

# Cauchy problem for viscous shallow water equations with a term of capillarity

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## Abstract

In this article, we consider the compressible Navier-Stokes equation with density dependent viscosity coefficients and a term of capillarity introduced formally by Van der Waals in [44]. This model includes at the same time the barotropic Navier-Stokes equations with variable viscosity coefficients, shallow-water system and the model introduced by Rohde in [39].

We first study the well-posedness of the model in critical regularity spaces with respect to the scaling of the associated equations. In a functional setting as close as possible to the physical energy spaces, we prove global existence of solutions close to a stable equilibrium, and local in time existence of solutions with general initial data. Uniqueness is also obtained.

## 1 Introduction

This paper is devoted to the Cauchy problem for the compressible Navier-Stokes equation with viscosity coefficients depending on the density and with a capillary term coming from the works of Van der Waals in [44]. He has formally written the link between the local and global expressions of the capilarity terms. Coquel, Rohde and theirs collaborators in [12], [39] have reactualized on a modern form the results of Van der Waals. Let  $\rho$  and  $u$  denote the density and the velocity of a compressible viscous fluid. As usual,  $\rho$  is a non-negative function and  $u$  is a vector valued function defined on  $\mathbb{R}^N$ . Then, the Navier-Stokes equation for compressible fluids endowed with internal capillarity studied in [39] reads:

$$(SW) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)Du) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = \kappa\rho\nabla D[\rho], \end{cases}$$

supplemented by the initial condition:

$$\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = \rho_0 u_0$$

and:

$$D[\rho] = \phi * \rho - \rho$$

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where  $\phi$  is chosen so that:

$$\phi \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi(x) dx = 1, \quad \phi \text{ even, and } \phi \geq 0.$$

Here  $Du = \frac{1}{2}(\nabla u + {}^t \nabla u)$  is the strain tensor,  $P(\rho)$  denotes the pressure and  $\mu, \lambda$  are the two Lamé viscosity coefficients. They depend in our case regularly on the density  $\rho$  and satisfy:  $\mu > 0$  and  $2\mu + N\lambda \geq 0$ .

Several physical models arise as a particular case of system  $(SW)$ :

- when  $\kappa = 0$   $(SW)$  represents compressible Navier-Stokes model with variable viscosity coefficients.
- when  $\kappa = 0$  and  $\mu(\rho) = \rho$ ,  $\lambda(\rho) = 0$ ,  $P(\rho) = \rho^2$ ,  $N = 2$  then  $(SW)$  describes the system of shallow-water.
- when  $\kappa \neq 0$  and  $\mu, \lambda$  are constant,  $(SW)$  reduce to the model studied by Rohde in [39].

One of the major difficulty of compressible fluid mechanics is to deal with the vacuum. The problem of existence of global solution in time for Navier-Stokes equations was addressed in one dimension for smooth enough data by Kazhikov and Shelukin in [32], and for discontinuous ones, but still with densities away from zero, by Serre in [41] and Hoff in [25]. Those results have been generalized to higher dimension by Matsumura and Nishida in [35] for smooth data close to equilibrium and by Hoff in the case of discontinuous data in [27, 28]. Concerning large initial data, Lions showed in [34] the global existence of weak solutions for  $\gamma \geq \frac{3}{2}$  for  $N = 2$  and  $\gamma \geq \frac{9}{5}$  for  $N = 3$ . Let us mention that Feireisl in [20] generalized the result to  $\gamma > \frac{N}{2}$  by establishing that we can obtain renormalized solution without imposing that  $\rho \in L^2_{loc}$ , for this he introduces the concept of oscillation defect measure evaluating the loss of compactness.

Other results provide the full range  $\gamma > 1$  under symmetries assumptions on the initial datum, see Jiang and Zhang [30]. All those results do not require to be far from the vacuum. However they rely strongly on the assumption that the viscosity coefficients are bounded below by a positive constant. This non physical assumption allows to get some estimates on the gradient of the velocity field.

The main difficulty when dealing with vanishing viscosity coefficients on vacuum is that the velocity cannot even be defined when the density vanishes and so we cannot use some properties of parabolicity of the momentum equation, see [10], [11] for results of strong solutions in finite time with vacuum.

The first result handling this difficulty for the weak solutions is due to Bresch, Desjardins and Lin in [7]. They show the existence of global weak solution for Korteweg system in choosing specific type of viscosity where  $\mu$  and  $\lambda$  are linked. The result was later improved by Bresch and Desjardins in [4] to include the case of vanishing capillarity ( $\kappa = 0$ ), but with an additional quadratic friction term  $r\rho u|u|$  (see also [6]). However, those estimates are not enough to treat the case without capillarity and friction effects  $\kappa = 0$  and  $r = 0$  (which corresponds to equation  $(SW)$  with  $\mu(\rho) = \rho$  and  $\lambda(\rho) = 0$ ).

The main difficulty, to prove the stability of  $(SW)$ , is to pass to the limit in the term  $\rho u \otimes u$  (which requires the strong convergence of  $\sqrt{\rho}u$ ). Note that this is easy when the

viscosity coefficients are bounded below by a positive constant. On the other hand, the new bounds on the gradient of the density make the control of the pressure term far simpler than in the case of constant viscosity coefficients (see the works of Lions, Feireisl [34], [20] where to get compactness results for the pressure is the main difficulty).

In [6] Bresch and Desjardins show a result of global existence of weak solution for the non isothermal Navier-Stokes system assuming density dependence of  $\mu(\rho)$  and  $\lambda(\rho)$ , considering perfect gas law with some cold pressure close to the vacuum, and the following relation:

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \quad (1.1)$$

Mellet and Vasseur by using the BD entropy, get in [36] a very interesting stability result. The interest of this result is to consider conditions where the viscosity coefficients vanish on the vacuum set. It includes the case  $\mu(\rho) = \rho$ ,  $\lambda(\rho) = 0$  (when  $N = 2$  and  $\gamma = 2$ , where we recover the Saint-Venant model for Shallow water). The key to the proof is a new energy inequality on the velocity and a gain of integrability, which allows to pass to the limit. Unfortunately, the construction of approximate solutions satisfying: energy estimates, BD mathematical entropy and Mellet-Vasseur estimates is far from being proven except in dimension one or with symmetry assumptions, see [37], [33], [24]. Note that approximate solutions construction process has been proposed in [3] satisfying energy estimates and BD mathematical entropy. This means that only global existence of weak solutions with some extra terms or cold pressure exists in dimension greater than 2.

The existence and uniqueness of local classical solutions for  $(SW)$  with smooth initial data such that the density  $\rho_0$  is bounded and bounded away from zero (i.e.,  $0 < \underline{\rho} \leq \rho_0 \leq M$ ) has been stated by Nash in [38]. Let us emphasize that no stability condition was required there.

On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [35]. Many works on the case of the one dimension have been devoted to the qualitative behavior of solutions for large time (see for example [25, 32]). Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [42], [43], [26], [31]. The most important result on the system of Navier-Stokes compressible isothermal comes from Danchin in [14], [17] who show the existence of global solution and uniqueness with initial data close from an equilibrium, and he obtain a similar result in finite time. The interest is that he works in *critical* Besov space (*critical* in the sense of the scaling of the equation). More precisely to speak roughly, he get strong solution with initial data in  $B_{2,1}^{\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-1} \times (B_{2,1}^{\frac{N}{2}-1})^N$ . Here compared with the result on Navier-Stokes incompressible, he needs to control the vacuum and the norm  $L^\infty$  of the density in the goal to use the parabolicity of the momentum equation and to have some properties of multiplier spaces. That's why Danchin works in Besov spaces with a third index  $r = 1$  for the density, and it's the same for the velocity as the equations are linked. In [18], R. Danchin generalize the previous result with large initial data on the density.

We generalize here the result of Danchin by considering general viscosity coefficient and by including this nonlocal capillarity term introduced in the works of Van der Waals. In

particular we have to check precisely how are coupled the equations of the linear part of the system to get global strong solution with small initial data. Indeed the behavior in low frequencies is crucial and we need to introduce the capillarity term in the linear part to control the low frequencies behavior. So we will get global strong solution with small initial data in  $B_{2,1}^{\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-1} \times (B_{2,1}^{\frac{N}{2}-1})^N$  for the system (SW) and we will improve the result of Danchin in [1] to get strong solution in finite time in Besov space  $B_{p,1}^{\frac{N}{p}} \times (B_{p,1}^{\frac{N}{p}-1})^N$  built on the space  $L^p$  with  $1 \leq p \leq N$ . To finish with, we will give a criterion of blow-up for these solutions. We can observe that our result is very close in dimension  $N = 2$  of the energy initial data particularly in the case of the results of Mellet and Vasseur. Indeed in their case,  $\rho_0^{\frac{1}{p}} u_0$  is assumed in  $L^p$  and  $\sqrt{\rho_0} u_0$  in  $L^2$ , we can check easily by interpolation that if  $\rho_0$  is bounded away then  $u_0$  is in  $(B_{p,1}^{\frac{N}{p}-1})^N$  when  $p > N$ . Moreover the condition on the initial density  $\rho_0 - \bar{\rho} \in B_{p,1}^N$  are close from these imposing by Bresch, Desjardins and Mellet, Vasseur. To conclude, our result improves too the case of strong solution for the shallow-water system, where Wang and Xu in [45] have got global existence in time for small initial data with  $h_0, u_0 \in H^{2+s}$  with  $s > 0$ .

### 1.1 Notations and main results

We will mainly consider the global well-posedness problem for initial data close enough to stable equilibria. Here we want to investigate the well-posedness problem of the system (SW) in critical spaces. By *critical*, we mean that we want to solve the system in functional spaces with norms is invariant by the changes of scales which leave (SW) invariant. In our case, we can easily check that, if  $(\rho, u)$  solves (SW), so does  $(\rho_\lambda, u_\lambda)$ , where:

$$\rho_\lambda(t, x) = \rho(\lambda^2 t, \lambda x) \quad \text{and} \quad u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$$

provided the pressure law  $P$  has been changed into  $\lambda^2 P$ .

**Definition 1.1** *We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation for all  $\lambda > 0$ :*

$$(\rho, u) \longrightarrow (\rho_\lambda, u_\lambda)$$

This suggests us to choose initial data  $(\rho_0, u_0)$  in spaces whose norm is invariant for all  $\lambda > 0$  by  $(\rho_0, u_0) \longrightarrow (\rho_0(\lambda \cdot), \lambda u_0(\lambda \cdot))$ . A natural candidate is the homogeneous Sobolev space  $\dot{H}^{N/2} \times (\dot{H}^{N/2-1})^N$ , but since  $\dot{H}^{N/2}$  is not included in  $L^\infty$ , we cannot expect to get  $L^\infty$  control on the density when  $\rho_0 \in \dot{H}^{N/2}$ , and in particular to avoid the vacuum. This is the reason why instead of the classical homogeneous Sobolev space, we will consider homogeneous Besov spaces  $B_{2,1}^{N/2} \times (B_{2,1}^{N/2-1})^N$  with the same derivative index. This allows to control the density from below and from above, without requiring more regularity on derivatives of  $\rho$ . In the sequel, we will work around a constant state, this motivates the following definition:

**Definition 1.2** *Let  $\bar{\rho} > 0, \bar{\theta} > 0$ . We will note in the sequel:  $q = \frac{\bar{\rho}-\bar{\theta}}{\bar{\rho}}$ .*

Let us first state a result of global existence and uniqueness of (SW) for initial data close to a equilibrium. In this theorem we will need to take in account the behavior in low and

high frequencies of the system, it will explain why the initial data are more regular than in the theorem on local strong solution.

**Theorem 1.1** *Let  $N \geq 2$ . Let  $\bar{\rho} > 0$  be such that:  $P'(\bar{\rho}) > 0$ ,  $\mu(\bar{\rho}) > 0$  and  $2\mu(\bar{\rho}) + \lambda(\bar{\rho}) > 0$ . There exist two positive constants  $\varepsilon_0$  and  $M$  such that if  $q_0 \in \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}$ ,  $u_0 \in B^{\frac{N}{2}-1}$  and:*

$$\|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}-1}} \leq \varepsilon_0$$

*then (SW) has a unique global solution  $(q, u)$  in  $E^{\frac{N}{2}}$  which satisfies:*

$$\|(q, u)\|_{E^{\frac{N}{2}}} \leq M(\|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}-1}}),$$

*for some  $M$  independent of the initial data where:*

$$E^{\frac{N}{2}} = [C_b(\mathbb{R}^+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}) \cap L^1(\mathbb{R}^+, \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})] \times [C_b(\mathbb{R}_+, B^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}^+, B^{\frac{N}{2}+1})]^N.$$

In the following theorem, we show the existence of strong solutions in finite time for large initial data in critical Besov space for the scaling of the equations. We note that we work on Besov space  $B_p^s$  with general index  $p$  on the integrability which improves the result of Danchin in [] and we will need to generalize some result on the heat equation with variable coefficients used by Danchin in [].

**Theorem 1.2** *Let  $p \in [1, +\infty[$ . Let  $q_0 \in B_p^{\frac{N}{p}}$  and  $u_0 \in B_p^{\frac{N}{p}-1}$ . Under the assumptions that  $\mu$  and  $\mu + 2\lambda$  are strictly bounded away zero on  $[\bar{\rho}(1 - 2\|q_0\|_{L^\infty}), \bar{\rho}(1 + 2\|q_0\|_{L^\infty})]$ , there exists a time  $T > 0$  such that the following results hold:*

1. *Existence: If  $p \in [1, 2N[$  then system (SW) has a solution  $(q, u)$  in  $F_p^{\frac{N}{p}}$  with:*

$$F_p^{\frac{N}{p}} = \tilde{C}_T(B_p^{\frac{N}{p}}) \times (L_T^1(B_p^{\frac{N}{p}+1}) \cap \tilde{C}_T(B_p^{\frac{N}{p}-1})).$$

2. *Uniqueness: If in addition  $1 \leq p \leq N$  then uniqueness holds in  $F_p^{\frac{N}{p}}$ .*

It turns out that our study of the linearization of (SW) leads also to the following continuation criterion:

**Theorem 1.3** *Assume that (SW) has a solution  $(q, u) \in C([0, T], B_{p,1}^{\frac{N}{p}} \times (B_{p,1}^{\frac{N}{p}})^N)$  on the time interval  $[0, T)$  which satisfies the following conditions:*

- *the function  $q$  is in  $L^\infty([0, T], B_{p,1}^{\frac{N}{p}})$  and  $\rho$  is bounded away from zero.*
- *we have  $\int_0^T \|\nabla u\|_{L^\infty} dt < +\infty$ .*

*Then  $(q, u)$  may be continued beyond  $T$ .*

The present paper is structured as follows. In the Section 2, we recall some basic facts about Littlewood-Paley decomposition and Besov spaces. In the Section 3 we prove the theorem 1.1 where we precisely show some estimates on the linear part of the system including the capillarity term. In the Section 4 we generalize some estimates on the heat equation with variable coefficient in Besov spaces and we show the theorem 1.2 and the theorem 1.2 on a condition of blow-up.

## 2 Littlewood-Paley theory and Besov spaces

### 2.1 Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any  $\varphi \in C^\infty(\mathbb{R}^N)$ , supported in  $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  such that:

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Denoting  $h = \mathcal{F}^{-1}\varphi$ , we then define the dyadic blocks by:

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy \quad \text{and} \quad S_l u = \sum_{k \leq l-1} \Delta_k u.$$

Formally, one can write that:  $u = \sum_{k \in \mathbb{Z}} \Delta_k u$ . This decomposition is called homogeneous Littlewood-Paley decomposition. Let us point out that the above formal equality holds in  $\mathcal{S}'(\mathbb{R}^N)$  modulo polynomials only.

### 2.2 Homogeneous Besov spaces and first properties

We can verify that  $\sum_{l \in \mathbb{Z}} \Delta_l u = u$  in  $\mathcal{S}'$  up to polynomials. This motivates the following definition:

**Definition 2.3** We denote by  $\dot{\mathcal{S}}'$  the space of temperate distributions  $u$  such that:

$$\lim_{l \rightarrow -\infty} S_l u = 0 \quad \text{in } \mathcal{S}'.$$

**Definition 2.4** For  $s \in \mathbb{R}$ ,  $p \in [1, +\infty]$ ,  $q \in [1, +\infty]$ , and  $u \in \mathcal{S}'(\mathbb{R}^N)$  we set:

$$\|u\|_{B_{p,q}^s} = \left( \sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q \right)^{\frac{1}{q}}.$$

We then define the space  $B_{p,q}^s$  as the subset of distribution  $u \in \dot{\mathcal{S}}'$  such that  $\|u\|_{B_{p,q}^s}$  is finite.

**Remark 1** In the sequel, we will use only Besov space  $B_{p,q}^s$  with  $q = 1$  and we will denote them by  $B_p^s$  or even by  $B^s$  if there is no ambiguity on the index  $p$ .

### 2.3 Hybrid Besov spaces and Chemin-Lerner spaces

Hybrid Besov spaces are functional spaces where regularity assumptions are different in low frequency and high frequency, see [14]. We are going to give the notation of these new spaces and give some of their main properties.

