Porous media, fast diffusion equations and the existence of global weak solution for the quasi-solution of compressible Navier-Stokes equations

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Abstract

We consider the compressible Navier Stokes equations for viscous and barotropic fluids with density dependent viscosity. The aim is to investigate mathematical properties of solutions of the Navier Stokes equations using solutions of the pressureless Navier Stokes equations, that we call *quasi solutions*. This regime corresponds to the limit of highly compressible flows. In this paper we are interested in proving the announced result in [23] concerning the existence of global weak solution for the quasi-solutions, we also observe that for some choice of initial data (irrotationnal) the quasi solutions verify the porous media, the heat equation or the fast diffusion equations in function of the structure of the viscosity coefficients. In particular it implies that it exists classical quasi-solutions in the sense that they are C^∞ on $(0,T)\times\mathbb{R}^N$ for any $T>0$. Finally we show the convergence of the global weak solution of compressible Navier-Stokes equations to the quasi solutions in the case of a vanishing pressure limit process. In particular for highly compressible equations the speed of propagation of the density is quasi finite when the viscosity corresponds to $\mu(\rho)=\rho^\alpha$ with $\alpha>1$. Furthermore the density is not far from converging asymptotically in time to the Barrenblatt solution of mass the initial density ρ_0.

1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho u)=0,\\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mathrm{div}(2\mu(\rho)D(u))-\nabla(\lambda(\rho)\mathrm{div}u)+\nabla P(\rho)=\rho f,\\ (\rho,u)_{/t=0}=(\rho_0,u_0).\end{cases}\tag{1.1}$$

Here $u=u(t,x)\in\mathbb{R}^N$ stands for the velocity field, $\rho=\rho(t,x)\in\mathbb{R}^+$ is the density and $D(u)=\frac{1}{2}(\nabla u+^t\nabla u)$ the strain tensor. The pressure P is such that $P(\rho)=a\rho^\gamma$ with $\gamma>1$ and $a>0$. We denote by $\mu(\rho)$ and $\lambda(\rho)$ the two-Lamé viscosity coefficients depending on the density and satisfying:

$$\mu(\rho)>0\quad 2\mu(\rho)+N\lambda(\rho)\geq 0.\tag{1.2}$$

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Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$. In this article, we are going to investigate the existence of global weak quasi solutions for the system (1.1), a notion which has been introduced in [21, 22, 17, 18] in order to prove the existence of global strong solution with large initial data on the rotational and irrotational part of the velocity for the scaling of the equation (we refer to the remark 4 for the definition of this notion of scaling when we assume that the initial density is far away from the vacuum).

Before entering in the heart of the topic and defining the notion of quasi-solutions, we would like to explain more in details the notion of invariance by scaling for compressible Navier-Stokes equations when we suppose that the density ρ_0 is in a Besov space. Up our knowledge these results are quite new and shall allow to understand the relation between the quasi solutions and the so called Barrenblatt solution for the porous media equation in terms of invariance by scaling.

Scaling of the equations

A natural way to understand the equations of fluid mechanics is to search for self similar solutions, it means that there is a scaling of the variables after which the system become stationary solutions. Precisely it holds when we set:

$$\begin{aligned} &\rho(t,x) = t^{-\alpha} F(xt^{-\beta}),\\ &u(t,x) = t^{-\alpha_1} G(xt^{-\beta}) \end{aligned} \tag{1.3}$$

The exponents α, α_1 and β are called *similarity exponents*, and functions F and G are the *self similar profiles*. In particular α and α_1 are the density contraction rate and β the space expansion rate. In the sequel we shall assume that $\mu(\rho) = \mu\rho^\theta$ and $\lambda(\rho) = \lambda\rho^\theta$ with $\theta \geq 0$. Simple calculus give when we set $\eta = xt^{-\beta}$:

$$\begin{aligned} &\partial_t \rho(t,x) = -t^{-\alpha-1}(\alpha F(\eta) + \beta \nabla F(\eta)\cdot\eta),\\ &\text{div}(\rho u) = t^{-\alpha-\alpha_1-\beta}\text{div}(GF)(\eta). \end{aligned}$$

Next we have:

$$\begin{aligned} &\rho\partial_t u = -t^{-\alpha-\alpha_1-1}(\alpha_1 F(\eta)G(\eta) + \beta\eta\cdot\nabla G(\eta)F(\eta)),\\ &\rho u\cdot\nabla u = t^{-\alpha-2\alpha_1-\beta}F(\eta)G(\eta)\cdot\nabla G(\eta),\\ &2\mu\text{div}(\rho^\theta Du) = 2\mu t^{-\theta\alpha-\alpha_1-2\beta}\text{div}(F^\theta(\eta)DG(\eta)),\\ &\nabla(\lambda(\rho)\text{div}u) = t^{-\theta\alpha-\alpha_1-2\beta}\nabla(F^\theta(\eta)\text{div}G(\eta)). \end{aligned}$$

and finally:

$$\nabla\rho^\gamma = \gamma t^{-\alpha\gamma-\beta}F^{\gamma-1}(\eta)\nabla F(\eta) = t^{-\alpha\gamma-\beta}\nabla(aF(\eta)^\gamma).$$

In order to ensure the existence of solution for the system(1.1) under the specific form introduced in (1.3) we need to assume that:

$$\begin{cases} \alpha+\alpha_1+\beta = \alpha+1,\\ \alpha+\alpha_1+1 = \alpha+2\alpha_1+\beta,\\ \alpha+\alpha_1+1 = \theta\alpha+\alpha_1+2\beta,\\ \alpha+\alpha_1+1 = \alpha\gamma+\beta, \end{cases}$$

which is equivalent to the following system:

$$\begin{cases} \alpha_1 + \beta = 1, \\ (\theta - 1)\alpha + 2\beta = 1, \\ \alpha(\gamma - 1) + \beta = \alpha_1 + 1. \end{cases}$$

The solution of the previous system is:

$$\alpha = \frac{-1}{\theta - \gamma}, \ \alpha_1 = \frac{1-\gamma}{2(\theta-\gamma)} \ \text{ et } \ \beta = \frac{2\theta - \gamma - 1}{2(\theta - \gamma)}. \tag{1.4}$$

With this choice on the parameter α, α_1 and β we finaly get the profile equation:

$$\begin{cases} \alpha F(\eta) + \beta \nabla F(\eta) \cdot \eta - \text{div}(GF)(\eta) = 0, \\ \alpha_1 F(\eta) G(\eta) + \beta \eta \cdot \nabla G(\eta) F(\eta) - F(\eta) G(\eta) \cdot \nabla G(\eta) + 2\mu \text{div}(F^\theta(\eta) DG(\eta)) \\ \qquad\qquad\qquad\qquad + \nabla(F^\theta(\eta) \text{div} G(\eta)) - \nabla(a F(\eta)^\gamma) = 0. \end{cases}$$

Remark 1 *A first remark consists in observing that there is no scaling invariance when $\theta = \gamma$.*

An other way to express this scaling invariance corresponds to consider a classical solution (ρ, u) of the system (1.1) and to check that the family:

$$(\rho_l, u_l)(t, x) = (l^\alpha \rho(lt, l^\beta x), l^{\alpha_1} \rho(lt, l^\beta x),$$

is a solution of (1.1) for any $l \in \mathbb{R}$ when α, α_1 and β verify (1.4).

Remark 2 *We shall say that a functional space E embedded in $\mathcal{S}'(\mathbb{R}^N) \times (\mathcal{S}'(\mathbb{R}^N)))^N$ is a critical space for (1.1) if the associated norm is invariant under the transformation $(\rho_0, u_0) \to ((\rho_0)_l, (u_0)_l)$ for any $l \in \mathbb{R}$. In particular we observe that:*

$$B_{2,\infty}^{\frac{N}{2} + \frac{2}{2\theta - \gamma - 1}} \times B_{2,\infty}^{\frac{N}{2} - 1 + 2\frac{\theta - 1}{2\theta - \gamma - 1}},$$

verifies a such property (we refer to [2] for the definition of Besov space) .

Remark 3 *Let us point out that when $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$ with $\gamma > 1$ (the case of the shallow-water equation) then $B_{2,\infty}^{\frac{N}{2} - \frac{2}{\gamma - 1}} \times B_{2,\infty}^{\frac{N}{2} - 1}$ is a critical space invariant by the scaling of the equation. In particular let us mention that it is the same scaling invariance for the initial velocity u_0 than for the incompressible Navier-Stokes equations.*
In the case of constant viscosity coefficients we observe that $B_{2,\infty}^{\frac{N}{2} - \frac{2}{\gamma+1}} \times B_{2,\infty}^{\frac{N}{2} - 1 + \frac{2}{\gamma + 1}}$ is a critical space for the scaling of the system (1.1).

Remark 4 *To finish we would like to mention as it has been observed in [9] that there is no invariance by scaling when we wish to work with density far from the vacuum, typically $\rho_0 = 1 + q_0$ with q_0 in a Besov space. However in [9] Danchin makes abstraction of the pressure term and define a other notion of critical space for the compressible Navier-Stokes equation when the initial density is far away from the vacuum. More precisely we can remark that (1.1) is invariant by the transformation:*

$$(\rho_0(x), u_0(x)) \to (\rho_0(lx), l u_0(lx)) \quad (\rho(t, x), u(t, x)) \to (\rho(l^2 t, lx), l u(l^2 t, lx))$$

up to a change of the pressure law P into l^2P. In particular $B^{\frac{N}{2}}_{2,1} \times B^{N-1}_{2,1}$ is norm invariant by the previous transformation and is critical for the initial data (q_0, u_0). This notion of critical space seems well adapted to the case of initial density far away from the vacuum and is also relevant at least for the initial velocity since $B^{\frac{N}{2}-1}_{2,1}$ is also critical for the incompressible Navier-Stokes equations. Let us recall that with this notion of critical space we have proved the existence of global strong solution with large initial data on the irrotational and rotational part of the velocity (see [17, 18, 20]) by involving the notion of quasi-solutions verifying regularizing effects.

Before coming back on this notion of scaling in order to make some link between self similar solutions for the porous media equation and the behavior of the quasi-solutions, we would like now to give precise assumptions on the viscosity coefficients with which we are going to work.

Condition on $\mu(\rho)$ and $\lambda(\rho)$

In this paper we are interested in extending the notion of quasi-solution developed in [21, 22, 17, 18] (where only shallow water coefficients were considered) for general viscosity coefficients following the algebraic equality discovered by Bresch and Desjardins in [6, 7]:

$$\lambda(\rho) = 2\rho\mu'(\rho) - 2\mu(\rho). \tag{1.5}$$

In the sequel we shall deal with the function $\varphi(\rho)$ and $f(\rho)$ defined by:

$$\varphi'(\rho) = \frac{2\mu'(\rho)}{\rho} \quad \text{and} \quad f'(\rho) = \sqrt{\rho}\varphi'(\rho).$$

Let us mention that the equality (1.5) implies that the viscosity coefficients are degenerated inasmuch as it imposes that $\mu(0) = \lambda(0) = 0$. With this choice of viscosity coefficients Bresch and Desjardins have obtained a remarkable new entropy for compressible Navier-Stokes equations (1.1) providing a $L^\infty_T(L^2(\mathbb{R}^N))$ control for any $T > 0$ on the gradient of the density (more precisely on $\sqrt{\rho}\nabla\varphi(\rho)$). In particular it allows them to prove the existence of global weak solution for a specific choice of pressure, more precisely what they describe as a cold pressure. Compared with the case of viscosity coefficient the pressure term is quite simple to deal with by using Sobolev embedding since we have uniform estimate on $\sqrt{\rho}\nabla\varphi(\rho)$ in $L^\infty_T(L^2(\mathbb{R}^N)$; however a new difficulty appears coming from the degenerescence of the viscosity coefficient. Indeed we lose the control of $\nabla u \in L^2((0,T)\times\mathbb{R}^N)$ what makes delicate the compactness study of the term $\rho u \otimes u$ (in particular via the classical energy estimates we have only a convergence in the sense of the measure) due to the existence of vacuum. In order to overcome this difficulty Mellet and Vasseur in [29] obtained new entropy on the velocity which provides a gain of integrability on the velocity. With this new ingredient they are able in [29] to prove the stability of the global weak solution for compressible Navier Stokes equations with such viscosity coefficients and with classical γ law pressure $P(\rho) = a\rho^\gamma$ with $a > 0$ and $\gamma > 1$. We are going to detail here the assumptions on the viscosity coefficients which allow to Mellet and Vasseur to obtain additional informations on the integrability of the velocity and we are going even to relax their hypothesis (it will be crucial in order to consider in the sequel quasi solution verifying porous media equations). We shall suppose the following inequalities on μ and λ, let $\nu_1 \in (0,1)$ and $\nu_2 > 0$ such that:

$$\begin{aligned} &|\lambda^{'}(\rho)| \leq \frac{1}{\nu_1}\mu^{'}(\rho), \\ &\nu_1\mu(\rho) \leq 2\mu(\rho) + N\lambda(\rho) \leq \nu_2\mu(\rho). \end{aligned} \tag{1.6}$$

Remark 5 *If we assume that $\mu(\rho) = \mu\rho^{\alpha}$ with $\alpha > 0$ then the relation (1.5) gives:*

$$\lambda(\rho) = 2(\alpha - 1)\mu\rho^{\alpha}, \tag{1.7}$$

and:

$$2\mu(\rho) + N\lambda(\rho) = 2(1 + N(\alpha - 1))\mu(\rho). \tag{1.8}$$

In this situation we have $\nu_1 = \nu_2 = 2(1 + N(\alpha - 1))$.

Following [29] let us briefly make some comments on the conditions (1.6).

Remark 6 *The condition (1.6) is crucial in order to obtain the estimates (3.39) and (3.38). In particular we can observe that the second condition in (1.6) is similar to the classical assumption on the Lamé coefficient $2\mu(\rho) + N\lambda(\rho) \geq 0$ when $\mu(\rho) = \mu\rho^{\alpha}$ and $\lambda(\rho)$ verifies (1.5).*

Remark 7 *The lower estimate in the second inequality in (1.6) is trivial when $\lambda(\rho) \geq 0$, while the upper estimate is trivial when $\lambda(\rho) \leq 0$. Together this provides:*

$$|\lambda(\rho)| \leq C\mu(\rho) \;\; \forall \rho > 0.$$

This inequality and the first inequality of (1.6) will be crucial for estimating the limit of $\nabla(\lambda(\rho_n)\text{div}u_n)$.

Remark 8 *Condition (1.6) and (1.5) implies that:*

$$\frac{N - 1 + \frac{\nu_1}{2}}{N\rho} \leq \frac{\mu^{'}(\rho)}{\mu(\rho)} \leq \frac{N - 1 + \frac{\nu_2}{2}}{N\rho} \;\; \forall \rho > 0.$$

It yields:

$$\begin{cases} C\rho^{1-\frac{1}{N}+\frac{\nu_1}{2N}} \leq \mu(\rho) \leq C\rho^{1-\frac{1}{N}+\frac{\nu_2}{2N}} \;\; \forall \rho > 1, \\ C\rho^{1-\frac{1}{N}+\frac{\nu_2}{2N}} \leq \mu(\rho) \leq C\rho^{1-\frac{1}{N}+\frac{\nu_1}{2N}} \;\; \forall \rho \leq 1, \end{cases} \tag{1.9}$$

We can now recall briefly the definition of the quasi solutions introduced in [17, 18, 21, 22] which roughly speaking are solutions of the compressible Navier-Stokes equations (1.1) where we have cancelled out the pressure term (in the sequel we shall give a more accurate definition).

Definition 1.1 *We say that (ρ, u) is a quasi solution if (ρ, u) verifies in distribution sense:*

$$\begin{cases} \frac{\partial}{\partial t}\rho + \text{div}(\rho u) = 0, \\ \frac{\partial}{\partial t}(\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho)\,\text{Du}) - \nabla(\lambda(\rho)\text{divu}) = 0, \\ (\rho, u)_{\ t=0} = (\rho_0, u_0) \end{cases} \tag{1.10}$$

As we explained previously this notion of quasi solution is interesting inasmuch as it allows to exhibit *large* initial velocity on the irrotational part in the sense of the scaling of the remark 4 (in particular we assume no vacuum on the initial density) providing global strong solution for compressible Navier-Stokes equation (see [17, 18, 20] for more details). This result is based on a strange phenomena on the quasi solution since these last one verify regularizing effects allowing to neglect in terms of scaling the pressure term in high frequencies. An other way to express the things is that the quasi solutions preserves a structure of irrotationality of the system when we choose irrotational initial data (it will be the case in the sequel), we shall say that the quasi solution, typically $u = -\nabla\varphi(\rho)$ is purely compressible.For other results on the existence of strong solution with critical initial data for variable viscosity coefficients we refer to [8, 15, 16].

We now are going to investigate the existence of such quasi solution for the viscosity coefficients verifying (1.5) when the initial data is assumed to be close from the vacuum, typically $\rho_0 \in L^1(\mathbb{R}^N)$. More precisely as in [17, 18, 20] we are going to search in a first time irrotational solution under the form $u(t,x) = \nabla c(t,x)$ for the system (1.10). Let us assume now to simplify that:

$$\mu(\rho) = \mu\rho^\alpha \;\text{ with } \alpha > 0 \;\text{ and }\; \lambda(\rho) = 2(\alpha-1)\mu\rho^\alpha, \tag{1.11}$$

with $\alpha \geq 1 - \frac{1}{N}$ in order to ensure the relation $2\mu(\rho) + N\lambda(\rho) > 0$. We observe here that $\mu(\rho)$ and $\lambda(\rho)$ verify the relation (1.5). In this case we will verify (see the theorem 1.1) that at least for suitable initial data on ρ_0 then it exists a explicit solution to the problem (1.10) written under the form $(\rho, -\nabla\varphi(\rho))$ with ρ verifying the porous media or the fast diffusion equation when $\alpha \neq 1$:

$$\begin{cases} \partial_t\rho - 2\mu\Delta\rho^\alpha = 0, \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{1.12}$$

In a very surprising way it means that the quasi solutions are directly related to the porous or the fast diffusion equations. Moreover we will show that when we work with highly compressible Navier-Stokes equations (which correspond to the case where a goes to 0 with $P(\rho) = a\rho^\gamma$), then the properties of porous media or fast diffusion equations are more or less preserved for the solution of (1.1) when a is small. Before giving more details on your results, let us give few words on the porous and fast diffusion equations for the reader which are not so familiar with these equations.

Porous media and fast diffusion equations

Let us consider the equation (1.12) when $2\mu = 1$ to simplify the notations; the case $\alpha > 1$ (the porous media equations) arises as a model of slow diffusion of a gas inside a porous container. Unlike the heat equation $\alpha = 1$, this equation exhibits finite speed of propagation in the sense that solutions associated to compactly supported initial data remain compactly supported in space variable at all times (see [33] and [1]). When $0 < \alpha < 1$, the opposite happens. Infinite speed of propagation occurs and solutions may even vanish in finite time. This problem is usually referred to as the fast diffusion equation.

Let us recall that there exists a theory of global unique solution for initial data $\rho_0 \in L^1(\mathbb{R}^N)$ (see the section 2 for some reminders). M. Pierre in [31] has extend this last one in obtaining the existence of unique global weak solution with bounded Radon measure as initial data. Let us mention also that the porous media equations are invariant by scaling, more precisely we can introduce a notion of self similarity (for more informations we refer to the chapter 16 of [33]). The notion of scaling consists in searching some solutions under the following form $\rho(t,x) = t^{-\gamma}F(\frac{x}{t^\beta})$, with γ and β to be determined. In our case γ and β have the form: $\gamma(\alpha-1)+2\beta = 1$, and F verifies the following equation:

$$\Delta F^\alpha + \beta\eta\dot{\nabla}F + \gamma F = 0.$$

In this case we said that ρ is a self similar solution of type I or a forward self similar solution. In particular it exists self similar solution such that the initial data is a Dirac mass (as in the theorem of M. Pierre in [31]) and such that for $t > 0$ this solution conserves a constant mass. This is the so-called Barrenblatt solutions (here $\alpha > 1$) that we can write under the following form:

$$U_m(t,x) = t^{-\gamma_1}F(\frac{x}{t^\beta}) \text{ with } F(x) = (C - \frac{(\alpha-1)\gamma_1}{2\alpha}|x|^2)_+^{\frac{1}{\alpha-1}}$$

with $C > 0$ and $\gamma_1 = \frac{N}{N(\alpha-1)+2}$, $\beta = \frac{1}{N(\alpha-1)+2}$. Here we have the conservation of the mass $\int U_m(t,x)dx = m$ with m depending on C and the initial data corresponds to the Dirac mass $m\delta_0$. Similarly when $m_c < \alpha < 1$ with $m_c = \max(0, \frac{N-2}{N})$ it exists also Barrenblatt solutions defined as follows:

$$U_m(t,x) = t^{-\gamma_1}F(xt^{-\beta}) \text{ with } F(x) = (C + \kappa_1|x|^2)_+^{\frac{-1}{\alpha-1}},$$

with $\kappa_1 = \frac{(1-\alpha)\gamma_1}{2N\alpha}$. We recall that asymptotically in time all the global weak solution with L^1 initial data converges to a Barrenblatt solution determined by his mass $\|u_0\|_{L^1}$ (we refer to Friedman and Kamin [14], Vázquez and Kamin [24, 25] and Dolbeault and Del Pino [12]). As we mentioned previously in the case of fast diffusion equation $0 < \alpha < 1$, infinite propagation occurs and solution may even vanish in finite time when α is in the interval $(0, m_c)$ with $m_c = \max(0, \frac{N-2}{N})$. In particular it implies a lost of the initial mass when ρ_0 is in L^1 (it implies also a lost of the regularity of the solution). We refer to [34] theorem 5.7 for a necessary condition of extinction, in particular the initial data belongs in an appropriate Marcinkewitz space $M_{p^*}(\mathbb{R}^N)$. Let us finished this subsection by mentioning that we shall recall more results on the porous and the fast diffusion equations in the section 2.

