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POROUS MEDIA EQUATIONS, FAST DIFFUSIONS EQUATIONS AND THE EXISTENCE OF GLOBAL WEAK SOLUTION FOR THE QUASI-SOLUTIONS OF COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In [3, 4, 5], we have developed a new tool called t*quasi solutions* which approximate in some sense the compressible Navier-Stokes equation. In particular it allows to obtain global strong solution for the compressible Navier-Stokes equations with *large* initial data on the irrotational part of the velocity (*large* in the sense that the smallness assumption is subcritical in terms of scaling, it turns out that in this framework we are able to obtain large initial data in the energy space in dimension $N = 2$). In this paper we are interesting in studying in details this notion of *quasi solution* and in particular proving global weak solution, we also observe that for some choice of initial data (irrotationnal) we obtain some quasi solutions verifying the porous medium equation, the heat equation or the fast diffusion equation in function of the structure of the viscosity coefficients. Finally we show the convergence of the global weak solution of compressible Navier-Stokes equations to the quasi solutions when the pressure vanishing.

1. Introduction.

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) - \operatorname{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = \rho f, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1}$$

Here $u = u(t, x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density and $D(u) = \frac{1}{2}(\nabla u + ^t\nabla u)$. The pressure P is such that $P(\rho) = a\rho^\gamma$ with $\gamma \geq 1$. We denote by $\mu(\rho) > 0$ and $2\mu(\rho) + N\lambda(\rho) > 0$ the viscosity coefficients of the fluid. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$.
In this paper we are interested in studying the notion of quasi-solution developed in [4, 5, 3] for general viscosity coefficients following the algebraic equality discovered by Bresch and Desjardin in [1]:

$$\lambda(\rho) = 2\rho\mu^{'}(\rho) - 2\mu(\rho). \tag{2}$$

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We shall define in the sequel the function $\varphi(\rho)$ by $\varphi^{'}(\rho)=\frac{2\mu^{'}(\rho)}{\rho}$. With this choice of viscosity coefficients Bresch and Desjardin have obtained a new entropy giving a L^2 control on the gradient of the density. It has permit to Mellet and Vasseur in [6] to prove the stability of the global weak solution for compressible Navier Stokes equations with such viscosity coefficients and with γ law pressure $P(\rho)=a\rho^{\gamma}$ with $a>0$ and $\gamma>1$. In the sequel we will work only with such viscosity coefficients verifying the relation (2).
This paper is devoted to prove the existence of quasi solutions for compressible Navier-Stokes equations with degenerate viscosity coefficients. Let recall the definition of quasi solutions introduced in [3, 4, 5].

Definition 1.1. We say that (ρ,u) is a quasi solution if (ρ,u) verifies in distribution sense:

$$\begin{cases}\dfrac{\partial}{\partial t}\rho+\mathrm{div}(\rho u)=0,\\ \dfrac{\partial}{\partial t}(\rho u)+\mathrm{div}(\rho u\otimes u)-\mathrm{div}(2\mu(\rho)\,\mathrm{Du})-\nabla(\lambda(\rho)\mathrm{divu})=0,\\ (\rho,u)_{\;t=0}=(\rho_0,u_0)\end{cases}\tag{3}$$

More precisely (ρ,u) is a weak solution of (3) on $[0,T]\times\mathbb{R}^N$with:

$$\begin{aligned}&\rho_0L^1(\mathbb{R}^N),\ \sqrt{\rho_0}\nabla\varphi(\rho_0)\in L^2(\mathbb{R}^N),\ \rho_0\geq 0,\\ &\sqrt{\rho_0}|u_0|\sqrt{\ln(1+|u_0|^2)}\in L^2(\mathbb{R}^N).\end{aligned}\tag{4}$$

if

- $\rho\in L^\infty_T(L^1(\mathbb{R}^N)),\ \sqrt{\rho}\nabla\varphi(\rho)\in L^\infty_T(L^2(\mathbb{R}^N)),\ \sqrt{\rho}u\in L^\infty_T(L^2(\mathbb{R}^N)),$
- $\sqrt{\mu(\rho)}\,\nabla u\in L^2((0,T)\times\mathbb{R}^N),\ \sqrt{\rho}|u|\sqrt{\ln(1+|u|^2)}\in L^\infty_T(L^2(\mathbb{R}^N)).$