**Notation 1** Let  $s, t \in \mathbb{R}$ . We set:

$$\|u\|_{\tilde{B}_{p,r}^{s,t}} = \left( \sum_{q \leq 0} (2^{qs} \|\Delta_q u\|_{L^p})^r \right)^{\frac{1}{r}} + \left( \sum_{q > 0} (2^{qt} \|\Delta_q u\|_{L^p})^r \right)^{\frac{1}{r}}.$$

We then define the space  $B_{p,q}^{s,t}$  as the subset of distribution  $u \in \dot{\mathcal{S}}'$  such that  $\|u\|_{B_{p,q}^{s,t}}$  is finite.

Let us recall a few product laws in Besov spaces which will be of constant use in the paper.

**Proposition 2.1** *For all  $s, t > 0$ ,  $1 \leq r, p \leq +\infty$ , the following inequality holds true:*

$$\|uv\|_{\tilde{B}_{p,r}^{s,t}} \leq C(\|u\|_{L^\infty}\|v\|_{\tilde{B}_{p,r}^{s,t}} + \|v\|_{L^\infty}\|u\|_{\tilde{B}_{p,r}^{s,t}}). \quad (2.2)$$

For all  $s_1, s_2, t_1, t_2 \leq \frac{N}{p}$  such that  $\min(s_1 + s_2, t_1 + t_2) > 0$  we have:

$$\|uv\|_{\tilde{B}_{p,r}^{s_1+t_1-\frac{N}{p}, s_2+t_2-\frac{N}{p}}} \leq C\|u\|_{\tilde{B}_{p,r}^{s_1,t_1}}\|v\|_{\tilde{B}_{p,\infty}^{s_2,t_2}}. \quad (2.3)$$

$$\|uv\|_{B_{p,r}^s} \leq C\|u\|_{B_{p,r}^s}\|v\|_{B_{p,\infty}^{\frac{N}{p}}} \quad \text{if } |s| < \frac{N}{p}. \quad (2.4)$$

For a proof of this proposition see [2]. The limit case  $s_1 + s_2 = t_1 + t_2 = 0$  in (2.3) is of interest. When  $p \geq 2$ , the following estimate holds true whenever  $s$  is in the range  $(-\frac{N}{p}, \frac{N}{p}]$  (see e.g. [40]):

$$\|uv\|_{B_{p,\infty}^{-\frac{N}{p}}} \leq C\|u\|_{B_{p,1}^s}\|v\|_{B_{p,\infty}^{-s}}. \quad (2.5)$$

The study of non stationary PDE's requires space of type  $L^\rho(0, T, X)$  for appropriate Banach spaces  $X$ . In our case, we expect  $X$  to be a Besov space, so that it is natural to localize the equation through Littlewood-Paley decomposition. But, in doing so, we obtain bounds in spaces which are not type  $L^\rho(0, T, X)$  (except if  $r = p$ ). We are now going to define the spaces of Chemin-Lerner in which we will work (see [8]), which are a refinement of the spaces  $L_T^\rho(B_{p,r}^s)$ .

**Definition 2.5** *Let  $\rho \in [1, +\infty]$ ,  $T \in [1, +\infty]$  and  $s_1, s_2 \in \mathbb{R}$ . We then denote:*

$$\|u\|_{\tilde{L}_T^\rho(\tilde{B}_{p,r}^{s_1,s_2})} = \left( \sum_{l \leq 0} 2^{lrs_1} (\|\Delta_l u(t)\|_{L_T^\rho(L^p)}^r)^{\frac{1}{r}} + \left( \sum_{l > 0} 2^{lrs_2} \left( \int_0^T \|\Delta_l u(t)\|_{L^p}^\rho dt \right)^{\frac{r}{\rho}} \right)^{\frac{1}{r}} \right).$$

We note that thanks to Minkowsky inequality we have:

$$\|u\|_{L_T^\rho(\tilde{B}_{p,r}^{s_1,s_2})} \leq \|u\|_{\tilde{L}_T^\rho(\tilde{B}_{p,r}^{s_1,s_2})} \quad \text{if } \rho \leq r, \quad \|u\|_{\tilde{L}_T^\rho(\tilde{B}_{p,r}^{s_1,s_2})} \leq \|u\|_{L_T^\rho(\tilde{B}_{p,r}^{s_1,s_2})} \quad \text{if } \rho \geq r.$$

We then define the space:

$$\tilde{L}_T^\rho(\tilde{B}_p^{s_1,s_2}) = \{u \in L_T^\rho(\tilde{B}_p^{s_1,s_2}) / \|u\|_{\tilde{L}_T^\rho(\tilde{B}_p^{s_1,s_2})} < \infty\}.$$

We denote moreover by  $\tilde{C}_T(\tilde{B}_p^{s_1,s_2})$  the set of those functions of  $\tilde{L}_T^\infty(\tilde{B}_p^{s_1,s_2})$  which are continuous from  $[0, T]$  to  $\tilde{B}_p^{s_1,s_2}$ .

**Remark 2** *It is esasy to generalize proposition 2.1 to  $\tilde{L}_T^\rho(\tilde{B}_p^{s_1,s_2})$ . The indices  $s, p$  behave just as in the stationary case whereas the time exponent  $\rho$  behaves according to Hólder inequality.*

Finally we need an estimate on the composition of functions in the spaces  $\tilde{L}_T^\rho(\tilde{B}_p^s)$ .

**Proposition 2.2** *Let  $s > 0$ ,  $r \in [1, +\infty]$  and  $F \in W_{loc}^{s+2, \infty}(\mathbb{R}^N)$  such that  $F(0) = 0$ . There exists a function  $C$  depending only on  $s, p, N$  and  $F$ , and such that:*

$$\|F(u)\|_{\tilde{L}_T^\rho(\tilde{B}_{p,r}^{s_1, s_2})} \leq C(\|u\|_{L_T^\infty(L^\infty)})\|u\|_{\tilde{L}_T^\rho(\tilde{B}_{p,r}^{s_1, s_2})}.$$

*If  $v, u \in \tilde{L}_T^\rho(B_p^{s_1, s_2}) \cap L_T^\infty(L^\infty)$  and  $G \in W_{loc}^{[s]+3, \infty}(\mathbb{R}^N)$  then  $G(u) - G(v)$  belongs to  $\tilde{L}_T^\rho(B_p^{s_1, s_2})$  and it exists a constant  $C$  depending only of  $s, p, N$  and  $G$  such that:*

$$\begin{aligned} \|G(u) - G(v)\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1, s_2})} &\leq C(\|u\|_{L_T^\infty(L^\infty)}, \|v\|_{L_T^\infty(L^\infty)})(\|v - u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1, s_2})}) \\ &\quad (1 + \|u\|_{L_T^\infty(L^\infty)} + \|v\|_{L_T^\infty(L^\infty)} + \|v - u\|_{L_T^\infty(L^\infty)}(\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1, s_2})} + \|v\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1, s_2})})). \end{aligned}$$

The proof is a adaptation of a theorem by J.Y. Chemin and H. Bahouri in [1]. We end this section by recalling some estimates in Besov spaces for transport and heat equations. For more details, the reader is referred to [8] and [16].

**Proposition 2.3** *Let  $(p, r) \in [1, +\infty]^2$  and  $s \in (-\min(\frac{N}{p}, \frac{N}{r}), \frac{N}{p} + 1)$ . Let  $u$  be a vector field such that  $\nabla u$  belongs to  $L^1(0, T; B_{p,r}^{\frac{N}{p}} \cap L^\infty)$ . Suppose that  $q_0 \in B_{p,r}^s$ ,  $F \in L^1(0, T, B_{p,r}^s)$  and that  $q \in L_T^\infty(B_{p,r}^s) \cap C([0, T]; \mathcal{S}')$  solves the following transport equation:*

$$\begin{cases} \partial_t q + u \cdot \nabla q = F, \\ q_{t=0} = q_0. \end{cases}$$

*Let  $U(t) = \int_0^t \|\nabla u(\tau)\|_{B_{p,r}^{\frac{N}{p}} \cap L^\infty} d\tau$ . There exists a constant  $C$  depending only on  $s, p$  and  $N$ , and such that for all  $t \in [0, T]$ , the following inequality holds:*

$$\|q\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq \exp^{CU(t)} \left( \|q_0\|_{B_{p,r}^s} + \int_0^t \exp^{-CU(\tau)} \|F(\tau)\|_{B_{p,r}^s} d\tau \right)$$

*If  $r < +\infty$  then  $q$  belongs to  $C([0, T]; B_{p,r}^s)$ .*

Actually, in [16], the proposition below is proved for non-homogeneous Besov spaces. The adaptation to homogeneous spaces is straightforward. Let us now give some estimates for the heat equation:

**Proposition 2.4** *Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, +\infty]^2$  and  $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ . Assume that  $u_0 \in B_{p,r}^s$  and  $f \in \tilde{L}_T^{\rho_2}(\tilde{B}_{p,r}^{s-2+2/\rho_2})$ . Let  $u$  be a solution of:*

$$\begin{cases} \partial_t u - \mu \Delta u = f \\ u_{t=0} = u_0. \end{cases}$$

*Then there exists  $C > 0$  depending only on  $N, \mu, \rho_1$  and  $\rho_2$  such that:*

$$\|u\|_{\tilde{L}_T^{\rho_1}(\tilde{B}_{p,r}^{s+2/\rho_1})} \leq C(\|u_0\|_{B_{p,r}^s} + \mu^{\frac{1}{\rho_2}-1} \|f\|_{\tilde{L}_T^{\rho_2}(\tilde{B}_p^{s-2+2/\rho_2})}).$$

*If in addition  $r$  is finite then  $u$  belongs to  $C([0, T], B_{p,r}^s)$ .*



### 3 Proof of the existence of solution for theorem 1.1

#### 3.1 Sketch of the Proof

In this section, we give the sketch of the proof of theorem 1.1 on the global existence result with small initial data. We will suppose that  $\rho$  is close to a constant state  $\bar{\rho} > 0$  to avoid the vacuum and to use the parabolicity of the momentum equation by getting a gain of derivatives on the velocity  $u$ . Let us rewrite the system (SW) in a non conservative form by using the definition 1.2.

$$\begin{cases} \partial_t q + u \cdot \nabla q + \operatorname{div} u = F, \\ \partial_t u + u \cdot \nabla u - \frac{\mu(\bar{\rho})}{\bar{\rho}} \Delta u - \frac{\mu(\bar{\rho}) + \lambda(\bar{\rho})}{\bar{\rho}} \nabla \operatorname{div} u + \bar{\delta} \nabla q - \kappa \bar{\rho} \phi * \nabla q = G, \end{cases} \quad (3.6)$$

with  $\bar{\delta} = (\kappa \bar{\rho} + P'(\bar{\rho}))$  and where we have:

$$F = -q \operatorname{div} u, \quad G = \mathcal{A}(\rho, u) + K(\rho) \nabla q,$$

with:

$$\begin{aligned} \mathcal{A}(\rho, u) &= \left[ \frac{\operatorname{div}(\mu(\rho) D(u))}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \Delta u \right] + \left[ \frac{\nabla((\mu(\rho) + \lambda(\rho)) \operatorname{div} u)}{\rho} - \frac{\mu(\bar{\rho}) + \lambda(\bar{\rho})}{\bar{\rho}} \nabla \operatorname{div} u \right], \\ K(\rho) &= \frac{\bar{\rho} P'(\rho)}{\rho} - P'(\bar{\rho}). \end{aligned}$$

For  $s \in \mathbb{R}$ , we denote  $\Lambda^s h = \mathcal{F}^{-1}(|\xi|^s \widehat{h})$ . We set now:  $d = \Lambda^{-1} \operatorname{div} u$  and  $\Omega = \Lambda^{-1} \operatorname{curl} u$  where  $d$  represents the compressible part of the velocity and  $\Omega$  the incompressible part. We rewrite the system (3.6) by using these previous notations on a linear form:

$$(SW2) \quad \begin{cases} \partial_t q + \Lambda d = F_1, \\ \partial_t d - \bar{\nu} \Delta d - \bar{\delta} \Lambda q + \bar{\kappa} \Lambda(\phi * q) = G_1 \\ \partial_t \Omega - \bar{\mu} \Delta \Omega = H_1 \\ u = -\Lambda^{-1} \nabla d - \Lambda \operatorname{div} \Omega \end{cases}$$

where:  $\bar{\mu} = \frac{\mu(\bar{\rho})}{\bar{\rho}}$ ,  $\bar{\lambda} = \frac{\lambda(\bar{\rho})}{\bar{\rho}}$ ,  $\bar{\nu} = 2\bar{\mu} + \bar{\lambda}$ ,  $\bar{\kappa} = \kappa \bar{\rho}$ . and:

$$F_1 = -q \operatorname{div} u - u \cdot \nabla q, \quad G_1 = -\Lambda^{-1} \operatorname{div}(G) \quad \text{and} \quad H_1 = -\Lambda^{-1} \operatorname{curl}(G).$$

The first idea would be to study the linear system associated to (SW2). We concentrate on the first two equations because the third equation is just a heat equation with a non linear term. The system we want to study reads:

$$\begin{cases} \partial_t q + \Lambda d = F', \\ \partial_t d - \bar{\nu} \Delta d - \bar{\delta} \Lambda q + \bar{\kappa} \Lambda(\phi * q) = G'. \end{cases}$$

This system has been studied by D. Hoff and K. Zumbrum in [29] in the case  $\bar{\kappa} = 0$ . There, they investigate the decay estimates, and exhibit the parabolic smoothing effect

on  $d$  and on the low frequencies of  $q$ , and a damping effect on the high frequencies of  $q$ . The problem is that if we focus on this linear system, it appears impossible to control the term of convection  $u \cdot \nabla q$  which is one derivative less regular than  $q$ . Hence we shall include the convection term in the linear system. We thus have to study:

$$(SW2)' \quad \begin{cases} \partial_t q + v \cdot \nabla q + \Lambda d = F, \\ \partial_t d + v \cdot \nabla d - \bar{\nu} \Delta d - \bar{\delta} \Lambda q + \bar{\kappa} \Lambda(\phi * q) = G, \end{cases}$$

where  $v$  is a function and we will precise its regularity in the next proposition. System  $(SW2)'$  has been studied in the case where  $\phi = 0$  by R. Danchin in [14], we then have to take into consideration the term coming from the capillarity and which play a important role in low frequencies. In the sequel we will assume  $\bar{\nu} > 0$  and  $\bar{\delta} - \bar{\kappa} \|\hat{\phi}\|_{L^\infty} \geq c > 0$ . We obtain then the following proposition.

**Proposition 3.5** *Let  $(q, d)$  a solution of the system  $(SW2)'$  on  $[0, T[$ ,  $1 - \frac{N}{2} < s \leq 1 + \frac{N}{2}$  and  $V(t) = \int_0^t \|v(\tau)\|_{B^{\frac{N}{2}+1}} d\tau$ . We have then the following estimate:*

$$\begin{aligned} \|(q, d)\|_{\tilde{B}^{s-1, s} \times B^{s-1}} + \int_0^t \|(q, d)(\tau)\|_{\tilde{B}^{s+1, s} \times B^{s+1}} d\tau \\ \leq C e^{CV(t)} (\|(q_0, d_0)\|_{\tilde{B}^{s-1, s} \times B^{s-1}} + \int_0^t e^{-CV(\tau)} \|(F, G)(\tau)\|_{\tilde{B}^{s-1, s} \times B^{s-1}} d\tau), \end{aligned}$$

where  $C$  depends only on  $\bar{\nu}$ ,  $\bar{\delta}$ ,  $\bar{\kappa}$ ,  $\phi$ ,  $N$  and  $s$ .

**Proof:**

Let  $(q, u)$  be a solution of  $(SW2)'$  and we set:

$$\tilde{q} = e^{-KV(t)} q, \quad \tilde{u} = e^{-KV(t)} u, \quad \tilde{F} = e^{-KV(t)} F \quad \text{and} \quad \tilde{G} = e^{-KV(t)} G. \quad (3.7)$$

We are going to separate the case of the low and high frequencies, which have a different behavior concerning the control of the derivative index for the Besov spaces. In this goal we will consider the two different expressions in low and high frequencies where  $l_0 \in \mathbb{Z}$ ,  $A$ ,  $B$  and  $K_1$  will be fixed later in the proof:

$$\begin{aligned} f_l^2 &= \bar{\delta} \|\tilde{q}_l\|_{L^2}^2 - \bar{\kappa}(\tilde{q}_l, \phi * \tilde{q}_l) + \|\tilde{d}_l\|_{L^2}^2 - 2K_1(\Lambda \tilde{q}_l, \tilde{d}_l) \quad \text{for } l \leq l_0, \\ f_l^2 &= \|\Lambda \tilde{q}_l\|_{L^2}^2 + A \|\tilde{d}_l\|_{L^2}^2 - \frac{2}{\bar{\nu}}(\Lambda \tilde{q}_l, \tilde{d}_l) \quad \text{for } l > l_0. \end{aligned} \quad (3.8)$$

In the first two steps, we show that  $K_1$  and  $A$  may be chosen such that:

$$2^{l(s-1)} f_l^2 \approx 2^{ls} \max(1, 2^{-l}) \|\tilde{q}_l\|_{L^2}^2 + 2^{l(s-1)} \|\tilde{d}_l\|_{L^2}^2, \quad (3.9)$$

and we will show the following inequality:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} f_l^2 + \alpha \min(2^{2l}, 1) f_l^2 &\leq C 2^{-l(s-1)} \alpha_l f_l (\|(\tilde{F}, \tilde{G})\|_{\tilde{B}^{s-1, s} \times B^{s-1}} \\ &\quad + V' \|(\tilde{q}, \tilde{d})\|_{\tilde{B}^{s-1, s} \times B^{s-1}}) - KV' f_l^2. \end{aligned} \quad (3.10)$$

where  $\sum_{l \in \mathbb{Z}} \alpha_l \leq 1$  and  $\alpha$  is a positive constant. This inequality enables us to get a decay for  $q$  and  $d$  which will be used to show a smoothing parabolic effect on  $d$ .