1.1 Main results

Let us now give your first result describing the link between quasi-solutions and the solutions of (1.12). Finally we obtain the following theorems.

Theorem 1.1 *Let $\mu(\rho) = \mu\rho^\alpha$ with $\alpha \geq 1 - \frac{1}{N}$ and $\lambda(\rho)$ verifying the relation (1.5). Let $\rho_0 \in L^1(\mathbb{R}^N)$ with $\rho_0 > 0$ and continuous and $u_0 = -\nabla\varphi(\rho_0)$. Then it exists a global weak solution of the system (1.10) of the form $(\rho, u = -\nabla\varphi(\rho))$ with (ρ, u) belonging in*

$C^\infty((0,+\infty)\times\mathbb{R}^N)\cap C([0,+\infty]\times\mathbb{R}^N)$ *and solving the following system almost everywhere :*

$$\begin{cases} \partial_t\rho - 2\Delta\mu(\rho) = 0, \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{1.13}$$

Furthermore we have:

$$\lim_{t\to+\infty} \|\rho(t) - U_m(t)\|_{L^1(\mathbb{R}^N)} = 0. \tag{1.14}$$

Convergence holds also in L^∞ norm:

$$\lim_{t\to+\infty} t^\beta \|\rho(t) - U_m(t)\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{1.15}$$

with $\beta = \frac{N}{N(\alpha-1)+2}$ and U_m the Barrenblatt of mass $m = \|\rho_0\|_{L^1(\mathbb{R}^N)}$. For every $p \in (1,+\infty)$ we have the following regularizing effect, $\rho(t,\cdot)$ belongs in $L^p(\mathbb{R}^N)$ and:

$$\|\rho(t)\|_{L^p(\mathbb{R}^N)} \leq Ct^{-\sigma_p}\|\rho_0\|^{\alpha_p}_{L^1(\mathbb{R}^N)},$$

with $\sigma_p = \frac{N(\alpha-1)+2p}{(N(\alpha-1)+2)p}$ and $\alpha_p = \frac{N(p-1)}{(N(\alpha-1)+2)p}$.

Remark 9 *Let us point out that we could have also global strong solution for more general viscosity coefficients with μ verifying the same conditions than the subsection (2.1.1). We refer to the section 2 for more details in this situation.*

Remark 10 *We shall remark in the proof of this theorem that any solution of* (1.12) *such that ρ is in $C^3((0,+\infty)\times\mathbb{R}^N)$ is a solution of (1.10) almost everywhere.*

Remark 11 *We can observe as in [19] that if we consider the compressible Navier-Stokes equation with a friction term $a\rho u$ and a pressure term of the form $2\mu a\rho^\alpha$ then the solution of the previous theorem 1.1 verify also a such system.*

Remark 12 *Let us mention that when α is in the interval $(0, m_c)$ with $m_c = \max(0, \frac{N-2}{N})$ (then this contradicts the Lamé condition on the viscosity coefficients) then it can appears a phenomena of extinction of the solution in finite time, in particular it implies a loss of the initial mass when ρ_0 is in L^1. A typical example is the solution:*

$$\rho(t,x) = c_\alpha\Big(\frac{T-t}{|x|^2}\Big)^{\frac{1}{1-\alpha}} \;\; \text{and} \;\; u(t,x) = -\frac{2\mu c_\alpha\alpha}{\alpha-1}\nabla\Big(\frac{|x|^2}{T-t}\Big),$$

with $c_\alpha^{1-\alpha} = 2(N - \frac{2}{1-\alpha})$. In particular we observe a blow-up behavior of u at time T.

We are going to give a general definition of global weak solution for the quasi solutions in the spirit of [29] including the case where the initial velocity is not necessary irrotational.

Definition 1.2 *We say that (ρ,u) is a global weak quasi solution if (ρ,u) verifies in distribution sense:*

$$\begin{cases} \frac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \frac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u\otimes u) - \mathrm{div}(2\mu(\rho)\,\mathrm{D}u) - \nabla(\lambda(\rho)\mathrm{div}u) = 0, \\ (\rho,u)_{\ t=0} = (\rho_0,u_0). \end{cases} \tag{1.16}$$

More precisely (ρ,u) is a weak solution of (1.10) on $[0,T]\times\mathbb{R}^N$

$$\rho_{/t=0}=\rho_0\geq 0,\ \ \rho u_{/t=0}=m_0. \tag{1.17}$$

with:

$$\begin{aligned}&\rho_0\in L^1(\mathbb{R}^N),\ \sqrt{\rho_0}\nabla\varphi(\rho_0)\in L^2(\mathbb{R}^N),\ \rho_0\geq 0,\\ &\sqrt{\rho_0}|u_0|(1+\sqrt{\ln(1+|u_0|^2)})\in L^2(\mathbb{R}^N).\end{aligned} \tag{1.18}$$

if

- $\rho\in L^\infty_T(L^1(\mathbb{R}^N),\ \sqrt{\rho}\nabla\varphi(\rho)\in L^\infty_T(L^2(\mathbb{R}^N)),\ \sqrt{\rho}u\in L^\infty_T(L^2(\mathbb{R}^N)),$
- $\sqrt{\mu(\rho)}\,\nabla u\in L^2((0,T)\times\mathbb{R}^N),\ \sqrt{\rho}|u|\sqrt{\ln(1+|u|^2)}\in L^\infty_T(L^2(\mathbb{R}^N)).$

with $\rho\geq 0$ and $(\rho,\sqrt{\rho}u)$ satisfying in distribution sense on $[0,T]\times\mathbb{R}^N$:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\sqrt{\rho}\sqrt{\rho}u)=0,\\ \rho(0,x)=\rho_0(x).\end{cases}$$

and if the following equality holds for all $\varphi(t,x)$ smooth test function with compact support such that $\varphi(T,\cdot)=0$:

$$\begin{aligned}\int_{\mathbb{R}^N}(\rho u)_0\cdot\varphi(0,\cdot)dx+\int_0^T\int_{\mathbb{R}^N}\sqrt{\rho}(\sqrt{\rho}u)\partial_t\varphi+\sqrt{\rho}u\otimes\sqrt{\rho}u:\nabla\varphi dx\\ -<2\mu(\rho)\,Du,\nabla\varphi>-<\lambda(\rho)\,\mathrm{div}u,\mathrm{div}\varphi>=0,\end{aligned} \tag{1.19}$$

where we give sense to the diffusion terms by rewriting him according to $\sqrt{\rho}$ and $\sqrt{\rho}u$:

$$<2\mu(\rho)\,Du,\nabla\varphi>=-\int\frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_j)\partial_{ii}\varphi_j dx\,dt-\int 2(\sqrt{\rho}u_j)\mu^{'}(\rho)\partial_i\sqrt{\rho}\partial_i\varphi_j dx\,dt$$
$$-\int\frac{\mu(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_j)\partial_{ji}\varphi_j dx\,dt-\int 2(\sqrt{\rho}u_i)\mu^{'}(\rho)\partial_j\sqrt{\rho}\partial_i\varphi_j dx\,dt$$

$$<\lambda(\rho)\,\mathrm{div}u,\mathrm{div}\varphi>=-\int\frac{\lambda(\rho)}{\sqrt{\rho}}(\sqrt{\rho}u_i)\partial_{ji}\varphi_j dx\,dt-\int 2(\sqrt{\rho}u_i)\lambda^{'}(\rho)\partial_i\sqrt{\rho}\partial_j\varphi_j dx\,dt$$

We assume also that μ and λ verify the conditions (1.5) and (1.6).

We obtain now a general result concerning the stability of the global weak solution for system (1.10) and a result of existence of global weak solution for general initial data of the form $(\rho_0,-\nabla\varphi(\rho_0))$ (in particularly ρ_0 is not assumed only strictly positive).

Theorem 1.2 *Assume that $\mu(\rho)$ and $\lambda(\rho)$ are two regular function of ρ verifying (1.5) and (1.6). Furthermore we shall set $g(x)=\frac{\mu(x)}{\sqrt{x}}$ and we assume that g is bijective and that g^{-1} is continuous on $(0,+\infty)$. When $2+N\leq\nu_1$, we assume in addition that g and $g^{'}$ are increasing on $(0,+\infty)$.*
Let (ρ_n,u_n) be a sequence of global weak solutions of system (1.10) satisfying entropy inequalities (3.36), (3.37) and (3.38) with initial data:

$$(\rho_n)_{/t=0}=\rho_0^n(x)\quad\textit{and}\quad(\rho_n u_n)_{/t=0}=\rho_0^n u_0^n(x)$$

with ρ_0^n and u_0^n such that:

$$\rho_0^n \geq 0, \;\; \rho_0^n \to \rho_0 \;\; in \, L^1(\mathbb{R}^N), \;\; \rho_0^n u_0^n \to \rho_0 u_0 \;\; in \, L^1(\mathbb{R}^N), \tag{1.20}$$

and satisfying the following bounds (with C constant independent on n):

$$\int_{\mathbb{R}^N} \rho_0^n \frac{|u_0^n|^2}{2} < C, \qquad \int_{\mathbb{R}^N} \sqrt{\rho_0^n} |\nabla\varphi(\rho_0^n)|^2 dx < C \tag{1.21}$$

and:

$$\int_{\mathbb{R}^N} \rho_0^n \frac{1 + |u_0^n|^2}{2} \ln(1 + |u_0^n|^2) dx < C. \tag{1.22}$$

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n} u_n)$ converges strongly to a global weak solution $(\rho, \sqrt{\rho} u)$ of (1.10) satisfying entropy inequalities (3.36), (3.37) and (3.38).
Furthermore the density ρ_n converges strongly to ρ in $C([0,T], L^{1+\alpha}_{loc}(\mathbb{R}^N))$ with $0 < \alpha < \nu_1$ when $N = 3$ and in $C([0,T], L^q_{loc}(\mathbb{R}^N))$ for any $q \geq 1$ when $N = 2$; $\sqrt{\rho_n} u_n$ converges strongly in $L^2(0,T,L^2_{loc})$ to $\sqrt{\rho} u$ and the momentum $m_n = \rho_n u_n$ converges strongly in $L^1(0,T,L^1_{loc}(\mathbb{R}^N))$, for any $T > 0$.

If we assume moreover that (ρ_0, u_0) verify the initial condition of the definition 1.2, that $u_0 = -\nabla\varphi(\rho)$ with $\mu(\rho) = \mu\rho^\alpha$ and that $\sqrt{\rho_0}|u_0|^{1+\frac{1}{p}}$ belongs in $L^2(\mathbb{R}^N)$ for p large enough, then it exists a global weak solution (ρ, u) of the system (1.10) where ρ is also the unique solution of the system (1.12) (see the theorem 2.4 for the existence of a unique global strong solution for the equation (1.12) with a L^1 initial data).

Remark 13 *Let us mention that the additional technical assumption on the viscosity μ remains quite natural since they are verified in the standard case $\mu(\rho) = \mu\rho^\alpha$ with $\alpha > 1 - \frac{1}{N}$. In particular we remark that this result extend the analysis of [29] to general viscosity coefficient, in particular we do not suppose that $\mu'(\rho) \geq c > 0$.*

Remark 14 *Let us emphasize that the condition (1.6) implies that we exclude the case of fast diffusion equation with $0 < \alpha < 1 - \frac{1}{N}$, in particular it forbids any phenomen of extinction and loss of mass $\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^N)}$.*

Remark 15 *Let us mention that our second result of existence of global weak solution when $u_0 = -\nabla\varphi(\rho_0)$ can be applied to the Barrenblatt solution by choosing $\rho_0 = U_m(\tau,\cdot)$ with $\tau > 0$. In this case we have then a free boundary problem with (ρ, u) C^∞ when $\rho > 0$ and $(\rho, u) = (0,0)$ when $\rho = 0$. In particular when $\mu(\rho) = \mu\rho^\alpha$ with $\alpha > 1$ we observe that the propagation of the support of the Barrenblatt is finite.*
More generally when we choose a initial density with compact support the support of the density remains bounded along the time when we deal with the case $\mu(\rho) = \mu\rho^\alpha$ with $\alpha > 1$. It means that quasi-solutions in this case have the same properties than the porous media solutions. We are related to a free boundary problem with on a side the solution which is C^∞ and on the other side the solution which is identically null. In particular it implies that we can not hope uniqueness as the velocity can take any value on the vacuum set, that is why it is natural to consider the momentum unknown $\sqrt{\rho} u$ as it is the case in the previous theorem.

Remark 16 *Let us emphasize on the fact that the problem of the existence of global weak solutions remains open in the general case (it means when u_0 is different from $-\nabla\varphi(\rho_0)$). Indeed in the previous theorem we prove the stability of the global weak solutions, however it seems quite complicated to construct approximate global weak solution which verify uniformly in n all the entropies. We have the same problem in the case of the compressible Navier Stokes problem where the problem remains also open.*

We are going to prove now the convergence of the global weak solution of the compressible Navier-Stokes equations to the quasi solutions when we consider a vanishing pressure process. More precisely let us consider the highly compressible Navier-Stokes system with ϵ going to 0:

$$\begin{cases} \partial_t\rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0, \\ \partial_t(\rho_\epsilon u_\epsilon) + \mathrm{div}(\rho_\epsilon u_\epsilon \otimes u_\epsilon) - \mathrm{div}(2\mu(\rho_\epsilon)D(u_\epsilon)) - \nabla(\lambda(\rho_\epsilon)\mathrm{div}u_\epsilon) + \epsilon\nabla\rho_\epsilon^\gamma = 0, \\ (\rho_\epsilon, u_\epsilon)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.23}$$

In the literature we find many result on the incompressible limit which corresponds to take $\epsilon = \frac{1}{\eta^2}$ with η the Mach number going to 0. For such results in the framework of the global weak solutions for the ill-prepared data we refer to the pioneering papers of Desjardins, Grenier, Lions and Masmoudi [10, 11, 27, 28]. Indeed in this last situation we can observe at least heuristically that the density ρ_ϵ converges to a constant 1 when we are working with initial density of the form $\rho_{0,\epsilon} = 1 + \epsilon q_{0,\epsilon}$ with $q_{0,\epsilon}$ uniformly bounded in appropriate space. By considering the mass equation, roughly speaking we can deduce that the limit solution of u_ϵ is incompressible. The main difficulty compared with the well prepared case corresponds to deal with the acoustic waves, in particular in order to overcome such difficulty the authors use Strichartz estimates.
In our case we are going to deal with the opposite situation when the solutions are highly compressible and converge to quasi-solution which are in some sense purely compressible since irrotationnal. In the case of constant viscosity it appears impossible to pass in the limit when ϵ goes to 0 since we lose the $L^\infty_T(L^\gamma(\mathbb{R}^N))$ estimate on ρ_ϵ coming from the pressure term (we conserve only the L^1 conservation of the mass which is not sufficient to pass to the limit since it does not provide enough compactness information). In our case with viscosity coefficients verifying the relation (1.5) we know that we have uniform estimate on $\sqrt{\rho_\epsilon}\nabla\varphi(\rho_\epsilon)$ in $L^\infty_T(L^2(\mathbb{R}^N))$ providing of the entropy (3.42) (see [6]) which will be sufficient in term of compactness to pass to the limit when ϵ goes to 0. Before giving our main result in this spirit, let us give a definition of global weak solution for the system (1.23) in the spirit of [29]:

Definition 1.3 *We say that (ρ, u) is a global weak solution of (1.1) if (ρ, u) verifies in distribution sense:*

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)\,\mathrm{Du}) - \nabla(\lambda(\rho)\mathrm{divu}) + \nabla\mathrm{P}(\rho) = 0, \\ (\rho, u)_{\ t=0} = (\rho_0, u_0), \end{cases} \tag{1.24}$$

with $P(\rho) = a\rho^\gamma$, $\gamma > 1$. More precisely (ρ, u) is a weak solution of (1.1) on $[0,T]\times\mathbb{R}^N$

$$\rho_{/t=0} = \rho_0 \geq 0, \ \ \rho u_{/t=0} = m_0. \tag{1.25}$$

with:
$$\begin{aligned}&\rho_0 \in L^1(\mathbb{R}^N)\cap L^\gamma(\mathbb{R}^N),\ \sqrt{\rho_0}\nabla\varphi(\rho_0)\in L^2(\mathbb{R}^N),\ \rho_0\geq 0,\\&\sqrt{\rho_0}|u_0|(1+\sqrt{\ln(1+|u_0|^2)})\in L^2(\mathbb{R}^N).\end{aligned}\tag{1.26}$$
if

- $\rho\in L^\infty_T(L^1(\mathbb{R}^N)),\ \sqrt{\rho}\nabla\varphi(\rho)\in L^\infty_T(L^2(\mathbb{R}^N)),\ \sqrt{\rho}u\in L^\infty_T(L^2(\mathbb{R}^N)),$
- $\sqrt{\mu(\rho)}\,\nabla u\in L^2((0,T)\times\mathbb{R}^N),\ \sqrt{\rho}|u|\sqrt{\ln(1+|u|^2)}\in L^\infty_T(L^2(\mathbb{R}^N)).$

with $\rho\geq 0$ *and* $(\rho,\sqrt{\rho}u)$ *satisfying in distribution sense on* $[0,T]\times\mathbb{R}^N$*:*
$$\begin{cases}\partial_t\rho+\operatorname{div}(\sqrt{\rho}\sqrt{\rho}u)=0,\\ \rho(0,x)=\rho_0(x).\end{cases}$$
and if the following equality holds for all $\varphi(t,x)$ *smooth test function with compact support such that* $\varphi(T,\cdot)=0$*:*
$$\begin{aligned}&\int_{\mathbb{R}^N}(\rho u)_0\cdot\varphi(0,\cdot)dx+\int_0^T\int_{\mathbb{R}^N}\sqrt{\rho}(\sqrt{\rho}u)\partial_t\varphi+\sqrt{\rho}u\otimes\sqrt{\rho}u:\nabla\varphi dx\\&+a\int_0^T\int_{\mathbb{R}^N}\rho^\gamma\operatorname{div}\varphi(s,x)dx-<2\mu(\rho)\,Du,\nabla\varphi>-<\lambda(\rho)\operatorname{div}u,\operatorname{div}\varphi>=0,\end{aligned}\tag{1.27}$$
where we give sense to the diffusion terms as in the definition 1.2. We assume also that μ *and* λ *verify the conditions (1.5) and (1.6).*

Theorem 1.3 *Let* $\gamma>1$*and* (ρ_0,u_0) *verifies the conditions of definition 1.3. Let us take the following assumptions on* γ*,* ν_1 *and* ν_2*:*

- $N=3$
  - $\nu_1\geq 2$*:*
  $$\frac{5}{6}+\frac{\nu_2}{12}<\gamma<2+\frac{\nu_1}{2}\quad \textit{if }\ N=3,$$
  $$\frac{5}{6}+\frac{\nu_2}{12}<\gamma<\frac{5}{6}+\frac{7}{12}\nu_1\quad \textit{if }\ N=3,$$
  - $0<\nu_1<2$*:*
  $$\frac{5}{6}+\frac{\nu_2}{12}<\gamma<\frac{(4-\nu_1)(1+\nu_1)}{2-\nu_1}\quad \textit{if }\ N=3,$$
  $$\frac{5}{6}+\frac{\nu_2}{12}<\gamma<\frac{5}{6}+\frac{7}{12}\nu_1\quad \textit{if }\ N=3,$$
- $N=2$
$$\frac{1}{4}+\frac{\nu_2}{8}<\gamma\quad \textit{if }\ N=2.$$

Then under these conditions we have as in the theorem 1.2 the stability of the global weak solutions for the system (1.1).
Assume that there exists global weak solution $(\rho_\epsilon, u_\epsilon)$ *verifying the definition of [29] with the conditions (1.5) and (1.6) on* $\mu(\rho)$ *and* $\lambda(\rho)$*. Then* $(\rho_\epsilon, u_\epsilon)$ *converges in distribution sense to a global weak quasi-solution* (ρ, u) *of the system (1.24) in the sense of the definition 1.2. Furthermore the density* ρ_ϵ *converges strongly to* ρ *in* $C([0,T], L^{1+\alpha}_{loc}(\mathbb{R}^N))$ *with* $0 < \alpha < \nu_1$ *when* $N = 3$ *and in* $C([0,T], L^q_{loc}(\mathbb{R}^N))$ *for any* $q \geq 1$ *when* $N = 2$*;* $\sqrt{\rho_\epsilon} u_\epsilon$ *converges strongly in* $L^2(0, T, L^2_{loc})$ *to* $\sqrt{\rho} u$ *and the momentum* $m_\epsilon = \rho_\epsilon u_\epsilon$ *converges strongly in* $L^1(0, T, L^1_{loc}(\mathbb{R}^N))$*, for any* $T > 0$