with $\rho\geq 0$ and $(\rho,\sqrt{\rho}u)$ satisfying in distribution sense on $[0,T]\times\mathbb{R}^N$:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\sqrt{\rho}\sqrt{\rho}u)=0,\\ \rho(0,x)=\rho_0(x).\end{cases}$$

and if the following equality holds for all $\varphi(t,x)$ smooth test function with compact support such that $\varphi(T,\cdot)=0$:

$$\begin{aligned}\int_{\mathbb{R}^N}(\rho u)_0\cdot\varphi(0,\cdot)dx+\int_0^T\int_{\mathbb{R}^N}\sqrt{\rho}(\sqrt{\rho}u)\partial_t\varphi+\sqrt{\rho}u\otimes\sqrt{\rho}u:\nabla\varphi dx\\ -<2\mu(\rho)\,Du,\nabla\varphi>-<\lambda(\rho)\,\mathrm{div}u,\mathrm{div}\varphi>=0,\end{aligned}\tag{5}$$

where we give sense to the diffusion terms by rewriting him according to $\sqrt{\rho}$ and $\sqrt{\rho}u$:

$$\begin{aligned}<2\mu(\rho)\,Du,\nabla\varphi>=-\int\frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_j)\partial_{ii}\varphi_j dx\,dt-\int 2(\sqrt{\rho}u_j)\mu^{'}(\rho)\partial_i\sqrt{\rho}\partial_i\varphi_j dx\,dt\\ -\int\frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_j)\partial_{ji}\varphi_j dx\,dt-\int 2(\sqrt{\rho}u_i)\mu^{'}(\rho)\partial_j\sqrt{\rho}\partial_i\varphi_j dx\,dt\end{aligned}$$

$$<\lambda(\rho)\,\mathrm{div}u,\mathrm{div}\varphi>=-\int\frac{\lambda(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_i)\partial_{ji}\varphi_j dx\,dt-\int 2(\sqrt{\rho}u_i)\lambda^{'}(\rho)\partial_i\sqrt{\rho}\partial_j\varphi_j dx\,dt$$

We assume also the same extra assumption on the viscosity coefficient 8-12 than in [6].

Remark 1. Here λ and μ verifies the condition (2), in particular we have classical energy estimates by multiplying the momentum equation by u except that we have no information on the density of the type $\rho^\gamma \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^N))$ as for compressible Navier-Stokes equation when ($P(\rho) = \rho^\gamma$ with $\gamma \geq 1$) because here $P(\rho) = 0$. However using the entropy discovered in [1] we can prove that $\sqrt{\rho}\nabla\varphi(\rho)$ belongs in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^N))$ with $\varphi^{'}(\rho) = \frac{2\mu^{'}(\rho)}{\rho}$ and that ρ belongs in $L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^N))$ by conservation of the mass. It will be sufficient to prove the stability of global weak solution and it explains the assumption of the definition 1.1.

Remark 2. Let us remark that by using this notion of quasi solution in [3] we obtain global strong solution with initial data small in subcritical space for the scaling of the equations. In this sense quasi solutions are good approximate in order to obtain global strong solution with large initial data in terms of scaling (in particular in dimension $N = 2$ we can choose large initial data in energy space).

We now are going to investigate the existence of such quasi solution for the viscosity coefficients verifying (2).More precisely as in [3] we are going to search in a first time irrotational solution under the form $u(t, x) = \nabla c(t, x)$. Let us assume now that:

$$\mu(\rho) = \mu\rho^\alpha \;\text{ with } \alpha > 0 \;\text{ and }\; \lambda(\rho) = 2(\alpha - 1)\mu\rho^\alpha, \tag{6}$$

