|^{2+\delta}}{2} dx < C, \tag{2.13}$$

*Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n}v_n, \sqrt{\rho_n}u_n)$ converges strongly to a weak solution $(\rho, \sqrt{\rho}v, \sqrt{\rho}u)$ of (1.1) satisfying entropy inequalities (3.23) and (3.24) (the density $\rho_n$ converges strongly in $C^0((0,T), L^{\frac{3}{2}}_{loc}(\Omega))$, $\sqrt{\rho_n}v_n$ converges strongly in $L^2(0,T,L^2_{loc})$ and the momentum $m_n = \rho_n v_n$ converges strongly in $L^1(0,T,L^1_{loc}(\Omega))$, for any $T > 0$).*

**Remark 3** *Let us emphasize on the importance of the gain of integrability on $v$. This is due to the specific structure of the diffusion on $v$ which is of the form* $\mathrm{div}(\rho\nabla v)$ *and not under the form* $\mathrm{div}(\rho\nabla v)$*. Following Mellet and Vasseur in [35] we obtain a supplementary entropy on the effective velocity $v$ which is a angular stone of the proof. It also shall play a crucial role in the theorem 2.4.*
*Let us emphasize that when $N = 3$, we have the restriction $1 < \gamma < 3$ essentially because in other case we are not able to derive a gain of integrability on $v$.*

**Remark 4** *Let us mention that the problem of the existence of global weak solutions remains open. Indeed in the previous theorem we prove the stability of the global weak solutions, however it seems very complicated to constructed approximate global weak solution which verify uniformly all the entropies. We have the same problem in the case of the shallow water system (see [35]).*

**Remark 5** *Let us point out that compared with [10] we prove the stability of the global weak solution without assuming that the test functions depend on the solution.*

**Remark 6** *Let us mention that unfortunately we are not able to prove the strong convergence of $\sqrt{\rho_n}u_n$ and $\nabla\sqrt{\rho_n}$ in $L^2_{loc}$ but only the strong convergence of $\sqrt{\rho_n}v_n$ in $L^2_{loc}$.*

**Remark 7** *We could also obtain exactly the same result for system (1.5).*

In the specific case of the system (1.1) we obtain new blow-up criterion in the case of the torus $\mathbb{T}^N$ which improves the results in [22]. In the sequel, we will set $m = \rho v$ and $q = \rho - 1$ and $q' = \ln\rho$. We start with recalling some results about the existence of strong solution (see [22]). Considering the unknown $q'$ leads to write the system under the following form:

$$\begin{cases} \partial_t q' + u\cdot\nabla q' + \mathrm{div}u = 0, \\ \partial_t u + u\cdot\nabla u - \mu\Delta u - \mu\nabla\mathrm{div}u - \mu\nabla q'\cdot D(u) + \nabla F(\rho) = \nabla\Delta q' + \frac{1}{2}\nabla(|\nabla q'|^2), \\ (q', u)_{\ t=0} = (\ln\rho_0, u_0), \end{cases} \tag{2.14}$$

with $F'(\rho) = \frac{P'(\rho)}{\rho}$. Let us recall a result coming from [22].

**Theorem 2.2 ([22])** *Let $N \geq 2$ and $p \in [1, +\infty[$. Assume that $P(\rho) = a\rho$ with $a > 0$. Furthermore we suppose that:*

$$q'_0 \in B^{\frac{N}{p}}_{p,\infty} \ \text{ and } \ u_0 \in B^{\frac{N}{p}-1}_{p,\infty}.$$

*There exists a time $T$ such that (2.14) has a unique solution $(q', u)$ on $(0, T)$ with:*

$$q' \in \widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty}), \ \text{ and } \ u \in \widetilde{L}^\infty_T(B^{\frac{N}{p}-1}_{p,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty}).$$

*Furthermore it exists $\epsilon_0$ such that if in addition $q'_0 \in B^{\frac{N}{2}-1}_{2,\infty}$ and:*

$$\|q'_0\|_{B^{\frac{N}{p}-1}_{p,\infty}\cap B^{\frac{N}{p}}_{p,\infty}} + \|u_0\|_{B^{\frac{N}{p}-1}_{p,\infty}} \leq \epsilon_0.$$

*then the solution* $(q^{'}, u)$ *is global and:*

$$q \in \widetilde{L}^{\infty}(B^{\frac{N}{p}-1}_{p,\infty} \cap B^{\frac{N}{p}}_{p,\infty}) \cap \widetilde{L}^{1}(B^{\frac{N}{p}+1}_{p,\infty} \cap B^{\frac{N}{p}+2}_{p,\infty}), \;\; \textit{and} \;\; u \in \widetilde{L}^{\infty}(B^{\frac{N}{p}-1}_{p,\infty}) \cap \widetilde{L}^{1}(B^{\frac{N}{p}+1}_{p,\infty}). \tag{2.15}$$

**Remark 8** *Let us point out that we only solve the system (2.14) which is equivalent to (1.3) only if we control the vacuum. In [22], we obtain the previous theorem only when* $p = 2$*, we are going briefly giving a sketch of the proof in the appendix.*

**Theorem 2.3 ([22])** *Let* $P$ *be a suitably smooth function of the density and* $1 \leq p < +\infty$*. Let* $u_0 \in B^{\frac{N}{p}-1+\epsilon^{'}}_{p,\infty}$ *with* $\epsilon^{'} > 0$ *and* $q^{'}_0 \in B^{\frac{N}{p}+\epsilon^{'}}_{p,\infty}$ *such that* $\rho_0 \geq c > 0$*.*
*There exists then a positive time* $T$ *such that system (1.1) has a unique solution* $(q^{'}, u)$ *with* $\rho$ *bounded away from* $0$ *and:*

$$q^{'} \in \widetilde{C}([0,T], B^{\frac{N}{p}+\epsilon^{'}}_{p,1}) \cap \widetilde{L}^{1}_{T}(\dot{B}^{\frac{N}{p}+2+\epsilon^{'}}_{p,1}), \;\; u \in \widetilde{C}([0,T]; B^{\frac{N}{p}-1+\epsilon^{'}}_{p,1}) \cap \widetilde{L}^{1}([0,T], B^{\frac{N}{p}+1+\epsilon^{'}}_{p,1}),$$

*and it exists* $C > 0$ *and* $\beta > 0$ *depending only on the physical coefficients such that:*

$$T \geq \frac{C}{(1 + \|u_0\|_{B^{\frac{N}{p}-1+\epsilon^{'}}_{p,\infty}} + \|\ln \rho_0\|_{B^{\frac{N}{p}+\epsilon^{'}}_{p,\infty}})^{\beta}} \tag{2.16}$$

**Remark 9** *We are going to give a sketch of the proof of this theorem in the appendix. In particular we will emphasize on the estimate (2.16).*

Let us give briefly the initial data space in which we will work in theorem 2.4.

**Definition 2.1** *We say that the initial data* $(\rho_0, u_0)$ *verify the conditions* $(\mathcal{H})$ *if:*

- $u_0 \in B^{\frac{N}{p}-1+\epsilon^{'}}_{p,\infty}$ *with* $\epsilon^{'} > 0$ *and* $q^{'}_0 \in B^{\frac{N}{p}+\epsilon^{'}}_{p,1}$ *for any* $1 \leq p < +\infty$ *and we assume that* $\rho_0 \geq c > 0$*,*
- $v_0 \in L^{\infty}$*,* $q_0 \in B^{1}_{p,\infty}$ *for any* $1 \leq p < +\infty$ *and the initial data are in the energy space, it means:*

$$\sqrt{\rho_0} u_0 \in L^2, \, \nabla \sqrt{\rho_0} \in L^2 \quad \textit{and} \quad \Pi(\rho_0) \in L^1.$$

*(we refer to the proposition 3.1 for the definition of* $\Pi$*)*

Let us give now our main result.

**Theorem 2.4** *Let* $P(\rho) = a\rho$ *with* $a > 0$*. We assume that* $(\rho_0, u_0)$ *follows the condition* $(\mathcal{H})$ *of the definition 2.1 and let* $(\rho, u)$ *the solution of system (1.1) on the interval* $(0, T)$*. We can then extend the solution beyond* $(0, T)$ *if:*

$$v \in L^p((0,T), L^q(\mathbb{T}^N)) \quad \textit{with} \;\; \frac{1}{p} + \frac{N}{2q} = \frac{1}{2} \;\; \textit{and} \;\; 2 \leq p < +\infty, \; N < q \leq +\infty, \tag{2.17}$$

*or*

$$v \in C([0,T], L^N(\mathbb{T}^N)) \tag{2.18}$$

*or if for any $\epsilon > 0$ arbitrary small:*

$$\frac{1}{\rho^\epsilon} 1_{\{|\rho| \leq \delta\}} \in L^\infty((0,T), L^1(\mathbb{T}^N)). \tag{2.19}$$

**Remark 10** *This result has to be considered like a Prodi-Serrin theorem on the effective velocity $v$. In terms of blow-up condition, he improves widely [22]. In fact the second condition could be improved as follows (we refer to the proof of theorem 2.4):*

$$\frac{1}{\rho^\epsilon} 1_{\{|\rho| \leq \delta\}} \in L^p((0,T), L^q(\mathbb{T}^N)) \quad \textit{with} \quad \frac{1}{p} + \frac{N}{2q} = \frac{1}{2}. \tag{2.20}$$

*Let us observe that compared with the compressible Navier-Stokes equation when the viscosity coefficients are constant, we do not need to control a $L^{p_1}((0,T), L^{q_1}(\mathbb{T}^N))$ norm on $\rho$ with suitable $(p_1, q_1)$ but to estimate the vacuum (it means $\frac{1}{\rho}$). In particular it means that the creation of vacuum is a main obstruction to the existence of global strong solutions.*

**Remark 11** *We would like to give some comments on the choice on the initial data. In particular we are working with subcritical initial data in order to have some information on the lifespan of the strong solution. Indeed it is classical than the time $T$ of existence is bounded by below in function of the initial data, this plays a crucial role in our proof.*

**Remark 12** *Let us mention that the condition (2.20) serves essentially to control the vacuum, we refer to the lemma 12 for more details.*

**Remark 13** *In fact the key of the proof correspond to get a gain of integrability on the effective velocity $v$, more precisely we are able to control for any $1 \leq p < +\infty$ $\rho^{\frac{1}{p}} v$ in $L^\infty((0,T), L^p(\mathbb{T}^N))$. Unfortunately it seems quite difficult to translate this information on $v$ only. We would like to mention a very interesting work of Mellet and Vasseur (see [37]) who gives a criterion for estimating the velocity $u$ in $L^p((0,T), L^q)$ space modulo some integrability control on the pressure terms and some bound by below on the viscosity coefficients. Let us point out that we can not apply the estimate of Mellet and Vasseur essentially because the viscosity coefficient is degenerate, it means $\mu(\rho) = \rho$. In particular this viscosity coefficient can not ben bounded by below due the apparition of eventual vacuum.*

**Remark 14** *We could probably extend this previous result for more general pressure terms. However it would requires additional informations on the integrability of the density or on the vacuum, i.e $\frac{1}{\rho} 1_{\{|\rho| \leq \delta\}}$ and maybe some restriction on the $\gamma$ if we choose a $\gamma$ law.*
*It would be also possible to deal with the euclidian space $\mathbb{R}^N$ (it does not change a lot, except that we need to be carreful when we apply interpolation argument.*

The paper is structured in the following way: in section 3 we prove some new entropies. In section 4, we give a few notation, some compactness results and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section 5 we prove theorem 2.1 and in section 6 we show the theorem 2.4. An appendix is devoted to justify rigorously the computations involving the effective velocity.

# 3 New entropies

## 3.1 Entropy for the system (1.5)

We now want to establish new entropy inequality for system (1.5) and (1.1). More precisely if we assume that $(\rho, u)$ are exact solutions of system (1.5), we obtain the following proposition.

**Proposition 3.1** *Assume that $(\rho, u)$ are exact solutions of system (1.5) with $P(\rho) = a\rho^\gamma$ ($\gamma \geq 1$) then for all $t > 0$:*

$$\begin{aligned}&\int_\Omega \big[\rho|u|^2(t,x) + \kappa|\nabla\sqrt{\rho}|^2(t,x) + \Pi(\rho)(t,x)\big]\,dx + \int_0^t\int_\Omega \nabla\ln\rho\cdot\nabla\rho^\gamma\,dxdt\\&+(\mu-\gamma)\int_0^t\int_\Omega \rho|\nabla u|^2 dxdt + \gamma\int_0^t\int_\Omega \rho|Du|^2 dxdt + \kappa\int_0^t\int_\Omega \rho(\partial_{ij}\ln\rho)^2(t,x)dtdx\\&\qquad\qquad\leq C\Big(\int_{\mathbb{R}^N}\big(\rho_0|v_0|^2(x) + \Pi(\rho_0(x)) + \kappa|\nabla\sqrt{\rho_0}|^2(t,x)\,dx\big).\end{aligned} \tag{3.21}$$

*with $\Pi(s) = s\int^s\int_0^s \frac{P(z)}{z^2}dz$.*

**Proof:** We now want to obtain this new entropy by taking profit of the specific structure of effective velocity involved in the system (1.5). It suffices to multiply the momentum equation in (1.5) by $v$, we then obtain:

$$\begin{aligned}&\int_\Omega \big(\rho(t,x)|v(t,x)|^2 + \Pi(\rho)(t,x)\big)dx + \int_0^t\int_\Omega \big(\mu\rho(t,x)|\nabla v|^2(t,x)\\&\quad+\frac{\kappa}{\mu}P''(\rho)|\nabla\rho|^2(t,x)\big)dtdx \leq C\Big(\int_\Omega \big(\rho_0(x)|v_0(x)|^2 + \Pi(\rho_0)(x)\big)dx\Big).\end{aligned}$$

By the previous inequality and (1.4) we obtain the desired result. $\square$

The following proposition comes from [36].