Remark 17 *Let us mention that the technical assumption on* ν_1*,* ν_2 *and* γ *are important in order to ensure a uniform gain of integrability on the velocity* u_ϵ *(as in the paper of Mellet and Vasseur [29]), more precisely we will see that we have a competition between the pressure and the viscosity.*

Remark 18 *We shall also emphasize on an important question which remains open; indeed when* $\mu(\rho) = \mu\rho^\alpha$ *and* $u_0 = -\nabla\varphi(\rho_0)$ *the limit solution* (ρ, u) *of* $(\rho_\epsilon, u_\epsilon)$ *when* ϵ *goes to* 0 *is a quasi solution of (1.13). However it is not clear how to prove the uniqueness of the quasi solutions in the class of the solutions giving by the definition 1.2 and in particular to show that this quasi solution is solution of the porous media equation for the density* ρ *when* $\alpha > 1$*. In the following corollary, we shall give properties of the solutions of (1.23) when we assume the uniqueness of the quasi solutions.*
Let us mention that in some case if we assume that the solution of the porous media is enough regular (typically the case of some solutions with initial data $\rho_0 = U_m(\tau, \cdot)$ *with* $\tau > 0$ *and* U_m *a suitable Barrenblatt solution) the uniqueness consists in proving a weak-strong uniqueness theorem since we can assume that the solution* $\big(U_m(T + \tau, \cdot), -\nabla\varphi(U_m(T + \tau, \cdot))\big)$ *is strong.*

Remark 19 *The second important remark consists to point out the fact that the question of global weak solution for the system (1.23) remains open, indeed Mellet and Vasseur have proved the stability of the global weak solution in [29]. However it seems not so easy to construct a regular sequel* (ρ_n, u_n) *approximating (1.23) (typically by a Friedrich process) and verifying uniformly all the entropies of [29] (3.41), (3.42) and (3.43).*

Corollary 1 *Let* $\gamma > 1$*,* ν_1 *and* ν_2 *with the hypothesis of theorem 1.3. Let* ρ_0 *and* u_0 *verifying the hypothesis of the theorem 1.3 with* $u_0 = -\nabla\varphi(\rho_0)$ *and* $\mu(\rho) = \mu\rho^\alpha$*,* $\lambda(\rho)$ *verifying also the hypothesis of theorem 1.3. We assume here that there exists a unique quasi solution of system (1.24) with such initial data, in particular the density of this quasi solution verifies (1.12). Then as in theorem 1.3* $(\rho_\epsilon, u_\epsilon)$ *converges in distribution sense to a global weak quasi-solution* (ρ, u) *of the system (1.24) such that* ρ *is solution of (1.12). Moreover:*

- *If* $\alpha > 1$*,* $\rho_0 \in L^\infty(\mathbb{R}^N)$ *and the support of* ρ_0 *is compact then we have for* $T > 0$ ρ_ϵ *converges strongly to* ρ *in* $C([0,T], L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N))$ *with* $1 < p < 1 + \nu_1$ *if* $N = 3$ *and in* $C([0,T], L^p(\mathbb{R}^N))$ *for any* $p > 1$*.*
  For all $\eta > 0$ *it exists* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon \leq \epsilon_0$ $\forall t \in [0,T]$ *we have:*

$$\rho_\epsilon(t, \cdot) = \rho(t, \cdot) + f_\epsilon(t, \cdot),$$

with $\rho(t,\cdot)$ with compact support for any $t\in[0,T]$ and we have $\|f_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}\leq\eta$ $\forall t\in[0,T]$.
In particular for all $\eta>0$ it exists $\epsilon_0>0$ such that for all $0<\epsilon\leq\epsilon_0$ we have:

$$\rho_\epsilon(t,\cdot)=\rho(t,\cdot)+f_\epsilon(t,\cdot),$$

with $\rho(t,\cdot)$ verifying for p as above:

$$\|\rho(t)\|_{L^p(\mathbb{R}^N)}\leq Ct^{-\sigma_p}\|\rho_0\|_{L^1(\mathbb{R}^N)}^{\alpha_p},$$

with $\sigma_p=\frac{N(\alpha-1)+2p}{(N(\alpha-1)+2)p}$ and $\alpha_p=\frac{N(p-1)}{(N(\alpha-1)+2)p}$, and we have $\|f_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^N)}\leq\eta$ for all $t\in[0,T]$.

- *If $\alpha\geq 1-\frac{1}{N}$, for all $\eta>0$ for all compact K it exists $T>0$, it exists $\epsilon_0>0$ such that for $0<\epsilon\leq\epsilon_0$ we have:*

$$\|\rho_\epsilon(t,\cdot)-U_m(t,\cdot)\|_{L^1(K)}\leq\eta\quad\forall t\in[T,2T].$$

Remark 20 *It is very surprising to observe that for ϵ small enough ρ_ϵ is subjected to a type of damping effect in $L^p(\mathbb{R}^N)$ with p choose as above up to a small remainder term in $L^p(\mathbb{R}^N)$. Let us point out that this effect is similar to the dispersion property for the Schrödingier or the waves equations. In [9, 15] we observe a damping effect on the density due to the role of the pressure, but in our case the pressure tends to disappear. As for the porous media equation this effect seems purely non linear and is exhibited because the particular structure of the viscosity coefficients.*

Remark 21 *Under the hypothesis of uniqueness of the quasi solution when $u_0=-\nabla\varphi(\rho_0)$ we show that the solution of the highly compressible Navier-Stokes equation are not so far to have a finite speed of propagation when we take a initial density with compact support. Indeed this is the case modulo a perturbation f_ϵ of small L^1 norm. Similarly modulo this hypothesis of uniqueness we expect a asymptotic convergence of ρ_ϵ to the Barrenblatt solution of L^1 norm $\|\rho_0\|_{L^1(\mathbb{R}^N)}$ modulo a small perturbation.*

The paper is structured in the following way: in section 2 we recall some important results on the porous media and the fast diffusions equations. In section 3 we adapted the entropy of [6] and [29] to the case of the quasi-solutions. In section 4, we give a few notation, some compactness results and briefly introduce the basic Fourier analysis techniques needed to prove our result. In section 4 we prove theorem 1.1 and in section 5 we show the theorem 1.2. In section 5.1 we conclude with the proof of the theorem 1.3 and the corollary 1. An appendix is postponed in order to prove rigorously some technical lemmas.

2 Important results on the porous media equation

for the sake of completeness for the reader which are not familiar with the porous and the fast diffusion equations, we are going to recall in this section some essential results on the porous media and the fast diffusion equations. The majority of them are directly issue from the excellent book [33], [34] from Vázquez. In this part in order to simplify the problem we shall only consider the following equation with $\alpha > 1 - \frac{1}{N}$:

$$\begin{cases} \partial_t \rho - 2\mu \Delta \rho^\alpha = 0, \\ \rho(0, \cdot) = \rho_0 \geq 0. \end{cases} \tag{2.28}$$

Let us start with the case where $\alpha > 1$

2.1 Porous Media, $\alpha > 1$

In the sequel we shall set $Q = (0, +\infty) \times \mathbb{R}^N$. Let us recall the notion of global strong solution for the equation (1.12) of the porous medium equation ($\alpha > 1$) (see [33] chapter 9 for more details).

Definition 2.4 *We say that a function $\rho \in C([0, +\infty), L^1(\mathbb{R}^N))$ positive is a strong L^1 solution of problem (2.28) if:*

- $\rho^\alpha \in L^1_{loc}(0, +\infty, L^1(\mathbb{R}^N))$ *and* $\rho_t, \Delta\rho^\alpha \in L^1_{loc}((0, +\infty) \times \mathbb{R}^N)$
- $\rho_t = \mu \Delta \rho^\alpha$ *in distribution sense.*
- $\rho(t) \to \rho_0$ *as* $t \to 0$ *in* $L^1(\mathbb{R}^N)$.

Let us mention (see [33] p197) that we have the following theorem of global strong solution, we are going to give a sketch of the proof of the existence and uniqueness of the L^1 solution (which is perfectly detailed in [33] Chapter 6 and 9). Indeed it will be important to understand this point for the proof of the last part of theorem 1.2.

Theorem 2.4 *Let $\alpha > 1$ For every non-negative function $\rho_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there exists a unique global strong solution $\rho \geq 0$ of (2.28). Moreover, $\partial_t \rho \in L^p_{loc}(Q)$ for $1 \leq p < \frac{(\alpha+1)}{\alpha}$ and:*

$$\partial_t \rho \geq -\frac{\rho}{(\alpha - 1)t} \quad \text{in } \mathcal{D}'(Q),$$
$$\|\partial_t \rho(t, \cdot)\|_{L^1(\mathbb{R}^N)} \leq \frac{2\|\rho_0\|_{L^1(\mathbb{R}^N)}}{(\alpha - 1)t}.$$

Let ρ_1 and ρ_2 be two strong solutions of (2.28) in $(0, T) \times \mathbb{R}^N$ then for every $0 \leq \tau < t$

$$\|\big(\rho_1(t, \cdot) - \rho_2(t, \cdot)\big)_+\|_{L^1(\mathbb{R}^N)} \leq \|\big(\rho_1(\tau, \cdot) - \rho_2(\tau, \cdot)\big)_+\|_{L^1(\mathbb{R}^N)}. \tag{2.29}$$

If ρ_1 and ρ_2 are two strong solution with initial data ρ_{01} and ρ_{02} with $\rho_{01} \leq \rho_{02}$ in $\mathbb{R}^N$, then $\rho_1 \leq \rho_2$ almost everywhere in $(0, +\infty) \times \mathbb{R}^N$.

Proof: The proof is decomposed in three steps. Let us recall that in the sequel the initial density is always positive.

Energy global weak solution on a bounded domain Ω

Let us mention that a global weak solution in our setting is given by the definition 5.4 of [33] and verify on the initial data $\rho_0 \in L^1(\Omega) \cap L^{1+\alpha}(\Omega)$. By considering $\rho_{0n} = \max(\rho_0 + \frac{1}{n}, n)$ which is strictly positive with boundary condition such that $\rho_n(t, \cdot) = \frac{1}{n}$ on the boundary of Ω, by the theory of the quasilinear equations (see Ladyzhenskaya et al [26] or Friedman in [13]) it exists global classical solutions which verify the energy estimates:

$$\int_0^T \int_\Omega |\nabla \rho_n^\alpha|^2 dxdt + \frac{1}{\alpha+1} \int_\Omega \rho_n^{\alpha+1} dxdt \leq \frac{1}{\alpha+1} \int_\Omega \rho_{0n}^{\alpha+1}.$$

Furthermore by applying the maximum principle we know that $(\rho_n)_{n\in\mathbb{N}}$ is a decreasing sequel so to converges everywhere to a limit ρ. Since ρ_n is uniform bounded in $L^\infty_T(L^{1+\alpha}(\Omega))$ for any $T > 0$, it implies that up to a subsequence ρ_n converge weakly to ρ in $L^\infty_T(L^{1+\alpha}(\Omega))$. Furthermore $\nabla \rho_n^\alpha$ is uniformly bounded in $L^2_T(L^2(\Omega))$ for any $T > 0$ it implies that up to a subsequence $\nabla \rho_n^\alpha$ converges to a limit ψ.
By applying the lemma 1 to ρ_n^α and the fact that ρ_n^α is uniformly bounded in $L^\infty_T(L^{1+\epsilon}(\Omega))$ with $\epsilon > 0$ and converges almost everywhere to ρ^α it shows that ρ_n^α converges strongly to ρ^α in $L^1_{loc}((0,T) \times \Omega)$. In particular we deduce that $\psi = \nabla \rho^\alpha$.

L^1 global weak solution on a bounded domain Ω

By the fundamental L^1 contraction principle which ensures that for any solution ρ_1, ρ_2 of the porous media in L^1 we have:

$$\|\rho_1(t) - \rho_2(t)\|_{L^1(\Omega)} \leq \|\rho_{01} - \rho_{02}\|_{L^1(\Omega)}. \tag{2.30}$$

The L^1 limit solution ρ of the equation (1.12) consists in considering the L^1 limit of an approximate energy solution ρ_n of (1.12) with ρ_{0n} converging to ρ_0 in $L^1(\Omega)$. By the estimate (2.30) we check easily that the limit ρ does not depend on the choice of the regularizing sequel ρ_{0n}. In addition we can verify that ρ is in $C([0, +\infty), L^1(\Omega))$ (see p129 in [33]). However an important question remains, is the limit solution ρ is a weak solution according the definition 5.2 of [33]. The answer is positive. Indeed for any approximative solution we can check that:

$$\int_\Omega |\rho_n - \rho_m|(t,x)\xi(x)dx + \int_0^t \int_\Omega \|\rho_n^\alpha(s,x) - \rho_m^\alpha(s,x)|dxds \leq \int_\Omega |\rho_{0n} - \rho_{0m}|(x)\xi(x)dx, \tag{2.31}$$

where ξ is the unique solution of the problem:

$$\Delta\xi = -1 \text{ in } \Omega, \;\; \xi = 0 \text{ on } \partial\Omega.$$

It implies that ρ_n^α is a Cauchy sequence in $L^1((0,T) \times \Omega)$ for any $T > 0$ and then converge strongly to ψ but ρ_n converge strongly in $L^\infty_T(L^1(\Omega)$ for any $T > 0$ then up to a subsequence rho_n converges almost everywhere to ρ and ρ_n^α up to a subsequence converges to ρ^α and ψ then $\psi = \rho^\alpha$ which implies that ρ is a very weak solution of (1.12) in the sense of the definition 5.2 of [33].

L^1 solution on $\mathbb{R}^N$

We start by approximating the initial data by setting:

$$\rho_{0n}(x) = \max(-n, \min(\rho_0(x), n)\chi(nx),$$

whit $\chi \in C_0^\infty(\mathbb{R}^N)$ such that the support of χ is embedded in the ball $B(0,2)$ and $\chi = 1$ on $B(0,1)$. We have seen that it exists global L^1 solutions for the homogeneous Cauchy-Dirichlet problem in bounded domain $\Omega_n = B(0,2n)$. It suffices to pass to the limit when n goes to infinity by using the same type of compactness argument.
Let us mention to finish that the uniqueness of the L^1 solution is a direct consequence of the L^1 contraction principle. □

Remark 22 *Let us recall that there exists global weak solution which are not classical it means not C^∞ even if the initial data is C^∞ (see a example due to Aronson in the problem 5.7 of [33]).*

We are now to recall the so called $L^1 - L^\infty$ smoothing effect (as for the dispersive equations), we refer to [33] p 202.

Theorem 2.5 *For every $t > 0$ we have:*

$$\rho(t,x) \leq C\|\rho_0\|^\sigma_{L^1(\mathbb{R}^N)} t^{-\beta},$$

with $\sigma = \frac{2}{N(\alpha-1)+2}$, $\beta = \frac{N}{N(\alpha-1)+2}$ and $C > 0$ depends only on α and N. The exponents are sharp.

Let us finnish by giving a more general theorem of existence of global strong solution for (2.28) with some properties on the solutions (see [33] p 204-205).

Theorem 2.6 *For every $\rho_0 \in L^1(\mathbb{R}^N)$ there exists a unique global strong solution of (2.28) such that $\rho \in C([0,+\infty), L^1(\mathbb{R}^N)) \cap L^\infty((\tau,+\infty) \times \mathbb{R}^N)$ with $\tau > 0$. Furthermore we have the following L^∞ estimate:*

$$|\rho(t,x)| \leq C\|\rho_0\|^\sigma_{L^1(\mathbb{R}^N)} t^{-\beta},$$

with $\sigma = \frac{2}{N(\alpha-1)+2}$, $\beta = \frac{N}{N(\alpha-1)+2}$ and $C > 0$ depends only on α and N. Moreover we have the following properties:

1. *The solutions are continuous functions of (t,x) in Q with a uniform modulus of continuity for $t \geq \tau > 0$.*
2. *The maximum principles holds.*
3. *if ρ_0 is strictly positive and continuous, then $\rho \in C^\infty(Q) \cap C(\bar{Q})$ and is a classical solution of (2.28).*
4. *For every $p \in (1,+\infty)$ we have the following regularizing effect, $\rho(t,\cdot)$ belongs in $L^p(\mathbb{R}^N)$ and:*
$$\|\rho(t)\|_{L^p(\mathbb{R}^N)} \leq Ct^{-\sigma_p}\|\rho_0\|^{\alpha_p}_{L^1(\mathbb{R}^N)},$$
with $\sigma_p = \frac{N(\alpha-1)+2p}{(N(\alpha-1)+2)p}$ and $\alpha_p = \frac{N(p-1)}{(N(\alpha-1)+2)p}$.

Let us conclude this section with two important theorem showing the finite speed of propagation for the porous media equation (see [33] p 210) and the time asymptotic behavior of the solution which converges to Barrenblatt solutions (see [35] p 69).

Proposition 2.1 *Let ρ be the global strong solution of (2.28) with initial data $\rho_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and assume that ρ_0 has a compact support then for every $t > 0$ the support of $\rho(t, \cdot)$ is a bounded set.*

Theorem 2.7 *Let $\rho(t, x)$ be the unique global strong solution of (2.28) with initial data $\rho_0 \in L^1(\mathbb{R}^N)$, $\rho_0 \geq 0$. Let U_m be the Barrenblatt with the same mass as ρ_0. Then we have:*

$$\lim_{t \to +\infty} \|\rho(t) - F_m(t)\|_{L^1(\mathbb{R}^N)} = 0. \tag{2.32}$$

Convergence holds also in L^∞ norm:

$$\lim_{t \to +\infty} t^\beta \|\rho(t) - F_m(t)\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{2.33}$$

with $\beta = \frac{N}{N(\alpha-1)+2}$.

Remark 23 *For more results in this direction we refer also to [12].*

Let us conclude this section by giving general results (essentially extracted form the chapter 9 from [33]) on porous media equation of the form:

$$\begin{cases} \partial_t \rho - 2\Delta\mu(\rho) = 0, \\ \rho(0, \cdot) = \rho_0. \end{cases} \tag{2.34}$$

2.1.1 General viscosity coefficients

We assume here that μ verifies the following assumptions:

- μ is a continuous and increasing function: $\mathbb{R} \to \mathbb{R}$ with $\mu(0) = 0$
- μ has at least linear growth at infinity in the sense that it exists $c >$ such that for large $|s|$ we have:
$$|\mu(s)| \geq c|s| > 0.$$

Definition 2.5 *A locally integrable function ρ defined in Q_T is said to be a weak solution of the problem if:*

1. *$\mu(\rho) \in L^2(0, T, H^1(\mathbb{R}^N))$*
2. *ρ satisfies the identity:*

$$\int\int_{Q_T} (\nabla\mu(\rho) \cdot \nabla\varphi - \rho\partial_t\varphi) dx dt = \int_{\mathbb{R}^N} \rho_0(x)\varphi(0, x) dx, \tag{2.35}$$

for any function $\varphi \in C^1(\bar{Q}_T)$ which vanishes for $t = T$ and has uniformly bounded support in the space variable.

We define $L_\mu(\mathbb{R}^N)$ by the set of measurable function ρ_0 such that $\mu(\rho_0) \in L^1(\mathbb{R}^N)$. We shall consider ψ the primitive of μ:

$$\psi(s) = \int_0^s \mu(\tau) d\tau.$$

Let $X_T = L_\mu(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $Y = L^\infty(Q_T) \cap L^1(Q_T)$ for $0 < T \leq +\infty$.

Theorem 2.8 *Let $\rho_0 \in X$. Then it exists a unique global weak solution defined in $(0, +\infty)$ and $\nabla \mu(\rho) \in L^2(Q_T)$. Moreover we also have $\rho \in L^\infty((0,T), X)$.*

2.2 Fast diffusion equations, $0 < \alpha < 1$

The situation is different in the case of fast diffusion equation $0 < \alpha < 1$, indeed in this case infinite propagation occurs and solution may even vanish in finite time. Let us mention that when α is in the interval $(0, m_c)$ with $m_c = \max(0, \frac{N-2}{N})$ then it can appears a phenomena of extinction of the solution in finite time. In particular it implies a lost of the initial mass when ρ_0 is in L^1 (it implies also a lost of the regularity of the solution). We refer to [34] theorem 5.7 for a necessary condition of extinction, in particular the solution belongs in an appropriate Marcinkewitz space $M_{p^*}(\mathbb{R}^N)$.
Let us mention that in the case $\alpha \in (m_c, 1)$ the situation is quite similar to the case of the porous media equation (except the infinite propagation speed) as the mass is preserved which implies no extinction in finite time. Moreover we have self similar solutions also discovered by Barrenblatt that we can write under the following form:

$$U_m(t,x) = t^{-\gamma_1} F(x t^{-\beta}) \;\text{ with }\; F(x) = (C + \kappa_1 |x|^2)_+^{\frac{-1}{\alpha-1}},$$

with $\kappa_1 = \frac{(1-\alpha)\gamma_1}{2N\alpha}$. In particular the proof of the most of the previous result in the last section are based on the existence of Barrenblatt solutions and on the maximum principle or in other word the L^1 contraction principle. This two fundamental point arise also in the case $\alpha \in (m_c, 1)$ which implies that we have the most of the result of the case $\alpha > 1$ exists also in this case (using essentially the same proof). In particular similarly to the case $\alpha > 1$ in the situation $\alpha_c < \alpha < 1$ the global strong solution converges asymptotically to a Barrenblatt solution and we have also regularizing effect $L^1 - L^\infty$. For more details in this situation we refer the reader to the excellent books of Vázquez [33, 34].