with $\alpha > 1 - \frac{1}{N}$ in order to insure the relation $2\mu(\rho) + N\lambda(\rho) > 0$. Furthermore we observe that $\mu(\rho)$ and $\lambda(\rho)$ verify the relation (2).
Let us now briefly recall the so-called porous medium and fast diffusion equations before explaining the link between the quasi-solutions and these solutions. More precisely the solutions of the following the nonlinear Cauchy problem:

$$\begin{cases} \partial_t\rho - \mu\Delta\rho^\alpha = 0, \\ \rho(0, \cdot) = \rho_0. \end{cases} \tag{7}$$

where α is a positive number which we assume different from one are solutions of the porous media equation or fast diffusion equations, here we assume that $\rho_0 \in L^1(\mathbb{R}^N)$ is nonnegative.The case $\alpha > 1$ (the porous media equations) arises as a model of slow diffusion of a gas inside a porous container. Unlike the heat equation $\alpha = 1$, this equation exhibits finite speed of propagation in the sense that solutions associated to compactly supported initial data remain compactly supported in space variable at all times (see [7]). When $0 < \alpha < 1$, the opposite happens. Infinite speed of propagation occurs and solutions may even vanish in finite time. This problem is usually referred to as the fast diffusion equation.
Let us recall the notion of global strong solution for the equation (7) of the porous medium equation ($\alpha > 1$) and of the fast diffusion equation ($0 < \alpha < 1$) (see [7] chapter 9 for more details and [8]).

Definition 1.2. We say that a function $\rho \in C([0, +\infty), L^1(\mathbb{R}^N))$ positive is a strong L^1 solution of problem (7) if:

- $\rho^\alpha \in L^1_{loc}(0, +\infty, L^1(\mathbb{R}^N))$ and $\rho_t, \Delta\rho^\alpha \in L^1_{loc}((0, +\infty) \times \mathbb{R}^N)$
- $\rho_t = \mu\Delta\rho^\alpha$ in distribution sense.
- $u(t) \to \rho_0$ as $t \to 0$ in $L^1(\mathbb{R}^N)$.

Let us mention (see [7]) that we have the following theorem:

Theorem 1.3. *Let $\alpha > 0$. For every $\rho_0 \in L^1(\mathbb{R}^N)$ positive there exists a unique global strong solution ρ positive of problem (7) such that $\rho \in C([0, +\infty), L^1(\mathbb{R}^N)) \cap L^\infty((\tau, +\infty) \times \mathbb{R}^N)$ for every $\tau > 0$.*

Let us recall that there exists global weak solution which are not classical it means not C^∞ even if the initial data is C^∞ (see a example due to Aronson in the problem 5.7 of [7]).

Remark 3. Let us mention that if the initial data is non-negative then the unique global weak solution is classical which means C^∞ on $(0,+\infty)\times\mathbb{R}^N$ (see the proposition 7.21 p 177 in [7]).

Let us now give your first result describing the link between quasi-solutions and the solutions of (7).

Theorem 1.4. *Let $\alpha > 1-\frac{1}{N}$. Assume that (ρ_0,u_0) verifies the assumptions of the definition 1.1, $\rho_0>0$ and $u_0=-\frac{2\mu\alpha}{\alpha-1}\nabla\rho_0^{\alpha-1}$ with $\alpha\neq 1$. It exists then a global weak solution solution of the system (3) of the form $(\rho,u=-\frac{2\mu\alpha}{\alpha-1}\nabla\rho^{\alpha-1})$ when $\alpha\neq1$ with (ρ,u) belonging in C^∞ on $(0,+\infty)\times\mathbb{R}^N$ and solving the following system almost everywhere :*

$$\begin{cases} \partial_t\rho-2\mu\Delta\rho^\alpha=0,\\ \rho(0,\cdot)=\rho_0. \end{cases} \tag{8}$$

When $\alpha=1$ with $(\rho_0,u_0=-2\mu\nabla\ln\rho_0)$, similarly we have particular global weak unique solution solution of the system (3) of the form $(\rho,u=-2\mu\nabla\ln\rho)$ solving the heat equation:

$$\begin{cases} \partial_t\rho-2\mu\Delta\rho=0,\\ \rho(0,\cdot)=\rho_0. \end{cases} \tag{9}$$

Remark 4. Let us point out that any solution of (7) such that ρ is in $C^3((0,+\infty)\times\mathbb{R}^N)$ is a classical solution of (3).In the case where $\rho=0$ the velocity is not defined when $0<\alpha<1$ that is why we assume that $u=0$ on the vacuum set. In other case we could give sense to ρu as in [6].