**Proposition 3.2** *Smooth solutions of system (1.5) satisfy the following inequality when $P(\rho) = a\rho^\gamma$ with $\gamma \geq 1$:*

$$\frac{d}{dt}\int_\Omega \rho\frac{|v|^{2+\delta}}{2+\delta} + \frac{\nu}{4}\int_\Omega \rho|v|^\delta|\nabla v|^2 dx \leq \Big(\int_\Omega \big(\rho^{2\gamma-1-\frac{\delta}{2}}\big)^{\frac{2}{2-\delta}}dx\Big)^{\frac{2}{2-\delta}}\Big(\int_\Omega \rho|v|^2 dx\Big)^{\frac{\delta}{2}}, \tag{3.22}$$

*for $\delta \in (0, \frac{1}{4})$.*

**Proof:** The proof follows exactly the same lines than in [35].

**Remark 15** *Let us mention that from proposition 3.1, we can easily show by interpolation that* $\rho^\gamma$ *is bounded in* $L^{\frac{5}{3}}((0,T)\times\Omega)$ *for* $N=3$ *(we refer to the lemma 3 for more details). In order to prove that* $\rho^{\frac{1}{2+\delta}}v$ *belongs in* $L^\infty((0,T),L^{2+\delta}(\Omega))$ *for* $\delta$ *small enough, it is necessary to control the integral* $\int_0^T\int_\Omega\big(\rho^{2\gamma-1-\frac{\delta}{2}}\big)^{\frac{2}{2-\delta}}dxdt$. *Since* $\rho$ *is bounded in* $L^{\frac{5}{3}}((0,T)\times\Omega)$*, a necessary condition is:*

$$2\gamma-1<\frac{5}{3}\gamma\Leftrightarrow\gamma<3.$$

*It explains in particular why in the theorem 2.1, we assume that* $\gamma<3$ *for* $N=3$.
*For* $N=2$*, we show that* $\rho$ *is bounded in* $L^r((0,T)\times\Omega)$ *for any* $1\leq r<2$*. In particular we have always* $2\gamma-1\leq 2\gamma$*, that is why we do not need any assumption on* $\gamma$ *for* $N=2$ *in the theorem 2.1.*

### 3.2 Entropy for the system (1.1)

By proceeding similarly we obtain the following propositions for system (1.1).

**Proposition 3.3** *Assume that* $(\rho,u)$ *are exact solutions of system (1.1) then for all* $t>0$*:*

$$\begin{aligned}&\int_\Omega\big(\rho|v|^2(t,x)+\Pi(\rho(t,x))\big)\,dx+\int_0^t\int_\Omega\nabla\ln\rho\cdot\nabla\rho^\gamma\,dxdt\\&\qquad+\kappa\int_0^t\int_\Omega\rho(\partial_{ij}\ln\rho)^2(t,x)dtdx\leq C(\int_\Omega\big(\rho_0|v_0|^2(x)+\Pi(\rho_0(x))\big)\,dx).\end{aligned}\tag{3.23}$$

**Proposition 3.4** *The smooth solutions of system 1.1 satisfy the following inequality when* $P(\rho)=a\rho^\gamma$ *with* $\gamma\geq1$*:*

$$\frac{d}{dt}\int_\Omega\rho\frac{|v|^{2+\delta}}{2+\delta}+\frac{\nu}{4}\int_\Omega\rho|v|^\delta|\nabla v|^2dx\leq\Big(\int_\Omega\big(\rho^{2\gamma-1-\frac{\delta}{2}}\big)^{\frac{2}{2-\delta}}dx\Big)^{\frac{2}{2-\delta}}\big(\int_\Omega\rho|v|^2dx\big)^{\frac{\delta}{2}},\tag{3.24}$$

*for* $\delta\in(0,\frac{1}{4})$.

**Remark 16** *We have the same remark than remark 15 for the system (1.1).*

## 4 Littlewood-Paley theory and Besov spaces

Throughout the paper, $C$ stands for a constant whose exact meaning depends on the context. The notation $A\lesssim B$ means that $A\leq CB$. For all Banach space $X$, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in $X$. For $p\in[1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in $X$ such that $t\to\|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\varphi\in C^\infty(\mathbb{R}^N)$, supported in the shell $\mathcal{C}=\{\xi\in\mathbb{R}^N/\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ and $\chi\in C^\infty(\mathbb{R}^N)$ supported in the ball $B(0,\frac{4}{3})$. $\varphi$ and $\chi$ are valued in $[0,1]$ such that:

$$\sum_{l\in\mathbb{Z}}\varphi(2^{-l}\xi)=1\ \text{ if }\ \xi\neq0.$$

We set $\mathcal{Q}^N = (0, 2\pi)^N$ and $\widetilde{\mathbb{Z}}^N = (\mathbb{Z}/1)^N$ the dual lattice associated to $\mathbb{T}^N$. We decompose now $u \in \mathcal{S}'(\mathbb{T}^N)$ into Fourier series:

$$u(x) = \sum_{\beta \in \widetilde{\mathbb{Z}}^N} \widehat{u}_\beta e^{i\beta \cdot x} \quad \text{with } \widehat{u}_\beta = \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} e^{-i\beta \cdot y} u(y) dy.$$

Denoting;

$$h_q(x) = \sum_{\beta \in \widetilde{\mathbb{Z}}^N} \varphi(2^{-q}\beta) e^{i\beta \cdot x},$$

one can now define the periodic dyadic blocks as:

$$\Delta_q u(x) = \sum_{\beta \in \widetilde{\mathbb{Z}}^N} \varphi(2^{-q}\beta) \widehat{u}_\beta e^{i\beta \cdot x} = \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} h_q(y) u(x-y) dy, \text{ for all } q \in \mathbb{Z}$$

and we have the following low frequency cut-off:

$$S_q u(x) = \widehat{u}_0 + \sum_{p \le q-1} \Delta_p u(x) = \sum_{\beta \in \widetilde{\mathbb{Z}}^N} \chi(2^{-q}\beta) \widehat{u}_\beta e^{i\beta \cdot x}.$$

It is obvious that $\Delta_p u = 0$ for negative enough $p$ and formally, one can write that:

$$u = \widehat{u}_0 + \sum_{k \in \mathbb{Z}} \Delta_k u.$$

This decomposition is called non-homogeneous Littlewood-Paley decomposition.
Furthermore we have the following proposition where $\widetilde{\mathcal{C}} = B(0, \frac{2}{3}) + \mathcal{C}$

**Proposition 4.5**

$$|k - k'| \ge 2 \implies \text{supp}\varphi(2^{-k}\cdot) \cap \text{supp}\varphi(2^{-k'}\cdot) = \emptyset, \tag{4.25}$$

$$k \ge 1 \implies \text{supp}\chi \cap \text{supp}\varphi(2^{-k}\cdot) = \emptyset, \tag{4.26}$$

$$|k - k'| \ge 5 \implies 2^{k'}\widetilde{\mathcal{C}} \cap 2^k \mathcal{C} = \emptyset. \tag{4.27}$$

## 4.1 Non homogeneous Besov spaces and first properties

**Definition 4.2** *For $s \in \mathbb{R}$, $p \in [1, +\infty]$, $q \in [1, +\infty]$, and $u \in \mathcal{S}'(\mathbb{T}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (|\widehat{u}_0|^q + \sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

*The non homogeneous Besov space $B^s_{p,q}$ is the set of temperate distribution $u$ such that $\|u\|_{\dot{B}^s_{p,q}} < +\infty$.*

**Remark 17** *The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the nonhomogeneous Hölder space $C^s$ and that $B^s_{2,2}$ is the nonhomogeneous space $H^s$.*

**Proposition 4.6** *The following properties holds:*

1. *If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*

2. *$B^{s'}_{p,r_1} \hookrightarrow B^s_{p,r}$ if $s' > s$ or if $s = s'$ and $r_1 \leq r$.*

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [7]).

**Proposition 4.7** *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}})\,. \tag{4.28}$$

- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,+\infty]^2$ such that: $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
  *if $s_1 + s_2 + N\inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:*

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{4.29}$$

  *when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r = 1$.*
  *If $s_1 + s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$ and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ then:*

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{4.30}$$

  *If $|s| < \frac{N}{p}$ for $p \geq 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:*

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{4.31}$$

**Remark 18** *In the sequel $p$ will be either $p_1$ or $p_2$ and in this case $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ if $p_1 \leq p_2$, resp $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ if $p_2 \leq p_1$.*

**Corollary 1** *Let $r \in [1,+\infty]$, $1 \leq p \leq p_1 \leq +\infty$ and $s$ such that:*

- *$s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$,*

- *$s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$ if $\frac{1}{p} + \frac{1}{p_1} > 1$,*

*then we have if $u \in B^s_{p,r}$ and $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$:*

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces $X$. In our case, we expect $X$ to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r=p$). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

**Definition 4.3** *Let $\rho\in[1,+\infty]$, $T\in[1,+\infty]$ and $s_1\in\mathbb{R}$. We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}=\big(\|\widehat{u}_0\|^r_{L^\rho_T(L^p)}+\sum_{l\in\mathbb{Z}}2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho(L^p)}\big)^{\frac{1}{r}}.$$

*We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ as the set of temperate distribution $u$ over $(0,T)\times\mathbb{R}^N$ such that $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}<+\infty$.*

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r})=\widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r})\cap\mathcal{C}([0,T],B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\leq\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\geq\rho,\quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\geq\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{ if }\ r\leq\rho.$$

**Remark 19** *It is easy to generalize proposition 4.7, to $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ spaces. The indices $s_1$, $p$, $r$ behave just as in the stationary case whereas the time exponent $\rho$ behaves according to Hölder inequality.*

In the sequel we will need of composition lemma in $\widetilde{L}^\rho_T(B^s_{p,r})$ spaces.

**Lemma 1** *Let $s>0$, $(p,r)\in[1,+\infty]$ and $u\in\widetilde{L}^\rho_T(B^s_{p,r})\cap L^\infty_T(L^\infty)$.*

1. *Let $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)\in\widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function $C$ depending only on $s$, $p$, $r$, $N$ and $F$ such that:*

$$\|F(u)\|_{\widetilde{L}^\rho_T(B^s_{p,r})}\leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

2. *Let $F\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0)=0$. Then $F(u)-F'(0)u\in\widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function $C$ depending only on $s$, $p$, $r$, $N$ and $F$ such that:*

$$\|F(u)-F'(0)u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}\leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|^2_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

Let us now give some estimates for the heat equation:

**Proposition 4.8** *Let $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$ and $1\leq\rho_2\leq\rho_1\leq+\infty$. Assume that $u_0\in B^s_{p,r}$ and $f\in\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})$. Let $u$ be a solution of:*

$$\begin{cases}\partial_t u-\mu\Delta u=f\\ u_{t=0}=u_0\,.\end{cases}$$

*Then there exists $C>0$ depending only on $N,\mu,\rho_1$ and $\rho_2$ such that:*

$$\|u-\widehat{u}_0\|_{\widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1}_{p,r})}\leq C\big((1+T^{\frac{1}{\rho_1}})\|u_0\|_{B^s_{p,r}}+\mu^{\frac{1}{\rho_2}-1}(1+T^{1+\frac{1}{\rho_1}-\frac{1}{\rho_2}})\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})}\big)\,.$$

*If in addition $r$ is finite then $u$ belongs to $C([0,T],B^s_{p,r})$.*

We would like to finish this section by giving very useful propositions of compactness that we shall often apply. We are going to recall the so-called Aubin-Lions theorem.

**Proposition 4.9** *Let $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$ be Banach spaces (with $X$ which is compactly imbedded in $B$) and $(f_n)_{n\in\mathbb{N}}$ a sequence bounded in $L^q((0,T),B)\cap L^1((0,T),X)$ (with $1<q\leq+\infty$) and $(\frac{d}{dt}f_n)_{n\in\mathbb{N}}$ bounded in $L^1((0,T),Y)$. Then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p((0,T),B)$ for any $1\leq p<q$.*

Let us recall now the theorem of Arzèla-Ascoli.

**Proposition 4.10** *Let $B$ and $X$ Banach spaces such that $B\hookrightarrow\hookrightarrow X$ is compact. Let $f_N$ be a sequence of functions $\bar{I}\to B$ (with $I$ an interval) uniformly bounded in $B$ and uniformly continuous in $X$. Then there exists $f\in C^0(\bar{I},B)$ such that $f_n\to f$ strongly in $f\in C^0(\bar{I},X)$ up to a subsequence.*

**Lemma 1** *Let $K$ a compact subset of $\mathbb{R}^N$ (with $N\geq 1$) and $v^\epsilon$ a sequel such that:*

- *$v^\epsilon$ is uniformly bounded in $L^{1+\alpha}(K)$ with $\alpha>0$,*
- *$v^\epsilon$ converge almost everywhere to $v$,*

*then $v^\epsilon$ converges strongly to $v$ in $L^1(K)$ with $v\in L^{1+\alpha}(K)$.*

**Proof**: First by the Fatou lemma $v$ is in $L^{1+\alpha}(K)$. Next we have for any $M>0$:

$$\int_K |v^\epsilon - v|dx \leq \int_{K\cap\{|v^\epsilon-v|\leq M\}}|v^\epsilon-v|dx+\int_{K\cap\{|v^\epsilon-v|\geq M\}}|v^\epsilon-v|dx. \tag{4.32}$$

We are dealing with the second member of the right hand side, by Hölder inequality and Tchebychev lemma we have for a $C>0$:

$$\begin{aligned}\int_{K\cap\{|v^\epsilon-v|\geq M\}}|v^\epsilon-v|dx &\leq (\int_K|v^\epsilon-v|^{1+\alpha}dx)^{\frac{1}{1+\alpha}}(\{|v^\epsilon-v|\geq M\})^{\frac{\alpha}{1+\alpha}},\\ &\leq \frac{C}{M^{\frac{\alpha}{1+\alpha}}}.\end{aligned} \tag{4.33}$$

In particular we have shown the strong convergence of $v^\epsilon$ to $v$, indeed from the inequality (2) it suffices to use the Lebesgue theorem for the first term on the right hand side and the estimate (4.33) with $M$ going to $+\infty$. □

# 5 Proof of the theorems 2.1

We now present the proof of theorem 2.1 inspired by [35]. To begin with, we need to make precise the assumptions on the initial data.

**Initial data:**

We recall that the initial data must satisfy (2.8), and (2.9) in order to take profit of the entropy inequalities from section 3:

- $\rho_0^n$ is bounded in $L^1(\Omega) \cap L^\gamma(\Omega)$, $\rho_0^n \geq 0$ a.e in $\Omega$,
- $\rho_0^n |u_0^n|^2$ is bounded in $L^1(\Omega)$,
- $\nabla\sqrt{\rho_0^n}$ is bounded in $L^2(\Omega)$,
- $\rho_0^n |v_0^n|^{2+\delta}$ is bounded in $L^1(\Omega)$.