3 Entropy inequality for the quasi-solutions and basic tools

3.1 Entropy for the quasi-solution of the system (1.10)

We now want to establish new entropy inequalities for system (1.10) by applying the entropy inequalities discovered in [6, 29]. More precisely if we assume that (ρ, u) are classical solutions of system (1.10), we obtain the following proposition.

Proposition 3.2 *Assume that (ρ,u) are classical solutions of system (1.10) then for all $t>0$ we have the two following entropy:*

$$\int_{\mathbb{R}^N}\frac{1}{2}\rho|u|^2(t,x)\,dx+\int_0^t\int_{\mathbb{R}^N}2\mu(\rho)|Du|^2dxdt+\int_0^t\int_{\mathbb{R}^N}\lambda(\rho)|\mathrm{div}u|^2dxdt \\ =\int_{\mathbb{R}^N}\rho_0|u_0|^2(x)\,dx. \tag{3.36}$$

$$\int_{\mathbb{R}^N}\frac{1}{2}\rho|u+\nabla\varphi(\rho)|^2(t,x)\,dx+\frac{1}{2}\int_0^t\int_{\mathbb{R}^N}\mu(\rho)|\nabla u-^t\nabla u|^2dxdt \\ =\int_{\mathbb{R}^N}\frac{1}{2}\rho_0|u_0+\nabla\varphi(\rho_0)|^2(x)\,dx. \tag{3.37}$$

Proof: In order to obtain (3.36) it suffices to multiplying the momentum equation by u and integrating over $(0,t)\times\mathbb{R}^N$.
Let us briefly recall the proof of the second entropy (3.37) introduced by Bresch and Desjardins in [6]. To this purpose, we have to study:

$$\frac{d}{dt}\int\big[\frac{1}{2}\rho|u|^2+\rho u\cdot\nabla\varphi(\rho)+\frac{1}{2}\rho|\nabla\varphi(\rho)|^2\big]dx.$$

Step 1:

First by the mass equation, we have:

$$\begin{aligned}\frac{1}{2}\int\frac{d}{dt}(\rho|\nabla\varphi(\rho)|^2)dx&=\int\rho\frac{d}{dt}\frac{|\nabla\varphi(\rho)|^2}{2}dx-\int\frac{|\nabla\varphi(\rho)|^2}{2}\mathrm{div}(\rho u)dx,\\ =-\int\rho\nabla u\nabla\varphi(\rho)\otimes\nabla\varphi(\rho)dx&+\int\rho^2\varphi^{'}(\rho)\Delta\varphi(\rho)\mathrm{div}udx+\int\rho|\nabla\varphi(\rho)|^2\mathrm{div}udx.\end{aligned}$$

Step 2

It remains to estimate the derivative of the cross product:

$$\begin{aligned}\frac{d}{dt}\int\rho u\cdot\nabla\varphi(\rho)dx&=\int\nabla\varphi(\rho)\cdot\frac{d}{dt}(\rho u)+\int\rho u\cdot\frac{d}{dt}\nabla\varphi(\rho)dx\\&=\int\nabla\varphi(\rho)\cdot\frac{d}{dt}(\rho u)-\int\mathrm{div}(\rho u)\varphi^{'}(\rho)\frac{d}{dt}\rho dx\\&=\int\nabla\varphi(\rho)\cdot\frac{d}{dt}(\rho u)+\int\mathrm{div}(\rho u)^2\varphi^{'}(\rho)dx.\end{aligned}$$

Multiplying the momentum equation by $\nabla\varphi(\rho)$, we get:

$$\begin{aligned}\int\nabla\varphi(\rho)\cdot\frac{d}{dt}(\rho u)=&-\int(2\mu(\rho)+\lambda(\rho))\Delta\varphi(\rho)\mathrm{div}udx+2\int\nabla u:\nabla\varphi(\rho)\otimes\nabla\mu(\rho)dx\\&-2\int\nabla\varphi(\rho)\cdot\nabla\mu(\rho)\mathrm{div}udx-\int\nabla\varphi(\rho)\mathrm{div}(\rho u\otimes u)dx,\end{aligned}$$

where we use the fact that:

$$\int \nabla(\lambda(\rho)\mathrm{div}u)\cdot\nabla\varphi(\rho)dx = -\int \lambda(\rho)\Delta\varphi(\rho)\mathrm{div}u dx,$$

and:

$$\begin{aligned}\int \mathrm{div}(2\mu(\rho)D(u))\cdot\nabla\varphi(\rho)dx &= \int \partial_j(\mu(\rho)\partial_j u_i)\partial_i\varphi(\rho)dx + \int \partial_j(\mu(\rho)\partial_i u_j)\partial_i\varphi(\rho)dx,\\
&= \int \partial_i(\mu(\rho)\partial_j u_i)\partial_j\varphi(\rho)dx + \int \partial_j(\mu(\rho)\partial_i u_j)\partial_i\varphi(\rho)dx,\\
&= \int \partial_i\mu(\rho)\partial_j u_i)\partial_j\varphi(\rho)dx - \int \partial_j\mu(\rho)\partial_i u_i\partial_j\varphi(\rho)dx\\
= -\int \mu(\rho)\partial_i u_i\partial_{jj}\varphi(\rho)dx &+ \int \partial_j\mu(\rho)\partial_i u_j\partial_i\varphi(\rho)dx - \int \partial_i\mu(\rho)\partial_j u_j\partial_i\varphi(\rho)dx\\
&\quad - \int \mu(\rho)\partial_j u_j\partial_{ii}\varphi(\rho)dx\\
= 2\int \nabla u : \nabla\mu(\rho)\otimes\nabla\varphi(\rho)dx &- 2\int \nabla\mu(\rho)\cdot\nabla\varphi(\rho)\mathrm{div}u dx - 2\int \mu(\rho)\Delta\varphi(\rho)\mathrm{div}u dx.\end{aligned}$$

Step 4

Sinceφ, μ and λ satisfies (1.5) and (1.6), then we obtain:

$$\frac{d}{dt}\Big(\int \rho u\cdot\nabla\varphi(\rho) + \rho\frac{|\nabla\varphi(\rho)|^2}{2}dx\Big) = -\int \nabla\varphi(\rho)\mathrm{div}(\rho u\otimes\rho u)dx + \int \varphi^{'}(\rho)(\mathrm{div}(\rho u))^2 dx.$$

Finally we have:

$$\begin{aligned}-\int &\nabla\varphi(\rho)\mathrm{div}(\rho u\otimes\rho u)dx + \int \varphi^{'}(\rho)(\mathrm{div}(\rho u))^2 dx\\
&= -\int \varphi^{'}(\rho)u\cdot\nabla\rho\mathrm{div}(\rho u) - \varphi^{'}(\rho)\nabla\rho(\rho u\cdot\nabla u) + \varphi^{'}(\rho)(\mathrm{div}(\rho u))^2 dx\\
&= \int \rho\varphi^{'}(\rho)\mathrm{div}u\,\mathrm{div}(\rho u) - \rho\varphi^{'}(\rho)\nabla\rho(u\cdot\nabla u)dx\\
&= \int \rho^2\varphi^{'}(\rho)(\mathrm{div}u)^2 + \rho\varphi^{'}(\rho)u\cdot\nabla\rho\mathrm{div}u - \rho\varphi^{'}(\rho)\nabla\rho(u\cdot\nabla u)dx,\end{aligned}$$

then by (1.5) and (1.6), we get:

$$\begin{aligned}-\int &\nabla\varphi(\rho)\mathrm{div}(\rho u\otimes\rho u)dx + \int \varphi^{'}(\rho)(\mathrm{div}(\rho u))^2 dx,\\
&= 2\int \rho h^{'}(\rho)(\mathrm{div}u)^2 + \nabla h(\rho)\cdot u\mathrm{div}u - \nabla(h(\rho))(u\cdot\nabla u)dx,\\
= 2\int &\rho\mu^{'}(\rho)(\mathrm{div}u)^2 - \mu(\rho)(\mathrm{div}u)^2 - \mu(\rho)u\cdot\nabla\mathrm{div}u dx + 2\int \mu(\rho)\partial_i u_j\partial_j u_i\\
&\quad + \mu(\rho)u\cdot\nabla\mathrm{div}u dx,\\
&= \int (2\rho\mu^{'}(\rho) - 2\mu(\rho))(\mathrm{div}u)^2 + 2\mu(\rho)\partial_i u_j\partial_j u_i dx\\
&= \int \lambda(\rho)(\mathrm{div}u)^2 + \int 2\mu(\rho)\partial_i u_j\partial_j u_i dx,\end{aligned}$$

whichgives:

$$\frac{d}{dt}\Big(\int \rho u\cdot\nabla\varphi(\rho)+\rho\frac{|\nabla\varphi(\rho)|^2}{2}dx\Big)$$
$$=\int\lambda(\rho)(\mathrm{div}u)^2dx+\int 2\mu(\rho)\partial_iu_j\partial_ju_idx,$$

Adding this equality and (3.36), and using the fact that:

$$\int 2\mu(\rho)|(u)|^2-\int 2\mu(\rho)\partial_iu_j\partial_ju_idx=\int 2\mu(\rho)(\frac{\partial_iu_j-\partial_ju_i}{2})^2,$$

we easily get (3.37). □ As in [29] we are also interested in getting a gain of integrability on the velocity. We have then the following proposition.

Proposition 3.3 *Assume that:*

$$2\mu(\rho)+N\lambda(\rho)\geq\nu\lambda(\rho)$$

for some $\nu\in(0,1)$ (which is a part of (1.6)). Then it exists $C>0$ such that smooth solutions of (1.10) satisfy the following inequality:

$$\frac{d}{dt}\int\rho\frac{1+|u|^2}{2}\ln(1+|u|^2)dx+\frac{\nu}{2}\int\mu(\rho)[1+\ln(1+|u|^2)|Du|^2dx\leq C\int\mu(\rho)|\nabla u|^2dx. \tag{3.38}$$

for any $\delta\in(0,2)$, and with $|\nabla u|^2=\sum_i\sum_j|\partial_iu_j|^2$.

Proof: Multiplying the momentum equation by $\big(1+\ln(1+|u|^2)\big)u$, we get:

$$\begin{aligned}&\int\rho\frac{d}{dt}\big[\frac{1+|u|^2}{2}\ln(1+|u|^2)\big]dx+\int\rho u\cdot\nabla\big(\frac{1+|u|^2}{2}\ln(1+|u|^2)\big)dx\\&+\int 2\mu(\rho)(1+\ln(1+|u|^2)|D(u)|^2dx+\int 2\mu(\rho)\frac{2u_iu_k}{1+|u|^2}\partial_ju_kD_{ij}(u)dx\\&+\int\lambda(\rho)(1+\ln(1+|u|^2)\mathrm{div}u|^2dx+\int\lambda(\rho)\frac{2u_iu_k}{1+|u|^2}\partial_iu_k\mathrm{div}udx=0.\end{aligned}$$

Since:

$$|\mathrm{div}u|^2\leq N|\nabla u|^2\ \text{ and }\ \nu\mu(\rho)\leq 2\mu(\rho)+N\lambda(\rho),$$

we obtain:

$$\begin{aligned}&\int\rho\frac{d}{dt}\big[\frac{1+|u|^2}{2}\ln(1+|u|^2)\big]dx+\int\rho u\cdot\nabla\big(\frac{1+|u|^2}{2}\ln(1+|u|^2)\big)dx\\&\quad+\nu\int\mu(\rho)(1+\ln(1+|u|^2)|D(u)|^2dx\leq C\int\mu(\rho)|\nabla u|^2dx.\end{aligned}\tag{3.39}$$

Moreover multiplying the mass equation by $\frac{1+|u|^2}{2}\ln(1+|u|^2)$ and integrating by parts, we have:

$$\int\frac{1+|u|^2}{2}\ln(1+|u|^2)\frac{d}{dt}\rho dx-\int\rho u\cdot\nabla\big(\frac{1+|u|^2}{2}\ln(1+|u|^2)\big)dx=0$$

We deduce that:

$$\frac{d}{dt}\int\rho\frac{1+|u|^2}{2}\ln(1+|u|^2)dx+\frac{\nu}{2}\int\mu(\rho)[1+\ln(1+|u|^2)]|D(u)|^2dx$$
$$\leq C\int\mu(\rho)|\nabla u|^2dx.$$

It concludes the proof. □

3.2 Entropy for the compressible Navier-Stokes equations

We are going now to consider the following system with $\mu(\rho)$ and $\lambda(\rho)$ verifying (1.5):

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho u)=0,\\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mathrm{div}(2\mu(\rho)Du)-\nabla(\lambda(\rho)\nabla u)+\epsilon\nabla P(\rho)=0,\\ (\rho,u)(0,\cdot)=(\rho_0,u_0).\end{cases}\tag{3.40}$$

Here P corresponds to the pressure and we shall consider a γ law $P(\rho)=a\rho^\gamma$ with $\gamma>1$ and $a>0$.

Proposition 3.4 *Assume that (ρ,u) are classical solutions of system (3.40) then for all $t>0$ we have the two following entropy:*

$$\begin{aligned}&\int_{\mathbb{R}^N}\big[\frac{1}{2}\rho|u|^2(t,x)+\frac{\epsilon a}{\gamma-1}\rho^\gamma(t,x)\big]\,dx+\int_0^t\int_{\mathbb{R}^N}2\mu(\rho)|Du|^2dxdt\\ &\quad+\int_0^t\int_{\mathbb{R}^N}\lambda(\rho)|\mathrm{div}u|^2dxdt=\int_{\mathbb{R}^N}\big[\rho_0|u_0|^2(x)+\frac{\epsilon a}{\gamma-1}\rho_0^\gamma(x)\big]\,dx.\end{aligned}\tag{3.41}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\big[\frac{1}{2}\rho|u+\nabla\varphi(\rho)|^2(t,x)+\frac{\epsilon a}{\gamma-1}\rho^\gamma(t,x)\big]\,dx+\frac{1}{2}\int_0^t\int_{\mathbb{R}^N}\mu(\rho)|\nabla u-{}^t\nabla u|^2dxdt\\ &+\epsilon a\int_0^t\int_{\mathbb{R}^N}\nabla\rho^\gamma\cdot\nabla\varphi(\rho)(s,x)ds\,dx=\int_{\mathbb{R}^N}\big[\frac{1}{2}\rho_0|u_0+\nabla\varphi(\rho_0)|^2(x)\\ &\qquad+\frac{\epsilon a}{\gamma-1}\rho_0^\gamma(t,x)\big]\,dx.\end{aligned}\tag{3.42}$$

Proof: We refer to [6] for the proof. □

Proposition 3.5 *Assume that:*

$$2\mu(\rho)+N\lambda(\rho)\geq\nu_1\lambda(\rho)$$

for some $\nu\in(0,1)$ (which is a part of (1.6)). Then it exists $C>0$ such that smooth solutions of (1.10) satisfy the following inequality:

$$\begin{aligned}&\frac{d}{dt}\int\rho\frac{1+|u|^2}{2}\ln(1+|u|^2)dx+\frac{\nu}{2}\int\mu(\rho)[1+\ln(1+|u|^2)|Du|^2dx\leq\\ &C\int\mu(\rho)|\nabla u|^2dx+C\epsilon^2\Big(\int\big(\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)}\big)^{\frac{2}{2-\delta}}dx\Big)^{\frac{2}{2-\delta}}\big(\int(\rho|u|^2+\rho)dx\big)^{\frac{\delta}{2}}.\end{aligned}\tag{3.43}$$

for any $\delta\in(0,2)$.

Proof: The proof follows the same lines than the lemma 3.2 in [29], for the sake of completeness let us adapt this proof to our situation. Multiplying the momentum equation by $\big(1+\ln(1+|u|^2)\big)u$, we get as in (3.39):

$$\begin{aligned}&\int \rho \frac{d}{dt}\big[\frac{1+|u|^2}{2}\ln(1+|u|^2)\big]dx + \int \rho u\cdot\nabla\big(\frac{1+|u|^2}{2}\ln(1+|u|^2)\big)dx\\ &+\nu_1\int \mu(\rho)(1+\ln(1+|u|^2)|D(u)|^2dx \leq -a\epsilon\int[1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma dx\\ &\qquad\qquad+C\int\mu(\rho)|\nabla u|^2dx.\end{aligned}\tag{3.44}$$

It remains to bound the right hand side. We have:

$$\begin{aligned}&|\epsilon\int[1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma dx|\\ &\leq \epsilon|\int\frac{2u_iu_k}{1+|u|^2}\partial_iu_k\rho^\gamma dx|+\epsilon|\int[1+\ln(1+|u|^2)]\mathrm{div}u\rho^\gamma dx|,\\ &\leq 2\epsilon(\int\mu(\rho)|\nabla u|^2dx)^{\frac{1}{2}}(\int\frac{\rho^{2\gamma}}{\mu(\rho)}dx)^{\frac{1}{2}}+\epsilon|\int[1+\ln(1+|u|^2)]\mathrm{div}u\rho^\gamma dx|.\end{aligned}$$

Let us deal with the last term on the right hand side:

$$\begin{aligned}&\epsilon|\int[1+\ln(1+|u|^2)]\mathrm{div}u\rho^\gamma dx|\leq\\ &\leq\epsilon(\int[1+\ln(1+|u|^2)]\mu(\rho)(\mathrm{div}u)^2dx)^{\frac{1}{2}}(\int[1+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{\mu(\rho)}dx)^{\frac{1}{2}},\\ &\qquad+\frac{\epsilon^2}{2\nu_1}(\int[1+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{\mu(\rho)}dx).\end{aligned}$$

We deduce that it exists $C>0$ such that:

$$\begin{aligned}|\epsilon\int[1+\ln(1+|u|^2)]u\cdot\nabla\rho^\gamma dx|\leq\int\mu(\rho)|\nabla u|^2dx+\frac{\nu_1}{2}(\int[1+\ln(1+|u|^2)]\mu(\rho)(\mathrm{div}u)^2dx)\\ +\frac{C\epsilon^2}{2\nu_1}(\int[2+\ln(1+|u|^2)]\frac{\rho^{2\gamma}}{\mu(\rho)}dx)\end{aligned}$$

where the last term satisfies (if $\delta\in(0,2)$) for a $C>0$:

$$\begin{aligned}\epsilon^2\int\frac{\rho^{2\gamma}}{h(\rho)}|u|^\delta dx&\leq\epsilon^2\Big(\int(\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)})^{\frac{2}{2-\delta}}dx\Big)^{\frac{2}{2-\delta}}(\int\rho[2+\ln(1+|u|^2)]^{\frac{2}{\delta}}dx)^{\frac{\delta}{2}},\\ &\leq C\epsilon^2\Big(\int(\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)})^{\frac{2}{2-\delta}}dx\Big)^{\frac{2}{2-\delta}}(\int(\rho|u|^2+\rho)dx)^{\frac{\delta}{2}}\end{aligned}$$

and the proposition follows. □

3.3 Basic results of compactness

We would like to finish this section by giving very useful propositions of compactness that we shall often apply. We are going to recall the so-called Aubin-Lions theorem.

Proposition 3.6 *Let $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$ be Banach spaces (with X which is compactly imbedded in B) and $(f_n)_{n\in\mathbb{N}}$ a sequence bounded in $L^q((0,T),B)\cap L^1((0,T),X)$ (with $1 < q \leq +\infty$) and $(\frac{d}{dt}f_n)_{n\in\mathbb{N}}$ bounded in $L^1((0,T),Y)$. Then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^p((0,T),B)$ for any $1 \leq p < q$.*

Let us recall now the theorem of Arzèla-Ascoli.

Proposition 3.7 *Let B and X Banach spaces such that $B \hookrightarrow\hookrightarrow X$ is compact. Let f_N be a sequence of functions $\bar{I} \rightarrow B$ (with I an interval) uniformly bounded in B and uniformly continuous in X. Then there exists $f \in C^0(\bar{I},B)$ such that $f_n \rightarrow f$ strongly in $f \in C^0(\bar{I},X)$ up to a subsequence.*

Lemma 1 *Let K a compact subset of $\mathbb{R}^N$ (with $N \geq 1$) and v^ϵ a sequel such that:*

- *v^ϵ is uniformly bounded in $L^{1+\alpha}(K)$ with $\alpha > 0$,*
- *v^ϵ converge almost everywhere to v,*

then v^ϵ converges strongly to v in $L^1(K)$ with $v \in L^{1+\alpha}(K)$.