Remark 5. We can observe as in [2] that if we consider the compressible Navier-Stokes equation with a friction term $a\rho u$ and a pressure of the form $2\mu a\rho^\alpha$ then the same solution than theorem 1.4 verify a such system.

Remark 6. We recognize here the so called equation of the porous medium when $\alpha>1$ and of the fast diffusion when $0<\alpha<1$. We refer for more details on the theory to the books of J-L Vázquez (see [7, 8]).

Proof of Theorem 1.4. Let us assume in a first time that the solution (ρ,u) of system (3) are classical, we are going to search solution under the form: $(\rho,-\frac{2\mu\alpha}{\alpha-1}\nabla\rho^{\alpha-1})$. The mass equation give us:

$$\partial_t\rho-2\mu\Delta\rho^\alpha=0 \tag{10}$$

Let us check that the second equation is compatible with the first and keep an irrotational structure. First we have:

$$\begin{aligned} &\partial_t(\rho u)=-\frac{2\mu\alpha}{\alpha-1}\partial_t(\rho\nabla\rho^{\alpha-1})=-2\partial_t\nabla\rho^\alpha.\\ &\text{div}(\rho u\otimes u)=\frac{4\mu^2\alpha^2}{(\alpha-\frac12)^2}(\Delta\rho^{\alpha-\frac12}\nabla\rho^{\alpha-\frac12}+\frac12\nabla|\nabla\rho^{\alpha-\frac12}|^2). \end{aligned} \tag{11}$$

Indeed we have:

$$\mathrm{div}(\rho u\otimes u)_j=\frac{4\mu^2\alpha^2}{(\alpha-1)^2}\sum_i\partial_i(\rho\partial_i\rho^{\alpha-1}\partial_j\rho^{\alpha-1})=4\alpha^2\mu^2\sum_i\partial_i(\rho^{2\alpha-3}\partial_i\rho\partial_j\rho)$$

$$=\frac{4\mu^2\alpha^2}{(\alpha-\frac{1}{2})^2}\sum_i\partial_i(\partial_i\rho^{\alpha-\frac{1}{2}}\partial_j\rho^{\alpha-\frac{1}{2}})=\frac{4\mu^2\alpha^2}{(\alpha-\frac{1}{2})^2}(\Delta\rho^{\alpha-\frac{1}{2}}\partial_j\rho^{\alpha-\frac{1}{2}}+\frac{1}{2}\partial_j|\nabla\rho^{\alpha-\frac{1}{2}}|^2)$$

Next we have:

$$-\mathrm{div}(2\mu\rho^\alpha Du)=\frac{4\alpha\mu^2}{\alpha-1}\mathrm{div}(\rho^\alpha\nabla\nabla\rho^{\alpha-1})=\frac{4\alpha\mu^2}{\alpha-1}(\rho^\alpha\nabla\Delta\rho^{\alpha-1}+\nabla\rho^\alpha\cdot\nabla\nabla\rho^{\alpha-1}).$$

$$-\nabla(\lambda(\rho)\mathrm{div}u)=2(\alpha-1)\mu^2\nabla(\rho^\alpha\mathrm{div}(\frac{2\alpha}{\alpha-1}\nabla\rho^{\alpha-1}))=4\alpha\mu^2\nabla(\rho^\alpha\Delta\rho^{\alpha-1}).\tag{12}$$