With those assumptions, and using the entropy inequalities (3.21), (3.22) and the mass equation, we have the following bounds:

$$\begin{aligned}
&\|\sqrt{\rho_n}\|_{L^\infty((0,T),L^2(\Omega))} \leq C,\\
&\|\rho_n\|_{L^\infty((0,T),L^\gamma(\Omega))} \leq C,\\
&\|\sqrt{\rho_n} u_n\|_{L^\infty((0,T),L^2(\Omega))} \leq C,\\
&\|\sqrt{\rho_n}\partial_{ij} \ln \rho_n\|_{L^2((0,T)\times\Omega)} \leq C,\\
&\|\sqrt{\rho_n}\nabla u_n\|_{L^2((0,T)\times\Omega)} \leq C,
\end{aligned} \tag{5.34}$$

and for $\delta$ small enough:

$$\begin{aligned}
&\|\rho_n^{\frac{\gamma}{2}-1}\nabla\rho_n\|_{L^2((0,T)\times\Omega)} \leq C,\\
&\|\rho_n|v_n|^{2+\delta}\|_{L^\infty((0,T),L^1(\Omega))} \leq C.
\end{aligned} \tag{5.35}$$

**Remark 20** *Let us point out that the gain of integrability on $v_n$ in (5.35) is a direct consequence of the remark 15.*

From the previous inequalities, the bounds (5.34) and (5.35) yields the following uniform bounds on the effective velocity $v_n$:

$$\begin{aligned}
&\|\sqrt{\rho_n}\nabla v_n\|_{L^2((0,T)\times\Omega)} \leq C,\\
&\|\nabla\sqrt{\rho_n}\|_{L^\infty(0,T;L^2(\Omega)} \leq C,\\
&\|\nabla\rho_n^{\frac{\gamma}{2}}\|_{L^2((0,T)\times\Omega)} \leq C,\\
&\|\rho_n|v_n|^{2+\delta}\|_{L^\infty((0,T),L^1(\Omega))} \leq C.
\end{aligned} \tag{5.36}$$

The proof of theorem 2.1 will be derived in three steps and follows the proof of [35]. In the first step, we deal with the strong convergence of the density (which enables us to treat the convergence of the pressure term). In the second step we prove the strong convergence of $\sqrt{\rho_n}v_n$ in $L^2_{loc}((0,T)\times\mathbb{R})$ (it allows us to give sense to the momentum product $\rho_n v_n \otimes u_n$) by taking advantage of the uniform gain of integrability on $v^n$ via the entropy inequality (3.22). Indeed it will suffice to use the lemma 1 after proving almost everywhere convergence via Sobolev injection. In this part, we also shall deal with the strong convergence in the distribution sense of the product $\sqrt{\rho_n}\sqrt{\rho_n}v_n$. The last step shows the convergence of the momentum term $\sqrt{\rho_n}v_n$ to $\sqrt{\rho}u + \frac{2\kappa}{\mu}\nabla\rho$.

**Step 1: Strong convergence of $\rho_n$**

The first step consists in proving the convergence almost everywhere of $\rho_n$ to a limit $\rho$ in order in the sequel to apply the lemma 1 for proving the strong convergence of $P(\rho_n)$ to $P(\rho)$.

**Lemma 2** *For any $T > 0$, $\rho_n$ converges up to a subsequence strongly to a limit $\rho$ in $L^p((0,T), L^q_{loc}(\Omega))$ with $1 \leq p < +\infty$ and $q = 3 - \alpha$ with $\alpha > 0$ small enough if $N = 3$ and $1 \leq q < +\infty$ if $N = 2$.*
*As a consequence up to a subsequence, $\sqrt{\rho_n}$ converge almost everywhere to $\rho$. Furthermore $\rho_n$ strongly converges to $\rho$ in $C(0,T; W^{1-\alpha,\frac{3}{2}}_{loc}(\Omega))$ with $\alpha > 0$ small enough.*

**Proof:** Indeed from (5.34) and (5.36), we know that $\nabla\sqrt{\rho_n}$ is bounded in $L^\infty((0,T), H^1(\Omega))$. Furthermore from the mass equation, we have:

$$\begin{aligned}\partial_t\sqrt{\rho_n} &= -\frac{1}{2}\sqrt{\rho_n}\mathrm{div}u_n - u_n\cdot\nabla\sqrt{\rho_n}\\ &= \frac{1}{2}\sqrt{\rho_n}\mathrm{div}u_n - \mathrm{div}(u_n\sqrt{\rho_n}),\end{aligned}$$

From (5.34) we obtain that $\partial_t\sqrt{\rho_n}$ is bounded in $L^2((0,T), H^{-1}(\Omega))$ for any $T > 0$. Thanks to Aubin-Lions Lemma 4.9, we conclude to the strong convergence in $L^p_{loc}((0,T), L^q(\Omega))$ of $\rho_n$ to a limit $\rho$ up to a subsequence. A direct consequence is that $\rho_n$ up to a subsequence converges almost everywhere to $\rho$.
We are now interesting in proving that $\rho_n$ strongly converges to $\rho$ in $C(0,T; W^{1-\alpha,\frac{3}{2}}_{loc}(\Omega))$ with $\alpha > 0$ small enough. To do this we are going to use the Arzelà-Ascoli proposition 4.10.Sobolev embedding implies that $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T; L^q(\Omega))$ for $q \in [2, +\infty[$ if $N = 2$ and $q \in [2, 6]$ if $N = 3$. It implies that $\rho_n$ is bounded (by interpolation for $N = 2$ and the fact that $\rho_n$ is bounded in $L^\infty((0,T), L^1(\Omega)))$ in $L^\infty(0,T; L^3(\Omega))$, and therefore:

$$\rho_n u_n = \sqrt{\rho_n}\sqrt{\rho_n}u_n \text{ is bounded in } L^\infty(0,T; L^{\frac{3}{2}}(\Omega)).$$

By the mass equation we deduce that $\partial_t\rho_n$ is bounded in $L^\infty(0,T; W^{-1,\frac{3}{2}}(\Omega))$. Moreover since $\nabla\rho_n = 2\sqrt{\rho_n}\nabla\sqrt{\rho_n}$, we also have that $\nabla\rho_n$, is bounded in $L^\infty(0,T; L^{\frac{3}{2}}(\Omega))$ and finally $\rho_n$ is bounded in $L^\infty(0,T; W^{1,\frac{3}{2}}(\Omega))$ (by using still an interpolation argument for proving that $\rho_n$ is bounded in $L^\infty(0,T; L^{\frac{3}{2}}(\Omega))$ ). Since $\partial_t\rho_n$ is bounded in $L^\infty((0,T), W^{-1,\frac{3}{2}}(\Omega))$ we show that $\rho_n$ is uniformly continuous on $(0,T)$ in $W^{-1,\frac{3}{2}}(\Omega)$. By interpolation between $W^{-1,\frac{3}{2}}(\Omega)$ and $W^{1,\frac{3}{2}}(\Omega)$, we obtain that $\rho_n$ is uniformly continuous on $(0,T)$ in $W^{1-\alpha,\frac{3}{2}}(\Omega)$. We conclude by using the Arzèla-Ascoli proposition 4.10 with $B = W^{1,\frac{3}{2}}_{loc}(\Omega)$ and $X = W^{1-\alpha,\frac{3}{2}}_{loc}(\Omega)$.

**Lemma 3** *The pressure $\rho_n^\gamma$ is uniformly bounded in $L^{\frac{5}{3}}((0,T)\times\Omega)$ when $N = 3$ and $L^r((0,T)\times\Omega)$ for all $r \in [1, 2[$ when $N = 2$. In particular, $\rho_n^\gamma$ converges to $\rho^\gamma$ strongly in $L^{p-\epsilon}_{loc}((0,T)\times\Omega)$ with $\epsilon > 0$ small enough with $p = \frac{5}{3}$ if $N = 3$ and $p = 2$ if $N = 2$.*

**Proof:** Combining inequalities (5.36) and (5.35) implies that $\rho_n^{\frac{\gamma}{2}}$ is uniformly bounded in $L^2(0,T; H^1(\Omega))$.

When $N=2$ by Sobolev embedding $\rho_n^{\frac{\gamma}{2}}$ is uniformly bounded in $L^2(0,T;L^p(\Omega))$ for all $p\in[2,\infty[$. We deduce that $\rho_n^\gamma$ is uniformly bounded in $L^1(0,T;L^{p'}(\Omega))\cap L^\infty(L^1(\Omega))$ for all $p'\in[1,+\infty[$, hence by interpolation $\rho_n^\gamma$ is bounded in $L^r((0,T)\times\Omega)$ for all $r\in[1,2[$. When $N=3$, we get by Sobolev embedding that $\rho_n^\gamma$ is uniformly bounded in $L^1(0,T;L^3(\Omega))$. As $\rho_n^\gamma$ is also uniformly bounded in $L^\infty((0,T),L^1(\Omega))$, by interpolation we have:

$$\|\rho_n^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)}\leq\|\rho_n^\gamma\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\Omega))}\|\rho_n^\gamma\|^{\frac{3}{5}}_{L^1(0,T;L^3(\Omega))}\leq M<+\infty$$

Hence $\rho_n^\gamma$ is bounded in $L^{\frac{5}{3}}((0,T)\times\Omega)$. By using that $\rho_n$ converges up to a subsequence almost everywhere to $\rho$ and the previous uniform bounds on $\rho_n^\gamma$ we conclude the proof via the use of the lemma 1. □

**Step 2: Convergence of $\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}v_n$ and $\sqrt{\rho_n}v_n$**

The strategy remains the same, it suffices to prove that in a certain way $\sqrt{\rho_n}v_n$ converges up to a subsequence almost everywhere to a limit $\sqrt{\rho}v$ and to use the lemma 1 by taking advantage of the gain of integrability on $v_n$ via (5.36).

**Lemma 4** *Up to a subsequence, the momentum $m_n=\rho_n v_n$ converges strongly in $L^{2-\epsilon}(0,T;L^p_{loc}(\Omega))$ (for any $\epsilon>0$ small enough) to some $m(x,t)$ for all $p\in[1,\frac{3}{2})$. It implies that:*

$$\rho_n v_n\rightarrow m\quad \textit{almost everywhere}\;\;(x,t)\in(0,T)\times\Omega.$$

**Remark 21** *We observe that we can define $v(t,x)=\frac{m(t,x)}{\rho(t,x)}$ outside the vacuum set $\{\rho(t,x)=0\}$, the only thing to check is to know if $m(t,x)$ is zero on the vacuum set.*

**Proof:** We have:

$$\rho_n v_n=\sqrt{\rho_n}\sqrt{\rho_n}v_n,$$

where $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $q\in[2,+\infty[$ if $N=2$ and $q\in[2,6]$ if $N=3$. Since $\sqrt{\rho_n}v_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$, it shows that $\rho_n v_n$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for all $q\in[1,\frac{3}{2}]$. Next we are going to prove that $m_n$ is bounded in Sobolev space in order to use Aubin-Lions proposition 4.9. We have:

$$\begin{aligned}\partial_i(\rho_n v_n^j)&=\rho_n\partial_i v_n^j+v_n^j\partial_i\rho_n\\&=\sqrt{\rho_n}\sqrt{\rho_n}\partial_i v_n^j+2\sqrt{\rho_n}v_n^j\partial_i\sqrt{\rho_n}.\end{aligned}$$

Using (5.36) the second term is bounded in $L^\infty(0,T;L^1(\Omega))$, while the first term is bounded in $L^2(0,T;L^q(\Omega))$ for all $q\in[1,\frac{3}{2}]$. Then $\nabla(\rho_n v_n)$ is bounded in $L^2(0,T;L^1_{loc}(\Omega))$. It implies that $\rho_n v_n$ is bounded in $L^2(0,T;W^{1,1}_{loc}(\Omega))$. It remains to obtain estimates on $\partial_t(\rho_n v_n)$, more precisely we have:

$$\text{for all compact } K,\ \partial_t(\rho_n v_n)\text{ is bounded in }\ L^2(0,T;W^{-2,\frac{4}{3}}(K)).\tag{5.37}$$

To show (5.40), we consider the momentum equation of system (1.7), first we observe that using (5.34) and (5.36):

$$\begin{aligned}\operatorname{div}(\sqrt{\rho_n}v_n\otimes\sqrt{\rho_n}u_n)&\in L^\infty(0,T;W^{-1,1}(K)),\\ \nabla\rho_n^\gamma&\in L^\infty(0,T;W^{-1,1}(K)).\end{aligned}$$

So we only have to deal with the terms $\operatorname{div}(\rho_n\nabla v_n)$. In this goal, we write:

$$\rho_n\nabla v_n=\nabla(\rho_n v_n)-v_n:\nabla\rho_n, \tag{5.38}$$

The second term in (5.38) can be written as:

$$v_n\nabla\rho_n=2\sqrt{\rho_n}v_n\nabla\sqrt{\rho_n},$$

which is bounded in $L^\infty(0,T;L^1(\Omega))$ in view of (5.36). The first term in (5.38) can be rewritten:

$$\nabla[\rho_n v_n]=\nabla[\sqrt{\rho_n}(\sqrt{\rho_n}v_n)],$$

which is bounded in $L^\infty(0,T;W^{-1,\frac{3}{2}}(\Omega))$ thanks to lemma 2.
It implies that $\rho_n\nabla v_n$ is bounded in $L^\infty(0,T;W^{-1,\frac{3}{2}}(K)+L^1(K))$, and by Sobolev embedding we have $L^1(K)\subset W^{-1,\frac{4}{3}}(K)$ and $W^{-1,\frac{3}{2}}(K)\subset W^{-1,\frac{4}{3}}(K)$. It suffices to show that $\operatorname{div}(\rho_n\,\nabla v_n)$ is bounded in $L^\infty(0,T;W^{-2,\frac{4}{3}}(K))$. We have final proved (5.40).
Since $\rho_n v_n$ and $\partial_t(\rho_n v_n)$ are respectively bounded in $L^2(0,T;W^{1,1}_{loc}(\Omega))$ and in $L^2(0,T;W^{-2,\frac{4}{3}}_{loc}(\Omega))$, by the Aubin-Lions proposition 4.9, $\rho_n v_n$ strongly converges up to a subsequence to a limit $m$ in $L^{2-\epsilon}(0,T;L^p(K))$ for all $p\in[1,\frac{3}{2})$ and any $\epsilon>0$ small enough. $\square$

**Lemma 5** *The quantity $\sqrt{\rho_n}v_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$ to $\frac{m}{\sqrt{\rho}}$ (defined to be zero when $\rho=0$).*
*In particular, we have $m(t,x)=0$ a.e on $\{\rho(t,x)=0\}$ and there exists a function $v(t,x)$ such that $m(t,x)=\rho(t,x)v(t,x)$ and:*

$$\begin{aligned}&\sqrt{\rho_n}v_n \textit{ is bounded in } L^\infty((0,T),L^{2+\alpha}(\Omega)) \textit{ for } \alpha>0 \textit{ small enough,}\\ &\sqrt{\rho_n}v_n\to\sqrt{\rho}v \quad \textit{strongly in } \ L^2_{loc}((0,T)\times\Omega).\end{aligned}$$