Proof: First by the Fatou lemma v is in $L^{1+\alpha}(K)$. Next we have for any $M > 0$:

$$\int_K |v^\epsilon - v|dx \leq \int_{K\cap\{|v^\epsilon - v|\leq M\}} |v^\epsilon - v|dx + \int_{K\cap\{|v^\epsilon - v|\geq M\}} |v^\epsilon - v|dx. \tag{3.45}$$

We are dealing with the second member of the right hand side, by Hölder inequality and Tchebychev lemma we have for a $C > 0$:

$$\begin{aligned}\int_{K\cap\{|v^\epsilon - v|\geq M\}} |v^\epsilon - v|dx &\leq (\int_K |v^\epsilon - v|^{1+\alpha}dx)^{\frac{1}{1+\alpha}}(\{|v^\epsilon - v| \geq M\})^{\frac{\alpha}{1+\alpha}},\\ &\leq \frac{C}{M^{\frac{\alpha}{1+\alpha}}}.\end{aligned} \tag{3.46}$$

In particular we have shown the strong convergence of v^ϵ to v, indeed from the inequality (3.45) it suffices to use the Lebesgue theorem for the first term on the right hand side and the estimate (3.46) with M going to $+\infty$. $\square$

Lemma 2 *Let $f \in \dot{H}^s$ with $s > 0$ and $f \in L^p + L^2$ with $1 \leq p < 2$. Then $f \in L^2$.*

Proof: Indeed we have as $f \in \dot{H}^s$:

$$\int_{\mathbb{R}^N} |\xi|^{2s}|\widehat{f}|^2 d\xi < +\infty,$$

so $\widehat{f}1_{\{|\widehat{f}|\geq 1\}} \in L^2(\mathbb{R}^N)$. And as $f = f_1 + f_2$ with $f_1 \in L^p(\mathbb{R}^N)$ and $f_2 \in L^2$. By using the Riesz-Thorin theorem, we know that $\widehat{f_1} \in L^q(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$. As $q \geq 2$ we then have $\widehat{f}1_{\{|\widehat{f}|\leq 1\}} \in L^2(\mathbb{R}^N)$. This concludes the proof. $\square$

4 Proof of theorem 1.1

Let us assume in a first time that the solution (ρ, u) of system (1.10) are classical, we are going to search solution under the form: $(\rho, -\nabla\varphi(\rho))$. The mass equation give us:

$$\partial_t\rho - \mathrm{div}(\rho\nabla\varphi(\rho)) = 0 \tag{4.47}$$

Since $\varphi^{'}(\rho) = \frac{2\mu^{'}(\rho)}{\rho}$ we get:

$$\partial_t\rho - 2\Delta\mu(\rho) = 0 \tag{4.48}$$

Let us check that the second equation is compatible with the first and keep an irrotational structure. First we have:

$$\partial_t(\rho u) = -\partial_t(\rho\nabla\varphi(\rho)) = -2\nabla\partial_t(\mu(\rho)).$$

$$\begin{aligned} -2\mathrm{div}(\mu(\rho)Du) &= 2\mathrm{div}(\mu(\rho)\nabla\nabla\varphi(\rho)), \\ &= 2\mu(\rho)\nabla\Delta\varphi(\rho) + 2\nabla\mu(\rho)\cdot\nabla\nabla\varphi(\rho). \end{aligned} \tag{4.49}$$

$$-\nabla(\lambda(\rho)\mathrm{div}u) = -\nabla(\lambda(\rho)\Delta\varphi(\rho)).$$

Next we have:

$$\begin{aligned} \mathrm{div}(\rho u\otimes u)_j &= \sum_i \partial_i(\rho u_i u_j) = \sum_i \partial_i(\rho\partial_i\varphi(\rho)\partial_j\varphi(\rho)), \\ &= \sum_i \partial_i(\rho\varphi^{'}(\rho)\partial_j\rho\partial_i\varphi(\rho)) = 2\sum_i \partial_i(\partial_j\mu(\rho)\partial_i\varphi(\rho)), \\ &= 2\Delta\varphi(\rho)\partial_j\mu(\rho) + 2(\nabla\varphi(\rho)\cdot\nabla\nabla\mu(\rho))_j. \end{aligned}$$

We have then:

$$\mathrm{div}(\rho u\otimes u) = 2\Delta\varphi(\rho)\nabla\mu(\rho) + 2\nabla\mu(\rho)\cdot\nabla\nabla\varphi(\rho). \tag{4.50}$$

Combining (4.49) and (4.50) we obtain:

$$\begin{aligned} \mathrm{div}(\rho u\otimes u) - 2\mathrm{div}(\mu(\rho)Du) &= 2\Delta\varphi(\rho)\nabla\mu(\rho) + 2\nabla\mu(\rho)\cdot\nabla\nabla\varphi(\rho) \\ &\quad + 2\mu(\rho)\nabla\Delta\varphi(\rho) + 2\nabla\mu(\rho)\cdot\nabla\nabla\varphi(\rho), \\ = 2\nabla(\mu(\rho)\Delta\varphi(\rho)) + 2\nabla(\nabla\mu(\rho)\cdot\nabla\varphi(\rho)). \end{aligned} \tag{4.51}$$

Finally using (4.49), (4.51) and the fact that $\lambda(\rho) + 2\mu(\rho) = 2\mu^{'}(\rho)$, we obtain:

$$\begin{aligned} &\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - 2\mathrm{div}(\mu(\rho)Du) - \nabla(\lambda(\rho)\mathrm{div}u) = \\ &-\nabla\big(2\partial_t\mu(\rho) - 2\mu(\rho)\Delta\varphi(\rho) - 2\nabla\mu(\rho)\cdot\nabla\varphi(\rho) - \lambda(\rho)\Delta\varphi(\rho)\big), \\ &-\nabla\big(2\partial_t\mu(\rho) - 2\rho\mu^{'}(\rho)\Delta\varphi(\rho) - 2\nabla\mu(\rho)\cdot\nabla\varphi(\rho)\big). \end{aligned} \tag{4.52}$$

Next we have since $\mu^{'}(\rho) = \frac{1}{2}\rho\varphi^{'}(\rho)$:

$$\begin{aligned} \Delta\mu(\rho) &= \sum_i \partial_{ii}\mu(\rho) = \sum_i \frac{1}{2}\partial_i(\rho\varphi^{'}(\rho)\partial_i\rho), \\ &= \frac{1}{2}\sum_i \partial_i(\rho\partial_i\varphi(\rho)), \\ &= \frac{1}{2}\rho\Delta\varphi(\rho) + \frac{1}{2}\nabla\rho\cdot\nabla\varphi(\rho). \end{aligned}$$

and:

$$\mu^{'}\rho)\Delta\mu(\rho) = \frac{1}{2}\rho\Delta\varphi(\rho) + \frac{1}{2}\mu^{'}(\rho)\nabla\rho\cdot\nabla\varphi(\rho),$$
$$= \frac{1}{2}\rho\Delta\varphi(\rho) + \frac{1}{2}\nabla\mu(\rho)\cdot\nabla\varphi(\rho). \tag{4.53}$$

In particular from (4.53) we have:

$$4\mu^{'}(\rho) = 2\rho\Delta\varphi(\rho) + 2\nabla\mu(\rho)\cdot\nabla\varphi(\rho) \tag{4.54}$$

Combining (4.55) and (4.54) we have:

$$\partial_t(\rho u) + \text{div}(\rho u\otimes u) - 2\text{div}(\mu(\rho)Du) - \nabla(\lambda(\rho)\text{div}u)$$
$$= -\nabla\big(2\mu^{'}(\rho)(\partial_t\rho - 2\Delta\mu(\rho))\big). \tag{4.55}$$

This concludes the proof inasmuch as via the above equation the momentum equation is compatible to the mass equation and must verify the equation (4.48).
But when we take initial density in L^1 non negative and continuous, we know via the theorem 2.6 that the unique global solution of (4.48) is classical and non negative. It justify in particular all the previous formal calculus and prove that $(\rho, u = -\nabla\varphi(\rho))$ is a classical solution of (1.10) with ρ verifying (4.48). It concludes the proof.
Furthermore the different properties on ρ are a direct consequence of theorem 2.4, 2.6 and 2.7.

5 Proof of the theorems 1.2

We now present the proof of theorem 1.1 extending to general viscosity coefficient the results of [29] in the case of the quasi solutions. Let us begin with recalling the assumptions on the initial data. Indeed we assume that it exits a sequence (ρ_n, u_n) of regular global weak solution verifying the system (1.24) (or at least an approximated system, typically by using Friedrich approximations).

Initial data:

In particular the initial data $\rho_0^n, u_0^n)$ must uniformly in n satisfy (1.25), and (1.26) in order to verify the entropy inequalities from section 3, more precisely we shall have:

- ρ_0^n is bounded in $L^1(\mathbb{R}^N)$, $\rho_0^n \geq 0$ a.e in $\mathbb{R}^N$,
- $\rho_0^n|u_0^n|^2$ is bounded in $L^1(\mathbb{R}^N)$,
- $\sqrt{\rho^n}\nabla\varphi(\rho_0^n) = \nabla f(\rho^n)$ is bounded in $L^2(\mathbb{R}^N)$,
- $\rho_0^n|u_0^n|^2\ln(1+|u_0^n|^2)$ is bounded in $L^1(\mathbb{R}^N)$.

With those assumptions, and using the entropy inequalities (3.36), (3.37) and the mass equation, we have the following bounds:

$$\begin{aligned}
&\|\sqrt{\rho_n}\|_{L^\infty((0,T),L^2(\mathbb{R}^N))} \leq C,\\
&\|\sqrt{\rho_n}u_n\|_{L^\infty((0,T),L^2(\mathbb{R}^N))} \leq C,\\
&\|\nabla f(\rho_n)\|_{L^\infty((0,T),L^2(\mathbb{R}^N))} \leq C,\\
&\|\sqrt{\rho_n}\nabla u_n\|_{L^2((0,T)\times\mathbb{R}^N)} \leq C,
\end{aligned} \tag{5.56}$$

and

$$\| \rho_n |u_n|^2 \ln(1+|u_n|^2) \|_{L^\infty((0,T),L^1(\mathbb{R}^N))} \leq C. \tag{5.57}$$

Remark 24 *Let us point out that the gain of integrability on u_n in (5.57) will be a direct consequence of a gain of integrability on the pressure with some restriction on γ, ν_1 and ν_2.*

The proof of theorem 1.1 will be derived in three steps and follows some arguments developed in [29]. In the first step, we deal with the strong convergence of the density $(\rho_n)_{n\in\mathbb{N}}$ which enables us to show the convergence almost everywhere of $(\rho_n)_{n\in\mathbb{N}}$ us to a subsequence. We shall also prove the strong convergence of a momentum sequel of the form $\sqrt{\rho_n}h(\rho_n)u_n$ with a function h to precise to a function $\sqrt{\rho}h(\rho)u$. In the second step we derive the strong convergence of $\sqrt{\rho_n}u_n$ to $\sqrt{\rho}u$ in $L^2_{loc}((0,T)\times\mathbb{R})$ (it allows us to give sense to the momentum product $\rho_n u_n \otimes u_n$) by taking advantage of the uniform gain of integrability on un via the entropy inequality (3.38). Indeed it will suffice to use the lemma 1 after proving almost everywhere convergence via Sobolev injection. In this part, we also shall deal with the strong convergence in the distribution sense of the product $\sqrt{\rho_n}\sqrt{\rho_n}u_n$. In the last step we will treat the diffusion term which will achieve the proof of the theorem 1.2.

Step 1: Convergence almost everywhere on ρ_n and $\rho_n u_n$

We are going to begin with proving a technical lemma giving uniforms estimates on ρ_n via the entropy (3.36) and (3.37).

Lemma 3 *When $N=2,3$ $\nabla(\frac{\mu(\rho_n)}{\sqrt{\rho_n}})$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for any $T>0$.*

- *When $N=3$ it implies that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$, $\frac{\lambda(\rho_n)}{\sqrt{\rho_n}}$ are uniformly bounded in $L^\infty((0,T),L^6(\mathbb{R}^N))$ for any $T>0$ which gives:*

$$\rho_n^{\frac{1}{6}+\frac{\nu_1}{2N}} \text{ is uniformly bounded in } L^\infty(0,T;L^6(\mathbb{R}^N)). \tag{5.58}$$

- *When $N=2$ we distinguish two cases:*
  - *$\nu_2 \geq 2$, ρ_n is uniformly bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $q\in[1,+\infty[$ and any $T>0$. It implies that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ and $\frac{\lambda(\rho_n)}{\sqrt{\rho_n}}$ are uniformly bounded in $L^\infty((0,T),L^q(K))$ for any compact K.*
  - *$0<\nu_2<2$, ρ_n is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $T>0$ and any $q\in[1,+\infty[$. It implies that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ and $\frac{\lambda(\rho_n)}{\sqrt{\rho_n}}$ are uniformly bounded in $L^\infty((0,T),L^q(K))$ for any compact K.*

Proof: When $N=3$, we observe that:

$$\begin{aligned}\nabla(\frac{\mu(\rho)}{\sqrt{\rho}}) &= 2\mu^{'}(\rho)\nabla\sqrt{\rho} - \frac{\mu(\rho)}{2\rho^{\frac{3}{2}}}\nabla\rho,\\ &= \frac{1}{2}\nabla f(\rho) - \frac{\mu(\rho)}{2\rho^{\frac{3}{2}}}\nabla\rho.\end{aligned}$$

and from conditions (1.5), (1.6) and the fact that $\mu(\rho)\geq 0$ we obtain:

$$2\mu^{'}(\rho)\rho=\lambda(\rho)+2\mu(\rho)=\frac{3\lambda(\rho)+2\mu(\rho)}{3}\geq\frac{\nu}{3}\mu(\rho). \tag{5.59}$$

We deduce that:

$$\begin{aligned}|\frac{\mu(\rho)}{2\rho^{\frac{3}{2}}}\nabla\rho|&\leq\frac{3}{\nu}|\frac{\mu^{'}(\rho)}{\sqrt{\rho}}||\nabla\rho|,\\&\leq\frac{3}{2\nu}|\varphi^{'}(\rho)\sqrt{\rho}||\nabla\rho|,\\&\leq\frac{3}{2\nu}|\nabla f(\rho)|.\end{aligned}$$

It provides that

$$|\nabla(\frac{\mu(\rho)}{\sqrt{\rho}})|\leq C|\nabla f(\rho)|.$$

It implies by energy estimates that:

$$\|\nabla(\frac{\mu(\rho_n)}{\sqrt{\rho_n}})\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}\leq C.$$

Sobolev's embedding ensured that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^6(\mathbb{R}^N))$. Next by using (1.9) we have:

$$\begin{aligned}&C\rho_n^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_1}{2N}}\leq\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\quad\text{when}\;\;\rho_n>1,\\&C\rho_n^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_2}{2N}}\leq\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\quad\text{when}\;\;\rho_n\leq 1.\end{aligned}$$

It implies in particular $\rho_n^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^\infty(0,T;L^6(\mathbb{R}^N))$ (this is due to the fact that $\rho_n 1_{\{\rho_n\leq 1\}}$ is uniformly bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))\cap L^\infty(0,T;L^\infty(\mathbb{R}^N))$ and that $3-\frac{6}{N}+\frac{2\nu_1}{N}\geq 1$ when $N=3$).

When $N=2$ by following the same proof than in the case $N=3$ we obtained that $\nabla(\frac{\mu(\rho)}{\sqrt{\rho}})$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))$ for any $T>0$. By using in a similar way (5.59) when $N=2$ and (1.9) we have:

$$\begin{aligned}&C\rho_n^{-1+\frac{\nu_1}{4}}|\nabla\rho_n|\leq 2|\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}||\nabla\rho_n|=|\nabla f(\rho_n)|\quad\text{when}\;\;\rho_n>1,\\&C\rho_n^{-1+\frac{\nu_2}{4}}|\nabla\rho_n|\leq 2|\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}||\nabla\rho_n|=|\nabla f(\rho_n)|\quad\text{when}\;\;\rho_n\leq 1.\end{aligned} \tag{5.60}$$

When $\nu_1\geq 2$, choosing $\psi\in C_0^\infty(\mathbb{R}^N)$ with $\psi=1$ on $B(0,1)$ and $\operatorname{supp}\psi$ included in $B(0,2)$ we have: $(1-\psi(\rho_n))\nabla\sqrt{\rho_n}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for any $T>0$. Since $\sqrt{\rho_n}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for any $T>0$ we deduce that $(1-\psi(\rho_n))\sqrt{\rho_n}$ is uniformly bounded in $L^\infty_T(H^1(\mathbb{R}^N))$ for any $T>0$. It implies that $(1-\psi(\rho_n))\rho_n$ is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $q\in[1,+\infty[$ by Sobolev embedding. Let us deal now with the term $\psi(\rho_n)\rho_n$ which is bounded in $L^\infty_T(L^1(\mathbb{R}^N))\cap L^\infty_T(L^\infty(\mathbb{R}^N))$, it proves that

ρ_n is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $q \in [1,+\infty[$. Via (1.9) It implies that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ and $\frac{\lambda(\rho_n)}{\sqrt{\rho_n}}$ are uniformly bounded in $L^\infty((0,T),L^q(\mathbb{R}^N))$ for any compact K.

Let us deal with the case $0<\nu_2<2$. By using (5.60) we show that $\nabla\big((1-\psi(\rho_n))\rho_n^{\frac{\nu_1}{4}}\big)$ is bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ and $(1-\psi(\rho_n))\rho_n^{\frac{\nu_1}{4}}$ is bounded in $L^\infty_T(L^{\frac{4}{\nu_1}}(\mathbb{R}^N))$. Since by Tchebytchev lemma $(1-\psi(\rho_n))\rho_n^{\frac{\nu_1}{4}}$ is strictly positive only on a set of finite measure it implies that $(1-\psi(\rho_n))\rho_n^{\frac{\nu_1}{4}}$ is also bounded in $L^\infty_T(L^2(\mathbb{R}^N))$. We deduce that $(1-\psi(\rho_n))\rho_n^{\frac{\nu_1}{4}}$ is bounded in $L^\infty_T(H^1(\mathbb{R}^N))$. By Sobolev embedding it yields that $(1-\psi(\rho_n))\rho_n^{\frac{\nu_1}{4}}$ is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $T>0$ and any $q\in[1,+\infty[$. Since $\psi(\rho_n)\rho_n$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$ for any $T>0$ we conclude that ρ_n is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $T>0$ and any $q\in[1,+\infty[$.

The proof is similar for $\frac{\lambda(\rho_n)}{\sqrt{\rho_n}}$ by using the remarks 7. □

Lemma 1 *If $\mu(\rho)$, $\lambda(\rho)$ satisfies (1.5), (1.6) and in addition we assume that $g(x)=\frac{\mu(x)}{\sqrt{x}}$ is a bijective function on $[0,+\infty)$ and that g^{-1} is continuous, then when we distinguish the two following cases, we have:*

- $\nu_1 \geq 2$
  1. *$\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty(0,T;H^1_{loc}(\mathbb{R}^N))$*
  2. *$\partial_t\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^2(0,T;W^{-1,2}_{loc}(\mathbb{R}^N))$.*

  As a consequence up to a subsequence (via the Cantor's diagonal process) $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges almost everywhere and strongly in $C([0,T],L^2_{loc}(\mathbb{R}^N)))$ to v. In particular it implies that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges up to a subsequence almost everywhere to v and we define ρ as follows:
  $$\rho = g^{-1}(v).$$
  It implies that ρ_n converge up to a subsequence almost everywhere to ρ.

- $0<\nu_1<2$
  Let us consider:
  $$\beta(\rho_n)=\psi(\rho_n)\sqrt{\rho_n}+(1-\psi(\rho_n))\rho_n^{\alpha_1},$$
  with $0<\alpha_1<\min(\frac{1}{3},\frac{\nu_1}{4N})$.
  1. *$\beta(\rho_n)$ is bounded in $L^\infty(0,T;H^1(\mathbb{R}^N))$*
  2. *$\partial_t\beta(\rho_n)$ is bounded in $L^2(0,T;W^{-1,1}(\mathbb{R}^N))$.*

  As a consequence up to a subsequence, $\beta(\rho_n)$ converge almost everywhere and strongly in $C([0,T],L^2_{loc}(\mathbb{R}^N)))$ to v. We define ρ by:
  $$\rho=\beta^{-1}(\rho).$$
  In particular we have that ρ_n converge up to a subsequence almost everywhere to ρ.

Furthermore ρ_n converge strongly to ρ in $C([0,T],L^{1+\alpha}_{loc}(\mathbb{R}^N))$ if $N=3$ with $\alpha>0$ small enough and in $C([0,T],L^p_{loc}(\mathbb{R}^N))$ for any $p\geq 1$ if $N=2$. This last result is under the following hypothesis:

- *When $2+N\leq\nu_1$, we assume that g and $g^{'}$ are increasing on $(0,+\infty)$.*

Remark 25 *Let us point out that we could weaken the last assumption on g when $\nu_1\geq 2$ by assuming that g and $g^{'}$ are only increasing in a neighbor of 0 and of $+\infty$. As mentioned above, let us point out that this last condition is quite natural since this is true when we set $\mu(\rho)=\mu\rho^\alpha$ with $\alpha\geq\frac{3}{2}$.*

Proof: Let $T>0$. We are going to distinguish two case when $\nu_1\geq 2$ and when $0<\nu_1<2$.