Finally we have from (11):

$$\mathrm{div}(\rho u\otimes u)=\frac{4\alpha\mu^2}{(\alpha-1)}\Delta\rho^{\alpha-1}\nabla\rho^\alpha+\frac{2\alpha^2\mu^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}+\frac{2\alpha^2\mu^2}{(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2.\tag{13}$$

by using the fact that:

$$\Delta\rho^{\alpha-\frac{1}{2}}=\sum_i\partial_i(\frac{\alpha-\frac{1}{2}}{\alpha-1}\partial_i\rho^{\alpha-1}\rho^{\frac{1}{2}})=\frac{\alpha-\frac{1}{2}}{\alpha-1}\rho^{\frac{1}{2}}\Delta\rho^{\alpha-1}+\frac{1}{2}(\alpha-\frac{1}{2})\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2.$$

$$\begin{aligned}\frac{4\alpha^2}{(\alpha-\frac{1}{2})^2}\Delta\rho^{\alpha-\frac{1}{2}}\nabla\rho^{\alpha-\frac{1}{2}}&=\frac{4\alpha^2}{(\alpha-\frac{1}{2})(\alpha-1)}\rho^{\frac{1}{2}}\Delta\rho^{\alpha-1}\nabla\rho^{\alpha-\frac{1}{2}}+\frac{2\alpha^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}},\\&=\frac{4\alpha}{(\alpha-1)}\Delta\rho^{\alpha-1}\nabla\rho^\alpha+\frac{2\alpha^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}\end{aligned}$$

Finally by combining (12) and (13) we obtain:

$$\begin{aligned}\mathrm{div}(\rho u\otimes u)-\mathrm{div}(2\rho^\alpha Du)&=\frac{4\alpha\mu^2}{(\alpha-1)}\nabla(\rho^\alpha\Delta\rho^{\alpha-1})+\frac{4\alpha\mu^2}{\alpha-1}\nabla\rho^\alpha\cdot\nabla\nabla\rho^{\alpha-1}\\&+\frac{2\mu^2\alpha^2}{(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}+\frac{2\mu^2\alpha^2}{(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2\end{aligned}\tag{14}$$

Now since we have:

$$\nabla\rho^\alpha\cdot\nabla\nabla\rho^{\alpha-1}=\frac{\alpha(\alpha-1)}{2(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2-\frac{\alpha(\alpha-1)}{2(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}$$

indeed it is due to the following calculus:

$$\begin{aligned}(\nabla\rho^\alpha\cdot\nabla\nabla\rho^{\alpha-1})_j&=\sum_i\partial_i\rho^\alpha\partial_{ij}\rho^{\alpha-1}\\&=\sum_i\frac{\alpha}{\alpha-\frac{1}{2}}\rho^{\frac{1}{2}}\partial_i\rho^{\alpha-\frac{1}{2}}\partial_i(\frac{\alpha-1}{\alpha-\frac{1}{2}}\rho^{-\frac{1}{2}}\partial_j\rho^{\alpha-\frac{1}{2}})\\&=\frac{\alpha(\alpha-1)}{(\alpha-\frac{1}{2})^2}\sum_i(\partial_i\rho^{\alpha-\frac{1}{2}}\partial_{ij}\rho^{\alpha-\frac{1}{2}}-\frac{1}{2}\rho^{-1}\partial_i\rho^{\alpha-\frac{1}{2}}\partial_j\rho^{\alpha-\frac{1}{2}}\partial_i\rho)\end{aligned}$$

we finally reduce (14) to the following equation:

$$\nabla\rho^\alpha\cdot\nabla\nabla\rho^{\alpha-1}=\frac{\alpha(\alpha-1)}{2(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2-\frac{\alpha(\alpha-1)}{2(\alpha-\frac{1}{2})}\rho^{\alpha-\frac{5}{2}}|\nabla\rho|^2\nabla\rho^{\alpha-\frac{1}{2}}$$

We finally have:

$$\operatorname{div}(\rho u\otimes u)-\operatorname{div}(2\rho^{\alpha}Du)=\frac{4\mu^2\alpha}{(\alpha-1)}\nabla(\rho^{\alpha}\Delta\rho^{\alpha-1})+\frac{4\mu^2\alpha^2}{(\alpha-\frac{1}{2})^2}\nabla|\nabla\rho^{\alpha-\frac{1}{2}}|^2 \tag{15}$$

Finally using (11), (12) and (15) we obtain:

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho u)+\operatorname{div}(\rho u\otimes u)-\operatorname{div}(2\rho^{\alpha}\,\mathrm{Du})-\nabla(\lambda(\rho)\mathrm{divu})=\\&=-2\mu\nabla(\partial_t\rho^{\alpha}-\frac{2\mu\alpha}{(\alpha-1)}\rho^{\alpha}\Delta\rho^{\alpha-1}-\frac{2\mu\alpha^2}{(\alpha-\frac{1}{2})^2}|\nabla\rho^{\alpha-\frac{1}{2}}|^2-2\mu\alpha\rho^{\alpha}\Delta\rho^{\alpha-1}),\\&=-2\alpha\mu\nabla\big(\rho^{\alpha-1}(\partial_t\rho-\frac{2\mu}{(\alpha-1)}\rho\Delta\rho^{\alpha-1}-2\mu\alpha\rho^{\alpha-2}|\nabla\rho|^2-2\mu\rho\Delta\rho^{\alpha-1})\big)\\&=-2\alpha\mu\nabla\big(\rho^{\alpha-1}(\partial_t\rho-2\mu\Delta\rho^{\alpha})\big).\end{aligned}$$

This concludes the proof inasmuch as via the above equation the momentum equation is compatible to the mass equation and verify the system (7). But when we take initial density in L^1 non negative, we know via the remark 3 that the unique global solution of (7) is classical and non negative. It justify in particular all the previous formal calculus and prove that $(\rho,u=-\frac{2\mu\alpha}{\alpha-1}\nabla\rho^{\alpha-1})$ is a classical solution of (3) with ρ verifying (7). It concludes the proof. □

Remark 7. More generally we have solution of the form $(\rho,-\nabla\varphi(\rho))$ with ρ verifying the more general porous media equation (we refer to [7] for such equations):

$$\partial_t\rho-\operatorname{div}(\rho\nabla\varphi(\rho))=0.$$

Remark 8. Let us mention that when α is in the interval $(0,m_c)$ with $m_c=\max(0,\frac{N-2}{N})$ then it can appears a phenomena of extinction of the solution in finite time, in particular it implies a lost of the initial mass when ρ_0 is in L^1 (it is not the case in our framework because $\alpha 1-\frac{1}{N}>m_c$). Let us recall that when $\alpha>1$ and ρ_0 belongs in L^1, it exists global unique weak solution and that the solution converges asymptotically to the so called Barrenblatt solution

$$U_m(t,x)=t^{-\gamma}F(\frac{x}{t^{\beta}})\ \text{ with }\ F(x)=(C-\frac{\alpha-1}{2\alpha}|x|^2)_+^{\frac{1}{\alpha-1}},$$

which are self similar.

Finally we obtain the following theorems.

Theorem 1.5. *Assume that we have a sequence (ρ_n,u_n) of global weak solutions of system (3) satisfying the entropies of [6]) with initial data ρ_0^n and u_0^n such that:*

$$\rho_0^n\geq 0,\ \ \rho_0^n\to\rho_0\ \text{ in } L^1(\mathbb{R}^N),\ \ \rho_0^n u_0^n\to\rho_0u_0\ \text{ in } L^1(\mathbb{R}^N), \tag{16}$$

and satisfy the following bounds (with C constant independent on n):

$$\int_{\mathbb{R}^N}\rho_0^n\frac{|u_0^n|^2}{2}<C,\ \int_{\mathbb{R}^N}\sqrt{\rho_0^n}|\nabla\varphi(\rho_0^n)|^2dx<C\ \text{ and }\int_{\mathbb{R}^N}\rho_0^n\frac{1+|u_0^n|^2}{2}\ln(1+|u_0^n|^2)dx<C. \tag{17}$$

Then, up to a subsequence, $(\rho_n,\sqrt{\rho_n}u_n)$ converges strongly to a weak solution $(\rho,\sqrt{\rho}u)$ of (3) satisfying entropy inequalities of [6].
Furthermore when we choose $(\rho_0,u_0)=(\rho_0,-\frac{2\mu\alpha}{\alpha-1}\nabla\rho_0^{\alpha-1})$ with $\alpha\neq 1$ it exists global weak quasi solutions in the sense of the definition 1.1.