*(Let us point out that $v$ is not uniquely defined on the vacuum set $\{\rho(t,x)=0\}$).*

**Proof:** First of all, since $\frac{m_n}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, Fatou's lemma yields:

$$\int\liminf\frac{m_n^2}{\rho_n}dx<+\infty.$$

In particular, we have $m(t,x)=0$ a.e. in $\{\rho(t,x)=0\}$. So if we define the limit velocity by $v(t,x)$ by setting $v(t,x)=\frac{m(t,x)}{\rho(t,x)}$ when $\rho(t,x)\neq0$ and $v(t,x)=0$ when $\rho(t,x)=0$, we have:

$$m(t,x)=\rho(t,x)v(t,x)$$

and

$$\int\frac{m^2}{\rho}dx=\int\rho|v|^2dx<+\infty.$$

For $\alpha$ small enough, we have:

$$\int_\Omega (\rho_n|v_n|^2)^{1+\alpha}dx \le (\int_\Omega \rho_n|v_n|^{2+\alpha}dx)^{\frac{2+2\alpha}{2+\delta}}(\int_\Omega \rho_n^{\delta\frac{\alpha+1}{\delta-2\alpha}}dx)^{\frac{\delta-2\alpha}{2+\delta}}.$$

We observe that when $\alpha$ goes to 0 then $\delta\frac{\alpha+1}{\delta-2\alpha}$ goes to 1. By using (5.35) and the fact that $\rho_n$ is a least bounded in $L^\infty((0,T),L^3(\Omega))$, we conclude that $\sqrt{\rho_n}v_n$ is in $L^\infty((0,T),L^{2+\alpha}(\Omega))$ for $\alpha$ small enough.
Moreover, Fatou's lemma implies that:

$$\begin{aligned}\int \rho|v|^{2+\delta}dx &\le \int \liminf \rho_n|v_n|^{2+\delta}dx\\ &\le \liminf \int \rho_n|v_n|^{2+\delta}dx,\end{aligned}$$

and so $\rho|v|^{2+\delta}$ is in $L^\infty(0,T;L^1(\Omega))$.
Next, since $m_n$ and $\rho_n$ converge almost everywhere, it is readily seen that in $\{\rho(t,x)\neq 0\}$, $\sqrt{\rho_n}v_n = \frac{m_n}{\sqrt{\rho_n}}$ converges almost everywhere to $\sqrt{\rho}u = \frac{m}{\sqrt{\rho}}$. The natural idea now consists in applying the lemma 1, we have only to be careful as $\sqrt{\rho_n}v_n$ converges a priori almost everywhere only on $A=\{\rho(t,x)=0\}$. In order to overcome this difficulty we are just going to decompose $\Omega$ as follows:

$$\Omega = ({}^cC_M^n\ A)\cup({}^cC_M^n\cap A)\cup C_M^n$$

with:

$$C_M^n = \{|\rho_n^{\frac{1}{2+\delta}}v_n|\ge M\}.$$

We have then:

$$\int_\Omega |\sqrt{\rho_n}v_n - \frac{m}{\sqrt{\rho}}|^2dx = \int_{{}^cC_M^n\ A)}\cdots dx + \int_{C_M^n}\cdots dx + \int_{({}^cC_M^n\cap A)}\cdots dx. \tag{5.39}$$

Concerning the two first terms on the right hand side, we easily prove that theses terms goes to 0 when $n$ goes to $+\infty$ by following the proof of the lemma 1. The only new term is the last one on the right hand side of (5.39) which deals with the set $A$. More precisely on $({}^cC_M^n\cap A)$ we have $\frac{m}{\sqrt{\rho}}=0$ and on $({}^cC_M^n\cap A)$:

$$|\sqrt{\rho_n}v_n - \frac{m}{\sqrt{\rho}}|^2 = |\sqrt{\rho_n}v_n|^2 \le M^2\rho_n^{1-\frac{2}{2+\delta}} \to 0 \ \text{ a.e.},$$

because $\rho_n$ converges to 0 on $A$. From the fact that $\rho_n$ converges almost everywhere on $({}^cC_M^n\cap A)$ and the bounded on $\rho_n$ in $L^\infty((0,T),L2(\Omega))$ by the lemma 1 we deduce that:

$$\int_{({}^cC_M^n\cap A)}|\sqrt{\rho_n}v_n - \frac{m}{\sqrt{\rho}}|^2dx \to 0.$$

It conclude the proof of the lemma. $\square$

**Lemma 6** *$\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}v_n$ converges in the distribution sense to $\sqrt{\rho_n}u\otimes\sqrt{\rho_n}v$.*

**Proof:** It suffices to observe that $\sqrt{\rho_n}u_n$ converges weakly in $L^2(0,T,L^2(\Omega))$ to $\sqrt{\rho}u$ and $\sqrt{\rho_n}v_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$ to $\sqrt{\rho}v$. We have then for all $\varphi\in C_0^\infty((0,T)\times\Omega)$:

$$\int_0^T\int_\Omega(\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}v_n)\varphi dxdt\to\int_0^T\int_\Omega(\sqrt{\rho}u\otimes\sqrt{\rho}v)\varphi dxdt.$$

**Lemma 7** *We have:*

$$\rho_n\nabla v_n\to\rho\nabla v\quad in\ \mathcal{D}',$$

**Proof:** Let $\phi$ be a test function, then:

$$\begin{aligned}\int\rho_n\nabla v_n\phi dxdt&=-\int\rho_n v_n\nabla\phi dxdt+\int v_n\nabla\rho_n\,\phi dxdt\\&=-\int\sqrt{\rho_n}(\sqrt{\rho_n}v_n)\nabla\phi dxdt+\int\sqrt{\rho_n}v_n\nabla(2\sqrt{\rho_n})\phi dxdt.\end{aligned}$$

Thanks to lemma 2 and 5, we know that $\sqrt{\rho_n}$ and $\sqrt{\rho_n}v_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$ what is enough to obtain the convergence of the first term.Next as $\sqrt{\rho_n}$ converges strongly to $\sqrt{\rho}$ in $L^2_{loc}((0,T)\times\Omega)$ and that $\nabla\sqrt{\rho_n}$ is bounded in $L^2_{loc}((0,T)\times\Omega)$, it implies that:

$$\nabla\sqrt{\rho_n}\to\nabla\sqrt{\rho}\quad L^2_{loc}((0,T)\times\Omega)-\text{weak}.$$

And we conclude for the second term as $\sqrt{\rho_n}v_n$ converges strongly in $L^2_{loc}((0,T)\times\Omega)$. □

**Step 3: Convergence of $\sqrt{\rho_n}v_n$ to $\sqrt{\rho}u+\frac{2\kappa}{\mu}\nabla\sqrt{\rho}$**

**Lemma 8** *Up to a subsequence, the momentum $(m_1)_n=\rho_n u_n$ converges strongly in $L^{2-\epsilon}(0,T;L^p_{loc}(\Omega))$ (for any $\epsilon>0$ small enough) to some $m_1(x,t)$ for all $p\in[1,\frac{3}{2})$. It implies that up to a subsequence:*

$$\rho_n u_n\to m\quad almost\ everywhere\ \ (x,t)\in(0,T)\times\Omega.$$

*The quantity $\sqrt{\rho_n}u_n$ strongly converges in $L^{2-\epsilon}_{loc}((0,T)\times\Omega)$ (for any $\epsilon>0$ small enough) to $\frac{m_1}{\sqrt{\rho}}$ (defined to be zero when $\rho=0$).*
*In particular, we have $m_1(t,x)=0$ a.e on $\{\rho(t,x)=0\}$ and there exists a function $u(t,x)$ such that $m(t,x)=\rho(t,x)u(t,x)$ (Let us point out that $u$ is not uniquely defined on the vacuum set $\{\rho(t,x)=0\}$).*
*The quantity $\nabla\rho_n$ strongly converges in $L^{2-\epsilon}((0,T),L^{\frac{3}{2}-\epsilon}_{loc}(\Omega))$ (for any $\epsilon>0$ small enough) to $\nabla\rho$. It implies that up to a subsequence $\nabla\sqrt{\rho_n}$ converges almost everywhere to $\nabla\sqrt{\rho}$ on $\{\rho\neq 0\}$.*
*Finally we have:*

$$\sqrt{\rho_n}v_n\to\sqrt{\rho}u+\frac{2\kappa}{\mu}\nabla\sqrt{\rho}=\sqrt{\rho}v\ \ in\ \mathcal{D}'(\Omega),$$

**Remark 22** *Let us mention that unfortunately we are not able to prove the strong convergence of $\sqrt{\rho_n}u_n$ and $\nabla\sqrt{\rho_n}$ in $L^2_{loc}$ but only the strong convergence of $\sqrt{\rho_n}v_n$ in $L^2_{loc}$.*

**Proof:** We have:

$$\rho_n u_n = \sqrt{\rho_n}\sqrt{\rho_n} u_n,$$

where $\sqrt{\rho_n}$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for $q \in [2,+\infty[$ if $N=2$ and $q \in [2,6]$ if $N=3$. Since $\sqrt{\rho_n}u_n$ is bounded in $L^\infty(0,T;L^2(\Omega))$, it shows that $\rho_n u_n$ is bounded in $L^\infty(0,T;L^q(\Omega))$ for all $q \in [1,\frac{3}{2}]$. Following the lemma 4 we are interested in proving that $\rho_n u_n$ is bounded in Sobolev space in order to use Aubin-Lions proposition 4.9. We have:

$$\begin{aligned}\partial_i(\rho_n u_n^j) &= \rho_n \partial_i u_n^j + u_n^j \partial_i \rho_n \\ &= \sqrt{\rho_n}\sqrt{\rho_n}\partial_i u_n^j + 2\sqrt{\rho_n} u_n^j \partial_i \sqrt{\rho_n}.\end{aligned}$$

Using (5.34) and (5.36) the second term is bounded in $L^\infty(0,T;L^1(\Omega))$, while the first term is bounded in $L^2(0,T;L^q(\Omega))$ for all $q \in [1,\frac{3}{2}]$. Then $\nabla(\rho_n u_n)$ is bounded in $L^2(0,T;L^1_{loc}(\Omega))$.

It implies that $\rho_n u_n$ is bounded in $L^2(0,T;W^{1,1}_{loc}(\Omega))$. It remains to obtain estimates on $\partial_t(\rho_n u_n)$, more precisely we have:

$$\text{for all compact } K,\ \partial_t(\rho_n u_n) \text{ is bounded in } L^2(0,T;W^{-2,\frac{3}{2}}(K)). \tag{5.40}$$

In order to prove (5.40), we deal with the momentum equation of system (1.1), first we observe that using (5.34) and (5.36):

$$\begin{aligned}\mathrm{div}(\sqrt{\rho_n}u_n \otimes \sqrt{\rho_n}u_n) &\in L^\infty(0,T;W^{-1,1}(K)),\\ \nabla\rho_n^\gamma &\in L^\infty(0,T;W^{-1,1}(K)).\end{aligned}$$

So we only have to deal with the terms $\mathrm{div}(\rho_n D u_n)$ and $\mathrm{div}K_n$. From the appendix, we recall that:

$$\mathrm{div}K_n = \mathrm{div}(\sqrt{\rho_n}\sqrt{\rho_n}\nabla\nabla\ln\rho_n).$$

Then by (5.34), we check that $\sqrt{\rho_n}\sqrt{\rho_n}\nabla\nabla\ln\rho_n$ is bounded in $L^2((0,T),L^{\frac{3}{2}}(\Omega))$, then $\mathrm{div}(\sqrt{\rho_n}\sqrt{\rho_n}\nabla\nabla\ln\rho_n)$ is bounded in $L^2((0,T),W^{-1,\frac{3}{2}}(\Omega))$.

It remains to treat the term $\mathrm{div}(\rho_n D u_n)$ which is bounded in $L^2((0,T),W^{-1,2}(\Omega))$. By Sobolev embedding we conclude that $\partial_t(\rho_n u_n)$ is uniformly bounded in $L^2((0,T),W^{-1,\frac{3}{2}}(\Omega))$.

Since $\rho_n u_n$ and $\partial_t(\rho_n u_n)$ are respectively bounded in $L^2(0,T;W^{1,1}_{loc}(\Omega))$ and in $L^2(0,T;W^{-2,\frac{3}{2}}_{loc}(\Omega))$, by the Aubin-Lions proposition 4.9, $\rho_n u_n$ strongly converges up to a subsequence to a limit $m_1$ in $L^{2-\epsilon}(0,T;L^p(K))$ for all $p \in [1,\frac{3}{2})$ and any $\epsilon > 0$ small enough.

To prove that $\sqrt{\rho_n}u_n$ converges strongly in $L^{2-\epsilon}_{loc}((0,T)\times\Omega)$ is a direct application of the lemma 1 in the spirit of the proof of the lemma 5.

In order to prove that $\nabla\rho_n$ strongly converges in $L^{2-\epsilon}((0,T),L^{\frac{3}{2}-\epsilon}_{loc}(\Omega))$, we are going to take advantage of the bounds on $\sqrt{\rho_n}\Delta\ln\rho_n$ in $L^2((0,T),L^2(\Omega))$. Indeed we have:

$$\Delta\rho_n = \sqrt{\rho_n}(\sqrt{\rho_n}\Delta\ln\rho_n) + 4|\nabla\sqrt{\rho_n}|^2,$$

we can show by (5.34) that $\Delta\rho_n$ is uniformly bounded in $L^\infty_T(L^1(\Omega)) + L^2((0,T),L^{\frac{3}{2}}(\Omega))$ and so in $L^2(0,T),L^1(\Omega))$. We easily prove that $\partial_t(\nabla\rho_n)$ is bounded in $L^\infty(W^{-2,\frac{3}{2}})$. B By Aubin Lions theorem, we deduce that $\nabla\rho_n$ strongly converges in $L^{2-\epsilon}((0,T),L^{\frac{3}{2}-\epsilon}_{loc}(\Omega))$.