$\bullet\nu_1\geq 2$

The first estimate is a direct consequence of the lemma 3 in the appendix. Indeed we know that $\nabla\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty_T(L^2(\mathbb{R}^N))$. Easily we deduce by lemma 3 that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty_T(L^2(K))$ for any compact K. Next we observe that:

$$\begin{aligned}\partial_t(\frac{\mu(\rho_n)}{\sqrt{\rho_n}})&=-\text{div}(\frac{\mu(\rho_n)}{\sqrt{\rho_n}}u_n)+\big(\mu^{'}(\rho_n)\sqrt{\rho_n}-\frac{3}{2}\frac{\mu(\rho_n)}{\sqrt{\rho_n}})\text{div}u_n,\\&=-\text{div}(\frac{\mu(\rho_n)}{\rho_n}\sqrt{\rho_n}u_n)+\big(\frac{\mu^{'}(\rho_n)\sqrt{\rho_n}}{\sqrt{\mu(\rho_n)}}-\frac{3}{2}\sqrt{\frac{\mu(\rho_n)}{\rho_n}}\big)\sqrt{\mu(\rho_n)}\text{div}u_n.\end{aligned}\tag{5.61}$$

Let us start with estimating the first term on the right hand side of (5.61). According to (1.9) we know that:

$$\frac{\psi(\rho_n)\mu(\rho_n)}{\rho_n}\leq C\psi(\rho_n)\rho_n^{\frac{\nu_1}{2N}-\frac{1}{N}}\;\;\text{when}\;\;\rho_n\leq 1.$$

Since $\frac{\nu_1}{2N}-\frac{1}{N}\geq 0$ it implies in particular that $\frac{\psi(\rho_n)\mu(\rho_n)}{\rho_n}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. In order to have local estimates, it suffices to deal with the region when $\rho_n>1$, it means $\frac{(1-\psi(\rho_n))\mu(\rho_n)}{\rho_n}$.
It suffices to use the lemma 3 which insures that when $N=3$ $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ belongs in $L^\infty_T(L^6(\mathbb{R}^N))$ and we deduce that $\frac{(1-\psi(\rho_n))\mu(\rho_n)}{\rho_n}$ is bounded in $L^\infty_T(L^6(\mathbb{R}^N))$. Finally we have obtain that $\frac{\mu(\rho_n)}{\rho_n}$ is bounded in $L^\infty_T(L^2_{loc}(\mathbb{R}^N))$ when $N=3$. It yields the uniform boundness of $\frac{\mu(\rho_n)}{\rho_n}\sqrt{\rho_n}u_n$ in $L^\infty_T(L^1_{loc}(\mathbb{R}^N))$ when $N=3$. A similar argument insure the same result when $N=2$.
By proceeding similarly we also prove that $\sqrt{\frac{\mu(\rho_n)}{\rho_n}}\sqrt{\mu(\rho_n)}\text{div}u_n$ is uniformly bounded in $L^\infty_T(L^1_{loc}(\mathbb{R}^N))$. Indeed following the same argument than for the previous term, we know that for $N=2,3$ via the lemma 3 $\sqrt{\frac{\mu(\rho_n)}{\rho_n}}$ is bounded in $L^\infty_T(L^{12}(K))$ for any compact K and since $\sqrt{\mu(\rho_n)}\text{div}u_n$ is bounded in $L^2_T(L^2(\mathbb{R}^N))$, Hölder(s inequality give the desired

result.
Let us now deal with the term $\frac{\mu^{'}(\rho_n)\sqrt{\rho_n}}{\sqrt{\mu(\rho_n)}}\sqrt{\mu(\rho_n)}\text{div}u_n$. A simple calculus using (1.5) give us:

$$\frac{\mu^{'}(\rho_n)\sqrt{\rho_n}}{\sqrt{\mu(\rho_n)}}=\frac{1}{2}(\frac{\lambda(\rho_n)}{\sqrt{\rho_n\mu(\rho_n)}}+2\sqrt{\frac{\mu(\rho_n)}{\rho_n}}).$$

We have only to deal with the term $\frac{\lambda(\rho_n)}{\sqrt{\rho_n\mu(\rho_n)}}$, the other one has been estimated. By the remarks 8 we know that it exists $C>0$ such that:

$$|\lambda(\rho)|\leq C\mu(\rho)\ \ \forall\rho>0.$$

It implies that:

$$|\frac{\lambda(\rho_n)}{\sqrt{\rho_n\mu(\rho_n)}}|\leq C\sqrt{\frac{\mu(\rho_n)}{\rho_n}}.$$

And as we know that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty_T(L^6_{loc}(\mathbb{R}^N))$ for $N=2,3$ via the lemma 3, it achieves the proof of the second estimates.
By the Ascoli's theorem, the fact that the application $u\rightarrow\phi u$ with $\phi\in C^\infty_0(\mathbb{R}^N)$ is compact from $H^1(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$ and the Cantor's diagonal process it entails that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges strongly up to a subsequence in $C([0,T],L^2_{loc}(\mathbb{R}^N)))$ to $v_1=\frac{\mu(\rho)}{\sqrt{\rho}}$ (and in particular in $L^2_{loc}((0,T)\times\mathbb{R}^N)$). We shall define ρ in the sequel by:

$$\rho=g^{-1}(\frac{\mu(\rho)}{\sqrt{\rho}}).$$

Furthermore an immediate consequence is that up to a subsequence $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converge almost everywhere to $\frac{\mu(\rho)}{\sqrt{\rho}}$. And since g^{-1} is continuous, it implies that up to a subsequence ρ_n converges almost everywhere to ρ.

- $0<\nu_1<2$

In this case we are going to study:

$$\beta(\rho_n)=\psi(\rho_n)\sqrt{\rho_n}+(1-\psi(\rho_n))\rho_n^{\alpha_1},$$

with α_1 to choose suitably. Let us start with dealing with $\psi(\rho_n)\sqrt{\rho_n}$, we have then:

$$\nabla(\psi(\rho_n)\sqrt{\rho_n})=\psi^{'}(\rho_n)\sqrt{\rho_n}\nabla\rho_n+\frac{\psi(\rho_n)}{\mu^{'}(\rho_n)}\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n$$

The first term is easily bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ via the entropy (3.37) since the support of $\psi^{'}$ is included in the shell $C(0,1,2)$.The second term is also bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ by observing that $\frac{\psi(\rho_n)}{\mu^{'}(\rho_n)}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ via (1.9), (1.5), (1.6) and $\nu_1\leq 2$. Indeed we have:

$$|\frac{\psi(\rho_n)}{\mu^{'}(\rho_n)}|\leq\psi(\rho_n)\rho_n^{\frac{1}{N}-\frac{\nu_1}{2N}}.$$

The conservation of mass provides that $\|\rho_n\|_{L^1} = \|\rho_0^n\|_{L^1}$which implies the $L^\infty(0,T;H^1)$ bound on $\psi(\rho_n)\sqrt{\rho_n}$. Next we observe that:

$$\partial_t(\psi(\rho_n)\sqrt{\rho_n}) = -\mathrm{div}(\psi(\rho_n)\sqrt{\rho_n}u_n) + (\frac{1}{2}\psi(\rho_n)\sqrt{\rho_n} - \psi^{'}(\rho_n)\rho_n^{\frac{3}{2}})\mathrm{div}u_n. \tag{5.62}$$

The first term on the right hand side is obviously bounded in $L^\infty_T(W^{-1,2}(\mathbb{R}^N))$. The two last terms in (5.62) are bounded in $L^2_T(W^{-1,2}(\mathbb{R}^N))$. It implies finally that $\psi(\rho_n)\sqrt{\rho_n}$ are uniformly bounded in $L^\infty_T(H^1(\mathbb{R}^N))$ and $\partial_t(\psi(\rho_n)\sqrt{\rho_n})$ in $L^2_T(W^{-1,2}(\mathbb{R}^N))$.
Let us deal now with the term $(1-\psi(\rho_n))\rho_n^{\alpha_1}$, we have then:

$$\nabla((1-\psi(\rho_n))\rho_n^{\alpha_1}) = - = \psi^{'}(\rho_n)\rho_n^{\alpha_1}\nabla\rho_n + (1-\psi(\rho_n))\frac{\rho_n^{\alpha_1-\frac{1}{2}}}{\mu^{'}(\rho_n)}\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n. \tag{5.63}$$

By (1.5), (1.6) and (1.9) we show that:

$$|(1-\psi(\rho_n))\frac{\rho_n^{\alpha_1-\frac{1}{2}}}{\mu^{'}(\rho_n)}| \leq (1-\psi(\rho_n))\rho_n^{\alpha_1-\frac{1}{2}+\frac{1}{N}-\frac{\nu_1}{2N}}.$$

Since $0 < \alpha_1 \leq \frac{\nu_1}{2N}$ it implies that $(1-\psi(\rho_n))\frac{\rho_n^{\alpha_1-\frac{1}{2}}}{\mu^{'}(\rho_n)}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ and $(1-\psi(\rho_n))\frac{\rho_n^{\alpha_1-\frac{1}{2}}}{\mu^{'}(\rho_n)}\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n$ is bounded in $L^\infty_T(L^2(\mathbb{R}^N))$. The first term on (5.63) is easy to deal with. Now we have:

$$\begin{aligned}\partial_t((1-\psi(\rho_n))\rho_n^{\alpha_1}) = -\mathrm{div}((1-\psi(\rho_n))\rho_n^{\alpha_1}u_n) - (-\psi^{'}(\rho_n)\frac{\rho_n^{\alpha_1+1}}{\sqrt{\mu(\rho_n)}} \\ + (\alpha_1-1)(1-\psi(\rho_n))\frac{\rho_n^{\alpha_1}}{\sqrt{\mu(\rho_n)}})\sqrt{\mu(\rho_n)}\mathrm{div}u_n.\end{aligned} \tag{5.64}$$

The first term on the right hand side of (5.64) is bounded in $L^\infty_T(W^{-1,1}(\mathbb{R}^N))$ since $(1-\psi(\rho_n))\rho_n^{\alpha_1}$ is bounded in $L^\infty(L^2(\mathbb{R}^N))$ because $\alpha_1 \leq \frac{1}{2}$. We have now according to (1.9)

$$|(1-\psi(\rho_n))\frac{\rho_n^{\alpha_1}}{\sqrt{\mu(\rho_n)}}| \leq (1-\psi(\rho_n))\rho_n^{\alpha_1-\frac{1}{2}+\frac{1}{2N}-\frac{\nu_1}{4N}}.$$

Since $\alpha_1 \leq \frac{1}{3}$ it implies that this term is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ which shows the bound of $(1-\psi(\rho_n))\frac{\rho_n^{\alpha_1}}{\sqrt{\mu(\rho_n)}}\sqrt{\mu(\rho_n)}\mathrm{div}u_n$ in $L^2_T(L^2(\mathbb{R}^N))$. The second term of 5.64 is easy to treat.
Finally we have proved that $(1-\psi(\rho_n))\rho_n^{\alpha_1}$ is uniformly bounded in $L^\infty_T(H^1(\mathbb{R}^N))$ and $\partial_t((1-\psi(\rho_n))\rho_n^{\alpha_1})$ is also in $L^2_T(W^{-1,2}(\mathbb{R}^N))$. In conclusion it shows that $\beta(\rho_n)$ is bounded in $L^\infty_T(H^1(\mathbb{R}^N))$ and $\partial_t(\beta(\rho_n))$ is bounded $L^2_T(W^{-1,1}(\mathbb{R}^N))$.
Thanks to the Ascoli's theorem and the Cantor's diagonal process it gives the strong convergence in

$$C([0,T],L^2_{loc}(\mathbb{R}^N))$$

of $\beta(\rho_n)$ to v_2. It implies in particular up a subsequence of the convergence almost everywhere of $\beta(\rho_n)$ to v_2. We shall define ρ in this situation by:

$$\rho = \beta^{-1}(\rho),$$

indeed β is inversible and we verify by continuity of β^{-1} that ρ_n converge up to a subsequence almost everywhere to ρ.

Strong convergence of ρ_n

We are now interested in proving the strong convergence of ρ_n to ρ. Let us deal with the first case $\nu_1 \geq 2$.

First case: $\nu_1 \geq 2$

For the moment we have only obtained strong convergence on $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ to $\frac{\mu(\rho)}{\sqrt{\rho}}$, let us translate this strong convergence on ρ_n. We are going to distinguish two different cases when $\nu_1 \geq 2$, let us start with the first one.

• We assume here that $(\frac{\mu(x)}{\sqrt{x}})'$, $\frac{\mu(x)}{\sqrt{x}}$ are increasing on $(0, +\infty)$. We have then the following lemma.

Lemma 4 *Let g_1 a regular function with $g_1(0) = 0$. When $x, y \geq 0$ and g_1, $g_1^{'}$ are increasing, we have:*

$$g_1(|x-y|) \leq |g_1(x) - g_1(y)|. \tag{5.65}$$

Proof: It suffices to study the function:

$$p(x) = g_1(x-y) - (g_1(x) - g_1(y)),$$

when $x \geq y$. We observe that for all $x \geq y \geq 0$:

$$p^{'}(x) = g_1^{'}(x-y) - g_1^{'}(x) \leq 0.$$

It implies that when $x \geq y$, $p^{'}$ is negative and p is decreasing on $[y, +\infty)$ with is equivalent to say that:

$$g_1(x-y) - (g_1(x) - g_1(y)) \leq p(y) = g_1(0) \;\; \forall x \geq y.$$

It implies that for all $x \geq y$ we have:

$$g_1(|x-y|) = g_1(x-y) \leq (g_1(x) - g_1(y)) = |g_1(x) - g_1(y)|.$$

By proceeding similarly when $0 \leq x \leq y$ we obtain (5.65). □

In particular since we assume that $(\frac{\mu(x)}{\sqrt{x}})'$ and $(\frac{\mu(x)}{\sqrt{x}})$ are increasing on $(0, +\infty)$ we deduce from the lemma 4:

$$\frac{\mu(|\rho-\rho_n|)}{\sqrt{|\rho-\rho_n|}} \leq |\frac{\mu(\rho_n)}{\sqrt{\rho_n}} - \frac{\mu(\rho)}{\sqrt{\rho}}|$$

Using (1.9) we obtain:

$$\begin{aligned} |\rho-\rho_n|^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_1}{2N}} &\leq \frac{1}{C}|\frac{\mu(\rho_n)}{\sqrt{\rho_n}} - \frac{\mu(\rho)}{\sqrt{\rho}}| \;\; \forall\, |\rho-\rho_n| > 1 \\ |\rho-\rho_n|^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_2}{2N}} &\leq \frac{1}{C}|\frac{\mu(\rho_n)}{\sqrt{\rho_n}} - \frac{\mu(\rho)}{\sqrt{\rho}}| \;\; \forall\, |\rho-\rho_n| \leq 1. \end{aligned} \tag{5.66}$$

Since $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges strongly to $\frac{\mu(\rho)}{\sqrt{\rho}}$ in $C([0,T],L^2_{loc})$ and $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$, $\frac{\mu(\rho)}{\sqrt{\rho}}$ are bounded via lemma 3 in $L^\infty_T(L^6(\mathbb{R}^N))$ when $N=3$ (it suffices to apply Fatou's lemma), we deduce by interpolation that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges strongly to $\frac{\mu(\rho)}{\sqrt{\rho}}$ in $C([0,T],L^{6-\alpha}_{loc})$ for any small $\alpha>0$. Similarly when $N=2$ we obtain by interpolation and via lemma 3 that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges strongly to $\frac{\mu(\rho)}{\sqrt{\rho}}$ in $C([0,T],L^p_{loc})$ for any $p>1$.
By using (5.66) we deduce that $(1-\psi(\rho-\rho_n))|\rho-\rho_n|^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_1}{2N}}$ and $\psi(\rho-\rho_n)|\rho-\rho_n|^{\frac{1}{2}-\frac{1}{N}+\frac{\nu_2}{2N}}$ converge strongly to 0 in $C([0,T],L^{6-\alpha}_{loc})$ for any small $\alpha>0$ when $N=3$ and in $C([0,T],L^p_{loc})$ for any $p>1$ when $N=2$. Since $\nu_1>0$ it yields that:

$$\sup_{t\in[0,T]}\||\rho-\rho_n|^{p(N)(\frac{1}{2}-\frac{1}{N})}1_{\{|\rho-\rho_n|>1\}}(t,\cdot)\|_{L^1(K)}\rightarrow_{n\rightarrow+\infty}0, \tag{5.67}$$

for any compact K with $p(N)=6-\alpha$ for any small $\alpha>0$ and $p(N)=p$ for any $p\in[1,+\infty[$ when $N=2$. And similarly we have:

$$\sup_{t\in[0,T]}\||\rho-\rho_n|^{p(N)(\frac{1}{2}-\frac{1}{N}+\frac{\nu_2}{2N})}1_{\{|\rho-\rho_n|>1\}}(t,\cdot)\|_{L^1(K)}\rightarrow_{n\rightarrow+\infty}0. \tag{5.68}$$

It implies since $\nu_2>0$ that ρ_n converges strongly to ρ in $C([0,T],L^{1+\alpha}_{loc}(\mathbb{R}^N))$ for $\alpha>0$ small enough when $N=3$ and ρ_n converges strongly to ρ in $C([0,T],L^p_{loc}(\mathbb{R}^N))$ for any $p\geq 1$.

- $\nu_2\leq N+2$

Let us deal now with the case when $(\frac{\mu(x)}{\sqrt{x}})'$ and $(\frac{\mu(x)}{\sqrt{x}})$ are not increasing on $(0,+\infty)$. By calculus and using (1.5) we have:

$$\begin{aligned}(\frac{\mu(\rho)}{\sqrt{\rho}})'&=\frac{\mu'(\rho)}{\sqrt{\rho}}-\frac{1}{2}\frac{\mu(\rho)}{\rho^{\frac{5}{2}}},\\&=\frac{1}{2}\frac{\lambda(\rho)+\mu(\rho)}{\rho^{\frac{3}{2}}}.\end{aligned}$$

Next by (1.9) we obtain:

$$\begin{aligned}&C\rho^{\frac{\nu_2}{2N}-\frac{1}{2}-\frac{1}{N}}\leq|(\frac{\mu(\rho)}{\sqrt{\rho}})'|\leq C\rho^{\frac{\nu_1}{2N}-\frac{1}{2}-\frac{1}{N}}\ \ \forall\rho\leq 1,\\&C\rho^{\frac{\nu_1}{2N}-\frac{1}{2}-\frac{1}{N}}\leq|(\frac{\mu(\rho)}{\sqrt{\rho}})'|\leq C\rho^{\frac{\nu_2}{2N}-\frac{1}{2}-\frac{1}{N}}\ \ \forall\rho>1.\end{aligned} \tag{5.69}$$

Assume that $\frac{\nu_2}{2N}-\frac{1}{2}-\frac{1}{N}\leq 0$ which is equivalent to $\nu_2\leq N+2$. In this case we obtain:

$$|(\frac{\mu(\rho)}{\sqrt{\rho}})'|\geq C\ \ \forall\rho\leq 1.$$

Let us recall that the derivative of the inverse function of $g(\rho)=\frac{\mu(\rho)}{\sqrt{\rho}}$ is:

$$(g^{-1})'(\rho)=\frac{1}{g'(g^{-1}(\rho))}.$$

In particular it means that the inverse function $g^{-1}(\rho)$ is Lipschitz on the region $\rho \leq 1$ and more generally on the region $\rho \leq M$, it provides then the following inequality for $C_M > 0$ depending on M (since when $\rho \leq M$ we have that $\frac{\mu(\rho)}{\sqrt{\rho}}$ is also bounded via (1.9) and the hypothesis $\frac{1}{2} + \frac{\nu_1}{2N} - \frac{1}{N} \geq 0$):

$$
\begin{aligned}
|\frac{\mu(\rho)}{\sqrt{\rho}} - \frac{\mu(\rho_n)}{\sqrt{\rho_n}}|1_{\{\rho\leq 1\}\cup\{\rho_n\leq 1\}}C_M &\geq |g^{-1}(\frac{\mu(\rho)}{\sqrt{\rho}}) - g^{-1}(\frac{\mu(\rho_n)}{\sqrt{\rho_n}})|1_{\{\rho\leq M\}\cup\{\rho_n\leq M\}} \\
&\geq |\rho-\rho_n|1_{\{\rho\leq 1\}\cup\{\rho_n\leq 1\}}.
\end{aligned}
\tag{5.70}
$$

We deduce the following estimate for any compact K:

$$
\begin{aligned}
\int_K |\rho_n(t,x)dx - \rho(t,x)|dx = &\int_K |\rho_n(t,x)dx - \rho(t,x)|1_{\{\rho>M\}\cup\{\rho_n>M\}}(t,x)dx \\
&+ \int_K |\rho_n(t,x) - \rho(t,x)|1_{\{\rho\leq M\}\cup\{\rho_n\leq M\}}(t,x)dx
\end{aligned}
\tag{5.71}
$$

The second term converges uniformly on $(0,T)$ to 0 when n goes to infinity via (5.70) applied to $M > 0$ and the strong convergence in $C([0,T], L^2_{loc}(\mathbb{R}^N))$ of $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ to $\frac{\mu(\rho)}{\sqrt{\rho}}$. Let us deal with the first term, since we know via the lemma 3 that ρ_n is uniformly bounded in $L^\infty_T(L^{1+\epsilon}_{loc}(\mathbb{R}^N))$ with $\epsilon > 0$ we have by Hölder's inequality and Tchebytchev lemma for $C > 0$:

$$
\begin{aligned}
&\int_K |\rho_n(t,x)dx - \rho(t,x)|1_{\{\rho>M\}\cup\{\rho_n>M\}}(t,x)dx \leq \\
&(\int_K |\rho_n(t,x)dx - \rho(t,x)|^{1+\epsilon}1_{\{\rho>M\}\cup\{\rho_n>M\}}(t,x)dx)^{\frac{1}{1+\epsilon}}|\{\rho>1\}\cup\{\rho_n>1\}|^{\frac{\epsilon}{1+\epsilon}}, \\
&\leq 2C\frac{\|\rho_0\|_{L^1(\mathbb{R}^N)}}{M}.
\end{aligned}
$$

This last term goes uniformly on $(0,T)$ to 0 in n when M goes to infinity. It show the desired result.