Theorem 1.6. *Assume that there exists global weak solution* $(\rho_\varepsilon, u_\varepsilon)$ *verifying the definition of* [6] *with the conditions* $(8)-(12)$ *on* μ *and* λ *of* [6] *and with the same restriction on* γ *as in* [6] *(see also theorem 2.1 in* [6]*) of the system:*

$$\begin{cases} \partial_t \rho_\varepsilon + \mathrm{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \mathrm{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \mathrm{div}(2\mu(\rho_\varepsilon) D(u_\varepsilon)) - \nabla(\lambda(\rho_\varepsilon)\mathrm{div} u_\varepsilon) + \varepsilon \nabla \rho_\varepsilon^\gamma = 0, \\ (\rho_\varepsilon, u_\varepsilon)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{18}$$

then $(\rho_\varepsilon, u_\varepsilon)$ *converges in distribution sense to a quasi-solution* (ρ, u) *when* ε *goes to* 0 *with initial data* (ρ_0, u_0)*. (Here* (ρ_0, u_0) *verifies the entropies of* [6]*).*

Proof of Theorem 1.5 and 1.6. Concerning the stability of global weak solution, assume the existence of a sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ of global weak solution in the sense of [6] then it suffices to observe that we have as in [1, 6] the following uniform bounds in n where we have multiplied the momentum equation by u_n and by $\nabla\varphi(\rho_n)$:

$$\begin{aligned} \int_{\mathbb{R}^N} \rho_n |u_n(t,x)|^2(t,x)\, dx + \int_0^t \int_{\mathbb{R}^N} \mu(\rho_n)|Du_n|^2 dxdt + \int_0^t \int_{\mathbb{R}^N} \lambda(\rho_n)|\mathrm{div} u_n|^2 dxdt \\ \leq \int_{\mathbb{R}^N} \rho_0^n |u_0^n|^2(x) dx. \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^N} \left[\rho_n |u_n(t,x)|^2 + \rho_n |\nabla\varphi(\rho_n)|^2(t,x)\right] dx \leq C(\int_{\mathbb{R}^N} \big(\rho_0^n |u_0^n|^2(x) \\ + \rho_0^n |\nabla\varphi(\rho_0^n)|^2(x)\big)\, dx). \end{aligned} \tag{19}$$

Finally as in [6] it remains to obtain a gain of integrability on the velocity u_n, it is obvious by multiplying the momentum equation by $(1+\ln(1+|u_n|^2))u_n$ and by bootstrap argument assuming the condition (11) as in [6]. In particular we show that $\rho_n \frac{1+|u_n|^2}{2} \ln(1+|u_n|^2)$ is uniformly bounded in $L^\infty((0,T), L^1(\mathbb{R}^N))$ for any $T>0$. And by the conservation of the mass ρ_n is uniform bounded in $L^\infty_T(L^1(\mathbb{R}^N))$ for any $T>0$. By using these different entropies as in [6] we can easily show the convergence in distribution sense of (ρ_n, u_n) to (ρ, u) via compactness argument.
We now want to prove the global existence of weak solution when $(\rho_0, u_0) = (\rho_0, -\frac{2\mu\alpha}{\alpha-1}\nabla\rho_0^{\alpha-1})$ with $\alpha \neq 1$. To do this it only suffices to construct a sequel of regular global weak solution (ρ_n, u_n) verifying uniformly in n the entropies of theorem 1.5 and to use the previous result. Indeed $\rho_n, \sqrt{\rho_n} u_n)$ must converge in distribution sense to a global weak solution $(\rho, \sqrt{\rho} u)$. Let us define (ρ_n, u_n) as the solutions of the theorem 1.4 with initial data $\rho_0^n = \rho_0 + \frac{1}{n} f$ where f is in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$ (for $\alpha \geq 1$), and is non-negative and we have $u_0^n = -\frac{2\mu\alpha}{\alpha-1}\nabla(\rho_0^n)^{\alpha-1}$. Then by theorem 1.4 it exists global regular weak solution (ρ_n, u_n) and $(\rho_0^n, \rho_0^n u_0^n)$ converges to (ρ_0, u_0) which concludes the proof of theorem 1.5.