It implies that up to a subsequence $\nabla\rho_n$ converges a.e to $\nabla\rho$ and prove that $\nabla\sqrt{\rho_n}$ converges almost everywhere to $\nabla\sqrt{\rho}$ on $\{\rho\neq 0\}$.
As $\sqrt{\rho_n}$ and $\sqrt{\rho_n}u_n$ strongly converge, it is easy to see that $\sqrt{\rho}v=\sqrt{\rho}v+\frac{\kappa}{\mu}\nabla\sqrt{\rho}$ which concludes the proof of the lemma. $\square$

# 6 Proof of theorem 2.4

Concerning the existence of strong solution on a finite interval $(0,T)$ in the theorem 2.4 with such initial data, we refer to [22] or to the appendix for a sketch of the proof. Let us mention that in the sequel we shall work in many case with unknowns of null average, in particular we will have to consider the variables $\rho(t,x)-\frac{1}{\mathbb{T}^N}\int_{\mathbb{T}^N}\rho(t,y)dy$, $v(t,x)-\frac{1}{\mathbb{T}^N}\int_{\mathbb{T}^N}v(t,y)dy$.For the simplicity we will omit this fastidious notation and we will only use the notation $\rho$ and $v$. Let us mention briefly that we control $\frac{1}{\mathbb{T}^N}\int_{\mathbb{T}^N}\rho(t,y)dy$ as $\frac{1}{\mathbb{T}^N}\int_{\mathbb{T}^N}\rho(t,y)dy=\frac{1}{\mathbb{T}^N}\int_{\mathbb{T}^N}\rho_0(y)dy$. It will be also the case for $\frac{1}{\mathbb{T}^N}\int_{\mathbb{T}^N}v(t,y)dy$ due to some control on the vacuum (indeed we observe that the momentum $\rho v$ is controled).
We now would like to focus on the result of blow-up of the theorem 2.4 and in particular to prove that the estimate (2.17) and (2.20) are sufficient in order. To do this, we shall derivate new regularizing effects on the density $\rho$ combining with some important gain of integrability on the effective velocity $v$. Indeed the key of the proof is to obtain enough integrability on $v$ (more precisely we will see that for any $1\leq p<+\infty$ $v$ belongs in $L^\infty((0,T),L^p(\mathbb{T}^N))$ in order to get regularizing effects on the density via the first equation of system 1.1 (which is a heat equation with a remainder term which depends on $v$).

**First step: Gain of integrability on $v$, for any $1\leq p<+\infty$ $v$ is in $L^\infty((0,T),L^p(\mathbb{T}^N))$**

In this part, we are going to prove that the effective velocity $v$ is in $L^\infty((0,T),L^p(\mathbb{T}^N))$ for any $1\leq p<+\infty$ due to our choice on the pressure $P$, it means $P(\rho)=a\rho$ with $a>0$. We also refer for a similar result to [37].

**Lemma 9** *Let $(\rho,v)$ our strong solution defined in the definition 2.1, then it exists $C(T)>0$ depending only on the initial data of theorem 2.4 (more precisely from the energy data conditions $\sqrt{\rho_0}u_0\in L^2(\mathbb{T}^N)$, $\Pi(\rho_0)\in L^1(\mathbb{T}^N)$ and $\nabla\sqrt{\rho_0}\in L^2(\mathbb{T}^N)$) and of the time $T$ such that:*

$$\Big(\frac{1}{p}\int_{\mathbb{T}^N}(\rho|v|^p)(t,x)dx+\int_0^t\int_{\mathbb{T}^N}\rho|v|^{p-2}|\nabla v|^2(s,x)dsdx \\ +(p-2)\int_0^t\int_{\mathbb{T}^N}\rho(\sum_i\partial_i(|v|^2)^2|v|^{p-4}(s,x)dtdx\Big)\leq C(T).$$

**Proof:** As in [28], we now want to obtain additional information on the integrability of $v$, and more precisely to show that $\rho^{\frac{1}{p}}v$ is in any $L^\infty_T(L^p(\mathbb{T}^N))$ with $1\leq p<+\infty$. To do it, we multiply the momentum equation of (1.1) by $v|v|^{p-2}$ and integrate over $\mathbb{T}^N$, we obtain then:

$$\begin{aligned}&\frac{1}{p}\int_{\mathbb{T}^N}\rho\partial_t(|v|^p)dx+\int_{\mathbb{T}^N}\rho u\cdot\nabla(\frac{|v|^p}{p})dx+\int_{\mathbb{T}^N}\rho|v|^{p-2}|\nabla v|^2dx\\&\qquad+(p-2)\int_{\mathbb{T}^N}\rho\sum_{i,j,k}v_jv_k\partial_iv_j\partial_iv_k|v|^{p-4}dx+\int_{\mathbb{T}^N}|v|^{p-2}v\cdot\nabla(a\rho)dx=0.\end{aligned}\tag{6.41}$$

Next we observe that:

$$\sum_{i,j,k}v_jv_k\partial_iv_j\partial_iv_k=\sum_i(\sum_j v_j\partial_iv_j)^2=\sum_i\big[\frac{1}{2}\partial_i(|v|^2)\big]^2=\frac{1}{4}|\nabla(|v|^2)|^2.$$

We get then as $\mathrm{div}(\rho u)=-\partial_t\rho$ and by using (6.41):

$$\begin{aligned}&\frac{1}{p}\int_{\mathbb{T}^N}\partial_t(\rho|v|^p)dx+\int_{\mathbb{T}^N}\rho|v|^{p-2}|\nabla v|^2dx+\frac{(p-2)}{4}\int_{\mathbb{T}^N}\rho|\nabla(|v|^2)|^2|v|^{p-4}dx\\&\qquad\qquad+\int_{\mathbb{T}^N}|v|^{p-2}v\cdot\nabla(a\rho)dx=0.\end{aligned}\tag{6.42}$$

We have then by integrating over $(0,t)$ with $0<t\leq T$:

$$\begin{aligned}&\frac{1}{p}\int_{\mathbb{T}^N}(\rho|v|^p)(t,x)dx+\int_0^t\int_{\mathbb{T}^N}\rho|v|^{p-2}|\nabla v|^2(s,x)dsdx\\&+\frac{(p-2)}{4}\int_0^t\int_{\mathbb{T}^N}\rho|\nabla(|v|^2)|^2|v|^{p-4}(s,x)dsdx\leq\frac{1}{p}\int_{\mathbb{T}^N}(\rho_0|v_0|^p)(x)dx\\&\qquad\qquad+|\int_0^t\int_{\mathbb{T}^N}|v|^{p-2}v\cdot\nabla a\rho(s,x)dsdx|.\end{aligned}\tag{6.43}$$

We now want to take the sup of the previous estimate on $(0,T)$, we have then:

$$\begin{aligned}&\sup_{t\in(0,T)}\Big(\frac{1}{p}\int_{\mathbb{T}^N}(\rho|v|^p)(t,x)dx+\int_0^t\int_{\mathbb{T}^N}\rho|v|^{p-2}|\nabla v|^2(s,x)dsdx\\&+\frac{(p-2)}{4}\int_0^t\int_{\mathbb{T}^N}\rho|\nabla(|v|^2)|^2|v|^{p-4}(s,x)dsdx\Big)\leq\frac{1}{p}\int_{\mathbb{T}^N}(\rho_0|v_0|^p)(x)dx\\&\qquad\qquad+\sup_{t\in(0,T)}|\int_0^t\int_{\mathbb{T}^N}|v|^{p-2}v\cdot\nabla(a\rho(s,x))dsdx|.\end{aligned}\tag{6.44}$$

By integration by parts we have:

$$\int_0^t\int_{\mathbb{T}^N}|v|^{p-2}v\cdot\nabla a\rho(s,x)dsdx=-a\int_0^t\int_{\mathbb{T}^N}\mathrm{div}(|v|^{p-2}v)\rho(s,x)dsdx.$$

We have then:

$$\mathrm{div}(|v|^{p-2}v)=|v|^{p-2}\mathrm{div}(v)+(p-2)|v|^{p-4}v\cdot(v\cdot\nabla v).$$

We now are interested in bounding the term $\sup_{t\in(0,T)}|\int_0^t\int_{\mathbb{T}^N}|v|^{p-2}v\cdot\nabla\rho(t,x)dtdx|$ in (6.43). To do this, we would like to apply Hölder's inequalities, indeed we have:

$$|\rho|v|^{p-4}v\cdot(v\cdot\nabla v)|\leq C|\sqrt{\rho}|v|^{\frac{p}{2}-1}\nabla v|\times|\rho^{\frac{1}{2}-\frac{1}{p}}|v|^{\frac{p}{2}-1}|\times|\rho^{\frac{1}{p}}|.$$

By Hölders inequalities we can prove that $|\rho|v|^{p-4}v\cdot(v\cdot\nabla v)|$ is in $L^1_t(L^1(\mathbb{T}^N)$ with:

$$\begin{aligned}\sup_{t\in(0,T)}\int_0^t\int_{\mathbb{T}^N}|\rho|v|^{p-4}v\cdot(v\cdot\nabla v)|dxds &\leq \sup_{t\in(0,T)}\big(\sqrt{t}\|\sqrt{\rho}|v|^{\frac{p}{2}-1}\nabla v\|_{L^2_t(L^2(\mathbb{T}^N))}\\ &\qquad\times\|\rho^{\frac{1}{2}-\frac{1}{p}}|v|^{\frac{p}{2}-1}|\|_{L^\infty_t(L^{\frac{2p}{p-2}}(\mathbb{T}^N))}\|\rho^{\frac{1}{p}}\|_{L^\infty_t(L^p(\mathbb{T}^N))}\big),\\ &\leq \sup_{t\in(0,T)}\big(\sqrt{t}\|\sqrt{\rho}|v|^{\frac{p}{2}-1}\nabla v\|_{L^2_t(L^2(\mathbb{T}^N))}\sup_{s\in(0,t)}\big(\int_{\mathbb{T}^N}\rho|v||^p(x)dx\big)^{\frac{1}{2}-\frac{1}{p}}\|\rho^{\frac{1}{p}}\|_{L^\infty_t(L^p(\mathbb{T}^N))}\big).\end{aligned}\tag{6.45}$$

Now we set:

$$\begin{aligned}A(t)=\big(\frac{1}{p}\int_{\mathbb{T}^N}(\rho|v|^p)(t,x)dx+\int_0^t\int_{\mathbb{T}^N}\rho|v|^{p-2}|\nabla v|^2(s,x)dsdx\\ +\frac{(p-2)}{4}\int_0^t\int_{\mathbb{T}^N}\rho|\nabla v|v|^{p-4}(s,x)dtdx\big).\end{aligned}$$

We have then from (6.44) and (6.45):

$$\sup_{t\in(0,T)}A(t)\leq C\big(1+\sqrt{T}(\sup_{t\in(0,T)}A(t))^{1-\frac{1}{p}}\|\rho_0\|^{\frac{1}{p}}_{L^\infty_T(L^1(\mathbb{T}^N))}\big)$$

We deduce then that $\sup_{t\in(0,T)}A(t)$ is finite. In particular, we obtain that $\rho^{\frac{1}{p}}v$ is in any $L^\infty_T(L^p(\mathbb{T}^N))$ for $1<p<+\infty$. $\square$

**Step 2: Regularizing effect on the density $\rho$**

**Lemma 10** *The density $\rho$ belongs in $L^\infty((0,T),B^1_{p,\infty})$ for any $N<p<+\infty$ and the nom in $L^\infty((0,T),B^1_{p,\infty})$ depends only on the initial energy data.*

**Proof:** We have then obtain that for any $1\leq p<+\infty$, $\rho^{\frac{1}{p}}v$ belongs to $L^\infty((0,T),L^p(\mathbb{T}^N))$. We now want to take into account this information to obtain regularizing effects on the density via the first equation of (1.1) (which is a heat equation):

$$\partial_t q-\frac{\kappa}{\mu}\Delta q=-\mathrm{div}(\rho v).\tag{6.46}$$

In the previous equation and In the sequel we assume that $q$ has zero average (it suffices just to consider the unknown $q-\frac{1}{T^N}\int_{\mathbb{T}^N}qdx$). Our goal is to transfer the information on the integrability of $v$ (which is a *subscaling* estimate) on the density $\rho$. More precisely we have by proposition 4.8 for any $1\leq p<+\infty$:

$$\|q\|_{\widetilde{L}^\infty((0,T),B^1_{p,\infty})}\leq C(T)(\|q_0\|_{\widetilde{L}^\infty(B^1_{p,\infty})}+\|\rho v\|_{\widetilde{L}^\infty_T(B^0_{p,\infty})}).\tag{6.47}$$

We now need to prove that $\rho v$ is in $\widetilde{L}^\infty((0,T),B^0_{p,\infty})$. As $\rho^{\frac{1}{p}}v$ is in $L^\infty((0,T),L^p(\mathbb{T}^N))$, we have then:

$$\begin{aligned}\|(\rho^{1-\frac{1}{p}})\rho^{\frac{1}{p}}v\|_{L^\infty((0,T),L^p(\mathbb{T}^N))}&\leq\|\rho^{\frac{1}{p}}v\|_{L^\infty((0,T),L^p(\mathbb{T}N))}\|\rho^{1-\frac{1}{p}}\|_{L^\infty_T(L^\infty(\mathbb{T}^N))},\\ &\leq C(p)\|\rho^{\frac{1}{p}}v\|_{L^\infty((0,T),L^p(\mathbb{T}N))}(1+\|q\|^{1-\frac{1}{p}}_{L^\infty_T(L^\infty(\mathbb{T}^N))}).\end{aligned}\tag{6.48}$$

By injecting (6.48) in (6.47) and using the fact that $L^p(\mathbb{T}^N)$ is embedded in $B^0_{p,\infty}$, we obtain that:

$$\begin{aligned}\|q\|_{\widetilde{L}^\infty((0,T),B^1_{p,\infty})} \leq C(p,T)(\|q_0\|_{\widetilde{L}^\infty(B^1_{p,\infty})} + \|\rho^{\frac{1}{p}}v\|_{L^\infty((0,T),L^p(\mathbb{T}^N))}\\ + \|\rho^{\frac{1}{p}}v\|_{L^\infty((0,T),L^p(\mathbb{T}^N))}\|q\|^{1-\frac{1}{p}}_{L^\infty((0,T),L^\infty(\mathbb{T}^N))}).\end{aligned}\tag{6.49}$$

By Besov embedding we know that $B^1_{p,\infty}$ is embedded in $L^\infty(\mathbb{T}^N)$ for $p\geq N$ large enough. We obtain by (6.50) that:

$$\|q\|_{\widetilde{L}^\infty_T(B^1_{p,\infty})} \leq C(T)(\|q_0\|_{\widetilde{L}^\infty(B^1_{p,\infty})} + \|\rho^{\frac{1}{p}}v\|_{L^\infty_T(L^p(\mathbb{T}^N))}\|\rho\|^{1-\frac{1}{p}}_{L^\infty_T(B^1_{p,\infty}))}).\tag{6.50}$$

We conclude then by bootstrap that $q$ is in $L^\infty_T(B^1_{p,\infty})$ for $p$ large enough (in fact $p>N$). □

**Step 3: Estimates on $\frac{1}{\rho}$ if we control $\frac{1}{\rho^\epsilon}$ in $L^\infty_T(L^1(\mathbb{T}^N))$**

Now we are interested in getting informations on the vacuum in order to express the regularity of $q' = \ln\rho$ in function of the regularity on $q$. To do that we are going to apply energy estimate on $\frac{1}{\rho}$.