### Case of low frequencies

Applying operator  $\Delta_l$  to the system  $(SW2)'$ , we obtain then in setting:  $\tilde{q}_l = \Delta_l \tilde{q}$  and  $\tilde{d}_l = \Delta_l d$  the following system:

$$\begin{cases} \frac{d}{dt} \tilde{q}_l + \Delta_l(v \cdot \nabla \tilde{q}) + \Lambda \tilde{d}_l = \tilde{F}_l - KV'(t) \tilde{q}_l, \\ \frac{d}{dt} d_l + \Delta_l(v \cdot \nabla \tilde{d}_l) - \bar{\nu} \Delta \tilde{d}_l - \bar{\delta} \Lambda \tilde{q}_l + \bar{\kappa} \Lambda(\phi * \tilde{q}_l) = \tilde{G}_l - KV'(t) \tilde{d}_l. \end{cases} \quad (3.11)$$

We set:

$$f_l^2 = \bar{\delta} \|\tilde{q}_l\|_{L^2}^2 + \|\tilde{d}_l\|_{L^2}^2 - 2K_1(\Lambda \tilde{q}_l, \tilde{d}_l) \quad (3.12)$$

for some  $K_1 \geq 0$  to be fixed hereafter and  $(\cdot, \cdot)$  noting the  $L^2$  inner product. To begin with, we consider the case where  $F = G = 0$ ,  $v = 0$  and  $K = 0$ . Taking the  $L^2$  scalar product of the first equation of (3.11) with  $\tilde{q}_l$  and of the second equation with  $\tilde{d}_l$ , we get the following two identities:

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|q_l\|_{L^2}^2 + (\Lambda d_l, q_l) = 0, \\ \frac{1}{2} \frac{d}{dt} \|d_l\|_{L^2}^2 + \bar{\nu} \|\Lambda d_l\|_{L^2}^2 - \bar{\delta} (\Lambda q_l, d_l) + \bar{\kappa} (\Lambda(\phi * q_l), d_l) = 0. \end{cases} \quad (3.13)$$

In the same way we have:

$$\frac{1}{2} \frac{d}{dt} (q_l, q_l * \phi) + (\Lambda d_l, \phi * q_l) = 0, \quad (3.14)$$

because we have by the theorem of Plancherel:

$$\left( \frac{d}{dt} q_l, q_l * \phi \right) = \left( \frac{d}{dt} \widehat{q_l}, \widehat{q_l \phi} \right) = \frac{1}{2} \frac{d}{dt} (\widehat{q_l}, \widehat{q_l \phi}) = \frac{1}{2} \frac{d}{dt} (q_l, q_l * \phi).$$

We want now get an equality involving  $\bar{\nu}(\Lambda d_l, q_l)$ . To achieve it, we apply  $\bar{\nu} \Lambda$  to the first equation of (3.11) and take the  $L^2$ -scalar product with  $d_l$ , then take the scalar product of the second equation with  $\Lambda q_l$  and sum both equalities, which yields:

$$\frac{d}{dt} (\Lambda q_l, d_l) + \|\Lambda d_l\|_{L^2}^2 - \bar{\delta} \|\Lambda q_l\|_{L^2}^2 + \bar{\kappa} \|\phi * \Lambda q_l\|_{L^2}^2 + \bar{\nu} (\Lambda^2 d_l, \Lambda q_l) = 0. \quad (3.15)$$

By linear combination of (3.13) and (3.15), we get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} f_l^2 + (\bar{\nu} - K_1) \|\Lambda d_l\|_{L^2}^2 + K_1 (\bar{\delta} \|\Lambda q_l\|_{L^2}^2 - \bar{\kappa} \|\phi * \Lambda q_l\|_{L^2}^2) \\ - \bar{\nu} K_1 (\Lambda^2 d_l, \Lambda q_l) = 0. \end{aligned} \quad (3.16)$$

And as we have assumed that:  $\delta - \bar{\kappa} \|\widehat{\phi}\|_{L^\infty} \geq c > 0$  we get:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + (\bar{\nu} - K_1) \|\Lambda d_l\|_{L^2}^2 + K_1 c \|q_l\|_{L^2}^2 - \bar{\nu} K_1 (\Lambda^2 d_l, \Lambda q_l) \leq 0. \quad (3.17)$$

Using spectral localization for  $d_l$  and convex inequalities, we find for every  $a > 0$ :

$$|(\Lambda^2 d_l, \Lambda q_l)| \leq \frac{a 2^{2l_0}}{2} \|\Lambda d_l\|_{L^2}^2 + \frac{1}{2a} \|\Lambda q_l\|_{L^2}^2.$$

By using the previous inequality and (3.16), we get:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + (\bar{\nu} - K_1 - \frac{a2^{2l_0}}{2}) \|\Lambda d_l\|_{L^2}^2 + (K_1 c - \frac{1}{2a}) \|\Lambda q_l\|_{L^2}^2 \leq 0. \quad (3.18)$$

From (3.12) and (3.18) we get by choosing  $a = \bar{\nu}$  and  $K_1 < \min(\frac{1}{2^{2l_0}}, \frac{\bar{\nu}}{2+2^{2l_0}\bar{\nu}^2})$ , then:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + \alpha 2^{2l} f_l^2 \leq 0, \quad (3.19)$$

for a constant  $\alpha$  depending only on  $\bar{\nu}$  and  $K_1$ .

In the general case where  $F$ ,  $G$ ,  $K$  and  $v$  are not zero, we have:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} f_l^2 + (\alpha 2^{2l} + KV') f_l^2 &\leq (\tilde{F}_l, \tilde{q}_l) + (\tilde{G}_l, \tilde{d}_l) - K(\Lambda \tilde{F}_l, \tilde{d}_l) - K(\Lambda \tilde{G}_l, \tilde{q}_l) - (\Delta_l(v \cdot \nabla \tilde{q}), \tilde{q}_l) \\ &\quad - (\Delta_l(v \cdot \nabla \tilde{d}), \tilde{d}_l) + K(\Lambda \Delta_l(v \cdot \nabla \tilde{q}), \tilde{d}_l) + (\Lambda \Delta_l(v \cdot \nabla \tilde{d}), \tilde{q}_l). \end{aligned}$$

Now we can use a lemma of harmonic analysis in [14] to estimate the last terms, and get the existence of a sequence  $(\alpha_l)_{l \in \mathbb{Z}}$  such that  $\sum_{l \in \mathbb{Z}} \alpha_l \leq 1$  and:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} f_l^2 + (\alpha 2^{2l} + KV') f_l^2 &\lesssim \alpha_l f_l 2^{-l(s-1)} (\|(\tilde{F}, \tilde{G})\|_{\tilde{B}^{s-1,s} \times B^{s-1}} \\ &\quad + V' \|(\tilde{q}, \tilde{d})\|_{\tilde{B}^{s-1,s} \times B^{s-1}}). \end{aligned} \quad (3.20)$$

### Case of high frequencies

We consider now the case where  $l \geq l_0 + 1$  and we recall that:

$$f_l^2 = \|\Lambda \tilde{q}_l\|_{L^2}^2 + A \|\tilde{d}_l\|_{L^2}^2 - \frac{2}{\bar{\nu}} (\tilde{q}_l, \tilde{d}_l).$$

For the sake of simplicity, we suppose here that  $F = G = 0$ ,  $v = 0$  and  $K = 0$ . We now want a control on  $\|\Lambda q_l\|_{L^2}^2$ , we apply the operator  $\Lambda$  to the first equation of (3.11), multiply by  $\Lambda q_l$  and integrate over  $\mathbb{R}^N$ , and similarly we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda q_l\|_{L^2}^2 + (\Lambda^2 d_l, \Lambda q_l) &= 0 \\ \frac{1}{2} \frac{d}{dt} \|d_l\|_{L^2}^2 + \bar{\nu} \|\Lambda d_l\|_{L^2}^2 - \bar{\delta} (\Lambda q_l, d_l) + \bar{\kappa} (\Lambda(\phi * q_l), d_l) &= 0. \\ \frac{d}{dt} (\Lambda q_l, d_l) + \|\Lambda d_l\|_{L^2}^2 - \bar{\delta} \|\Lambda q_l\|_{L^2}^2 + \bar{\kappa} \|\phi * \Lambda q_l\|_{L^2}^2 + \bar{\nu} (\Lambda^2 d_l, \Lambda q_l) &= 0. \end{aligned} \quad (3.21)$$

By linear combination of (3.21) we have:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + \frac{1}{\bar{\nu}} \|\Lambda q_l\|_{L^2}^2 + (A\bar{\nu} - \frac{1}{\bar{\nu}}) \|\Lambda d_l\|_{L^2}^2 - A\bar{\delta} (\Lambda q_l, d_l) + A\bar{\kappa} (\Lambda(\phi * q_l), d_l) = 0. \quad (3.22)$$

With:

$$| -A\bar{\delta} (\Lambda q_l, d_l) + A\bar{\kappa} (\Lambda(\phi * q_l), d_l) | \leq A(\bar{\delta} + \bar{\kappa} \|\widehat{\phi}\|_{L^\infty}) |(\Lambda q_l, d_l)|$$

We have now by using Young inequalities for all  $a > 0$ :

$$\frac{1}{2} \frac{d}{dt} f_l^2 + 2^{2l_0} (A\bar{\nu} - \frac{1}{\bar{\nu}} - \frac{1}{2a}) \|d_l\|_{L^2}^2 + (\frac{1}{\bar{\nu}} - \frac{a}{2}) \|\Lambda q_l\|_{L^2}^2 \leq 0. \quad (3.23)$$

So by choosing:  $a = \frac{1}{\bar{\nu}A}$  and  $A > \max(\frac{2}{\bar{\nu}}, 1)$  there exists a constant  $\alpha$  such that for  $l \geq l_0 + 1$  we have:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + \alpha f_l^2 \leq 0. \quad (3.24)$$

In the general case where  $F, G, H, K$  and  $v$  are not necessarily zero, we use a lemma of harmonic analysis in [14] to control the convection terms. We finally get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} f_l^2 + (\alpha + KV') f_l^2 &\lesssim \alpha_l f_l 2^{-l(s-1)} (\|(\tilde{F}, \tilde{G})\|_{\tilde{B}^{s-1,s} \times B^{s-1}} \\ &\quad + V' \|(\tilde{q}, \tilde{d})\|_{\tilde{B}^{s-1,s} \times B^{s-1}}). \end{aligned} \quad (3.25)$$

This finish the proof of (3.8) and (3.10).

### The damping effect

We are now going to show that inequality (3.10) entails a decay for  $q$  and  $d$ . In fact we get a parabolic decay for  $d$ , while  $q$  has a behavior similar to a transport equation. Using  $h_l^2 = f_l^2 + \delta^2$ , integrating over  $[0, t]$  and then having  $\delta$  tend to 0, we infer:

$$\begin{aligned} f_l(t) + \alpha \min(2^{2l}, 1) \int_0^t f_l(\tau) d\tau &\leq f_l(0) + C 2^{-l(s-1)} \int_0^t \alpha_l(\tau) \|(\tilde{F}(\tau), \tilde{G}(\tau))\|_{\tilde{B}^{s-1,s} \times B^s} d\tau \\ &\quad + \int_0^t V'(\tau) (C 2^{-l(s-1)} \alpha_l(\tau) \|(\tilde{q}, \tilde{d})\|_{\tilde{B}^{s-1,s} \times B^s} - K f_l(\tau)) d\tau. \end{aligned} \quad (3.26)$$

Thanks to (3.9), we have by taking  $K$  large enough :

$$\sum_{l \in \mathbb{Z}} (C 2^{-l(s-1)} \alpha_l(\tau) \|(\tilde{q}, \tilde{d})\|_{\tilde{B}^{s-1,s} \times B^s} - K f_l(\tau)) \leq 0,$$

By multiplying (3.26) by  $2^{l(s-1)}$  and by using the last inequality, we conclude after summation on  $\mathbb{Z}$ , that:

$$\begin{aligned} \|\tilde{q}(t)\|_{\tilde{B}^{s-1,s}} + \|\tilde{d}\|_{\tilde{B}^{s-1}} + \alpha \int_0^t \|\tilde{q}(\tau)\|_{\tilde{B}^{s-1,s}} d\tau &+ \sum_{l \in \mathbb{Z}} \int_0^t \alpha 2^{l(s-1)} \min(2^{2l}, 1) \\ &\times \|\tilde{d}_l(\tau)\|_{L^2} d\tau \lesssim \|(\tilde{q}_0, \tilde{d}_0)\|_{\tilde{B}^{s-1,s} \times B^{s-1}} + \int_0^t \|(\tilde{F}, \tilde{G})\|_{\tilde{B}^{s-1,s} \times B^{s-1}} d\tau. \end{aligned} \quad (3.27)$$

### The smoothing effect

Once stated the damping effect for  $q$ , it is easy to get the smoothing effect on  $d$  by considering the last two equations where the term  $\Lambda q$  is considered as a source term . Thanks to (3.27), it suffices to prove it for high frequencies only. We therefore suppose in this subsection that  $l \geq l_0$  for a  $l_0$  big enough. We set  $g_l = \|\tilde{d}_l\|_{L^2}$  and by using the previous inequalities, we have:

$$\frac{1}{2} \frac{d}{dt} \|\tilde{d}_l\|_{L^2}^2 + \bar{\nu} \|\Lambda \tilde{d}_l\|_{L^2}^2 - \bar{\delta}(\Lambda \tilde{q}_l, \tilde{d}_l) + \bar{\kappa}(\Lambda(\phi * \tilde{q}_l), \tilde{d}_l) = \tilde{G}_l \cdot \tilde{d}_l - KV'(t) \|\tilde{d}_l\|_{L^2}^2.$$

We get finally with  $\alpha > 0$ :

$$\frac{1}{2} \frac{d}{dt} g_l^2 + \alpha 2^{2l} g_l^2 \leq g_l (\|\Lambda \tilde{q}_l\|_{L^2} + \|\tilde{G}_l\|_{L^2}) + g_l V'(t) (C \alpha_l 2^{-l(s-1)} \|\tilde{d}\|_{B^{s-1}} - K g_l).$$

We therefore get in using standard computations:

$$\begin{aligned} \sum_{l \geq l_0} 2^{l(s-1)} \|\tilde{d}_l(t)\|_{L^2} + \alpha \int_0^t \sum_{l \geq l_0} 2^{l(s+1)} \|\tilde{d}_l(\tau)\|_{L^2} d\tau &\leq \|d_0\|_{B^{s-1}} + \int_0^t \|\tilde{G}(\tau)\|_{B^{s-1}} d\tau \\ &+ \int_0^t \sum_{l \geq l_0} 2^{ls} \|\tilde{q}_l(\tau)\|_{L^2} + CV(t) \sup_{\tau \in [0, t]} (\|\tilde{d}(\tau)\|_{B^{s-1}}). \end{aligned}$$

Using the above inequality and (3.27), we have:

$$\begin{aligned} \int_0^t \sum_{l \geq l_0} 2^{l(s+1)} \|\tilde{d}_l(\tau)\|_{L^2} d\tau &\lesssim (1 + V(t)) (\|q_0\|_{\tilde{B}^{s-1, s}} + \|d_0\|_{B^{s-1}}) \\ &+ \int_0^t (\|\tilde{F}(\tau)\|_{\tilde{B}^{s-1, s}} + \|\tilde{G}(\tau)\|_{B^{s-1}}) d\tau. \end{aligned} \tag{3.28}$$

Combining that last inequality (3.28) with (3.27), we achieve the proof of proposition 3.5.  $\square$

### 3.2 Proof of the existence for theorem 1.1

This section is devoted to the proof of the theorem 1.1. The principle of the proof is a classical one. We want to construct a sequence  $(q^n, u^n)_{n \in \mathbb{N}}$  of approximate solutions of the system (SW), and we will use the proposition 3.5 to get some uniform bounds on  $(q^n, u^n)_{n \in \mathbb{N}}$ . We will conclude by stating some properties of compactness, which will guarantee that up to an extraction,  $(q^n, u^n)_{n \in \mathbb{N}}$  converges to a solution  $(q, u)$  of the system (SW).