Second case $0 < \nu_1 < 2$

In this case it suffices to apply exactly the same argument than in the case $\nu_1 \geq N+2$ except at the place we consider not $\frac{\mu(\rho)}{\sqrt{\rho}}$ but $\beta(\rho)$. It concludes the proof of the lemma. □

Lemma 5 *Let $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\psi = 1$ on $B(0,1)$ and supp ψ is included in $B(0,2)$. We are going distinguish two cases:*

- *When $\nu_1 \geq 2$ we set:*

$$
v_n = \psi(\rho_n)\mu(\rho_n)u_n + (1-\psi(\rho_n))\rho_n u_n,
$$

we have:

  - *v_n is uniformly bounded in $L^2(0,T,W^{1,1}(K)$ for any compact K.*

- $\partial_t v_n$ *is uniformly bounded in* $L^2(0,T,W^{-2,1}(K)$ *for any compact* K.

Up to a subsequence, the sequel v_n *converges strongly in* $L^2(0,T;L^p_{loc}(\mathbb{R}^N))$ *to some* $v(t,x)$ *for all* $p\in[1,\frac{3}{2})$. *In particular:*

$$(\psi(\rho_n)\mu(\rho_n)+(1-\psi(\rho_n)))u_n\to v \quad \textit{almost everywhere } (x,t)\in(0,T)\times\mathbb{R}^N.$$

Note that we can already define $u(t,x)=\frac{v(t,x)}{\psi(\rho)\mu(\rho)+(1-\psi(\rho))\rho}$ *outside the vacuum set* $\{\rho(t,x)=0\}$, *but we do not know yet whether* $v(t,x)$ *is zero on the vacuum set (in particular if there is no concentration phenomena for* v *on* $\{\rho(t,x)=0\}$*).*

- *When* $0<\nu_1<2$ *we consider:*

$$v_n=\psi(\rho_n)\rho_n^{\beta_1+1}u_n+(1-\psi(\rho_n))\rho_n^{\beta+1}u_n,$$

with $\beta\le\frac{-1}{N}$ *and* β_1 *verifying the following assumptions:*

$$\begin{aligned}&\beta_1\ge 1,\\ &\beta_1+\frac{1}{N}-\frac{\nu_2}{2N}\ge 0,\\ &\beta_1-\frac{1}{2}+\frac{1}{2N}-\frac{\nu_2}{4N}\ge 0\end{aligned}\tag{5.72}$$

we have:

- v_n *is uniformly bounded in* $L^2(0,T,W^{1,1}(K)$ *for any compact* K.
- $\partial_t v_n$ *is uniformly bounded in* $L^2(0,T,W^{-2,1}(K)$ *for any compact* K.

Up to a subsequence, the sequel v_n *converges strongly in* $L^2(0,T;L^p_{loc}(\mathbb{R}^N))$ *to some* $v(t,x)$ *for all* $p\in[1,\frac{3}{2})$. *In particular:*

$$(\psi(\rho_n)\rho_n^{\beta_1+1}+(1-\psi(\rho_n))\rho_n^{\beta+1})u_n\to v \quad \textit{almost everywhere } (x,t)\in(0,T)\times\mathbb{R}^N.$$

Note that we can already define $u(t,x)=\frac{v(t,x)}{\psi(\rho)\rho^{\beta_1+1}+(1-\psi(\rho))\rho^{\beta+1}}$ *outside the vacuum set* $\{\rho(t,x)=0\}$, *but we do not know yet whether* $v(t,x)$ *is zero on the vacuum set (in particular if there is no concentration phenomena for* v *on* $\{\rho(t,x)=0\}$*).*

Proof: Let us start with proving that v_n is uniformly bounded in $L^2(0,T,W^{1,1}(\mathbb{R}^N)$.

• We are going to start with the case $\nu_1\ge 2$.
Let us prove that v_n^2 is bounded in $L^2(0,T,W^{1,1}(K)$ for any compact K. We have then:

$$\begin{aligned}&\partial_i((1-\psi(\rho_n))\rho_n u_{nj})=-\frac{\psi'(\rho_n)}{\mu'(\rho_n)}\rho_n\,\frac{\mu'(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}\\ &+\frac{(1-\psi(\rho_n))}{\mu'(\rho_n)}\,\frac{\mu'(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}+\frac{(1-\psi(\rho_n))\rho_n}{\sqrt{\mu(\rho_n)}}\sqrt{\mu(\rho_n)}\partial_i u_{nj}.\end{aligned}\tag{5.73}$$

We observe that by using (1.9), (1.5) and $\nu_1\ge 2$ that $\frac{(1-\psi(\rho_n))}{\mu'(\rho_n)}$ is uniformly bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. It implies that the second term $\frac{(1-\psi(\rho_n))}{\mu'(\rho_n)}\,\frac{\mu'(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}$ is bounded by Hólder's inequality in $L^2_T(L^1(\mathbb{R}^N))$. The first term on the right hand side of (5.73) is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by using the fact that the support of ψ' is included in the shell $C(0,1,2)$. The last term is also bounded in $L^2_T(L^1(\mathbb{R}^N))$. Indeed by (1.9) and $\nu_2 \geq 2$ it suffices to observe that:

$$\frac{(1-\psi(\rho_n))\rho_n}{\sqrt{\mu(\rho_n)}} \leq C\sqrt{\rho_n},$$

which implies that $\frac{(1-\psi(\rho_n))\rho_n}{\sqrt{\mu(\rho_n)}}$ is bounded in $L^\infty_T(L^2(\mathbb{R}^N))$.
Finally we have seen that ∇v_n^2 is bounded in $L^2_T(L^1(\mathbb{R}^N))$. And since we see easily that v_n^2 is also bounded in $L^\infty_T(L^1(\mathbb{R}^N))$, it implies that v_n^2 is bounded in $L^2_T(W^{1,1}(\mathbb{R}^N))$.
Let us deal now with estimating $\partial_t v_n^2$. A simple calculus gives for any regular function g:

$$\begin{aligned}
&\partial_t(g(\rho_n)\rho_n u_n) = -g(\rho_n)\rho_n u_n\cdot\nabla u_n - g'(\rho_n)\rho_n^2 u_n \mathrm{div} u_n - u_n \mathrm{div}(g(\rho_n)\rho_n u_n)\\
&+2\mathrm{div}(g(\rho_n)\mu(\rho_n)Du_n) - 2\nabla g(\rho_n)\cdot\mu(\rho_n)Du_n + \nabla(g(\rho_n)\lambda(\rho_n)\mathrm{div}u_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad - \nabla g(\rho_n)\,\lambda(\rho_n)\mathrm{div}u_n, \qquad (5.74)\\
&= -\mathrm{div}(g(\rho_n)\rho_n u_n\otimes u_n) - g'(\rho_n)\rho_n^2 u_n\mathrm{div}u_n + 2\mathrm{div}(g(\rho_n)\mu(\rho_n)Du_n)\\
&-2\nabla g(\rho_n)\cdot\mu(\rho_n)Du_n + \nabla(g(\rho_n)\lambda(\rho_n)\mathrm{div}u_n) - \nabla g(\rho_n)\,\lambda(\rho_n)\mathrm{div}u_n.
\end{aligned}$$

When we apply the previous formula to $g(\rho_n) = (1-\psi(\rho_n))\sqrt{\frac{\mu(\rho_n)}{\rho_n}}$, we have to estimate all the terms on the right hand side of (5.74), it comes for $T>0$:

$$g(\rho_n)\rho_n u_n\otimes u_n = (1-\psi(\rho_n))\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}u_n. \qquad (5.75)$$

By Hölder's inequality we obtain that $(1-\psi(\rho_n))\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}u_n$ is bounded in $L^\infty_T(L^1(\mathbb{R}^N))$.
Next we have:

$$g'(\rho_n)\rho_n^2 u_n\mathrm{div}u_n = -\psi'(\rho_n)\frac{\rho_n^{\frac{3}{2}}}{\sqrt{\mu(\rho_n)}}\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\mathrm{div}u_n \qquad (5.76)$$

This term is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality since ψ' is supported in the shell $C(0,1,2)$ which implies that $\psi'(\rho_n)\frac{\rho_n^{\frac{3}{2}}}{\sqrt{\mu(\rho_n)}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. Similarly we have:

$$\begin{aligned}
g(\rho_n)\mu(\rho_n)Du_n &= (1-\psi(\rho_n))\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}Du_n,\\
&= (1-\psi(\rho_n))\sqrt{\frac{\mu(\rho_n)}{\sqrt{\rho_n}}}\rho_n^{\frac{1}{4}}\sqrt{\mu(\rho_n)}Du_n
\end{aligned} \qquad (5.77)$$

According lemma 3 we deduce easily that $(1-\psi(\rho_n))\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}Du_n$ is bounded in $L^2_T(L^1(K))$ for any compact K. Next we have:

$$2\nabla g(\rho_n)\cdot\mu(\rho_n)Du_n = -2\psi'(\rho_n)\nabla\rho_n\cdot\mu(\rho_n)Du_n. \qquad (5.78)$$

This term is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality since ψ' is supported in the shell $C(0,1,2)$. The two last term in (5.74) are similar to treat. (5.75), (5.76), (5.77) and (5.78) implies that $\partial_t v_n^2$ is uniformly bounded in $L^2_T(W^{-1,1}(K))$ for any compact K.

• Case $0 < \nu_1 < 2$.

In this case we are going to consider:

$$v_n^2 = (1-\psi(\rho_n))\rho_n^\beta \rho_n u_n.$$

Let us start with proving that ∇v_n belongs in $L^2_T(L^1(K))$ for any compact K. We have then:

$$\begin{aligned}
&\partial_i((1-\psi(\rho_n))\rho_n^\beta\rho_n u_{nj}) = -\frac{\psi^{'}(\rho_n^{\beta+1})}{\mu^{'}(\rho_n)}\rho_n \,\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}\\
&+\frac{(1-\psi(\rho_n))\rho_n^\beta}{\mu^{'}(\rho_n)}\,\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}+\frac{(1-\psi(\rho_n))\rho_n^{\beta+1}}{\sqrt{\mu(\rho_n)}}\sqrt{\mu(\rho_n)}\partial_i u_{nj}.
\end{aligned}\tag{5.79}$$

By using (1.6) and (1.5), we obtain that:

$$\begin{aligned}
\mu^{'}(\rho)&=\frac{1}{2\rho}\frac{2\mu(\rho)+N\lambda(\rho)}{N}+(1-\frac{1}{N})\frac{\mu(\rho)}{\rho},\\
&\geq (1-\frac{1}{N}+\frac{\nu_1}{2N})\frac{\mu(\rho)}{\rho}.
\end{aligned}$$

We deduce that according to (1.9) that:

$$\begin{aligned}
|\frac{(1-\psi(\rho_n))\rho_n^\beta}{\mu^{'}(\rho_n)}|&\leq C|\frac{(1-\psi(\rho_n))\rho_n^{\beta+1}}{\mu(\rho_n)}|,\\
&\leq C|\frac{(1-\psi(\rho_n))\rho_n^{\beta+1}}{\rho_n^{1-\frac{1}{N}+\frac{\nu_1}{2N}}}|,\\
&\leq C|(1-\psi(\rho_n))\rho_n^{\beta+\frac{1}{N}-\frac{\nu_1}{2N}}|.
\end{aligned}$$

Since $\beta\leq\frac{-1}{N}$ we deduce that $\frac{(1-\psi(\rho_n))\rho_n^\beta}{\mu^{'}(\rho_n)}$ is uniformly bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. In particular it implies that $\frac{(1-\psi(\rho_n))\rho_n^\beta}{\mu^{'}(\rho_n)}\,\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$. Since the support of $\psi^{'}$ is included in the shell $C(0,1,2)$ we easily observe that the first term on the right hand side of (5.79) is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$. The last term is also bounded in $L^2_T(L^1(\mathbb{R}^N))$ since $frac(1-\psi(\rho_n))\rho_n^{\beta+1}\sqrt{\mu(\rho_n)}$ belongs in $L^\infty_T(L^2(\mathbb{R}^N))$ by using (1.9), $\beta\leq-\frac{1}{N}$ and the lemma 3.
Finally we have seen that ∇v_n^2 is bounded in $L^2_T(L^1(\mathbb{R}^N))$. And since we see easily that v_n^2 is also bounded in $L^\infty_T(L^1(\mathbb{R}^N))$, it implies that v_n^2 is bounded in $L^2_T(W^{1,1}(\mathbb{R}^N))$.

Let us deal now with estimating $\partial_t v_n^2$. It suffices to deal with the formula (5.74) and replacing $g(\rho_n)$ by $(1-\psi(\rho_n))\rho_n^\beta$. Let us start with the first term of (5.74):

$$g(\rho_n)\rho_n u_n\otimes u_n=(1-\psi(\rho_n))\rho_n^\beta\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}u_n.\tag{5.80}$$

Since $\beta \leq -\frac{1}{N}$ it implies that $(1-\psi(\rho_n))\rho_n^\beta$ is L^∞ bounded and then by Hölder's inequality we obtain that $(1-\psi(\rho_n))\rho_n^\beta\sqrt{\rho_n}u_n \otimes \sqrt{\rho_n}u_n$ is bounded in $L^\infty_T(L^1(\mathbb{R}^N))$. Next we have:

$$\begin{aligned} g^{'}(\rho_n)\rho_n^2 u_n \mathrm{div} u_n = -\psi^{'}(\rho_n)\frac{\rho_n^{\frac{3}{2}+\beta}}{\sqrt{\mu(\rho_n)}}\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\mathrm{div}u_n, \\ + \beta(1-\psi(\rho_n))\frac{\rho_n^{\frac{1}{2}+\beta}}{\sqrt{\mu(\rho_n)}}\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\mathrm{div}u_n. \end{aligned} \tag{5.81}$$

The first term is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality since $\psi^{'}$ is supported in the shell $C(0,1,2)$ which implies that $\psi^{'}(\rho_n)\frac{\rho_n^{\frac{3}{2}+\beta}}{\sqrt{\mu(\rho_n)}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. The second term is also bounded in $L^2_T(L^1(\mathbb{R}^N))$ since $(1-\psi(\rho_n))\frac{\rho_n^{\frac{1}{2}+\beta}}{\sqrt{\mu(\rho_n)}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ by using (1.9) and the fact that $\beta \leq \frac{-1}{N}$.
Similarly we have:

$$\begin{aligned} g(\rho_n)\mu(\rho_n)Du_n = (1-\psi(\rho_n))\sqrt{\mu(\rho_n)}\rho_n^\beta\sqrt{\mu(\rho_n)}Du_n, \\ = (1-\psi(\rho_n))\sqrt{\frac{\mu(\rho_n)}{\sqrt{\rho_n}}}\rho_n^{\frac{1}{4}+\beta}\sqrt{\mu(\rho_n)}Du_n \end{aligned} \tag{5.82}$$

According lemma 3 we deduce easily that $(1-\psi(\rho_n))\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}Du_n$ is bounded in $L^2_T(L^1(K))$ for any compact K. The last term gives:

$$\begin{aligned} 2\nabla g(\rho_n)\cdot\mu(\rho_n)Du_n = -2\psi^{'}(\rho_n)\rho_n^\beta\nabla\rho_n\cdot\mu(\rho_n)Du_n \\ + \beta(1-\psi(\rho_n))\frac{\rho_n^{\beta-\frac{1}{2}}\sqrt{\mu(\rho_n)}}{\mu^{'}(\rho_n)}\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n\cdot\sqrt{\mu(\rho_n)}Du_n. \end{aligned} \tag{5.83}$$

The first term is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality since $\psi^{'}$ is supported in the shell $C(0,1,2)$ and we proceed similarly for the second by observing that $(1-\psi(\rho_n))\frac{\rho_n^{\beta-\frac{1}{2}}\sqrt{\mu(\rho_n)}}{\mu^{'}(\rho_n)}$ is L^∞ bounded via (1.5), (1.6) and (1.9). The two last term in (5.74) are similar to treat. (5.80), (5.81), (5.82) and (5.83) implies that $\partial_t v_n^2$ is uniformly bounded in $L^2_T(W^{-1,1}(K))$ for any compact K.

Let us deal now with the term v_n^1 . We are going to distinguish two cases.

- Case $\nu_1 \geq 2$.

We have:

$$\psi(\rho_n)\mu(\rho_n)u_n = \frac{\psi(\rho_n)\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}u_n,$$

where via the lemma 3 $\frac{\psi(\rho_n)\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ and $\sqrt{\rho_n}u_n$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))$ which implies that $\psi(\rho_n)\mu(\rho_n)u_n$ is bounded in

$L^\infty(0,T;L^2(\mathbb{R}^N))$.
Next we have:

$$\begin{aligned}\partial_i(\psi(\rho_n)\mu(\rho_n)u_{nj}) &= \psi(\rho_n)\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}\partial_i u_{nj} + \psi(\rho_n)\mu^{'}\rho_n)\partial_i\rho_n u_{nj},\\ &= \psi(\rho_n)\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}\partial_i u_{nj} + \frac{\psi(\rho_n)\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}.\end{aligned}$$

By entropy inequality (3.37) we know that $\frac{\psi(\rho_n)\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ which implies that $\frac{\psi(\rho_n)\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$.
Let us deal now with the term $\psi(\rho_n)\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}\partial_i u_{nj}$, we know that $\sqrt{\mu(\rho_n)}\partial_i u_{nj}$ is uniformly bounded in $L^2_T(L^2(\mathbb{R}^N))$. Next we know that $\psi(\rho_n)\sqrt{\mu(\rho_n)}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ which provides on $\psi(\rho_n)\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}\partial_i u_{nj}$ a $L^2_T(L^2(\mathbb{R}^N))$ bound. Hence for any compact K:

$$\nabla(\psi(\rho_n)\mu(\rho_n)u_n) \text{ is bounded in } L^2(0,T;L^1(K))$$

In particular we have obtained that for all compact K:

$$v^1_n = \psi(\rho_n)\mu(\rho_n)u_n \text{ is bounded in } L^2(0,T;W^{1,1}(K)).$$

We are now going to estimate $\partial_t v^1_n$, it suffices to estimate each term on the right hand side of (5.74) by replacing $g(\rho_n)$ by $\psi(\rho_n)\frac{\mu(\rho_n)}{\rho_n}$.
We have then:

$$g(\rho_n)\rho_n u_n\otimes u_n = \psi(\rho_n)\frac{\mu(\rho_n)}{\rho_n}\sqrt{\rho_n}u_n\otimes\sqrt{\rho_n}u_n, \tag{5.84}$$

By Hölder's inequality we obtain that $\psi(\rho_n)\mu(\rho_n)u_n\otimes u_n$ is bounded in $L^\infty_T(L^1(\mathbb{R}^N))$. Indeed we have used the fact that by (1.9) and since $\nu_2\geq 2$ then $\psi(\rho_n)\frac{\mu(\rho_n)}{\rho_n}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$.
Next we have:

$$\begin{aligned}g^{'}(\rho_n)\rho_n^2 u_n\mathrm{div}u_n &= (\psi^{'}(\rho_n)\mu(\rho_n)+\psi(\rho_n)\mu^{'}(\rho_n))\frac{\rho_n^{\frac{3}{2}}}{\sqrt{\mu(\rho_n)}}\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\mathrm{div}u_n,\\ &= (\psi^{'}(\rho_n)\sqrt{\mu(\rho_n)}\rho_n^{\frac{3}{2}}+\psi(\rho_n)\frac{\mu^{'}(\rho_n)\rho_n^{\frac{3}{2}}}{\sqrt{\mu(\rho_n)}})\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\mathrm{div}u_n\end{aligned} \tag{5.85}$$

The first term is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality since $\psi^{'}$ is supported in the shell $C(0,1,2)$ which implies that $\psi^{'}(\rho_n)\sqrt{\mu(\rho_n)}\rho_n^{\frac{3}{2}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. The second term is also bounded in $L^2_T(L^1(\mathbb{R}^N))$ because using (1.9) we observe that $\psi(\rho_n)\frac{\mu^{'}(\rho_n)\rho_n^{\frac{3}{2}}}{\sqrt{\mu(\rho_n)}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$.
The third term gives:

$$\begin{aligned}g(\rho_n)\mu(\rho_n)Du_n &= \psi(\rho_n)\frac{\mu^{\frac{3}{2}}(\rho_n)}{\rho_n}\sqrt{\mu(\rho_n)}Du_n,\\ &= \psi(\rho_n))\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\frac{\mu(\rho_n)}{\rho_n}}\sqrt{\mu(\rho_n)}Du_n\end{aligned} \tag{5.86}$$

By (1.9) and $\nu_1 \geq 2$ we know that $\sqrt{\frac{\mu(\rho_n)}{\rho_n}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$, since via lemma 3 $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty_T(L^6(K))$ for any compact K we deduce that $\psi(\rho_n)\frac{\mu(\rho_n)}{\rho_n}\mu(\rho_n)Du_n$ is uniformly bounded in $L^2_T(L^1(K))$ for any compact K.
Next we have:

$$\begin{aligned}2\nabla g(\rho_n)\cdot\mu(\rho_n)Du_n &= 2\psi^{'}(\rho_n)\nabla\rho_n\cdot\mu(\rho_n)Du_n\\&+2\psi(\rho_n)\sqrt{\mu(\rho_n)\rho_n}\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n\cdot\sqrt{\mu(\rho_n)}Du_n.\end{aligned}\tag{5.87}$$

The first term is easily bounded in $L^2_T(L^1(\mathbb{R}^N))$ since the support of $\psi^{'}$ is included in $C(0,1,2)$. The second term is also bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality because via (1.9) we deduce that $\psi(\rho_n)\sqrt{\mu(\rho_n)\rho_n}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. The two last term in (5.74) are similar to treat. (5.84), (5.85), (5.86) and (5.87) implies that $\partial_t v_n^1$ is uniformly bounded in $L^2_T(W^{-1,1}(K))$ for any compact K.