**Lemma 11** *Let $\epsilon>0$ and assume that $\frac{1}{\rho^\epsilon}$ belongs in $L^\infty((0,T),L^1(\mathbb{T}^N))$ then for any $1\leq p<+\infty$, $\frac{1}{\rho}$ is bounded in $L^\infty((0,T),L^p(\mathbb{T}^N))$. Furthermore this bound depends only on $T$ and the energy initial data.*

**Proof:** We now would like to obtain estimates on $\frac{1}{\rho}$, to do this we will consider the first equation of (1.1) that we multiply by $-\frac{1}{\rho^p}$ with $p\geq 2$:

$$\frac{1}{p-1}\partial_t(\frac{1}{\rho^{p-1}}) - \frac{\kappa}{\mu(p-1)}\Delta(\frac{1}{\rho^{p-1}}) + \frac{4p\kappa}{\mu(p-1)^2}|\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}})|^2 = \frac{1}{\rho^p}\mathrm{div}(\rho v),\tag{6.51}$$

indeed we have:

$$\frac{\kappa}{\mu\rho^p}\Delta\rho = -\frac{\kappa}{\mu(p-1)}\Delta(\frac{1}{\rho^{p-1}}) + \frac{4p\kappa}{\mu(p-1)^2}|\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}})|^2.$$

Our goal is now to integrate over $(0,t)\times\mathbb{T}^N$ the equality (6.51), we obtain then:

$$\begin{aligned}\frac{1}{p-1}\int_{\mathbb{T}^N}\frac{1}{\rho^{p-1}(t,x)}dx + \frac{4p\kappa}{\mu(p-1)^2}\int_0^t\int_{\mathbb{T}^N}|\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}(s,x)})|^2dsdx\\ \leq \frac{1}{p-1}\int_{\mathbb{T}^N}\frac{1}{\rho_0^{p-1}(x)}dx + \Big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\mathrm{div}(\rho v)(s,x)dsdx\Big|.\end{aligned}\tag{6.52}$$

We now pass to the sup on $(0,T)$, and we have:

$$\begin{aligned}\sup_{t\in(0,T)}\Big(\frac{1}{p-1}\int_{\mathbb{T}^N}\frac{1}{\rho^{p-1}(t,x)}dx + \frac{4p\kappa}{\mu(p-1)^2}\int_0^t\int_{\mathbb{T}^N}|\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}(s,x)})|^2dsdx\Big)\\ \leq \frac{1}{p-1}\int_{\mathbb{T}^N}\frac{1}{\rho_0^{p-1}(x)}dx + \sup_{t\in(0,T)}\Big(\Big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\mathrm{div}(\rho v)(s,x)dsdx\Big|\Big).\end{aligned}\tag{6.53}$$

By integration by parts we get:

$$|\int_0^t \int_{\mathbb{T}^N} \frac{1}{\rho^p} \mathrm{div}(\rho v)(s,x) ds dx| = |p \sum_i \int_0^t \int_{\mathbb{T}^N} \frac{1}{\rho^p} \partial_i \rho \, v^i(s,x) ds dx|.$$

We recall that:

$$|\frac{1}{\rho^p} \partial_i \rho \, v^i|(t,x) = |\frac{1}{\rho^{\frac{p}{2}+\frac{1}{2}}} \partial_i \rho| \times |\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}| \times |v^i|(t,x). \tag{6.54}$$

In the sequel we only shall deal with the case $N=3$, the other cases are similar to treat. Now as on the left side of (6.53), we have information on $\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}$ in $L^\infty((0,T), L^2(\mathbb{T}^N))$ and on $\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}})$ in $L^2((0,T), L^2(\mathbb{T}^N))$ we can assume by Sobolev embedding and interpolation that:

$$\left(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}} - \frac{1}{|T^N|} \int_{T^N} \frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}} dx\right) \in L^k_T(L^q(\mathbb{T}^N)) \quad \text{with } \frac{1}{q} + \frac{2}{3k} = \frac{1}{2} \text{ with } k \in [2,+\infty], q \in [2,6]. \tag{6.55}$$

We have then:

$$\begin{aligned} \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L^k_T(L^q(\mathbb{T}^N))} &\leq \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|^{1-\frac{2}{p}}_{L^\infty_T(L^2(\mathbb{T}^N))} \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|^{\frac{2}{p}}_{L^2_T(L^6(\mathbb{T}^N))}, \\ &\leq \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|^{1-\frac{2}{p}}_{L^\infty_T(L^2(\mathbb{T}^N))} \big(\|\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}})\|_{L^2_T(L^2(\mathbb{T}^N))} + \sqrt{T} \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L^\infty_T(L^2(\mathbb{T}^N))} |T^N|^{\frac{1}{2}}\big)^{\frac{2}{p}}. \end{aligned} \tag{6.56}$$

We now want to take advantage of the fact that $\frac{1}{\rho^\epsilon}$ belongs to $L^\infty_T(L^1(\mathbb{T}^N))$ for $\epsilon$ arbitrary small. Indeed the idea is now to use interpolation results in order to absorb the term on the right side of (6.53) by the left side. More precisely we have by (6.55) that:

$$\frac{1}{\rho^\epsilon} \in L^\infty_T(L^1(\mathbb{T}^N)) \quad \text{and} \quad \frac{1}{\rho^\epsilon} \in L_T^{k(\frac{p-1}{2\epsilon})}(L^{q(\frac{p-1}{2\epsilon})}(\mathbb{T}^N)), \tag{6.57}$$

with:

$$\|\frac{1}{\rho^\epsilon}\|_{L_T^{k(\frac{p-1}{2\epsilon})}(L^{q(\frac{p-1}{2\epsilon})}(\mathbb{T}^N))} = \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|^{\frac{2\epsilon}{(p-1)}}_{L^k_T(L^q(\mathbb{T}^N))}, \tag{6.58}$$

Now by interpolation we can show that:

$$\|\frac{1}{\rho^\epsilon}\|_{L^\alpha_T(L^\beta(\mathbb{T}^N))} \leq \|\frac{1}{\rho^\epsilon}\|^{1-\theta}_{L^\infty_T(L^1(\mathbb{T}^N))} \|\frac{1}{\rho^\epsilon}\|^\theta_{L_T^{k(\frac{p-1}{2\epsilon})}(L^{q(\frac{p-1}{2\epsilon})}(\mathbb{T}^N))}, \tag{6.59}$$

with $\frac{1}{\alpha} = \frac{2\epsilon\theta}{k(p-1)}$ and $\frac{1}{\beta} = (1-\theta) + \frac{2\epsilon\theta}{q(p-1)}$ and $0 \leq \theta \leq 1$.
We check easily that:

$$\|\frac{1}{\rho^\epsilon}\|_{L^\alpha_T(L^\beta(\mathbb{T}^N))} = \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|^{\frac{p-1}{2\epsilon}}_{L_T^{\frac{2\epsilon\alpha}{p-1}}(L^{\frac{2\epsilon\beta}{p-1}}(\mathbb{T}^N))}. \tag{6.60}$$

From (6.58), (6.59) and (6.60), we obtain finally that:

$$\|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|^{\frac{p-1}{2\epsilon}}_{L_T^{\frac{2\epsilon\alpha}{p-1}}(L^{\frac{2\epsilon\beta}{p-1}}(\mathbb{T}^N))} \leq \|\frac{1}{\rho^\epsilon}\|^{1-\theta}_{L^\infty_T(L^1(\mathbb{T}^N))} \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L^k_T(L^q(\mathbb{T}^N))}^{\frac{2\epsilon\theta}{(p-1)}}, \tag{6.61}$$

or:

$$\|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L_T^{\frac{2\epsilon\alpha}{p-1}}(L^{\frac{2\epsilon\beta}{p-1}}(\mathbb{T}^N))} \leq \|\frac{1}{\rho^\epsilon}\|_{L_T^\infty(L^1(\mathbb{T}^N))}^{\frac{2\epsilon}{(p-1)}(1-\theta)} \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L_T^k(L^q(\mathbb{T}^N))}^{(\frac{2\epsilon}{(p-1)})^2\theta}. \tag{6.62}$$

We recall that:

$$\frac{2\epsilon\alpha}{p-1} = \frac{k}{\theta} \quad \text{and} \quad \frac{2\epsilon\beta}{p-1} = \frac{2\epsilon q}{(1-\theta)(p-1)q+2\epsilon\theta}. \tag{6.63}$$

By using (6.54), (6.56) and (6.62) we obtain by Hölder'sinequelities:

$$\begin{aligned} \Big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\operatorname{div}(\rho v)(t,x)dtdx\Big| \leq C\|\frac{1}{\rho^{\frac{p}{2}+\frac{1}{2}}}\partial_i\rho\|_{L_T^2(L^2(\mathbb{T}^N))} \\ \times \|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L_T^{k(1+\alpha_1)}(L^{q(1+\alpha_2)}(\mathbb{T}^N))}\|v\|_{L_T^{k^1}(L^{q^1}(\mathbb{T}^N))}. \end{aligned} \tag{6.64}$$

with $\alpha_1>0$, $\alpha_2>0$ arbitrary small, $\theta$ depending on $\alpha_1>0$ and $\alpha_2$ (such that $0<\theta<1$ and $\theta$ goes to 1 when $\alpha_1$ and $\alpha_2$ go to 0) and $k^1$, $k^2$ such that:

$$\frac{1}{2}+\frac{1}{k(1+\alpha_1)}+\frac{1}{k^1}=1 \quad \text{and} \quad \frac{1}{2}+\frac{1}{q(1+\alpha_2)}+\frac{1}{k^2}=1.$$

From (6.56), (6.64) and (6.62) we have finaly:

$$\begin{aligned} \Big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\operatorname{div}(\rho v)(t,x)dtdx\Big| \leq C\big(\sup_{t\in(0,T)} B(t)\big)^{1+(\frac{2\epsilon}{(p-1)})^2\theta} \\ \|\frac{1}{\rho^\epsilon}\|_{L_T^\infty(L^1(\mathbb{T}^N))}^{\frac{2\epsilon}{(p-1)}(1-\theta)}\|v\|_{L_T^{k^1}(L^{q^1}(\mathbb{T}^N))}, \end{aligned} \tag{6.65}$$

with:

$$B^2(t) = \big(\frac{1}{p-1}\int_{\mathbb{T}^N}\frac{1}{\rho^{p-1}(t,x)}dx + \frac{4p\kappa}{\mu(p-1)^2}\int_0^t\int_{\mathbb{T}^N}|\nabla(\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}(t,x)})|^2dtdx\big).$$

When $(p-1)>2\epsilon$ we can absorb the term $\big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\operatorname{div}(\rho v)(t,x)dtdx\big|$ because from the previous inequality, we have:

$$\big(\sup_{t\in(0,T)} B(t)\big)^2 \leq C\big(1+\big(\sup_{t\in(0,T)} B(t)\big)^{1+(\frac{2\epsilon}{(p-1)})^2\theta}\big)\|v\|_{L_T^{k^1}(L^{q^1}(\mathbb{T}^N))}\big),$$

with $1+(\frac{2\epsilon}{(p-1)})^2\theta<2$. Indeed we can show easily that $v$ is bounded in $L_T^{k^1}(L^{q^1}(\mathbb{T}^N))$. To do this, we recall that $\rho^{\frac{1}{p}}v$ is bounded in $L_T^\infty(L^p(\mathbb{T}^N))$ for $p$ enough big. As we control $\frac{1}{\rho^\epsilon}$ in $L_T^\infty(L^1(\mathbb{T}^N))$ and that:

$$v = \frac{1}{\rho^{\frac{1}{p}}}\rho^{\frac{1}{p}}v,$$

It means that $v$ is bounded in $L_T^\infty(L^{\frac{p}{2}}(\mathbb{T}^N))$ for $p$ enough big and then $v$ is bounded in $L_T^{k^1}(L^{q^1}(\mathbb{T}^N))$.
We have then proved that for any $p>0$, $\frac{1}{\rho^p}$ is bounded in $L_T^\infty(L^1(\mathbb{T}^N))$. $\square$

**Step 3 bis: Estimates on $\frac{1}{\rho}$ if we control $v$ in $L^{p_1}_T(L^{q_1}(\mathbb{T}^N))$ or in $C([0,T],L^N(\mathbb{T}^N)$ with $\frac{1}{p_1}+\frac{N}{2q_1}=\frac{1}{2}$ and $2\leq p_1<+\infty$, $3<q_1\leq\infty$**

**Lemma 12** *Let $\epsilon>0$ and we assume that $v$ belongs in $L^{p_1}_T(L^{q_1}(\mathbb{T}^N))$ or in $C([0,T],L^N(\mathbb{T}^N)$ with $\frac{1}{p_1}+\frac{N}{2q_1}=\frac{1}{2}$ and $2\leq p_1<+\infty$, $3<q_1\leq\infty$. Then $\frac{1}{\rho}$ is bounded in $L^\infty((0,T),L^p(\mathbb{T}^N))$. Furthermore this bound depends only on $T$ and the energy initial data.*

**Proof:** For the sake of the simplicity we are only dealing with the case $N=3$, indeed the case $N=2$ is similar. It suffices only to bound $\big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\mathrm{div}(\rho v)(t,x)dtdx\big|$ as follows, we recall that we have from (6.56):

$$\|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L^{p_2}((0,T),L^{q_2}(\mathbb{T}^N))}\leq C(T)B(T),$$

with $\frac{1}{q_2}+\frac{2}{3p_2}=\frac{1}{2}$. By using (6.54) and the previous inequality we have:

$$\big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\mathrm{div}(\rho v)(t,x)dtdx\big|\leq C(T)\|\frac{1}{\rho^{\frac{p}{2}+\frac{1}{2}}}\partial_i\rho\|_{L^2_T(L^2(\mathbb{T}^N))}\|\frac{1}{\rho^{\frac{p}{2}-\frac{1}{2}}}\|_{L^{p_2}((0,T),L^{q_2}(\mathbb{T}^N))}\|v\|_{L^{p_1}((0,T),L^{q_1}(\mathbb{T}^N))}.$$