#### First step: Building the sequence $(q^n, u^n)_{n \in \mathbb{N}}$

We start with the construction of the sequence  $(q^n, u^n)_{n \in \mathbb{N}}$ , in this goal we use the Friedrichs operators  $(J_n)_{n \in \mathbb{N}}$  defined by:

$$J_n g = \mathcal{F}^{-1} (1_{B(\frac{1}{n}, n)} \hat{g}),$$

where  $\mathcal{F}^{-1}$  is the inverse Fourier transform. Let us consider the approximate system:

$$\begin{cases} \partial_t q^n + J_n (J_n u^n \cdot \nabla J_n q^n) + \Lambda J_n d^n = F^n \\ \partial_t d^n + J_n (J_n u^n \cdot \nabla J_n d^n) - \bar{\nu} \Delta J_n d^n - \bar{\delta} \Lambda J_n q^n - \bar{\kappa} \phi * \Lambda J_n q^n = G^n \\ \partial_t \Omega^n - \bar{\nu} \Delta J_n \Omega^n = H^n \\ u^n = -\Lambda^{-1} \nabla d^n - \Lambda^{-1} \operatorname{div} \Omega^n \\ (q^n, d^n, \Omega^n)_{t=0} = (J_n q_0, J_n d_0, J_n \Omega_0) \end{cases} \tag{3.29}$$

with:

$$\begin{aligned} F^n &= -J_n((J_n q^n) \operatorname{div} J_n u^n), \\ G^n &= J_n \Lambda^{-1} \operatorname{div} [\mathcal{A}(\varphi(\bar{\rho}(1 + J_n q^n)), J_n u^n) + K(\varphi(\bar{\rho}(1 + J_n q^n)) \nabla q^n)], \\ H^n &= J_n \Lambda^{-1} \operatorname{curl} [\mathcal{A}(\varphi(\bar{\rho}(1 + J_n q^n)), J_n u^n) + K(\varphi(\bar{\rho}(1 + J_n q^n)) \nabla q^n)]. \end{aligned}$$

where  $\varphi$  is a smooth function verifying  $\varphi(s) = s$  for  $\frac{1}{n} \leq s \leq n$  and  $\varphi \geq \frac{1}{4}$ . We want to show that (3.29) is only an ordinary differential equation in  $L^2 \times L^2 \times L^2$ . We can observe easily that all the source term in (3.29) turn out to be continuous in  $L^2 \times L^2 \times L^2$ . As a example, we consider the term  $J_n \mathcal{A}(\varphi(\bar{\rho}(1 + J_n q^n)), J_n u^n)$ . We have then by Plancherel theorem:

$$\begin{aligned} \|J_n(\frac{\operatorname{div}(\mu(\varphi(\bar{\rho}(1 + J_n q^n)) D J_n u^n))}{\varphi(\bar{\rho}(1 + J_n q^n))})\|_{L^2} &\leq n \|\mu(\varphi(\bar{\rho}(1 + J_n q^n)) D J_n u^n)\|_{L^2} \\ &\quad \times \left\| \frac{1}{\varphi(\bar{\rho}(1 + J_n q^n))} \right\|_{L^\infty}, \\ &\leq 4M_n n^2 \|u^n\|_{L^2}. \end{aligned}$$

where  $M_n = \|\mu(\varphi(\bar{\rho}(1 + J_n q^n))\|_{L^\infty}$ . According to the Cauchy-Lipschitz theorem, a unique maximal solution exists in  $C([0, T_n]; L^2)$  with  $T_n > 0$ . Moreover, since  $J_n = J_n^2$  we show that  $(J_n q^n, J_n d^n, J_n \Omega^n)$  is also a solution and then by uniqueness we get that  $(J_n q^n, J_n u^n) = (q^n, u^n)$ . This implies that  $(q^n, d^n, \Omega^n)$  is solution of the following system:

$$\begin{cases} \partial_t q^n + J_n(u^n \cdot \nabla q^n) + \Lambda d^n = F_1^n \\ \partial_t d^n + J_n(u^n \cdot \nabla d^n) - \bar{\nu} \Delta d^n - \bar{\delta} \Lambda q^n - \bar{\kappa} \phi * \Lambda q^n = G_1^n \\ \partial_t \Omega^n - \bar{\nu} \Delta \Omega^n = H_1^n \\ u^n = -\Lambda^{-1} \nabla d^n - \Lambda^{-1} \operatorname{div} \Omega^n \\ (q^n, d^n, \Omega^n)_{/t=0} = (J_n q_0, J_n d_0, J_n \Omega_0) \end{cases} \quad (3.30)$$

and:

$$\begin{aligned} F_1^n &= -J_n(q^n \operatorname{div} u^n), \\ G_1^n &= J_n \Lambda^{-1} \operatorname{div} [\mathcal{A}(\varphi(\bar{\rho}(1 + q^n)), u^n) + K(\varphi(\bar{\rho}(1 + q^n)) \nabla q^n)], \\ H_1^n &= J_n \Lambda^{-1} \operatorname{curl} [\mathcal{A}(\varphi(\bar{\rho}(1 + q^n)), u^n) + K(\varphi(\bar{\rho}(1 + q^n)) \nabla q^n)]. \end{aligned}$$

And the system (3.30) is again an ordinary differential equation in  $L_n^2$  with:  $L_n^2 = \{g \in L^2(\mathbb{R}^N) / \operatorname{supp} \hat{g} \subset B(\frac{1}{n}, n)\}$ . Due to the Cauchy-Lipschitz theorem again, a unique maximal solution exists in  $C^1([0, T'_n]; L_n^2)$  with  $T'_n \geq T_n > 0$ .

## Second step: Uniform estimates

In this part, we want to get uniform estimates independent of  $T$  on  $\|(q^n, u^n)\|_{E_T^{\frac{N}{2}}}$  for all  $T < T'_n$ . This will show that  $T'_n = +\infty$  by Cauchy-Lipchitz because the norms  $\|\cdot\|_{E^{\frac{N}{2}}}$  and  $L^2$  are equivalent on  $L_n^2$ . Let us set:

$$\begin{aligned} E(0) &= \|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}}}, \\ E(q, u, t) &= \|q\|_{L_t^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|q\|_{L_t^\infty(B^{\frac{N}{2}-1})} + \|q\|_{L_t^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})} + \|q\|_{L_t^\infty(B^{\frac{N}{2}+1})}, \end{aligned}$$

and:  $\bar{T}_n = \sup\{t \in [0, T'_n), E(q^n, u^n, t) \leq 3CE(0)\}$ .  $C$  corresponds to the constant in the proposition 3.5 and as  $C > 1$  we have  $3C > 1$  so by continuity we have  $\bar{T}_n > 0$ . We are going to prove that  $\bar{T}_n = T'_n$  for all  $n \in \mathbb{N}$  and we will conclude that  $\forall n \in \mathbb{N} T'_n = +\infty$ . To achieve it, one can use the proposition 3.5 to the system (3.30) to obtain uniform bounds, so we by setting  $V_n(t) = \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})}$  we have:

$$\begin{aligned} \|(q^n, u^n)\|_{E_T^{\frac{N}{2}}} &\leq C e^{CV_n(t)} (\|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}}} + \int_0^T e^{-CV_n(\tau)} (\|F_1^n(\tau)\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} \\ &\quad + \|G_1^n(\tau)\|_{B^{\frac{N}{2}-1}} + \|H_1^n(\tau)\|_{B^{\frac{N}{2}-1}}) d\tau.) \end{aligned}$$

Therefore, it is only a matter of proving appropriate estimates for  $F_1^n$ ,  $G_1^n$  and  $H_1^n$  by using properties of continuity on the paraproduct. We estimate now  $\|F_1^n\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$  by using proposition 2.1 and 2.2:

$$\|F_1^n\|_{L_T^1(B^{\frac{N}{2}-1, \frac{N}{2}})} \leq C \|q^n\|_{L_T^\infty(B^{\frac{N}{2}-1, \frac{N}{2}})} \|\operatorname{div} u^n\|_{L_T^1(B^{\frac{N}{2}})},$$

We now want to estimate  $G_1^n$ :

$$\|\mathcal{A}(\varphi(\bar{\rho}(1 + q^n)), u^n)\|_{L_T^1(B^{\frac{N}{2}-1})} \leq C \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}),$$

We can verify that  $K$  fulfills the hypothesis of the proposition 2.2, so we get:

$$\|K(\varphi(\bar{\rho}(1 + q^n)) \nabla q^n\|_{L_T^1(B^{\frac{N}{2}-1})} \leq C \|q^n\|_{L_T^2(B^{\frac{N}{2}})}^2 \|q^n\|_{L_T^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})},$$

Moreover we recall that according to proposition 2.1:

$$\|q^n\|_{L_T^2(B^{\frac{N}{2}})}^2 \leq \|q^n\|_{L_T^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q^n\|_{L_T^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})}.$$

We proceed similarly to estimate  $\|H_1^n\|_{L_T^1(B^{\frac{N}{2}-1})}$  and finally we have:

$$\begin{aligned} \|F_1^n\|_{L_T^1(B^{\frac{N}{2}-1})} + \|G_1^n\|_{L_T^1(B^{\frac{N}{2}-1})} + \|H_1^n\|_{L_T^1(B^{\frac{N}{2}-1})} &\leq 2C(E^2(q^n, u^n, T) \\ &\quad + E^3(q^n, u^n, T)), \end{aligned}$$

whence:  $\|(q^n, u^n)\|_{E_T^{\frac{N}{2}}} \leq C e^{C^2 3E(0)} E(0) (1 + 18CE(0)(1 + 3E(0)))$ .

We want now to obtain:  $e^{3C^2 E(0)} (1 + 18CE(0)(1 + 3E(0))) \leq 2$  for this it suffices to choose  $E(0)$  small enough, let  $E(0) < \varepsilon$  such that:

$$1 + 18CE(0)(1 + 3E(0)) \leq \frac{3}{2} \quad \text{and} \quad e^{3C^2 E(0)} \leq \frac{4}{3}.$$

So we get  $\bar{T}_n = T'_n$  as  $\forall T$  such that  $T < \bar{T}_n$ :  $E(q^n, u^n, T) \leq 2CE(0)$ . We have then  $\bar{T}_n = T'_n$ , because if  $\bar{T}_n < T'_n$  we know that  $E(q^n, u^n, \bar{T}_n) \leq 2CE(0)$  and so by continuity for  $\bar{T}_n + \varepsilon$  with  $\varepsilon$  small enough we obtain again  $E(q^n, u^n, \bar{T}_n + \varepsilon) \leq 3CE(0)$ . This stands in contradiction with the definition of  $\bar{T}_n$ . Assume now that  $\bar{T}_n = T'_n < +\infty$  but we know that:

$$E(q^n, u^n, T'_n) \leq 3CE(0).$$



As  $\|q_n\|_{L_{T_n}^\infty(\tilde{B}^{\frac{N}{2}})} < +\infty$  and  $\|u_n\|_{L_{T_n}^\infty(\tilde{B}^{\frac{N}{2}-1})} < +\infty$ , it implies that  $\|q_n\|_{L_{T_n}^\infty(L_n^2)} < +\infty$  and  $\|u_n\|_{L_{T_n}^\infty(L_n^2)} < +\infty$ , so by Cauchy-Lipschitz theorem, one may continue the solution beyond  $T_n'$  which contradicts the definition of  $T_n'$ . Finally the approximate solution  $(q^n, u^n)_{n \in \mathbb{N}}$  is global in time.

### Second step: existence of a solution

In this part, we shall show that, up to an extraction, the sequence  $(q^n, u^n)_{n \in \mathbb{N}}$  converges in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$  to a solution  $(q, u)$  of  $(SW)$  which has the desired regularity properties. The proof lies on compactness arguments. To start with, we show that the time first derivative of  $(q^n, u^n)$  is uniformly bounded in appropriate spaces. This enables us to apply Ascoli's theorem and get the existence of a limit  $(q, u)$  for a subsequence. Now, the uniform bounds of the previous part provide us with additional regularity and convergence properties so that we may pass to the limit in the system.

It is convenient to split  $(q^n, u^n)$  into the solution of a linear system with initial regular data and forcing term, and the discrepancy to that solution. More precisely, we denote by  $(q_L^n, u_L^n)$  the solution to:

$$\begin{cases} \partial_t q_L^n + \operatorname{div} u_L^n = 0 \\ \partial_t u_L^n - \mathcal{A} u_L^n + \nabla q_L^n = 0 \\ (q_L^n, u_L^n)_{/t=0} = (J_n q_0, J_n u_0) \end{cases} \quad (3.31)$$

where:  $\mathcal{A} = \bar{\mu} \Delta + (\bar{\lambda} + \bar{\mu}) \nabla \operatorname{div}$  and we set  $(\bar{q}^n, \bar{u}^n) = (q^n - q_L^n, u^n - u_L^n)$ . Obviously, the definition of  $(q_L^n, u_L^n)_{/t=0}$  entails:

$$(q_L^n)_{/t=0} \rightarrow q_0 \text{ in } \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}, (u_L^n)_{/t=0} \rightarrow u_0 \text{ in } \tilde{B}^{\frac{N}{2}-1}.$$

The proposition 2.4 insures that  $(q_L^n, u_L^n)$  converges to the solution  $(q_L, u_L)$  of the linear system associated to (3.31) in  $E^{\frac{N}{2}}$ . We now have to prove the convergence of  $(\bar{q}^n, \bar{u}^n)$ . This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

**Lemma 1**  $(q^n, u^n)_{n \in \mathbb{N}}$  is uniformly bounded in  $C^{\frac{1}{2}}(\mathbb{R}^+; B^{\frac{N}{2}-1}) \times (C^{\frac{1}{4}}(\mathbb{R}^+; B^{\frac{N}{2}-\frac{3}{2}}))^N$ .

**Proof:**

In all the proof, we will note u.b for uniformly bounded. We first prove that  $\frac{\partial}{\partial t} q^n$  is u.b in  $L^2(\mathbb{R}^+, B^{\frac{N}{2}-1})$ , which yields the desired result for  $q^n$ . Let us observe that  $q^n$  verifies the following equation:

$$\frac{\partial}{\partial t} q^n = \operatorname{div} u^n - J_n(u^n \cdot \nabla q^n) - J_n(q^n \operatorname{div} u^n).$$

According to the first part,  $(u_n)_{n \in \mathbb{N}}$  is u.b in  $L^2(B^{\frac{N}{2}})$ , so we can conclude that  $\frac{\partial}{\partial t} q^n$  is u.b in  $L^2(B^{\frac{N}{2}-1})$ . Indeed we have:

$$\|J_n(q^n \operatorname{div} u^n)\|_{L^2(B^{\frac{N}{2}-1})} \leq \|u^n\|_{L^2(B^{\frac{N}{2}})} \|q^n\|_{L^\infty(B^{\frac{N}{2}})},$$

and we get the same estimate for  $\|J_n(u^n \cdot \nabla q^n)\|_{L^2(B^{\frac{N}{2}-1})}$ . Let us prove now that  $\frac{\partial}{\partial t} d^n$  is u.b in  $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}}) + L^4(B^{\frac{N}{2}-\frac{3}{2}})$  and that  $\partial_t \Omega^n$  is u.b in  $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$  (which gives the required result for  $u^n$  in using the relation  $u^n = -\Lambda^{-1} \nabla d^n - \Lambda^{-1} \operatorname{div} \Omega^n$ ). We recall that:

$$\begin{aligned} \frac{\partial}{\partial t} d^n &= J_n(u^n \cdot \nabla d^n) + J_n \Lambda^{-1} \operatorname{div} [\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + J_n(K(\varphi(\bar{\rho}(1+q^n))) \nabla q^n)] \\ &\quad + \bar{\nu} \Delta d^n + \bar{\delta} \Lambda q^n - \bar{\kappa} \phi * \Lambda q^n, \\ \frac{\partial}{\partial t} \Omega^n &= J_n \Lambda^{-1} \operatorname{curl} [\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + J_n(K(\varphi(\bar{\rho}(1+q^n))) \nabla q^n)] + \bar{\mu} \Delta \Omega^n. \end{aligned}$$

Results of step one and an interpolation argument yield uniform bounds for  $u^n$  in  $L^\infty(B^{\frac{N}{2}-1}) \cap L^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}})$ , we infer in proceeding as for  $\frac{\partial}{\partial t} q^n$  that:

$$\begin{aligned} A_n &= J_n(u^n \cdot \nabla d^n) + J_n \Lambda^{-1} \operatorname{div} [\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + J_n(K(\varphi(\bar{\rho}(1+q^n))) \nabla q^n)] + \bar{\nu} \Delta d^n \\ &\quad \text{is u.b in } L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}}). \end{aligned}$$

Using the bounds for  $q^n$  in  $L^2(B^{\frac{N}{2}}) \cap L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})$ , we get  $q^n$  u.b in  $L^4(B^{\frac{N}{2}-\frac{1}{2}})$  by using proposition 2.1. We thus have  $J_n(K(\varphi(\bar{\rho}(1+q^n))) \nabla q^n$  u.b in  $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$ . Using the bounds for  $u^n$  in  $L^\infty(B^{\frac{N}{2}-1}) \cap L^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}})$  we finally get  $A_n$  is u.b in  $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$ . To conclude  $\phi * \Lambda q^n$  is u.b in  $L^4(B^{\frac{N}{2}-\frac{3}{2}})$ , so  $\frac{\partial}{\partial t} d^n$  is u.b in  $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}}) + L^4(B^{\frac{N}{2}-\frac{3}{2}})$ . The case of  $\frac{\partial}{\partial t} \Omega^n$  goes along the same lines. As the terms corresponding to  $\Lambda q^n$  and  $\phi * \Lambda \bar{q}^n$  do not appear, we simply get  $\partial_t \Omega^n$  u.b in  $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$ .  $\square$

We can now turn to the proof of the existence of a solution and using Ascoli theorem to get strong convergence. We proceed similarly to the theorem of Aubin-Lions.