We are now going to prove that if we have some global weak solution $(\rho_\varepsilon, u_\varepsilon)$ for the system (18) in the sense of the definition in [6], then these global weak solution converge in distribution sense to a quasi-solution with initial data (ρ_0, u_0). It suffices then to obtain the same uniform entropies in ε than in [6] for the sequel

$(\rho_\varepsilon, u_\varepsilon)$. Similarly we have:

$$\int_{\mathbb{R}^N} \big[\rho_\varepsilon |u_\varepsilon(t,x)|^2(t,x) + \frac{\varepsilon}{\gamma-1}\rho_\varepsilon^\gamma\big]\, dx + \int_0^t \int_{\mathbb{R}^N} \mu(\rho_\varepsilon)|Du_\varepsilon|^2 dxdt$$
$$+ \int_0^t \int_{\mathbb{R}^N} \lambda(\rho_\varepsilon)|\mathrm{div} u_\varepsilon|^2 dxdt \leq \int_{\mathbb{R}^N} \big[\rho_0|u_0|^2(x) + \frac{\varepsilon}{\gamma-1}\rho_0^\gamma\big] dx.$$

$$\int_{\mathbb{R}^N} \big[\rho_\varepsilon |u_\varepsilon(t,x)|^2 + \rho_\varepsilon |\nabla\varphi(\rho_\varepsilon)|^2(t,x)\big]\, dx + \varepsilon \int_0^t \int_{\mathbb{R}^N} \nabla\varphi(\rho_\varepsilon)\cdot\nabla\rho_\varepsilon^\gamma dxdt$$
$$\leq C(\int_{\mathbb{R}^N} \big(\rho_0|u_0|^2(x) + \rho_0|\nabla\varphi(\rho_0)|^2(x) + \frac{\varepsilon}{\gamma-1}\rho_0^\gamma(x)\big)\, dx). \tag{20}$$

By using the lemma 3.2 of [6] with $\forall\delta\in(0,2)$, we have that:

$$\int_{\mathbb{R}^N} \rho_\varepsilon \frac{1+|u_\varepsilon|^2}{2}\ln(1+|u_\varepsilon|^2)(t,x)dx + \nu\int_0^t \int_{\mathbb{R}^N} \mu(\rho_\varepsilon)(1+\ln(1+|u_\varepsilon|^2))|Du_\varepsilon|^2(t,x)dxdt$$
$$\leq C\int_0^t \int_{\mathbb{R}^N} \mu(\rho_\varepsilon)|\nabla u_\varepsilon|^2(t,x)dxdt + C_\delta\varepsilon\int_0^t \big(\int_{\mathbb{R}^N} \frac{\rho_\varepsilon^{2\gamma-\frac{\delta}{2}}}{\mu(\rho_\varepsilon)}\big)^{\frac{2}{2-\delta}} dx\big)dt. \tag{21}$$

We can easily observe that via the energy estimates the right hand side of (21) is uniformly bounded in ε. The last step corresponds to use the same compactness argument than in [6] to show that $(\rho_\varepsilon, u_\varepsilon)$ converges in distribution sense to a quasi-solution (ρ, u) when ε goes to 0 with initial data (ρ_0, u_0). Let us point out that $\varepsilon\rho_\varepsilon^\gamma$ goes to 0 in distribution sense. Indeed it suffice to observe that ρ_ε and $\sqrt{\rho_\varepsilon}\nabla\varphi(\rho_\varepsilon)$ are uniformly bounded in ε respectively in $L^\infty((0,T), L^1(\mathbb{R}^N)$ and $L^\infty((0,T), L^2(\mathbb{R}^N)$ for any $T>0$. In particular by Sobolev embedding and interpolation (for γ not so large as in [6]) we obtain that ρ_ε^γ with $\alpha>0$ is uniformly bounded in L^1_{loc} what means that $\varepsilon\rho_\varepsilon^\gamma$ goes to 0. □

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