Indeed we have:

$$\frac{1}{2}+\frac{1}{p_2}+\frac{1}{p_1}=1\ \text{ and }\ \frac{1}{2}+\frac{1}{q_2}+\frac{1}{q_1}=1,$$

and:

$$\frac{1}{p_1}+\frac{3}{2q_1}=\frac{1}{2}-\frac{1}{p_2}+\frac{3}{2}(\frac{1}{2}-\frac{1}{q_2})=\frac{1}{2}+\frac{3}{4}-\frac{3}{2}(\frac{1}{q_2}+\frac{2}{3p_2})=\frac{1}{2}.$$

We finally have:

$$\begin{aligned}\big|\int_0^t\int_{\mathbb{T}^N}\frac{1}{\rho^p}\mathrm{div}(\rho v)(t,x)dtdx\big|&\leq C(T)\big(\sup_{t\in(0,T)}B(t)\big)^2\|v\|_{L^p((0,T),L^q(\mathbb{T}^N))},\\&\leq C(T)T^{\frac{\epsilon}{(p(p+\epsilon)}}\big(\sup_{t\in(0,T)}B(t)\big)^2\|v\|_{L^{p+\epsilon}((0,T),L^q(\mathbb{T}^N))}.\end{aligned}$$

This term can be absorbed in (6.53) for $T$ small enough. To obtain the result on a general interval $(0,T)$ it suffices to use a density argument, indeed we can easily write $v$ under the following form:

$$v=v_1+v_2,$$

with $\|v_1\|_{L^{p_1}((0,T),L^{q_1}(\mathbb{T}^N))}$ small enough in function on $T$ and $v_2$ belongs in $L^{p_1+\epsilon}((0,T),L^{q_1}(\mathbb{T}^N))$ with $\epsilon>0$. By repeating the previous procedure, it is sufficient for concluding. Indeed the smallness on $v_1$ ensures the possibility to use a bootstrap argument and the gain of time integrability on $v_2$ allows us to develop a Gronwall argument. We proceed exactly in the same way when $v$ belongs in $C((0,T),L^3(\mathbb{T}^N))$. $\square$

**Final blow-up argument**

**Lemma 13** *Under the assumption of theorem 2.4, $\ln\rho$ belongs in $B^{\frac{N}{p_1}+\epsilon}_{p_1,+\infty}$ for $\epsilon>0$ small enough and $3<p_1<6$. Furthermore the norm of $\ln\rho$ in $B^{\frac{N}{p_1}+\epsilon}_{p_1,+\infty}$ depends only on the initial energy data.*

**Proof:** To do this it suffice to recall that $\nabla\sqrt{\rho}$ is in $L^\infty((0,T),L^2(\mathbb{T}^N))$, and as $\frac{1}{\rho^\alpha}$ is in $L^\infty((0,T),L^1(\mathbb{T}^N))$ for any $1<\alpha<+\infty$ we deduce that $\nabla\frac{1}{\rho^p}$ belongs in any $L^\infty((0,T),L^{2-\epsilon}(\mathbb{T}^N))$ for any $1\leq p<+\infty$ and any $\epsilon>0$ small enough. It implies in particular since $\frac{1}{\rho^\alpha}$ is in $L^\infty((0,T),L^1(\mathbb{T}^N))$ for any $1<\alpha<+\infty$ that for any $1\geq p<+\infty$, $\frac{1}{\rho^p}$ belongs in any $L^\infty((0,T),W^{1,2-\epsilon}(\mathbb{T}^N))$ for any $\epsilon>0$ small enough.
For the sake of simplicity, we are only dealing with the case $N=3$, the case $N=2$ is similar. Since for any $1\geq p<+\infty$, $\frac{1}{\rho^p}$ belongs in any $L^\infty((0,T),W^{1,2-\epsilon}(\mathbb{T}^N))$ for any $\epsilon>0$ small enough, by Besov embedding and interpolation in non homogeneous Besov space we have that for any $1\geq p<+\infty$, $\frac{1}{\rho^p}$ is in $L^\infty((0,T),B^0_{6-\epsilon,1})$ for any $\epsilon>0$ small enough.

We are now interested in proving that $\nabla\ln\rho$ belongs in $L^\infty((0,T),B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty})$ for a certain $p_1>N$ and a small $\epsilon>0$. We are going to use Bony papraproduct (for the classical notation we refer to [3, 23]), more precisely we have for any $1\leq p<+\infty$ and any $\epsilon>0$ small enough:

$$\|T_{\frac{1}{\rho}}\nabla\rho\|_{B^{-\epsilon}_{p,\infty}}\leq\|\frac{1}{\rho}\|_{B^{-\epsilon}_{\infty,\infty}}\|\nabla\rho\|_{B^0_{p,\infty}}\leq C(1+\|\frac{1}{\rho}\|_{B^0_{q,\infty}})\|\nabla\rho\|_{B^0_{p,\infty}}.$$

Indeed by Besov embedding for $q$ large enough and by interpolation we have a control on $\frac{1}{\rho}$ in $B^{-\epsilon}_{\infty,\infty}$.
Similarly for the same reasons we have:

$$\|T_{\nabla\rho}\frac{1}{\rho}\|_{B^{-\epsilon}_{p,\infty}}\leq\|\nabla\rho\|_{B^{-\epsilon}_{\infty,\infty}}\|\frac{1}{\rho}\|_{B^0_{p,\infty}}\leq C(1+\|\nabla\rho\|_{B^0_{q,\infty}})\|\frac{1}{\rho}\|_{B^0_{p,\infty}}.$$

The only delicate point is to treat carefully the remainder term $R(\nabla\rho,\frac{1}{\rho})$, we have then for any $1\leq p<+\infty$ and any $\epsilon>0$ small enough:

$$\|R(\nabla\rho,\frac{1}{\rho})\|_{B^0_{p,\infty}}\leq C\|\nabla\rho\|_{B^0_{p_1,\infty}}\|\frac{1}{\rho}\|_{B^0_{6-\epsilon,1}},$$

with $\frac{1}{p_1}=\frac{1}{6-\epsilon}+\frac{1}{p}$.

In conclusion we have prove that for $3<p_1<6$, $\nabla\rho$ is in $L^\infty_T(B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty})$ for any $\epsilon>0$ small enough. We can easily observe that $\frac{1}{\rho}$ is in $L^\infty((0,T),B^0_{p_1,\infty})$ since it is in $L^\infty((0,T),L^{p_1}(\mathbb{T}^N))$. It concludes the proof of the lemma. □

The idea now consists in proving that $u$ and $\ln\rho$ are bounded respectively in $L^\infty((0,T),B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty}$ and in $L^\infty((0,T),B^{\frac{N}{p_1}+\epsilon}_{p_1,\infty}$. By using the theorem 2.2 and 2.3, we will be able to extend the strong solution $(\rho,u)$ on an interval $(0,T+\alpha)$ with $\alpha>0$.

From the previous lemma, it remains only to bound $u$ in $L^\infty((0,T),B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty}$. We recall that we have obtained a bound on $\rho^{\frac{1}{p}}v$ and $\frac{1}{\rho}$ in $L^\infty((0,T),L^p(\mathbb{T}^N))$ for any $1\leq p<+\infty$ and that $\nabla\ln\rho\in L^\infty((0,T),B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty})$. In particular we easily verify that $v$ is in any $L^\infty((0,T),L^p(\mathbb{T}^N)$ for $1\leq p<+\infty$. As $u=v-\frac{\kappa}{\mu}\nabla\ln\rho$, we obtain easily by Besov embedding that $u$ is bounded in in $L^\infty((0,T),B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty})$. To summarize what we have obtained, we have:

$$u \in L^\infty((0,T), B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty}) \quad \text{and} \quad \ln\rho \in L^\infty((0,T), B^{\frac{N}{p_1}+\epsilon}_{p_1,\infty}). \tag{6.66}$$

An important fact is that we easily can prove that if $u_0 \in B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty}$ and $\ln\rho_0 \in B^{\frac{N}{p_1}+\epsilon}_{p_1,\infty}$ then the system (1.1) has a strong solution $(\rho,u)$ on $(0,T^{'})$ with $T^{'}$ bounded by below as follows:

$$T^{'} \geq \frac{C}{(1+\|u_0\|_{B^{\frac{N}{p_1}-1+\epsilon}_{p_1,\infty}} + \|\ln\rho_0\|_{B^{\frac{N}{p_1}+\epsilon}_{p_1,\infty}})^\beta},$$

with $C,\beta > 0$. We refer to [22] for the proof. This is an easy consequence of the fact that the initial data are chosen subcritical this fact is well-known for the incompressible Navier-Stokes equations).
It means that there exists a time $T^{'} \geq c > 0$, where $c$ depends only on the physical coefficients (typically the pressure term, the viscosity and the capillary coefficient) and the subcritical initial data. We can construct by theorem 2.2 a solution $(\rho_1,u_1)$ on $(T-\alpha, T-\alpha+T^{'})$ with initial data $(\rho(T-\alpha), u(T-\alpha))$ (here $\alpha < T^{'}$). The only difficulty is to prove that on $(T-\alpha,T)$ we have:

$$(\rho_1,u_1) = (\rho,u).$$

To do this, it suffices only to use the uniqueness part of theorem 1 in [22] or the theorem 2.2. It concludes the proof of theorem 2.4. $\square$

# 7 Appendix

In this appendix, we only want to detail the computation on the Korteweg tensor and to give a brief sketch of the proof of the theorem 2.2.

**Lemma 14**

$$\mathrm{div}K = \kappa\mathrm{div}(\rho\nabla\nabla\ln\rho) = \kappa\mathrm{div}(\rho D(\nabla\ln\rho)).$$

**Proof:** By calculus, we obtain then:

$$\begin{aligned}(\mathrm{div}K)_j &= \big(\nabla\Delta\rho - \mathrm{div}(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho)\big)_j,\\ &= \partial_j\Delta\rho - \frac{1}{\rho}\Delta\rho\,\partial_j\rho - \frac{1}{2\rho}\partial_j|\nabla\rho|^2 + \frac{1}{\rho^2}|\nabla\rho|^2\partial_j\rho,\end{aligned} \tag{7.67}$$

Next we have:

$$\Delta\rho = \rho\Delta\ln\rho + \frac{1}{\rho}|\nabla\rho|^2.$$

We have then:

$$\begin{aligned}\partial_j\Delta\rho - \frac{1}{\rho}\Delta\rho\partial_j\rho &= \partial_j(\rho\,\Delta\ln\rho + \frac{1}{\rho}|\nabla\rho|^2) - \Delta\ln\rho\,\partial_j\rho - \frac{1}{\rho^2}|\nabla\rho|^2\partial_j\rho,\\ &= \rho\partial_j\Delta\ln\rho + \frac{1}{\rho}\partial_j(|\nabla\rho|^2) - \frac{2}{\rho^2}|\nabla\rho|^2\partial_j\rho.\end{aligned} \tag{7.68}$$

Putting the expression of (7.68) in (7.67), we obtain:

$$(\mathrm{div} K)_j = \partial_j \Delta\rho + \frac{1}{2\rho}\partial_j(|\nabla\rho|^2) - \frac{1}{\rho^2}|\nabla\rho|^2\partial_j\rho. \tag{7.69}$$

Next by calculus, we have:

$$\begin{aligned}\frac{1}{2\rho}\partial_j(|\nabla\rho|^2) - \frac{1}{\rho^2}|\nabla\rho|^2\partial_j\rho &= \sum_i(\partial_i \ln\rho\partial_{ij}\rho - (\partial_i\ln\rho)^2\partial_j\rho),\\ &= \sum_i \partial_i \ln\rho\, \rho\partial_{i,j}\ln\rho,\\ &= \frac{\rho}{2}\nabla(|\ln\rho|^2)_j.\end{aligned} \tag{7.70}$$

Finally by using (7.75) and (7.69), we obtain:

$$\mathrm{div} K = \rho(\nabla\Delta(\ln\rho) + \frac{\rho}{2}\nabla(|\nabla\ln\rho|^2)).$$

We now want to prove that we can rewrite (7) under the form of a viscosity tensor. To see this, we have:

$$\begin{aligned}\mathrm{div}(\rho\nabla(\nabla\ln\rho))_j &= \sum_i \partial_i(\rho\partial_{ij}\ln\rho),\\ &= \sum_i[\partial_i\rho\partial_{ij}\ln\rho + \rho\partial_{iij}\ln\rho],\\ &= \rho(\Delta\nabla\ln\rho)_j + \sum_i \rho\partial_i\ln\rho\partial_j\partial_i\ln\rho),\\ &= \rho(\Delta\nabla\ln\rho)_j + \frac{\rho}{2}(\nabla(|\nabla\ln\rho|^2))_j,\\ &= \mathrm{div}K.\end{aligned}$$

We have then:

$$\mathrm{div}K = \kappa\mathrm{div}(\rho\nabla\nabla\ln\rho) = \kappa\mathrm{div}(\rho D(\nabla\ln\rho)).$$

□

We are now going to give a sketch of the proof of the theorem 2.2.

## Proof of the theorem 2.2

We are interested in giving here a sketch of the proof of the theorem 2.2. As a first step, we shall study the linear part of the system (2.14) about constant reference density, that is:

$$(N) \qquad \begin{cases}\partial_t q + \mathrm{div} u = F,\\ \partial_t u - a\Delta u - b\nabla\mathrm{div} u - c\nabla\Delta q = G,\end{cases}$$

We want to prove a priori estimates in Chemin-Lerner spaces for system $(N)$ with the following hypotheses on $a, b, c, d$ which are constant:

$$0 < c_1 \leq a < M_1 < \infty,\ 0 < c_2 \leq a + b < M_2 < \infty \ \text{ and } \ 0 < c_3 \leq c < M_3 < \infty.$$

This system has been studied by Danchin and Desjardins in [16], the following proposition uses exactly the same type of arguments (see [22] for the proof).