**Theorem 3.4** *Let  $X$  a compact metric space and  $Y$  a complete metric space. Let  $A$  be an equicontinuous part of  $C(X, Y)$ . Then we have the two equivalent proposition:*

1.  $A$  is relatively compact in  $C(X, Y)$
2.  $A(x) = \{f(x); f \in A\}$  is relatively compact in  $Y$

We need to localize because we have some result of compactness for the local Sobolev space. Let  $(\chi_p)_{p \in \mathbb{N}}$  be a sequence of  $C_0^\infty(\mathbb{R}^N)$  cut-off functions supported in the ball  $B(0, p+1)$  of  $\mathbb{R}^N$  and equal to 1 in a neighborhood of  $B(0, p)$ .

For any  $p \in \mathbb{N}$ , lemma 1 tells us that  $((\chi_p q^n, \chi_p u^n))_{n \in \mathbb{N}}$  is uniformly equicontinuous in  $C(\mathbb{R}^+; B^{\frac{N}{2}-1} \times (B^{\frac{N}{2}-\frac{3}{2}})^N)$ . By using Ascoli's theorem we just need to show that  $((\chi_p q^n(t, \cdot), \chi_p u^n(t, \cdot))_{n \in \mathbb{N}}$  is relatively compact in  $B^{\frac{N}{2}-1} \times (B^{\frac{N}{2}-\frac{3}{2}})^N \forall t \in [0, p]$ . Let us observe now that the application  $u \rightarrow \chi_p u$  is compact from  $\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}} = B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}$  into  $\dot{H}^{\frac{N}{2}-1}$ , and from  $B^{\frac{N}{2}-1} \cap B^{\frac{N}{2}-\frac{3}{2}}$  into  $\dot{H}^{\frac{N}{2}-\frac{3}{2}}$ . Next we apply Ascoli's theorem to the family  $((\chi_p q^n, \chi_p u^n))_{n \in \mathbb{N}}$  on the time interval  $[0, p]$ . By using Cantor's diagonal process we obtain a distribution  $(q, u)$  belonging to  $C(\mathbb{R}^+; \dot{H}^{\frac{N}{2}-1} \times (\dot{H}^{\frac{N}{2}-\frac{3}{2}})^N)$  and a subsequence (which we still denote by  $(q^n, u^n)_{n \in \mathbb{N}}$ ) such that, for all  $p \in \mathbb{N}$ , we have:

$$(\chi_p q^n, \chi_p u^n) \rightarrow_{n \rightarrow +\infty} (\chi_p q, \chi_p u) \text{ in } C([0, p]; \dot{H}^{\frac{N}{2}-1} \times (\dot{H}^{\frac{N}{2}-\frac{3}{2}})^N) \quad (3.32)$$

This obviously entails that  $(q^n, u^n)$  tends to  $(q, u)$  in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$ .

Coming back to the uniform estimates of step one, we moreover get that  $(q, u)$  belongs to:

$$L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}} \times (B^{\frac{N}{2}+1})^N) \cap L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}} \times (B^{\frac{N}{2}+1})^N)$$

and to  $C^{\frac{1}{2}}(\mathbb{R}^+; B^{\frac{N}{2}-1}) \times (C^{\frac{1}{4}}(\mathbb{R}^+; B^{\frac{N}{2}-\frac{3}{2}})^N)$ . Obviously, we have the bounds provided of the first step.

Let us now prove that  $(q, u)$  solves the system (SW), we first recall that  $(q^n, u^n)$  solves the following system:

$$\begin{cases} \partial_t q^n + J_n(u^n \cdot \nabla q^n) + \operatorname{div} u^n = -J_n(q^n \operatorname{div} u^n) \\ \partial_t u^n - \bar{\nu} \Delta u^n + \bar{\delta} \nabla q^n - \bar{\kappa} \phi * \nabla q^n + J_n(u^n \cdot \nabla u^n) + J_n(K(\varphi(\bar{\rho}(1 + q^n)) \nabla q^n) \\ \quad + J_n(\mathcal{A}(\varphi(\bar{\rho}(1 + q^n)), u^n)) = 0 \end{cases}$$

The only problem is to pass to the limit in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$  in the non linear terms. This can be done by using the convergence results coming from the uniform estimates (3.32). As it is just a matter of doing tedious verifications, we show as a example the case of the term  $J_n(K(\varphi(\bar{\rho}(1 + q^n))) \nabla q^n)$  and  $J_n(\mathcal{A}(\varphi(\bar{\rho}(1 + q^n)), u^n))$ . We decompose:

$$J_n(K(\varphi(\bar{\rho}(1 + q^n))) \nabla q^n) - K(\rho^n) \nabla q^n = J_n(K(\varphi(\bar{\rho}(1 + q^n))) \nabla q^n) - K(\varphi(\bar{\rho}(1 + q))) \nabla q.$$

(Note that for  $n$  big enough, we have  $K(\varphi(\bar{\rho}(1 + q^n))) = K(\rho^n)$  as we control  $\|\rho^n\|_{L^\infty}$  and  $\|\frac{1}{\rho^n}\|_{L^\infty}$ ). Next we have:

$$J_n(K(\varphi(\bar{\rho}(1 + q^n))) \nabla q^n) - K(\varphi(\bar{\rho}(1 + q))) \nabla q = J_n A_n + (J_n - I) K(\varphi(\bar{\rho}(1 + q))) \nabla q,$$

where  $A_n = K(\varphi(\bar{\rho}(1 + q^n))) \nabla q^n - K(\varphi(\bar{\rho}(1 + q))) \nabla q$ .

We have then  $(J_n - I) K(\varphi(\bar{\rho}(1 + q))) \nabla q$  tends to zero as  $n \rightarrow +\infty$  due to the property of  $J_n$  and the fact that  $K(\varphi(\bar{\rho}(1 + q))) \nabla q$  belongs to  $L^\infty(B^{\frac{N}{2}-1}) \hookrightarrow L^\infty(L^q)$  for some  $q \geq 2$ . Choose  $\psi \in C_0^\infty([0, T) \times \mathbb{R}^N)$  and  $\varphi' \in C_0^\infty([0, T) \times \mathbb{R}^N)$  such that  $\varphi' = 1$  on  $\operatorname{supp} \psi$ , we have:

$$| \langle (J_n - I) K(\varphi(\bar{\rho}(1 + q))) \nabla q, \psi \rangle | \leq \|\varphi' K(\varphi(\bar{\rho}(1 + q))) \nabla q\|_{L^\infty(L^2)} \|(J_n - I) \psi\|_{L^2},$$

because  $L_{loc}^q \hookrightarrow L_{loc}^2$  and we conclude by the fact that  $\|(J_n - I) \psi\|_{L^2} \rightarrow 0$  as  $n$  tends to  $+\infty$ . Next:

$$\langle J_n A_n, \psi \rangle = I_n^1 + I_n^2,$$

with:

$$\begin{aligned} I_n^1 &= \langle (K(\varphi(\bar{\rho}(1 + q^n))) - K(\varphi(\bar{\rho}(1 + q)))) \nabla q^n, J_n \psi \rangle \\ I_n^2 &= \langle K(\varphi(\bar{\rho}(1 + q))) \nabla (q^n - q), J_n \psi \rangle. \end{aligned}$$

We have then:

$$I_n^1 \leq \|\varphi' q^n\|_{L^\infty(B^{\frac{N}{2}})} \|\varphi' (q^n - q)\|_{L^\infty(\dot{H}^{\frac{N}{2}-1})} \|\psi\|_{L^\infty},$$

Indeed we just use the fact that  $\varphi' B^{\frac{N}{2}-1}$  and  $\varphi' \dot{H}^{\frac{N}{2}-1}$  are embedded in  $L^2$ . Next we conclude as we have seen that  $q^n \rightarrow_{n \rightarrow +\infty} q$  in  $C_{loc}(H_{loc}^{\frac{N}{2}-1})$ . So we obtain:

$$I_n^1 \rightarrow_{n \rightarrow +\infty} 0 \quad \text{in } \mathcal{D}'((0, T^*) \times \mathbb{R}^N).$$

We proceed similarly for  $I_n^2$ . We concentrate us now on the term  $J_n(\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n))$ . Let  $\varphi' \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$  and  $p \in \mathbb{N}$  be such that  $\text{supp} \varphi' \subset [0, p] \times B(0, p)$ . We use the decomposition for  $n$  big enough:

$$\begin{aligned} \varphi' J_n \mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) - \varphi' \mathcal{A}(\rho, u) &= \varphi' \chi_p \mathcal{A}(\varphi(\bar{\rho}(1+q^n)), \chi_p(u^n - u)) \\ &\quad + \varphi' \mathcal{A}(\chi_p \varphi(\bar{\rho}(1+q^n)) - \chi_p \bar{\rho}(1+q), u). \end{aligned}$$

According to the uniform estimates and (3.32),  $\chi_p(u^n - u)$  tends to 0 in  $L^1([0, p]; \dot{H}^{\frac{N}{2}+1})$  by interpolation so that the first term tends to 0 in  $L^1(\dot{H}^{\frac{N}{2}-1})$  and we conclude for the second term in remarking that  $\frac{\varphi}{\rho_n}$  tends to  $\frac{\varphi}{\rho}$  as  $\rho_n$  in  $L^\infty(L^\infty \cap \dot{H}^{\frac{N}{2}})$ . The other nonlinear terms can be treated in the same way.  $\square$

## 4 Proof of the existence for theorem 1.2

We will proceed similarly to the proof of the theorem 1.1. It means we have to get a sequence of smooth solutions  $(q^n, u^n)_{n \in \mathbb{N}}$  to system  $(SW)$  on a bounded interval  $[0, T^n]$  which may depend on  $n$ . We will exhibit a positive lower bound  $T$  for  $T_n$  and prove uniform estimates for the smooth solutions  $(q^n, u^n)_{n \in \mathbb{N}}$ , and we will conclude by using compactness arguments. We begin with the study of the linear part associated to the system  $(SW)$ .

### 4.1 Estimates for parabolic system with variable coefficients

To avoid condition of smallness on the initial density data as in [], it is crucial to study very precisely the following parabolic system with variable coefficient which is obtained by linearizing the momentum equation:

$$\begin{cases} \partial_t u + v \cdot \nabla u + u \cdot \nabla w - b(\mu \Delta u + (\lambda + \mu) \nabla \text{div} u) = f, \\ u|_{t=0} = u_0. \end{cases} \quad (4.33)$$

Above  $u$  is the unknown function. We assume that  $u_0 \in B_{p,1}^s$  with  $1 \leq p \leq +\infty$  and  $f \in L^1(0, T; B_{p,1}^s)$ , that  $v$  and  $w$  are time dependent vector-fields with coefficients in  $L^1(0, T; B_{p,1}^{\frac{N}{p}+1})$ , that  $b$  is bounded by below by a positive constant  $\underline{b}$  and that  $a = b - 1$  belongs to  $L^\infty(0, T; B_{2,1}^{\frac{N}{2}})$ . We will need of the following proposition to get some uniform estimates on the smooth solutions  $(q^n, u^n)_{n \in \mathbb{N}}$ . This proposition generalize a result of Danchin in [] and the fact that we are able to take into account the variable coefficient for heat equation, will allow us to consider large initial data on the density in critical space.

**Proposition 4.6** *Let  $\underline{\nu} = \underline{b} \min(\mu, \lambda + 2\mu)$  and  $\bar{\nu} = \mu + |\lambda + \mu|$ . Assume that  $s \in (-\frac{N}{p}, \frac{N}{p}]$ . Let  $m \in \mathbb{Z}$  be such that  $b_m = 1 + S_m a$  and  $1 - S_m a$  satisfies for  $c$  small enough (depending only on  $N$  and on  $s$ ):*

$$\inf_{(t,x) \in [0,T] \times \mathbb{R}^N} b_m(t, x) \geq \frac{\underline{b}}{2} \quad \text{and} \quad \|a - S_m a\|_{\tilde{L}^\infty(0,T; B_{p,1}^{\frac{N}{p}})} \leq c \frac{\underline{\nu}}{\bar{\nu}}. \quad (4.34)$$

There exist two constants  $C$  and  $\kappa$  (with  $C$ , depending only on  $N$  and on  $s$ , and  $\kappa$  universal) such that by setting:

$$V(t) = \int_0^t \|v\|_{B_{p,1}^{\frac{N}{p}+1}} d\tau, \quad W(t) = \int_0^t \|w\|_{B_{p,1}^{\frac{N}{p}+1}} d\tau, \quad \text{and} \quad Z_m(t) = 2^{2m} \bar{\nu}^2 \underline{\nu}^{-1} \int_0^t \|a\|_{B_{p,1}^{\frac{N}{p}}}^2 d\tau,$$

we have for all  $t \in [0, T]$ ,

$$\begin{aligned} \|u\|_{\tilde{L}^\infty((0,T) \times B_{p,1}^s)} + \kappa \underline{\nu} \|u\|_{\tilde{L}^1((0,T) \times B_{p,1}^{s+2})} &\leq e^{C(V+W+Z_m)(t)} (\|u_0\|_{B_{p,1}^s} \\ &+ \int_0^t e^{-C(V+W+Z_m)(\tau)} \|f(\tau)\|_{B_{p,1}^s} d\tau). \end{aligned}$$

**Remark 3** Let us stress the fact that if  $a \in \tilde{L}^\infty((0, T) \times B_{p,1}^{\frac{N}{p}})$  then assumption (4.34) is satisfied for  $m$  large enough. Indeed, according to Bernstein inequality, we have:

$$\|a - S_m a\|_{L^\infty((0,T) \times \mathbb{R}^N)} \leq \sum_{q \geq m} \|\Delta_q a\|_{L^\infty((0,T) \times \mathbb{R}^N)} \lesssim \sum_{q \geq m} 2^{q \frac{N}{p}} \|\Delta_q a\|_{L^\infty(L^p)}.$$

The right-hand side is the remainder of a convergent series hence tends to zero when  $m$  goes to infinity. For a similar reason, the other inequality is satisfied for  $m$  large enough.

**Proof:**

Let us first rewrite (4.33) as follows:

$$\partial_t u + v \cdot \nabla u + u \cdot \nabla w - b_m(\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u) = f + E_m - u \cdot \nabla w, \quad (4.35)$$

with  $E_m = (\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u)(\operatorname{Id} - S_m)a$ . Note that, because  $-\frac{N}{p} < s \leq \frac{N}{p}$ , the error term  $E_m$  may be estimated by:

$$\|E_m\|_{B_{p,1}^s} \lesssim \|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|D^2 u\|_{B_{p,1}^s}. \quad (4.36)$$

and we have:

$$\|u \cdot \nabla w\| \lesssim \|\nabla w\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p,1}^s}. \quad (4.37)$$

Now applying  $\Delta_q$  to equation (4.35) yields:

$$\begin{aligned} \frac{d}{dt} u_q + v \cdot \nabla u_q - \mu \operatorname{div}(b_m \nabla u_q) - (\lambda + \mu) \nabla(b_m \operatorname{div} u_q) &= f_q + E_{m,q} \\ &- \Delta_q(u \cdot \nabla w) + R_q + \tilde{R}_q, \end{aligned} \quad (4.38)$$

where we denote by  $u_q = \Delta_q u$  and with:

$$\begin{aligned} R_q &= [v^j, \Delta_q] \partial_j u, \\ \tilde{R}_q &= \mu(\Delta_q(b_m \Delta u) - \operatorname{div}(b_m \nabla u_q)) + (\lambda + \mu)(\Delta_q(b_m \nabla \operatorname{div} u) - \nabla(b_m \operatorname{div} u_q)). \end{aligned}$$

Next multiplying both sides by  $|u_q|^{p-2}u_q$ , integrating by parts and using Hölder's inequalities, we get:

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \|u_q\|_{L^p}^p - \frac{1}{p} \int |u_q|^p \operatorname{div} v dx + \int \mu b_m (|\nabla u_q|^2 |u_q|^{p-2} + \frac{p-2}{4} |\nabla |u_q|^2|^2 |u_q|^{p-4}) dx \\ & + (\lambda + \mu) b_m (|\operatorname{div} u_q|^2 |u_q|^{p-2} + (p-2) |u_q|^{p-4} \operatorname{div} u_q \sum_{i,j} \partial_j u_q^i u_q^i u_q^j) dx \leq \|u_q\|_{L^p}^{p-1} (\|f_q\|_{L^p} \\ & + \|\Delta_q E_m\|_{L^p} + \|\Delta_q (u \cdot \nabla w)\|_{L^p} + \|R_q\|_{L^p} + \|\tilde{R}_q\|_{L^p}), \end{aligned}$$

Next by using lemma in [], Young's inequality and the fact that  $\mu \geq 0$  and  $\lambda + 2\mu \geq 0$ , we get:

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|u_q\|_{L^p}^p + \frac{\nu \underline{b}(p-1)}{p^2} 2^{2q} \|u_q\|_{L^p}^p & \leq \|u_q\|_{L^p}^{p-1} (\|f_q\|_{L^p} + \|E_{m,q}\|_{L^p} + \|\Delta_q (u \cdot \nabla w)\|_{L^p} \\ & + \frac{1}{p} \|u_q\|_{L^p} \|\operatorname{div} u\|_{L^\infty} + \|R_q\|_{L^p} + \|\tilde{R}_q\|_{L^p}), \end{aligned}$$

which leads, after time integration to:

$$\begin{aligned} \|u_q\|_{L^p} + \frac{\nu \underline{b}(p-1)}{p} 2^{2q} \int_0^t \|u_q\|_{L^p} d\tau & \leq \|\Delta_q u_0\|_{L^p} + \int_0^t (\|f_q\|_{L^p} + \|E_{m,q}\|_{L^p}) d\tau \\ & + \int_0^t (\|\Delta_q (u \cdot \nabla w)\|_{L^p} + \frac{1}{p} \|u_q\|_{L^p} \|\operatorname{div} u\|_{L^\infty} + \|R_q\|_{L^p} + \|\tilde{R}_q\|_{L^p}) d\tau, \end{aligned} \quad (4.39)$$

For commutators  $R_q$  and  $\tilde{R}_q$ , we have the following estimates (see lemma 4 and 5 in the appendix)

$$\|R_q\|_{L^p} \lesssim c_q 2^{-qs} \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|u\|_{B_{p,1}^s}, \quad \|\tilde{R}_q\|_{L^p} \lesssim c_q \bar{\nu} 2^{-qs} \|S_m a\|_{B_{p,1}^{\frac{N}{p}+1}} \|Du\|_{B_{p,1}^s}, \quad (4.40)$$

where  $(c_q)_{q \in \mathbb{Z}}$  is a positive sequence such that  $\sum_{q \in \mathbb{Z}} c_q = 1$ , and  $\bar{\nu} = \mu + |\lambda + \mu|$ . Note that, using Bernstein inequality, we have:

$$\|S_m a\|_{B_{p,1}^{\frac{N}{p}+1}} \lesssim 2^m \|a\|_{B_{p,1}^{\frac{N}{p}}}$$

Hence, plugging these latter estimates and (4.36), (4.37) in (4.39), then multiplying by  $2^{qs}$  and summing up on  $q \in \mathbb{Z}$ , we discover that, for all  $t \in [0, T]$ :

$$\begin{aligned} \|u\|_{L_t^\infty(B_{p,1}^s)} + \frac{\nu \underline{b}(p-1)}{p} \|u\|_{L_t^1(B_{p,1}^{s+2})} & \leq \|u_0\|_{B_{p,1}^s} + \|f\|_{L_t^1(B_{p,1}^s)} + C \int_0^t (\|v\|_{B_{p,1}^{\frac{N}{p}}} \\ & + \|w\|_{B_{p,1}^{\frac{N}{p}+1}}) \|u\|_{B_{p,1}^s} d\tau + C \bar{\nu} \int_0^t (\|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p,1}^{s+2}} + 2^m \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p,1}^{s+1}}) d\tau, \end{aligned}$$

for a constant  $C$  depending only on  $N$  and  $s$ . Let  $X(t) = \|u\|_{L_t^\infty(B_{p,1}^s)} + \nu \underline{b} \|u\|_{L_t^1(B_{p,1}^{s+2})}$ . Assuming that  $m$  has been chosen so large as to satisfy:  $C \bar{\nu} \|a - S_m a\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq \underline{\nu}$ , and

by interpolation, we have:

$$C \bar{\nu} \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p,1}^{s+2}} \leq \kappa \underline{\nu} + \frac{C^2 \bar{\nu}^2 2^{2m}}{4 \kappa \underline{\nu}} \|a\|_{B_{p,1}^{\frac{N}{p}}}^2 \|u\|_{B_{p,1}^s},$$

we end up with:

$$X(t) \leq \|u_0\|_{B_{p,1}^s} + \|f\|_{L_t^1(B_{p,1}^s)} + C \int_0^t (\|v\|_{B_{p,1}^{\frac{N}{p}}} + \|w\|_{B_{p,1}^{\frac{N}{p}+1}} + \frac{\bar{\nu}^2}{\underline{\nu}} 2^{2m} \|a\|_{B_{p,1}^{\frac{N}{2}}}) X d\tau$$

Grönwall lemma then leads to the desired inequality.  $\square$

**Remark 4** *The proof of the continuation criterion (theorem 1.3) relies on a better estimate which is available when  $u = v = w$ . In fact, by arguing as in the proof of the previous proposition and by making use of inequality (7.70) instead of (7.68), one can prove that under conditions (4.34), there exists constants  $C$  and  $\kappa$  such that:*

$$\forall t \in [0, T], \|u\|_{L_t^\infty(B_{p,1}^s)} + \kappa \underline{\nu} \|u\|_{L_t^1(B_{p,1}^{s+2})} \leq e^{C(U+Z_m)(t)} (\|u_0\|_{B_{p,1}^s} + \int_0^t e^{-C(U+Z_m)(\tau)} \|f(\tau)\|_{B_{p,1}^s} d\tau) \quad \text{with} \quad U(t) = \int_0^t \|\nabla u\|_{L^\infty} d\tau.$$

Proposition 4.6 fails in the limit case  $s = -\frac{N}{p}$ . One can however state the following result which will be the key to the proof of uniqueness.

**Proposition 4.7** *Under condition (4.34), there exists two constants  $C$  and  $\kappa$  (with  $c, C$ , depending only on  $N$ , and  $\kappa$  universal) such that we have:*

$$\|u\|_{L_t^\infty(B_{p,\infty}^{-\frac{N}{p}})} + \kappa \underline{\nu} \|u\|_{\tilde{L}_t^1(B_{p,\infty}^{2-\frac{N}{p}})} \leq 2e^{C(V+W)(t)} (\|u_0\|_{B_{p,\infty}^{-\frac{N}{p}}} + \|f\|_{\tilde{L}_t^1(B_{p,\infty}^{\frac{N}{p}})}),$$

whenever  $t \in [0, T]$  satisfies:

$$\bar{\nu}^2 t \|a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})}^2 \leq c 2^{-2m} \underline{\nu}. \quad (4.41)$$

## Proof

We just point out the changes that have to be done compare to the proof of proposition 4.6. The first one is that instead of (4.36) and (4.37), we have in accordance with proposition 2.1:

$$\begin{aligned} \|E_m\|_{\tilde{L}_t^1(B_{p,\infty}^{-\frac{N}{p}})} &\lesssim \|a - S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \|D^2 u\|_{\tilde{L}_t^1(B_{p,\infty}^{-\frac{N}{p}})}, \\ \|u \cdot w\|_{B_{p,\infty}^{-\frac{N}{p}}} &\lesssim \|u\|_{B_{p,\infty}^{-\frac{N}{p}}} \|\nabla u\|_{B_{p,1}^{\frac{N}{p}}}. \end{aligned}$$

The second change concerns the estimates of commutator  $R_q$  and  $\tilde{R}_q$ . According to inequality (7.69), we now have for all  $q \in \mathbb{Z}$ :

$$\|R_q\|_{L^p} \lesssim 2^{q\frac{N}{p}} \|v\|_{B_{p,1}^{\frac{N}{p}+1}} \|u\|_{B_{p,\infty}^{-\frac{N}{p}}}, \quad \|\tilde{R}_q\| \lesssim \bar{\nu} 2^{q\frac{N}{p}} \|S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}+1})} \|Du\|_{\tilde{L}_t^1(B_{p,\infty}^{-\frac{N}{p}})}. \quad (4.42)$$

Plugging all these estimates in (4.39) then taking the supremum over  $q \in \mathbb{Z}$ , we get:

$$\begin{aligned} \|u\|_{L_t^\infty(B_{p,\infty}^{-\frac{N}{p}})} + 2\underline{\nu} \|u\|_{\tilde{L}_t^1(B_{p,\infty}^{2-\frac{N}{p}})} &\leq \|u_0\|_{B_{p,1}^{-\frac{N}{p}}} + C_i n t_0^t (\|v\|_{B_{p,1}^{\frac{N}{p}+1}} + \|w\|_{B_{p,1}^{\frac{N}{p}+1}}) \|u\|_{B_{p,\infty}^{-\frac{N}{p}}} d\tau \\ &+ C \bar{\nu} (\|a - S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \|u\|_{\tilde{L}_t^1(B_{p,\infty}^{2-\frac{N}{p}})} + 2^m \|a\|_{L_t^\infty(B_{p,1}^{\frac{N}{p}})} \|u\|_{\tilde{L}_t^1(B_{p,\infty}^{1-\frac{N}{p}})} + \|f\|_{\tilde{L}_t^1(B_{p,\infty}^{-\frac{N}{p}})}). \end{aligned}$$

Using that:

$$\|u\|_{\tilde{L}_t^1(B_{p,\infty}^{1-\frac{N}{p}})} \leq \sqrt{t} \|u\|_{\tilde{L}_t^1(B_{p,\infty}^{2-\frac{N}{p}})}^{\frac{1}{2}} \|u\|_{L_t^\infty(B_{p,\infty}^{\frac{N}{p}})}^{\frac{1}{2}},$$

and taking advantage of assumption (4.34) and (4.41), it is now easy to complete the proof.  $\square$

## 4.2 The mass conservation equation

We now focus on the mass equation associated to (SW)

$$\begin{aligned} \partial_t a + v \cdot \nabla a &= (1+a) \operatorname{div} v, \\ a|_{t=0} &= a_0. \end{aligned} \quad (4.43)$$

**Proposition 4.8** *Assume that  $a_0 \in B_{2,1}^{\frac{N}{2}}$ ,  $v \in L^1(0, T; B_{2,1}^{\frac{N}{2}+1})$  and that  $a \in \tilde{L}_T^\infty(B_{2,1}^{\frac{N}{2}})$  satisfies (4.43). Let  $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{2,1}^{\frac{N}{2}+1}} d\tau$ . There exists a constant  $C$  depending only on  $N$  such that for all  $t \in [0, T]$  and  $m \in \mathbb{Z}$ , we have:*

$$\|a\|_{\tilde{L}_t^\infty(B_{2,1}^{\frac{N}{2}})} \leq e^{CV(t)} \|a_0\|_{B_{2,1}^{\frac{N}{2}}} + e^{CV(t)} - 1, \quad (4.44)$$

$$\sum_{l \geq m} 2^{l\frac{N}{2}} \|\Delta_l a\|_{L_t^\infty(L^2)} \leq \sum_{l \geq m} 2^{l\frac{N}{2}} \|\Delta_l a_0\|_{L^2} + (1 + \|a_0\|_{B_{2,1}^{\frac{N}{2}}})(e^{CV(t)} - 1), \quad (4.45)$$

$$\sum_{l \geq m} 2^{l\frac{N}{2}} \|\Delta_l(a - a_0)\|_{L_t^\infty(L^2)} \leq (1 + \|a_0\|_{B_{2,1}^{\frac{N}{2}}})(e^{CV(t)} - 1) + C 2^l \|a_0\|_{B_{2,1}^{\frac{N}{2}}} \int_0^t \|v\|_{B_{2,1}^{\frac{N}{2}}} d\tau. \quad (4.46)$$

## 4.3 The proof of existence for theorem 1.2

We smooth out the data as follows:

$$q_0^n = S_n q_0, \quad u_0^n = S_n u_0 \quad \text{and} \quad f^n = S_n f.$$

Now according to [], one can solve (SW) with smooth initial data  $(q_0^n, u_0^n, f^n)$  on a time interval  $[0, T_n]$ . Let  $\varepsilon > 0$ , we get solution checking:

$$q^n \in C([0, T_n], B_{p,1}^{\frac{N}{p}+\varepsilon}) \quad \text{and} \quad u^n \in C([0, T_n], B_{p,1}^{\frac{N}{p}-1+\varepsilon}) \cap \tilde{L}^1([0, T_n], B_{p,1}^{\frac{N}{p}+1+\varepsilon}). \quad (4.47)$$

### Uniform Estimates for $(q^n, u^n)_{n \in \mathbb{N}}$

Let  $T_n$  be the lifespan of  $(q_n, u_n)$ , that is the supremum of all  $T > 0$  such that (SW) with initial data  $(q_0^n, u_0^n)$  has a solution which satisfies (4.47). Let  $T$  be in  $(0, T_n)$ , we aim at getting uniform estimates in  $E_T$  for  $T$  small enough. For that, we need to introduce the solution  $u_L^n$  to the linear system:

$$\partial_t u_L^n - \mathcal{A} u_L^n = f^n, \quad u_L^n(0) = u_0^n.$$



Now, the vectorfield  $\tilde{u}^n = u^n - u_L^n$  satisfies the parabolic system:

$$\begin{cases} \partial_t \tilde{u}^n + u_L^n \cdot \nabla \tilde{u}^n + (1 + a^n) \mathcal{A} \tilde{u}^n = a^n \mathcal{A} u_L^n - u_L^n \cdot \nabla u_L^n - \nabla g(a^n), \\ \tilde{u}^n(0) = 0, \end{cases} \quad (4.48)$$

which has been studied in proposition 4.6. Define  $m \in \mathbb{Z}$  by:

$$m = \inf \left\{ p \in \mathbb{Z} : 2\bar{\nu} \sum_{l \geq p} 2^{l \frac{N}{2}} \|\Delta_l a_0\|_{L^2} \leq c\bar{\nu} \right\} \quad (4.49)$$

where  $c$  is small enough positive constant (depending only  $N$ ) to be fixed hereafter.

Let:

$$\bar{b} = 1 + \sup_{x \in \mathbb{R}^N} a_0(x), \quad A_0 = 1 + 2\|a_0\|_{B^{\frac{N}{2}}}, \quad U_0 = \|u_0\|_{B^{\frac{N}{2}}} + \|f\|_{L^1(B^{\frac{N}{2}-1})},$$

and  $\tilde{U}_0 = 2CU_0 + 4C\bar{\nu}A_0$  (where  $C$  stands for a large enough constant depending only  $N$  which will be determined when applying proposition 2.1 and 4.6 in the following computations.) We assume that the following inequalities are fulfilled for some  $\eta > 0$ :

$$\begin{aligned} (\mathcal{H}_1) \quad & \|a^n - S_m a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq c\bar{\nu}^{-1}, \\ (\mathcal{H}_2) \quad & C\bar{\nu}^2 T \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}^2 \leq 2^{-2m} \underline{\nu}, \\ (\mathcal{H}_3) \quad & \frac{1}{2}\bar{b} \leq 1 + a^n(t, x) \leq 2\bar{b} \text{ for all } (t, x) \in [0, T] \times \mathbb{R}^N, \\ (\mathcal{H}_4) \quad & \|a^n\|_{\tilde{L}^\infty(B_{p,1}^{\frac{N}{p}})} \leq A_0, \\ (\mathcal{H}_5) \quad & \|u_L^n\|_{\tilde{L}^1(B_{p,1}^{\frac{N}{p}+1})} \leq \eta, \\ (\mathcal{H}_6) \quad & \|\tilde{u}^n\|_{\tilde{L}^\infty(B_{p,1}^{\frac{N}{p}-1})} + \underline{\nu} \|\tilde{u}^n\|_{\tilde{L}^1(B_{p,1}^{\frac{N}{p}+1})} \leq \tilde{U}_0 \eta, \end{aligned}$$

Remark that since:

$$1 + S_m a^n = 1 + a^n + (S_m a^n - a^n),$$

assumptions  $(\mathcal{H}_1)$  and  $(\mathcal{H}_3)$  combined with the embedding  $B_{p,1}^{\frac{N}{p}} \hookrightarrow L^\infty$  insure that:

$$\inf_{(t,x) \in [0,T] \times \mathbb{R}^N} (1 + S_m a^n)(t, x) \geq \frac{1}{4}\bar{b}. \quad (4.50)$$

We are going to prove that under suitable assumptions on  $T$  and  $\eta$  (to be specified below) if condition  $(\mathcal{H}_1)$  to  $(\mathcal{H}_6)$  are satisfied. Since all those conditions depend continuously on the time variable and are strictly satisfied initially, a basic boobstrap argument insures that  $(\mathcal{H}_1)$  to  $(\mathcal{H}_6)$  are indeed satisfied for  $T$ . First we shall assume that  $\eta$  satisfies:

$$C(1 + \underline{\nu}^{-1} \tilde{U}_0) \eta \leq \log 2 \quad (4.51)$$

so that denoting  $\tilde{U}^n(t) = \int_0^t \|\tilde{u}^n\|_{B_{p,1}^{\frac{N}{p}+1}} d\tau$  and  $U_L^n(t) = \int_0^t \|u_L^n\|_{B_{p,1}^{\frac{N}{p}+1}} d\tau$ , we have, according to  $(\mathcal{H}_5)$  and  $(\mathcal{H}_6)$ :

$$e^{C(U_L^n + \tilde{U}^n)(T)} < 2 \text{ and } e^{C(U_L^n + \tilde{U}^n)(T)} - 1 \leq \frac{C}{\log 2} (U_L^n + \tilde{U}^n)(T) \leq 1. \quad (4.52)$$

In order to bound  $a^n$  in  $\tilde{L}_T^\infty(B^{\frac{N}{2}})$ , we apply proposition (2.3) and get:

$$\|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} < 1 + 2\|a_0\|_{B^{\frac{N}{2}}} = A_0. \quad (4.53)$$

Hence  $(\mathcal{H}_4)$  is satisfied with a strict inequality. Next, applying proposition 2.4 yields:

$$\|u^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})} \leq U_0, \quad (4.54)$$

$$\kappa\nu\|u_L^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})} \leq \sum_{l \in \mathbb{Z}} 2^{l(\frac{N}{p}-1)} (1 - e^{-\kappa\nu 2^{2l}T}) (\|\Delta_l u_0\|_{L^p} + \|\Delta_l f\|_{L^1(\mathbb{R}^+, L^p)}). \quad (4.55)$$

Hence taking  $T$  such that:

$$\sum_{l \in \mathbb{Z}} 2^{l(\frac{N}{p}-1)} (1 - e^{-\kappa\nu 2^{2l}T}) (\|\Delta_l u_0\|_{L^p} + \|\Delta_l f\|_{L^1(\mathbb{R}^+, L^p)}) < \kappa\eta\nu, \quad (4.56)$$

insures that  $(\mathcal{H}_5)$  is strictly verified.