**Proposition 7.11** *Let* $1 \leq r \leq +\infty$ , $0 \leq s \leq 1$, $(q_0, u_0) \in B^{\frac{N}{2}+s}_{2,r} \times (B^{\frac{N}{2}-1+s}_{2,r})^N$, *and* $(F, G) \in \widetilde{L}^1_T(B^{\frac{N}{2}+s}_{2,r}) \times (\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r}))^N$.
*Let* $(q, u) \in (\widetilde{L}^1_T(B^{\frac{N}{2}+s+2}_{2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{2}+s}_{2,r})) \times ((\widetilde{L}^1_T(B^{\frac{N}{2}+s+1}_{2,r}))^N \cap (\widetilde{L}^\infty_T(B^{\frac{N}{2}+s-1}_{2,r})^N)$ *be a solution of the system* $(N)$, *then there exists a universal constant* $C$ *such that:*

$$\|(\nabla q, u)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1+s}_{2,r})\cap\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s}_{2,r})} \leq C(\|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}+s}_{2,2}} + \|(\nabla F, G)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r})}).$$

We now are going to prove the existence of strong solutions in critical space for system (2.14). In particular we recall that the main interest of theorem 2.2 is to allow discontinuous initial data for the density, such that we can authorize discontinuous interfaces.

**Existence of solutions**

We use a standard scheme:

1. We will use a classical iterative scheme to constructed a sequence of approximated solutions $(q^n, u^n)$ on a bounded interval $[0, T]$ which depend not on $n$. We will get uniform estimates on $(q^n, u^n)$ in:

$$E_T = \big(\widetilde{C}_T(B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty})\big) \times \big(\widetilde{C}_T(B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})\big).$$

2. We will prove that the sequence $(q^n, u^n)$ is of Cauchy and converges to a solution of (2.14).

**First step**

We smooth out the data as follows:

$$q^n_0 = S_n q_0, \;\; u^n_0 = S_n u_0 \;\; \text{and} \;\; f^n = S_n f.$$

Note that we have:

$$\forall l \in \mathbb{Z}, \;\; \|\Delta_l q^n_0\|_{L^p} \leq \|\Delta_l q_0\|_{L^p} \;\; \text{and} \;\; \|q^n_0\|_{B^{\frac{N}{p}}_{p,\infty}} \leq \|q_0\|_{B^{\frac{N}{p}}_{p,\infty}},$$

and similar properties for $u^n_0$ and $f^n$, a fact which will be used repeatedly during the next steps. Now, according [25], one can solve (2.14) with the smooth data $(q^n_0, u^n_0, f^n)$. We get a solution $(q^n, u^n)$ on a non trivial time interval $[0, T_n]$ such that:

$$q^n \in \widetilde{C}([0, T_n), B^{\frac{N}{p}}_{p,1}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,1}) \;\; \text{and} \;\; u^n \in \widetilde{C}([0, T_n), B^{\frac{N}{p}-1}_{p,1}) \cap \widetilde{L}^1_{T_n}(B^{\frac{N}{p}+1}_{p,1}). \tag{7.71}$$

**Uniform bounds**

Let:

$$q^n = q_L + \bar{q}^n,\ u^n = u_L + \bar{u}^n,$$

where $(q_L, u_L)$ stands for the solution of:

$$\begin{cases} \partial_t q_L + \mathrm{div} u_L = 0, \\ \partial_t u_L - \mathcal{A} u_L - \kappa \nabla(\Delta q_L) = 0, \end{cases} \tag{7.72}$$

supplemented with initial data:

$$q_L(0) = q_0\ ,\ u_L(0) = u_0.$$

Using the proposition 7.11, we obtain the following estimates on $(q_L, u_L)$ for all $T > 0$:

$$q_L \in \widetilde{C}([0,T], B^{\frac{N}{p}}_{p,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty}) \text{ and } u_L \in \widetilde{C}([0,T], B^{\frac{N}{p}-1}_{p,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty}).$$

We let $(\bar{q}^0, \bar{u}^0) = (0,0)$. We now want study the behavior of $(\bar{q}_n, \bar{u}_n)$ where $(\bar{q}_n, \bar{u}_n)$ are the solution of the following system:

$$(N_1) \qquad \begin{cases} \partial_t \bar{q}^n + \mathrm{div}(\bar{u}^n) = F_{n-1}, \\ \partial_t \bar{u}_n - \mathcal{A}\bar{u}_n - \kappa\nabla(\Delta \bar{q}^n) = G_{n-1}, \\ (\bar{q}_n, \bar{u}_n)_{t=0} = (0,0), \end{cases}$$

where:

$$\begin{aligned} F_{n-1} =& - u^{n-1} \cdot \nabla q^{n-1}, \\ =& - u_L \cdot \nabla q_L - \bar{u}^{n-1} \cdot \nabla q_L - u^L \cdot \nabla \bar{q}^{n-1} - \bar{u}^{n-1} \cdot \nabla \bar{q}^{n-1}, \\ G_{n-1} =& - (u^{n-1})^* . \nabla u^{n-1} + \mu \nabla q^{n-1} \cdot D u^{n-1} + \lambda \nabla q^{n-1}\, \mathrm{div} u^{n-1} + \frac{1}{2}\nabla(|\nabla q^{n-1}|^2) - K \nabla q^{n-1}. \end{aligned}$$

**1) First Step , Uniform Bound**

Let $\epsilon$ be a small parameter and choose $T$ small enough so that by using the estimate of proposition 7.11 we have:

$$(\mathcal{H}_\epsilon) \qquad \begin{aligned} &\|u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty})} + \|q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty})} \leq \epsilon, \\ &\|u_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1}_{p,\infty})} + \|q_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty})} \leq A_0. \end{aligned}$$

We are going to show by induction that:

$$(\mathcal{P}_n), \qquad \|(\bar{q}^n, \bar{u}^n)\|_{F_T} \leq \sqrt{\epsilon}.$$

for $\epsilon$ small enough with:

$$F_T = \big(\widetilde{C}([0,T], B^{\frac{N}{p}}_{p,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty})\big) \times \big(\widetilde{C}([0,T], B^{\frac{N}{p}-1}_{p,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty})\big).$$

As $(\bar{q}^0,\bar{u}^0)=(0,0)$ the result is true for $n=0$. We now suppose $(\mathcal{P}_{n-1})$ (with $n\geq 1$) true and we are going to show $(\mathcal{P}_n)$. Applying proposition 7.11 we have:

$$\|(\bar{q}^n,\bar{u}^n)\|_{F_T}\leq C\|(\nabla F_{n-1},G_{n-1})\|_{\widetilde{L}^1_T(B^{\frac{N}{p}-1}_{p,\infty})}. \tag{7.73}$$

Bounding the right-hand side may be done by applying proposition 4.7, lemma 1 and corollary 1. We begin with treating the case of $\|F_{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,\infty})}$, we have then:

$$\|u_L\cdot\nabla q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,\infty})}\leq\|u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty})}\|q_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty})}+\|q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty})}\|u_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1}_{p,\infty})}.$$

Similarly we obtain:

$$\|u_L\cdot\nabla\bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,\infty})}\leq\|u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty})}\|\bar{q}^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty})}+\|\bar{q}^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{p}+\frac{3}{2}}_{p,\infty})}\|u_L\|_{\widetilde{L}^4_T(B^{\frac{N}{p}-\frac{1}{2}}_{p,\infty})},$$

$$\|\bar{u}^n\cdot\nabla q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,\infty})}\leq\|\bar{u}^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{p}+\frac{1}{2}}_{p,\infty})}\|q_L\|_{\widetilde{L}^4_T(B^{\frac{N}{p}+\frac{1}{2}}_{p,\infty})}+\|q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty})}\|\bar{u}^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1}_{p,\infty})},$$

and:

$$\|\bar{u}^{n-1}\cdot\nabla\bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,\infty})}\leq\|\bar{u}^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,\infty})}\|\bar{q}^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty})}+\|\bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,\infty})}\|\bar{u}^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1}_{p,\infty})}.$$

By using the previous inequalities and $(\mathcal{H}_\epsilon)$, we obtain that:

$$\|F_n\|_{L^1_T(B^{\frac{N}{p}})}\leq C(2A_0\epsilon+2\epsilon^{\frac{3}{2}}+2\sqrt{\epsilon}\epsilon^{\frac{1}{4}}+2\epsilon). \tag{7.74}$$

Next we want to control $\|G_n\|_{\widetilde{L}^1(B^{\frac{N}{p}-1}_{p,\infty})}$. According to propositions 4.7, corollary 1 and 7.11, we have:

$$\begin{aligned}
\|(u^{n-1})^*.\nabla u^{n-1}\|_{L^1_T(B^{\frac{N}{p}-1}_{p,\infty})}&\lesssim\|u^{n-1}\|_{L^{\frac{4}{3}}_T(B^{\frac{N}{p}+\frac{1}{2}}_{p,\infty})}\|u^{n-1}\|_{L^4_T(B^{\frac{N}{p}-\frac{1}{2}}_{p,\infty})},\\
\|\nabla(|\nabla q^{n-1}|^2)\|_{\widetilde{L}^1_T(B^{\frac{N}{p}-1}_{p,\infty})}&\lesssim\||\nabla q^{n-1}|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,\infty})},\\
&\lesssim\|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{p}+\frac{1}{2}}_{p,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{p}-\frac{1}{2}}_{p,\infty})},\\
&\lesssim\|q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{p}+\frac{3}{2}}_{p,\infty})}\|q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{p}+\frac{1}{2}}_{p,\infty})}.
\end{aligned}$$

We proceed similarly for the other terms and we obtain by using (7.73) and the different previous inequalities:

$$\|(\bar{q}_{n+1},\bar{u}_{n+1})\|_{F_T}\leq C\sqrt{\epsilon}(\sqrt{\epsilon}A_0+\sqrt{\epsilon}+\epsilon^{\frac{1}{4}}). \tag{7.75}$$

By taking $T$ and $\epsilon$ small enough we have $(\mathcal{P}_{n+1})$, so we have shown by induction that $(q^n,u^n)$ is bounded in $F_T$.
The rest of the proof is standard and consists in using compactness arguments for proving the existence. The uniqueness is also classical and we refer to [22]. □

We are now going to give a sketch of the proof of the theorem 2.3 by especially emphasizing on the estimate (2.16).

## Proof of the theorem 2.3

We are proving the estimate (2.16). It only suffices to recall a more precise estimate on the heat equation as in the proposition 4.8, indeed we have with the same hypothesis as in the proposition 4.8 (see [3]) for a constant $C > 0$:

$$\begin{aligned}&\|u\|_{\widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1}_{p,r})} \leq C\big(T^{\frac{1}{\rho_1}}\|u_0\|_{B^s_{p,r}} + \mu^{\frac{1}{\rho_2}-1}T^{1+\frac{1}{\rho_1}-\frac{1}{\rho_2}})\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})}\big)\\ &+\big[\sum_l \big(\frac{1-e^{-C\mu T\rho_1 2^{2l}}}{C\mu\rho_1}\big)^{\frac{r}{\rho_1}}(2^{ls}\|\Delta_l u_0\|_{L^p})^r\big]^{\frac{1}{r}} + \big[\sum_l \big(\frac{1-e^{-C\mu T\rho_2' 2^{2l}}}{C\mu\rho_2'}\big)^{\frac{r}{\rho_2}}(2^{l(s-2+\frac{2}{\rho_2})}\|\Delta_l u_0\|_{L^p})^r\big]^{\frac{1}{r}}\end{aligned} \tag{7.76}$$

with $\rho_2' = (1+\frac{1}{\rho_1}-\frac{1}{\rho_1})^{-1}$. For using estimate (7.75), it is only a matter of proving smallness assumptions on $q_L$ and $u_L$ in critical Besov space. Let us remark that$q_L$ and $u_L$ verify an heat equation (to see this it just suffices to introduces the effective velocity $v_L = u_L + \frac{\kappa}{\mu}\nabla q_L$).
By the estimates (7.76) when $f = 0$, we easily show that:

$$\begin{aligned}\|u_L\|_{\widetilde{L}^{\rho_1}_T(B^{\frac{N}{p}-1+2/\rho_1}_{p,1})} &\leq C\big(T^{\frac{1}{\rho_1}}\|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} + \big[\sum_l T^{\frac{1}{\rho_1}}2^{\frac{2l}{\rho_1}}(2^{l(\frac{N}{p}-1)}\|\Delta_l u_0\|_{L^p})\big]\\ &\quad + T^{\frac{1}{\rho_1}}\|q_0\|_{B^{\frac{N}{p}}_{p,1}} + \big[\sum_l T^{\frac{1}{\rho_1}}2^{\frac{2l}{\rho_1}}(2^{l(\frac{N}{p})}\|\Delta_l q_0\|_{L^p})\big]\big),\\ &\leq C\big(T^{\frac{1}{\rho_1}}\|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} + T^{\frac{1}{\rho_1}}\big[\sum_l (2^{l(\frac{N}{p}-1+\frac{2}{\rho_1})}\|\Delta_l u_0\|_{L^p})\big]\\ &\quad + T^{\frac{1}{\rho_1}}\|q_0\|_{B^{\frac{N}{p}}_{p,1}} + T^{\frac{1}{\rho_1}}\big[\sum_l (2^{l(\frac{N}{p}+\frac{2}{\rho_1})}\|\Delta_l q_0\|_{L^p})\big]\big),\end{aligned} \tag{7.77}$$

Here we recall that $u_0$ belongs in $B^{\frac{N}{p}-1+\epsilon'}_{p,\infty}$ which is embedded in $B^{\frac{N}{2}-1}_{p,1}$. By choosing $\rho_1$ large enough such that $\frac{2}{\rho_1} < \epsilon'$, we have shown that:

$$\|u_L\|_{\widetilde{L}^{\rho_1}_T(B^{\frac{N}{p}+2/\rho_1}_{p,1})} \leq CT^{\frac{1}{\rho_1}}(\|u_0\|_{B^{\frac{N}{p}-1+\epsilon'}_{p,\infty}} + \|q_0\|_{B^{\frac{N}{p}+\epsilon'}_{p,\infty}}). \tag{7.78}$$

In particular by choosing $T$ small enough as in (2.16) we are able to show that $u_L$ is small in $\widetilde{L}^{\rho_1}_T(B^{\frac{N}{p}+\frac{2}{\rho_1}}_{p,1})$. By interpolation we are then able to show that $u_L\cdot\nabla u_L$ is sufficiently small in $\widetilde{L}^1_T(B^{\frac{N}{p}+\epsilon'}_{p,\infty})$. To see this, it suffices to apply paraproduct law. We obtain the same type of estimate on $q_L$ which allows to show that $\nabla P(q_L)$ is small in appropriate norm.
In particular it is sufficient on $(0,T)$ for applying bootstrap estimate in (7.75). $\square$

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