Since  $(\mathcal{H}_1)$ ,  $(\mathcal{H}_2)$  and (4.50) are satisfied, proposition 4.6 may be applied, we get:

$$\begin{aligned} & \|\tilde{u}^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})} + \nu\|\tilde{u}^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})} \\ & \leq C e^{C(U_L^n + \tilde{U}^n)(T)} \int_0^T (\|a^n \mathcal{A} u_L^n\|_{B_{p,1}^{\frac{N}{p}-1}} + \|u_L^n \cdot \nabla u_L^n\|_{B_{p,1}^{\frac{N}{p}-1}} + \|\nabla g(a^n)\|_{B_{p,1}^{\frac{N}{p}-1}} + Rohde) dt. \end{aligned}$$

By taking advantage of proposition 2.1 and 2.2, we end up with:

$$\begin{aligned} & \|\tilde{u}^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})} + \nu\|\tilde{u}^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})} \leq C e^{C(U_L^n + \tilde{U}^n)(T)} \\ & \times (C\|u_L^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})} (\bar{\nu}\|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})} + \|u_L^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}-1})}) + C_g T \|a^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}})}) dt. \end{aligned} \quad (4.57)$$

with  $C = C(N)$  and  $C_g = (N, g, \underline{b}, \bar{b})$ . Now, using assumptions  $(\mathcal{H}_4)$ ,  $(\mathcal{H}_5)$  and  $(\mathcal{H}_6)$ , and inserting (4.52) in (4.57) gives:

$$\|\tilde{u}^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1})} + \|\tilde{u}^n\|_{L_T^1(B_{p,1}^{\frac{N}{p}+1})} \leq 2C(\bar{\nu}A_0 + U_0)\eta + 2C_g T A_0,$$

hence  $(\mathcal{H}_6)$  is satisfied with a strict inequality provided:  $C_g T < C\bar{\nu}\eta$ .

We now have to check whether  $(\mathcal{H}_1)$  is satisfied with strict inequality. For that we apply (??) which yields for all  $m \in \mathbb{Z}$ ,

$$\sum_{l \geq m} 2^{l\frac{N}{2}} \|\Delta_l a^n\|_{L_T^\infty(L^2)} \leq \sum_{l \geq m} 2^{l\frac{N}{2}} \|\Delta_l a_0\|_{L^2} + (1 + \|a_0\|_{B_{2,1}^{\frac{N}{2}}}) (e^{C(U_L^n + \tilde{U}^n)(T)} - 1). \quad (4.58)$$

Using (??) and  $(\mathcal{H}_5)$ ,  $(\mathcal{H}_6)$ , we thus get:

$$\|a^n - S_m a^n\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \sum_{l \geq m} 2^{l\frac{N}{2}} \|\Delta_l a_0\|_{L^2} + \frac{C}{\log 2} (1 + \|a_0\|_{B_{2,1}^{\frac{N}{2}}}) (1 + \nu^{-1} \tilde{L}_0) \eta.$$

Hence  $(\mathcal{H}_1)$  is strictly satisfied provided that  $\eta$  further satisfies:

$$\frac{C}{\log 2}(1 + \|a_0\|_{B_{p,1}^{\frac{N}{2}}})(1 + \underline{\nu}^{-1}\tilde{L}_0)\eta < \frac{c\underline{\nu}}{2\bar{\nu}}. \quad (4.59)$$

Next according to  $(\mathcal{H}_4)$ , condition  $(\mathcal{H}_2)$  is satisfied provided:

$$T < \frac{2^{-2m}\underline{\nu}}{C\bar{\nu}^2 A_0^2}.$$

In order to check whether  $(\mathcal{H}_3)$  is satisfied, we use the fact that:

$$a^n - a_0 = S_m(a^n - a_0) + (Id - S_m)(a^n - a_0) + \sum_{l>n} \Delta_l a_0,$$

whence, using  $B_{p,1}^{\frac{N}{p}} \hookrightarrow L^\infty$  and assuming (with no loss of generality) that  $n \geq m$ ,

$$\begin{aligned} \|a^n - a_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} &\leq C(\|S_m(a^n - a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} + \|a^n - S_m a^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \\ &\quad + 2 \sum_{l \geq m} 2^{\frac{lN}{p}} \|\Delta_l a_0\|_{L^p}). \end{aligned}$$

Changing the constant  $c$  in the definition of  $m$  and in (4.59) if necessary, one can, in view of the previous computations, assume that:

$$C(\|a^n - S_m a^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} + 2 \sum_{l \geq m} 2^{\frac{lN}{p}} \|\Delta_l a_0\|_{L^p}) \leq \frac{b}{4}.$$

As for the term  $\|S_m(a^n - a_0)\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})}$ , it may be bounded according to inequality (??):

$$\|S_m(a^n - a_0)\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq (1 + \|a_0\|_{B_{2,1}^{\frac{N}{2}}})(e^{C(\tilde{U}^n + U_L^n)(T)} - 1) + C2^{2m}\sqrt{T}\|a_0\|_{B_{2,1}^{\frac{N}{2}}}\|u^n\|_{L_T^2(B_{2,1}^{\frac{N}{2}})}.$$

Note that under assumptions  $(\mathcal{H}_5)$ ,  $(\mathcal{H}_6)$ , (4.51) and (4.59) (and changing  $c$  if necessary), the first term in the right-hand side may be bounded by  $\frac{b}{8}$ .

Hence using interpolation, (4.54) and the assumptions (4.51) and (4.59), we end up with:

$$\|S_m(a^n - a_0)\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}})} \leq \frac{b}{8} + C2^m\sqrt{T}\|a_0\|_{B_{2,1}^{\frac{N}{2}}}\sqrt{\eta(U_0 + \tilde{U}_0\eta)(1 + \underline{\nu}^{-1}\tilde{U}_0)}.$$

Assuming in addition that  $T$  satisfies:

$$C2^m\sqrt{T}\|a_0\|_{B_{2,1}^{\frac{N}{2}}}\sqrt{\eta(U_0 + \tilde{U}_0\eta)(1 + \underline{\nu}^{-1}\tilde{U}_0)} < \frac{b}{8}, \quad (4.60)$$

and using the assumption  $\underline{b} \leq 1 + a_0 \leq \bar{b}$  yields  $(\mathcal{H}_3)$  with a strict inequality.

One can now conclude that if  $T < T^n$  has been chosen so that conditions (4.56) and (4.60) are satisfied (with  $\eta$  verifying (4.51) and (4.59), and  $m$  defined in (4.49) and  $n \geq m$  then  $(a^n, u^n)$  satisfies  $(\mathcal{H}_1)$  to  $(\mathcal{H}_6)$ , thus is bounded independently of  $n$  on  $[0, T]$ .

We still have to state that  $T^n$  may be bounded by below by the supremum  $\bar{T}$  of all times  $T$  such that (4.56) and (4.60) are satisfied. This is actually a consequence of the uniform bounds we have just obtained. Indeed, by combining all these informations, one can prove that if  $T^n < \bar{T}$  then  $(a^n, u^n)$  is actually in:

$$\tilde{L}_{T^n}^\infty(B_{2,1}^{\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}+1}) \times (\tilde{L}_{T^n}^\infty(B_{2,1}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}}) \cap L_{T^n}^1(B_{2,1}^{\frac{N}{2}+1} \cap B_{2,1}^{\frac{N}{2}+2}))^N$$

hence may be continued beyond  $\bar{T}$ .

## 2) Existence of a solution

The existence of a solution stems from compactness properties for the sequence  $(q^n, u^n)_{n \in \mathbb{N}}$  by using some results of type Ascoli as in the proof of theorem 1.1.

**Lemma 2** *The sequence  $(\partial_t \bar{q}^n, \partial_t \bar{u}^n)_{n \in \mathbb{N}}$  is uniformly bounded for some  $\alpha > 1$  in:*

$$L^2(0, T; \tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1}) \times (L^\alpha(0, T; \tilde{B}_{p,1}^{\frac{N}{p}-1, \frac{N}{p}-2}))^N.$$

**Proof:**

Throughout the proof, we will extensively use that  $\tilde{L}_T^\rho(B_{p,1}^s) \hookrightarrow L_T^\rho(B_{p,1}^s)$ . The notation u.b will stand for uniformly bounded. We have:

$$\begin{aligned} \partial_t q^n &= -u^n \cdot \nabla q^n - (1 + q^n) \operatorname{div} u^n, \\ \partial_t \bar{u}^n &= -u^n \cdot \nabla u^n - q^n \mathcal{A}(\rho^n, \bar{u}^n) - K(q^n) \nabla q^n + \frac{1}{n} \nabla \Delta \bar{q}^n. \end{aligned} \quad (4.61)$$

We start with show that  $\partial_t \bar{q}^n$  is u.b in  $L^2(0, T; \tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1})$ . Since  $u^n$  is u.b in  $L_T^2(B_{p,1}^{\frac{N}{p}})$  and  $\nabla q^n$  is u.b in  $L_T^\infty(B_{p,1}^{\frac{N}{p}-1})$ , then  $u^n \cdot \nabla q^n$  is u.b in  $L_T^2(\tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1})$ . Similar arguments enable us to conclude for the term  $(1 + q^n) \operatorname{div} u^n$  which is u.b in  $L_T^2(\tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1})$  because  $q^n$  is u.b in  $L_T^\infty(B_{p,1}^{\frac{N}{p}})$  and  $\operatorname{div} u^n$  is u.b in  $L_T^2(B_{p,1}^{\frac{N}{p}-1})$ .

Let us now study  $\partial_t \bar{u}^{n+1}$ . According to step one and to the definition of  $u_L^n$ , the term  $\mathcal{A} \bar{u}^{n+1}$  is u.b in  $L^2(B_{p,1}^{\frac{N}{p}-2})$ . Since  $u^n$  is u.b in  $L^\infty(B_{p,1}^{\frac{N}{p}-1})$  and  $\nabla u^n$  is u.b in  $L^2(B_{p,1}^{\frac{N}{p}-1})$ , so  $u^n \cdot \nabla u^n$  is u.b in  $L^2(B_{p,1}^{\frac{N}{p}-2})$  thus in  $L^2(\tilde{B}_{p,1}^{\frac{N}{p}-1, \frac{N}{p}-2})$ . Moreover we have  $q^n$  is u.b in  $L^\infty(B_{p,1}^{\frac{N}{p}})$  and  $q^n$  is u.b in  $L^\infty$ , so by proposition 2.2  $\nabla K_0(q^n)$  is u.b in  $L^\infty(B_{p,1}^{\frac{N}{p}-1})$  thus in  $L^2(\tilde{B}_{p,1}^{\frac{N}{p}-1, \frac{N}{p}-2})$ . This concludes the lemma.  $\square$

Now, let us turn to the proof of the existence of a solution for the system (SW). We want now use some results of type Ascoli to conclude and using the properties of compactness of the lemma 2. According lemma 2,  $(q^n, u^n)_{n \in \mathbb{N}}$  is u.b in:

$$C^{\frac{1}{2}}([0, T]; \tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1}) \times (C^{1-\frac{1}{\alpha}}([0, T]; \tilde{B}_{p,1}^{\frac{N}{p}-1, \frac{N}{p}-2}))^N,$$

thus is uniformly equicontinuous in  $C([0, T]; \tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1}) \times (\tilde{B}_{p,1}^{\frac{N}{p}-1, \frac{N}{p}-2})^N$ . On the other hand we have the following result of compactness, for any  $\phi \in C_0^\infty(\mathbb{R}^N)$ ,  $s \in \mathbb{R}$ ,  $\delta > 0$  the application  $u \rightarrow \phi u$  is compact from  $B_{p,1}^s$  to  $\tilde{B}_{p,1}^{s, s-\delta}$ . Applying Ascoli's theorem, we infer that up to an extraction  $(q^n, u^n)_{n \in \mathbb{N}}$  converges in  $\mathcal{D}'([0, T] \times \mathbb{R}^N)$  to a limit  $(\bar{q}, \bar{u})$  which belongs to:

$$C^{\frac{1}{2}}([0, T]; \tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1}) \times (C^{1-\frac{1}{\alpha}}([0, T]; \tilde{B}_{p,1}^{\frac{N}{p}-1, \frac{N}{p}-2}))^N.$$

Let  $(q, u) = (\bar{q}, \bar{u}) + (q_0, u_L)$ . Using again uniform estimates of step one and proceeding as, we gather that  $(q, u)$  solves  $(SW)$  and belongs to:

$$\bar{\rho} + \tilde{L}_T^\infty(\tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1}) \times (\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}+1}) \cap \tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1}))^N.$$

Applying proposition, we get the continuity results:  $\rho - \bar{\rho} \in C([0, T], \tilde{B}_{p,1}^{\frac{N}{p}, \frac{N}{p}-1})$  and  $u \in C([0, T], B_{p,1}^{\frac{N}{p}-1})$ .  $\square$

## 5 Proof of the uniqueness for theorem 1.1 and 1.2

In the following theorem, we show the uniqueness for theorem 1.2 and in the sequel we will explain how to adapt for the uniqueness in theorem 1.1.

**Theorem 5.5** *Let  $N \geq 2$ , and  $(q_1, u_1)$  and  $(q_2, u_2)$  be solutions of  $(SW)$  with the same data  $(q_0, u_0)$  on the time interval  $[0, T^*)$ . Assume that for  $i = 1, 2$ :*

$$q_i \in C([0, T^*), B_{N,1}^1) \quad \text{and} \quad u_i \in (C([0, T^*), B_{N,1}^0) \cap L_{loc}^1([0, T^*), B_{N,1}^2))^N.$$

*then  $(q_1, u_1) = (q_2, u_2)$  on  $[0, T^*)$ .*

Let  $(q_1, u_1), (q_2, u_2)$  belong to  $E^{\frac{N}{p}}$  with the same initial data (we can then easily check by embedding that  $(q_1, u_1), (q_2, u_2)$  verify the hypothesis of theorem 5.5). We set  $(\delta q, \delta u) = (q_2 - q_1, u_2 - u_1)$ . We can then write the system  $(SW)$  as follows:

$$\begin{cases} \frac{\partial}{\partial t} \delta q + u_2 \cdot \nabla \delta q = H_1, \\ \frac{\partial}{\partial t} \delta u - \bar{\nu} \Delta \delta u = H_2 \end{cases} \quad (5.62)$$

with:

$$\begin{aligned} H_1 &= -\operatorname{div} \delta u - \delta u \cdot \nabla q_1 - \delta q \operatorname{div} u_2 - q_1 \operatorname{div} u, \\ H_2 &= -\bar{\delta} \nabla \delta q - \bar{\kappa} \phi * \nabla \delta q - u_2 \cdot \nabla \delta u - \delta u \cdot \nabla u_1 + \mathcal{A}(q_1, \delta u) + \mathcal{A}(\delta q, u_2). \end{aligned}$$

Due to the term  $\delta u \cdot \nabla q^1$  in the right-hand side of the first equation, we loose one derivative when estimating  $\delta q$ : one only gets bounds in  $L^\infty(B_{N,1}^0)$ . Now, the right hand-side of the second equation contains a term of type  $\mathcal{A}(\delta q, u_2)$  so that the loss of one derivative for  $\delta q$  entails a loss of one derivative for  $\delta u$ . Therefore, getting bounds in:

$C(\mathbb{R}^+; B_{N,1}^{-1}) \cap L^1(\mathbb{R}^+; B_{N,1}^1)$  for  $\delta u$  is the best that one can hope.

Unfortunately in our case, the above heuristic fails because we have reached some limit cases for the product laws. Indeed the product  $\delta u \cdot \nabla u_1$  does not map  $B_{N,1}^0 \times B_{N,1}^0$  into  $B_{N,1}^{-1}$  but in the somewhat *larger space*  $B_{N,\infty}^{-1}$ . At this point, we could try instead to get bounds for  $\delta u$  in:  $C([0, T^*]; B_{N,\infty}^{-1}) \cap L_{loc}^1([0, T^*]; B_{N,\infty}^1)$ , but we then have to face the lack of control on  $\delta u$  in  $L^1(0, T; L^\infty)$  (because  $B_{N,\infty}^1$  is not imbedded in  $L^\infty$ ) so that we run into troubles when estimating  $\delta u \cdot \nabla q_1$ . The key to that difficulty relies on logarithmic interpolation inequality and the following Osgood lemma.

**Lemma 3** *Let  $F$  be a measurable positive function and  $\gamma$  a positive locally integrable function, each defined on the domain  $[t_0, t_1]$ . Let  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  be a continuous nondecreasing function, with  $\mu(0) = 0$ . Let  $a \geq 0$ , and assume that for all  $t \in [t_0, t_1]$ ,*

$$F(t) \leq a + \int_{t_0}^t \gamma(s) \mu(F(s)) ds.$$

*If  $a > 0$ , then:*

$$-\mathcal{M}(F(t)) + \mathcal{M}(a) \leq \int_{t_0}^t \gamma(s) ds, \quad \text{where } \mathcal{M}(x) = \int_x^1 \frac{ds}{\mu(s)}.$$

*If  $a = 0$  and  $\mathcal{M}(0) = +\infty$ , then  $F = 0$ .*

**Proof of the theorem 5.5:**

Fix an integer  $m$  such that:

$$1 + \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} S_m a^1 \geq \frac{b}{2} \quad \text{and} \quad \|1 - S_m a^1\|_{\tilde{L}^\infty(B_{p,1}^{\frac{N}{p}})} \leq c \frac{\underline{\nu}}{\bar{\nu}}, \quad (5.63)$$

and define  $T_1$  as the supremum of all positive time such that:

$$t \leq T \quad \text{and} \quad t \bar{\nu}^2 \|a^1\|_{\tilde{L}^\infty(B_{p,1}^{\frac{N}{p}})}^2 \leq c 2^{-2m} \underline{\nu}. \quad (5.64)$$

Remark that the proposition 2.3 ensures that  $a^1$  belongs to  $\tilde{C}_T(B_{2,1}^1)$  so that the above two assumptions are satisfied if  $m$  has been chosen large enough. For bounding  $\delta a$  in  $L_T^\infty(B_{N,\infty}^0)$ , we apply proposition 2.3 with  $r = +\infty$  and  $s = 0$ . We get  $\forall t \in [0, T]$ :

$$\|\delta a(t)\|_{B_{N,\infty}^0} \leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} \|\delta a \operatorname{div} u^2 + \delta u \cdot \nabla a^1 + (1 + a^1) \operatorname{div} \delta u\|_{B_{N,\infty}^0} d\tau,$$

hence using that the product of two functions maps  $B_{N,\infty}^0 \times B_{p,1}^{\frac{N}{p}}$  in  $B_{N,\infty}^0$ , and applying Gronwall lemma,

$$\|\delta a(t)\|_{B_{N,\infty}^0} \leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} (1 + \|a^1\|_{B_{N,1}^1}) \|\delta u\|_{B_{N,1}^1} d\tau. \quad (5.65)$$

Next, using proposition 4.7 combined with proposition 2.1 and 2.2 in order to bound the nonlinear terms, we get for all  $t \in [0, T_1]$ :

$$\|\delta u\|_{\tilde{L}_T^1(B_{N,\infty}^1)} \leq C e^{C(U^1+U^2)(t)} \int_0^t (1 + \|a^1\|_{B_{N,1}^1} + \|a^2\|_{B_{N,1}^1} + \|u^2\|_{B_{N,1}^1}) \|\delta a\|_{B_{N,\infty}^0} d\tau. \quad (5.66)$$

In order to control the term  $\|\delta u\|_{B_{N,1}^1}$  which appears in the right-hand side of (5.65), we make use of the following logarithmic interpolation inequality whose proof may be found in [?], page 120:

$$\|\delta u\|_{L_t^1(B_{N,\infty}^1)} \lesssim \|\delta u\|_{\tilde{L}_t^1(B_{N,\infty}^1)} \log \left( e + \frac{\|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^2)}}{\|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)}} \right). \quad (5.67)$$

Because  $u^1$  and  $u^2$  belong to  $\tilde{L}_T^\infty(B_{N,1}^0) \cap L_T^1(B_{N,1}^2)$ , the numerator in the right-hand side may be bounded by some constant  $C_T$  depending only on  $T$  and on the norms of  $u^1$  and  $u^2$ . Therefore inserting (5.65) in (5.66) and taking advantage of (5.67), we end up for all  $t \in [0, T_1]$  with:

$$\begin{aligned} \|\delta u\|_{\tilde{L}_T^1(B_{N,\infty}^1)} &\leq C(1 + \|a^1\|_{\tilde{L}_T^\infty(B_{N,1}^1)}) \\ &\int_0^t (1 + \|a^1\|_{B_{N,1}^1} + \|a^2\|_{B_{N,1}^1} + \|u^2\|_{B_{N,1}^1}) \|\delta u\|_{\tilde{L}_\tau^1(B_{N,\infty}^1)} \log(e + C_T \|\delta u\|_{\tilde{L}_\tau^1(B_{N,\infty}^1)}^{-1}) d\tau. \end{aligned}$$

Since the function  $t \rightarrow \|a^1(t)\|_{B_{N,1}^1} + \|a^2(t)\|_{B_{N,1}^1} + \|u^2(t)\|_{B_{N,1}^2}$  is integrable on  $[0, T]$ , and:

$$\int_0^1 \frac{dr}{r \log(e + C_T r^{-1})} = +\infty$$

Osgood lemma yields  $\|\delta u\|_{\tilde{L}_T^1(B_{N,\infty}^1)} = 0$ . Note that the definition of  $m$  depends only on  $T$  and that (5.63) is satisfied on  $[0, T]$ . Hence, the above arguments may be repeated on  $[T_1, 2T_1]$ ,  $[2T_1, 3T_1]$ , etc. until the whole interval  $[0, T]$  is exhausted. This yields uniqueness on  $[0, T]$ .

## 6 Continuation criterion

In this section, we prove theorem 1.3. So we assume that we are given a solution  $(a, u)$  to (SW) which belongs to  $E_{T'}$  for all  $T' < T$  and such that conditions 1, 2 and 3 of theorem 1.3 are satisfied. Fix an integer  $m$  such that conditions (4.34) and (??) are fulfilled. Remark that  $u$  satisfies:

$$\partial_t u + u \cdot \nabla u - (1 + a) \mathcal{A} u = f - \nabla g(a), \quad u|_{t=0} = u_0.$$

Hence, taking advantage of remark and of proposition 2.1, we get for some constant  $C$  depending only on  $N$ , and all  $t \in [0, T]$ ,

$$\begin{aligned} &\|u\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}-1})} + \kappa \underline{\nu} \|u\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}+1})} \\ &\quad C \int_0^t (\|\nabla u\|_{L^\infty} + 2^{2m} \underline{\nu}^{-1} \bar{\nu}^2 \|a\|_{B_{p,1}^{\frac{N}{p}}}^2) d\tau \\ &\leq e \left( \|u_0\|_{B_{p,1}^{\frac{N}{p}-1}} + \|f\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}-1})} + C \int_0^t \|a\|_{B_{p,1}^{\frac{N}{p}}}^2 d\tau \right). \end{aligned}$$

This yields a bound on  $\|u\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}-1})}$  and on  $\|u\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}+1})}$  depending only on the data and on  $m, \underline{\nu}, \bar{\nu}, \|a\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}})}$  and  $\|\nabla u\|_{L_T^1(L^\infty)}$ .

Of course due to  $\|a\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})}$  and proposition 2.3, we also have  $\|a\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}$ . By replacing  $\|\Delta_q a_0\|_{L^p}$  and  $\|\Delta_q u_0\|_{L^p}$  by  $\|\Delta_q a\|_{L_T^\infty(L^p)}$  and  $\|\Delta_q u\|_{L_T^\infty(L^p)}$  in the definition of  $m$  and in the lower bounds (4.56) and (4.60) that we have obtained for the existence time, we obtain an  $\varepsilon > 0$  such that  $(SW)$  with data  $a(T-\varepsilon)$ ,  $u(T-\varepsilon)$  and  $f \cdot + (T-\varepsilon)$  has a solution on  $[0, 2\varepsilon]$ . Since the solution  $(a, u)$  is unique on  $[0, T)$ , this provides a continuation of  $(a, u)$  beyond  $T$ .

## 7 Appendix

This section is devoted to the proof of commutator estimates which have been used in section 2 and 3. They are based on paradifferential calculus, a tool introduced by J.-M. Bony in [2]. The basic idea of paradifferential calculus is that any product of two distributions  $u$  and  $v$  can be formally decomposed into:

$$uv = T_u v + T_v u + R(u, v) = T_u v + T'_v u$$

where the paraproduct operator is defined by  $T_u v = \sum_q S_{q-1} u \Delta_q v$ , the remainder operator  $R$ , by  $R(u, v) = \sum_q \Delta_q u (\Delta_{q-1} v + \Delta_q v + \Delta_{q+1} v)$  and  $T'_v u = T_v u + R(u, v)$ . Inequalities (4.40) and (4.42) are consequence of the following lemma:

**Lemma 4** *Let  $\sigma \in (-\frac{N}{p}, \frac{N}{p} + 1]$ . There exists a sequence  $c_q \in l^1(\mathbb{Z})$  such that  $\|c_q\|_{l^1} = 1$  and a constant  $C$  depending only on  $N$  and  $\sigma$  such that:*

$$\forall q \in \mathbb{Z}, \quad \|[v \cdot \nabla, \Delta_q]a\|_{L^p} \leq C c_q 2^{-q\sigma} \|\nabla v\|_{B_{p,1}^{\frac{N}{p}}} \|a\|_{B_{p,1}^\sigma}. \quad (7.68)$$

In the limit case  $\sigma = -\frac{N}{p}$ , we have:

$$\forall q \in \mathbb{Z}, \quad \|[v \cdot \nabla, \Delta_q]a\|_{L^p} \leq C c_q 2^{q\frac{N}{p}} \|\nabla v\|_{B_{p,1}^{\frac{N}{p}}} \|a\|_{B_{p,\infty}^{-\frac{N}{p}}}. \quad (7.69)$$

Finally, for all  $\sigma > 0$ , there exists a constant  $C$  depending only on  $N$  and on  $\sigma$  and a sequence  $c_q \in l^1(\mathbb{Z})$  with norm 1 such that:

$$\forall q \in \mathbb{Z}, \quad \|[v \cdot \nabla, \Delta_q]v\|_{L^p} \leq C c_q 2^{-q\sigma} \|\nabla v\|_{B_{p,1}^{\frac{N}{p}}} \|v\|_{B_{p,1}^\sigma}. \quad (7.70)$$

**Proof:**

Inequality (7.68) has been proved in ([19]), lemma A1 under the hypothesis (which does not play any role if  $\sigma > \frac{N}{p}$ ) that  $\operatorname{div} v = 0$ . It is based on the decomposition:

$$[v \cdot \nabla, \Delta_q]a = [T_{v_j}, \Delta_q] \partial_j a + T'_{\Delta_q \partial_j a} v^j - \Delta_q T_{\partial_j a} v^j - \Delta_q R(\partial_j a, v^j). \quad (7.71)$$



In the case  $\sigma = -\frac{N}{p}$ , the computations made in [19] show that the  $L^p$  norm of the first three terms in (7.71) may be bounded by  $C2^{q\frac{N}{p}} \|\nabla v\|_{B_{p,1}^{\frac{N}{p}}} \|a\|_{B_{p,\infty}^{-\frac{N}{p}}}$ .

For bounding the last term, we use the following classical result of continuity for the remainder (see [40]):

$$\|R(f, g)\|_{B_{2,\infty}^{-\frac{N}{2}}} \lesssim \|f\|_{B_{2,\infty}^{-s}} \|g\|_{B_{2,1}^s} \quad (7.72)$$

which holds for all real number  $s$ . This yields (7.69). The proof of (7.70) relies on a similar arguments. The details are left to the reader.  $\square$

**Lemma 5** *Let  $\alpha \in (1 - \frac{N}{p}, 1]$ ,  $k \in \{1, \dots, N\}$  and  $R_q = \Delta_q(a\partial_k w) - \partial_k(a\Delta_q w)$ . There exists  $c = c(\alpha, N, \sigma)$  such that:*

$$\sum_q 2^{q\sigma} \|R_q\|_{L^p} \leq C \|a\|_{B_{p,1}^{\frac{N}{p}+\alpha}} \|w\|_{B_{p,1}^{\sigma+1-\alpha}} \quad (7.73)$$

whenever  $-\frac{N}{2} < \sigma \leq \alpha + \frac{N}{2}$ .

In the limit case  $\sigma = -\frac{N}{2}$ , we have for some constant  $C = C(\alpha, N)$ :

$$\sup_q 2^{-q\frac{N}{2}} \|R_q\|_{L^p} \leq C \|a\|_{B_{2,1}^{\frac{N}{2}+\alpha}} \|w\|_{B_{2,\infty}^{-\frac{N}{2}+1-\alpha}}. \quad (7.74)$$

## Proof

The proof is almost the same as the one of lemma A3 in [17]. It is based on Bony's decomposition which enables us to split  $R_q$  into:

$$R_q = \underbrace{\partial_k[\Delta_q, T_a]w}_{R_q^1} - \underbrace{\Delta_q T_{\partial_k a} w}_{R_q^2} + \underbrace{\Delta_q T_{\partial_k w} w}_{R_q^3} + \underbrace{\Delta_q R(\partial_k w, a)}_{R_q^4} - \underbrace{\partial_k T_{\Delta_q w} a}_{R_q^5}.$$

Using the fact that:  $R_q^1 = \sum_{q'=q-4}^{q+4} \partial_k[\Delta_q, S_{q'-1} a] \Delta_{q'} w$ , and the mean value theorem, we readily get under the hypothesis that  $\alpha \leq 1$ ,

$$\sum_q 2^{q\sigma} \|R_q^1\|_{L^p} \lesssim \|\nabla a\|_{B_{\infty,1}^{\alpha-1}} \|w\|_{B_{p,1}^{\sigma+1-\alpha}}. \quad (7.75)$$

Standard continuity results for the paraproduct insure that  $R_q^2$  satisfies (7.75) and that:

$$\sum_q 2^{q\sigma} \|R_q^1\|_{L^p} \lesssim \|\nabla w\|_{B_{\infty,1}^{\sigma-\alpha-\frac{N}{p}}} \|a\|_{B_{p,1}^{\frac{N}{p}+\alpha}}. \quad (7.76)$$

provided  $\sigma - \alpha - \frac{N}{p} \leq 0$ . Next, standard continuity result for the remainder insure that under the hypothesis  $\sigma > -\frac{N}{p}$ , we have:

$$\sum_q 2^{q\sigma} \|R_q^1\|_{L^p} \lesssim \|\nabla w\|_{B_{p,1}^{\sigma-\alpha}} \|a\|_{B_{p,1}^{\frac{N}{p}+\alpha}}. \quad (7.77)$$

For bounding  $R_q^5$  we use the decomposition:  $R_q^5 = \sum_{q' \geq q-3} \partial_k (S_{q'+2} \Delta_q w \Delta_{q'} a)$ , which leads (after a suitable use of Bernstein and Hölder inequalities) to:

$$2^{q\sigma} \|R_q^5\|_{L^p} \lesssim \sum_{q' \geq q-2} 2^{(q-q')(\alpha + \frac{N}{p} - 1)} 2^{q(\sigma+1-\alpha)} \|\Delta_q w\|_{L^p} 2^{q'(\frac{N}{p} + \alpha)} \|\Delta_{q'} a\|_{L^p}.$$

Hence, since  $\alpha + \frac{N}{p} - 1 > 0$ , we have:

$$\sum_q 2^{q\sigma} \|R_q^5\|_{L^p} \lesssim \|\nabla w\|_{B_{p,1}^{\sigma+1-\alpha}} \|a\|_{B_{p,1}^{\frac{N}{p} + \alpha}}.$$

Combining this latter inequality with (7.75), (7.76) and (7.77), and using the embedding  $B_{p,1}^{\frac{N}{2}} \hookrightarrow B_{\infty,1}^{r-\frac{N}{p}}$  for  $r = \frac{N}{p} + \alpha - 1$ ,  $\sigma_\alpha$  completes the proof of (7.73).

The proof of (7.74) is almost the same: for bounding  $R_q^1$ ,  $R_q^2$ ,  $R_q^3$  and  $R_q^5$ , it is just a matter of changing  $\sum_q$  into  $\sup_q$ . As for  $R_q^4$  we use (7.72). □

**Remark 5** For proving proposition 4.7, we shall actually use the following non-stationary version of inequality (7.74):

$$\sup_q 2^{-q\frac{N}{p}} \|R_q\|_{L_T^1(L^p)} \leq C \|a\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p} + \alpha})} \|w\|_{\tilde{L}_T^1(B_{p,\infty}^{-\frac{N}{p} + 1 - \alpha})},$$

which may be easily proved by following the computations of the previous proof, dealing with the time dependence according to Hölder inequality.

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