- Case $0<\nu_1<2$.

We are going to work with:

$$v_n^1=\psi(\rho_n)\rho_n^{\beta_1}\rho_n u_n.$$

We have then:

$$\psi(\rho_n)\rho_n^{\beta_1}\rho_n u_n=\psi(\rho_n)\rho_n^{\beta_1+\frac{1}{2}}\sqrt{\rho_n}u_n.$$

Since $\beta_1\geq\frac{-1}{2}$, it implies that $\psi(\rho_n)\rho_n^{\beta_1}\rho_n u_n$ is bounded in $L^\infty_T(L^2(\mathbb{R}^N))$.
Next we have:

$$\begin{aligned}\partial_i(\psi(\rho_n)\rho_n^{\beta_1+1}u_{nj}) &= \frac{\psi(\rho_n)\rho_n^{\beta_1+1}}{\sqrt{\mu(\rho_n)}}\sqrt{\mu(\rho_n)}\partial_i u_{nj},\\&+\beta_1\psi(\rho_n)\frac{\rho_n^{\beta_1}}{\mu^{'}(\rho_n)}\frac{\mu^{'}(\rho_n)}{\sqrt{\rho_n}}\partial_i\rho_n\sqrt{\rho_n}u_{nj}+\psi^{'}(\rho_n)\rho_n^{\beta_1+1}\partial_i\rho_n u_{nj}.\end{aligned}\tag{5.88}$$

Using (1.9), (1.5) and (1.6) we show that:

$$|\psi(\rho_n)\frac{\rho_n^{\beta_1}}{\mu^{'}(\rho_n)}|\leq\psi(\rho_n)\rho_n^{\beta_1+\frac{1}{N}-\frac{\nu_2}{2N}}.$$

Since $\beta_1+\frac{1}{N}-\frac{\nu_2}{2N}\geq 0$ it implies that $\psi(\rho_n)\frac{\rho_n^{\beta_1}}{\mu^{'}(\rho_n)}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. We deduce that the second term on the right hand side of (5.88) is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$. The third term is easy to treat by using that the support of $\psi^{'}$ is a shell $C(0,1,2)$.
Let us deal with the first term of (5.88). By using (1.9) we get:

$$|\frac{\psi(\rho_n)\rho_n^{\beta_1+1}}{\sqrt{\mu(\rho_n)}}|\leq\psi(\rho_n)\rho_n^{\beta_1+\frac{1}{2}+\frac{1}{2N}-\frac{\nu_2}{4N}}.$$

Since $\beta_1+\frac{1}{2}+\frac{1}{2N}-\frac{\nu_2}{4N}\geq 0$ it provides a L^∞ bound on $\frac{\psi(\rho_n)\rho_n^{\beta_1+1}}{\sqrt{\mu(\rho_n)}}$ which insures the $L^2_T(L^2(\mathbb{R}^N)$ bound of $\frac{\psi(\rho_n)\rho_n^{\beta_1+1}}{\sqrt{\mu(\rho_n)}}\sqrt{\mu(\rho_n)}\partial_i u_{nj}$. Hence for any compact K:

$$\nabla(\psi(\rho_n)\rho_n^{\beta_1+1}u_n)\ \text{ is bounded in }\ L^2(0,T;L^1(K))$$

In particular we have obtained that for all compact K:

$$v_n^1 = \psi(\rho_n)\rho_n^{\beta_1+1}u_n \text{ is bounded in } L^2(0,T;W^{1,1}(K)).$$

We are now going to estimate $\partial_t v_n^1$, it suffices to estimate each term on the right hand side of (5.74) by replacing $g(\rho_n)$ by $\psi(\rho_n)\rho_n^{\beta_1}$.
We have then:

$$g(\rho_n)\rho_n u_n \otimes u_n = \psi(\rho_n)\rho_n^{\beta_1-1}\sqrt{\rho_n}u_n \otimes \sqrt{\rho_n}u_n, \tag{5.89}$$

By Hölder's inequality and the fact that $\beta_1 \geq 1$ we obtain that $\psi(\rho_n)\rho_n^{\beta_1}u_n \otimes u_n$ is bounded in $L^\infty_T(L^1(\mathbb{R}^N))$. Next we have:

$$\begin{aligned} g'(\rho_n)\rho_n^2 u_n \text{div} u_n &= (\psi'(\rho_n)\rho_n^{\beta_1} + \beta_1\psi(\rho_n)\rho_n^{\beta_1-1})\frac{\rho_n^{\frac{3}{2}}}{\sqrt{\mu(\rho_n)}}\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\text{div}u_n, \\ &= (\psi'(\rho_n)\frac{\rho_n^{\beta_1+\frac{3}{2}}}{\sqrt{\mu(\rho_n)}} + \beta_1\psi(\rho_n)\frac{\rho_n^{\beta_1+\frac{1}{2}}}{\sqrt{\mu(\rho_n)}})\sqrt{\rho_n}u_n\sqrt{\mu(\rho_n)}\text{div}u_n \end{aligned} \tag{5.90}$$

The first term is bounded in $L^2_T(L^1(\mathbb{R}^N))$ by Hölder's inequality since ψ' is supported in the shell $C(0,1,2)$ which show that $\psi'(\rho_n)\frac{\rho_n^{\beta_1+\frac{3}{2}}}{\sqrt{\mu(\rho_n)}}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$. The second term is also bounded in $L^2_T(L^1(\mathbb{R}^N))$ because using (1.9) we observe that:

$$|\psi(\rho_n)\frac{\rho_n^{\beta_1+\frac{1}{2}}}{\sqrt{\mu(\rho_n)}}| \leq C\psi(\rho_n)\rho_n^{\beta_1+\frac{1}{2N}-\frac{\nu_2}{4N}}.$$

It provides a L^∞ bounds on $\psi(\rho_n)\frac{\rho_n^{\beta_1+\frac{1}{2}}}{\sqrt{\mu(\rho_n)}}$ since $\beta_1 + \frac{1}{2N} - \frac{\nu_2}{4N} \geq 0$.
The third term gives:

$$g(\rho_n)\mu(\rho_n)Du_n = \psi(\rho_n)\rho_n^{\beta_1}\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}Du_n, \tag{5.91}$$

By (1.9) we prove that $\psi(\rho_n)\rho_n^{\beta_1}\sqrt{\mu(\rho_n)}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N)$ and it yields that $\psi(\rho_n)\rho_n^{\beta_1}\sqrt{\mu(\rho_n)}\sqrt{\mu(\rho_n)}Du_n$ is uniformly bounded in $L^2_T(L^2(\mathbb{R}^N))$. Next we have:

$$\begin{aligned} &2\nabla g(\rho_n)\cdot\mu(\rho_n)Du_n = 2\psi'(\rho_n)\rho_n^{\beta_1}\nabla\rho_n\cdot\mu(\rho_n)Du_n \\ &+ 2\psi(\rho_n)\frac{\rho_n^{\beta_1-\frac{1}{2}}\sqrt{\mu(\rho_n)}}{\mu'(\rho_n)}\frac{\mu'(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n\cdot\sqrt{\mu(\rho_n)}Du_n. \end{aligned} \tag{5.92}$$

The first term is easily bounded in $L^2_T(L^1(\mathbb{R}^N))$ since the support of ψ' is included in $C(0,1,2)$. In order to deal with the second term we observe that via (1.5), (1.6) and (1.9):

$$\begin{aligned} |\psi(\rho_n)\frac{\rho_n^{\beta_1-\frac{1}{2}}\sqrt{\mu(\rho_n)}}{\mu'(\rho_n)}| &= |\psi(\rho_n)\frac{\rho_n^{\beta_1-\frac{1}{2}}}{\sqrt{\mu'(\rho_n)}}\sqrt{\frac{\mu(\rho_n)}{\mu'(\rho_n)}}|, \\ &\leq C\psi(\rho_n)\rho_n^{\beta_1-\frac{1}{2}+\frac{1}{2N}-\frac{\nu_2}{4N}}. \end{aligned}$$

It implies that $\psi(\rho_n)\frac{\rho_n^{\beta_1-\frac{1}{2}}\sqrt{\mu(\rho_n)}}{\mu'(\rho_n)}$ is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ since $\beta_1-\frac{1}{2}+\frac{1}{2N}-\frac{\nu_2}{4N}\geq 0$ and we deduce that the second term is bounded in $L^2_T(L^1(\mathbb{R}^N))$. The two last term in (5.74) are similar to treat.
(5.89), (5.90), (5.91) and (5.92) implies that $\partial_t v_n^2$ is uniformly bounded in $L^2_T(W^{-1,1}(K))$ for any compact K. □

Step 2: Convergence of $\sqrt{\rho_n}u_n$ and $\rho_n u_n$

In the sequel we shall define $h(\rho)$ as follows:

$$\begin{aligned}
&h(\rho)=\psi(\rho)\frac{\mu(\rho)}{\sqrt{\rho}}+(1-\psi(\rho))\sqrt{\rho}\quad \text{if}\;\; \nu_1\geq 2,\\
&h(\rho)=\psi(\rho)\rho^{\beta_1+\frac{1}{2}}+(1-\psi(\rho))\rho^{\beta+\frac{1}{2}}\quad \text{if}\;\; 0<\nu_1<2.
\end{aligned}\tag{5.93}$$

Here ψ, β and β_1 verify the hypothesis of lemma 5.

Lemma 2 *The quantity $\sqrt{\rho_n}u_n$ strongly converges in $L^2_{loc}((0,T)\times\Omega)$ to $\frac{v}{h(\rho)}$ (which is null when $v=0$).*
In particular, we have $v(t,x)=0$ a.e on $\{\rho(t,x)=0\}$ and there exists a function $u(t,x)$ such that $v(t,x)=\sqrt{\rho}(t,x)h(\rho)(t,x)u(t,x)$ and:

$$\begin{aligned}
&\sqrt{\rho_n}u_n\rightarrow\sqrt{\rho}u\quad \textit{strongly in}\;\; L^2_{loc}((0,T)\times\mathbb{R}^N),\\
&\rho_n u_n\rightarrow\rho u\quad \textit{strongly in}\;\; L^1(0,T;L^1_{loc}(\mathbb{R}^N)).
\end{aligned}$$

Remark 26 *Here u is not uniquely defined on the vacuum set $\{\rho(t,x)=0\}$. We will set $u=0$ on the vacuum set $\{\rho(t,x)=0\}$.*

Proof: Since $\frac{v_n}{h(\rho_n)}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))$, Fatou's lemma implies that:

$$\int\liminf\frac{v_n^2}{h(\rho_n)^2}dx<+\infty.$$

We deduce that $v(t,x)=0$ a.e. in $\{\rho(t,x)=0\}$ since $h(\rho)=0$ when $\rho=0$. We can define the limit velocity by $u(t,x)$ with $u(t,x)=\frac{v(t,x)}{\sqrt{\rho(t,x)}h(\rho(t,x))}$ when $\rho(t,x)\neq 0$ and $u(t,x)=0$ on $\{\rho(t,x)=0\}$. In particular this last point implies that there is no concentration effect of $\rho_n u_n\otimes u_n$ on the set $\{\rho=0\}$. And for all $t>0$:

$$\int_{\mathbb{R}^N}\frac{v^2(t,x)}{h^2(\rho)(t,x)}dx=\int_{\mathbb{R}^N}\rho(t,x)|u(t,x)|^2dx<+\infty.$$

Furthermore applying the Fatou's lemma once more, we obtain:

$$\begin{aligned}
\int\rho|u|^2\ln(1+|u|^2)dx&\leq\int\liminf\rho_n|u_n|^2\ln(1+|u_n|^2)dx\\
&\leq\liminf\int\rho_n|u_n|^2\ln(1+|u_n|^2)dx,
\end{aligned}$$

which yields $\rho|u|^2\ln(1+|u|^2)\in L^\infty(0,T;L^1(\mathbb{R}^N))$.
Next, since v_n and ρ_n converge almost everywhere, we know that in $\{\rho(t,x)\neq 0\}$, $\sqrt{\rho_n}u_n=\frac{v_n}{h(\rho_n)}$ converges almost everywhere to $\sqrt{\rho}u=\frac{v}{h(\rho)}$. In particular it implies that:

$$\begin{aligned}
&\sqrt{\rho_n}u_n 1_{\{\{|u_n|\leq M\}\cap\{\rho>0\}}\rightarrow\sqrt{\rho}u1_{\{|u|\leq M\}}\quad\text{almost everywhere.}\\
&\sqrt{\rho_n}u_n 1_{\{\{|u_n|\leq M\}\cap\{\rho=0\}}\leq M\sqrt{\rho_n}\rightarrow 0\quad\text{almost everywhere.}
\end{aligned}\tag{5.94}$$

Following the argument of the proof of the lemma 1 for any compact K we have:

$$\begin{aligned}
\int_{(0,T)\times K}|\sqrt{\rho_n}u_n-\sqrt{\rho}u|^2dxdt&\leq\int_{(0,T)\times K}|\sqrt{\rho_n}u_n1_{\{|u_n|\leq M\}}-\sqrt{\rho}u1_{\{|u|\leq M\}}|^2dxdt\\
&+2\int_{(0,T)\times K}|\sqrt{\rho_n}u_n1_{\{|u_n|\geq M\}}|^2dxdt+2\int_{(0,T)\times K}|\sqrt{\rho}u1_{\{|u|\geq M\}}|^2dxdt,
\end{aligned}\tag{5.95}$$

Let us deal with the first term of (5.95) we have then:

$$\begin{aligned}
&\int_{(0,T)\times K}|\sqrt{\rho_n}u_n1_{\{|u_n|\leq M\}}-\sqrt{\rho}u1_{\{|u|\leq M\}}|^2dxdt\leq\\
&\qquad\int_{\{(0,T)\times K\}\cap\{\sqrt{\rho_n}\leq c\}}|\sqrt{\rho_n}u_n1_{\{|u_n|\leq M\}}-\sqrt{\rho}u1_{\{|u|\leq M\}}|^2dxdt\\
&\qquad+\int_{\{(0,T)\times K\}\cap\{\sqrt{\rho_n}>c\}}|\sqrt{\rho_n}u_n1_{\{|u_n|\leq M\}}-\sqrt{\rho}u1_{\{|u|\leq M\}}|^2dxdt
\end{aligned}\tag{5.96}$$

The first term of (5.96) converges to 0 when n goes to $+\infty$ via the theorem of dominated convergence.
Now let us recall that via the inequality (1.9) we have:

$$\rho_n^{1-\frac{1}{N}-\frac{1}{2}+\frac{\nu}{N}}\leq\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\ \text{ when }\ \rho_n\geq 1.$$

Since $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty_T(L^6(\mathbb{R}^N))$ for $N=3$ (see the lemma 3) we deduce that $\sqrt{\rho_n}$ is uniformly bounded in $L^\infty_T(L^{2+4\nu}(\mathbb{R}^N))$. It allows to deal with the second term of (5.96) for $N=3$ by Hölder's inequality and Tchebytchev lemma with c going to infinity.
For $N=2$ using the inequality (1.9) and the fact that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty_T(L^p_{loc}(\mathbb{R}^N))$ for any $p>1$ when $N=2$ (see the lemma 3), we deal similarly with the second term of (5.95) via Hölder's inequality and Tchebytchev lemma.
Finally, we take advantage of the gain of integrability on the velocity provided by the entropy (3.38):

$$\int|\sqrt{\rho_n}u_n1_{\{|u_n|\geq M\}}|^2dxdt\leq\frac{1}{\ln(1+M^2)}\int\rho_n|u_n|^2\ln(1+|u_n|^2)dxdt.$$

Similarly we have:

$$\int|\sqrt{\rho}u1_{\{|u|\geq M\}}|^2dxdt\leq\frac{1}{\ln(1+M^2)}\int\rho|u|^2\ln(1+|u|^2)dxdt.$$

Combining all the previous estimate, it yields:

$$\limsup_{n\to+\infty}\int|\sqrt{\rho_n}u_n-\sqrt{\rho}u|^2dxdt\leq\frac{C}{\ln(1+M^2)}$$

for all $M>0$, and the lemma follows by taking $M\to+\infty$.

• Let us prove now the strong convergence of $\rho_n u_n$ to ρu. Since $\sqrt{\rho_n}u_n$ converges strongly in $L^2_{loc}((0,T)\times\mathbb{R}^N)$ to $\sqrt{\rho}u$ and that via the lemma 1 $\sqrt{\rho_n}$ converges also in $L^2_{loc}((0,T)\times\mathbb{R}^N)$ to $\sqrt{\rho}$ it implies that $\rho_n u_n$ converges strongly in $L^1_{loc}((0,T)\times\mathbb{R}^N)$ to ρu. □

Step 3: Convergence of the diffusion terms

Lemma 3 *We have the convergence in distribution sense up to subsequence for any* $T>0$*:*

$$\mu(\rho_n)\nabla u_n\to\mu(\rho)\nabla u\quad in\ \mathcal{D}'((0,T)\times\mathbb{R}^N),$$
$$\mu(\rho_n)^t\nabla u_n\to\mu(\rho)^t\nabla u\quad in\ \mathcal{D}'((0,T)\times\mathbb{R}^N),$$

and:

$$\lambda(\rho_n)\text{div}u_n\to\lambda(\rho)\text{div}u\quad in\ \mathcal{D}'((0,T)\times\mathbb{R}^N).$$

Proof: Let ϕ be a test function, then:

$$\begin{aligned}\int\mu(\rho_n)\nabla u_n\phi dxdt&=-\int\mu(\rho_n)u_n\nabla\phi dxdt+\int u_n\nabla\mu(\rho_n)\phi\,dxdt\\&=-\int\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}u_n\nabla\phi\,dxdt+\int\sqrt{\rho_n}u_n\frac{\mu'(\rho_n)}{\sqrt{\rho_n}}\nabla\rho_n\phi\,dxdt.\\&=-\int\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}u_n\nabla\phi\,dxdt+\frac{1}{2}\int\sqrt{\rho_n}u_n\nabla f(\rho_n)\phi\,dxdt.\end{aligned}\tag{5.97}$$

From lemma 3 in the appendix we know that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is uniformly bounded in $L^\infty(0,T;L^6_{loc}(\mathbb{R}^N))$. Furthermore via the inequality (1.9) and the convergence almost everywhere from ρ_n to ρ we know that $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ converges almost everywhere to $\frac{\mu(\rho)}{\sqrt{\rho}}$ (defined to be zero on the vacuum set). Therefore by the lemma 1, it converges strongly in $L^2_{loc}((0,T)\times\mathbb{R}^N)$ to $\frac{\mu(\rho)}{\sqrt{\rho}}$. This point is enough to prove the convergence of the first term as $\sqrt{\rho_n}u_n$ converges strongly.

Next since $\nabla f(\rho_n)$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^N))$, up to a subsequence $\nabla f(\rho_n)$ converges weakly to v in $L^2_{loc}((0,T)\times\mathbb{R}^N)$. In addition by Sobolev embedding we know that $f(\rho_n)$ is bounded in $L^\infty(0,T;L^6_{loc}(\mathbb{R}^N)))$. Since $f(\rho_n)$ converges almost everywhere (f is a continuous function) to $f(\rho)$, it converges strongly in $L^2_{loc}((0,T)\times\mathbb{R}^N)$ by using the lemma 1. It follows that:

$$\nabla f(\rho_n)\to\nabla f(\rho)\quad L^2_{loc}((0,T)\times\mathbb{R}^N)-\text{weak}.$$

It concludes the proof for the second term of (5.97).

A similar argument holds for $\mu(\rho_n)^t\nabla u_n$ and $\lambda(\rho_n)\text{div}u_n$ inasmuch as we have $|\lambda(\rho)|\leq C\mu(\rho)$ and $|\lambda'(\rho)|\leq C\mu'(\rho)$ via (1.6) and the remark 8.

Global existence when $u_0=-\nabla\varphi(\rho_0)$ when $\mu(\rho)=\mu\rho^\alpha$ with $\alpha>1-\frac{1}{N}$

The first thing to observe is that $\mu(\rho)=\mu\rho^\alpha$ with $\alpha>1-\frac{1}{N}$ verifies the hypothesis of theorem 1.2 for the stability of the global weak solution and in particular (1.6).Now it suffices to construct a sequence of global regular solution (ρ_n,u_n) verifying the hypothesis of the theorem 1.2, in particular the uniform bound via the entropy (3.36), (3.37) and (3.38) and the following properties:

- ρ_0^n converges strongly to ρ_0 in $L^1(\mathbb{R}^N)$.
- $\rho_0^n u_0^n$ converges strongly to $\rho_0 u_0$ in $L^1(\mathbb{R}^N)$.

Let κ a function belonging in the Schwarz space $\mathcal{S}(\mathbb{R}^N)$ with $\kappa>0$ and $\int_{\mathbb{R}^N}\kappa dx=1$. We define κ_n by:

$$\kappa_n=n^N\kappa(n\cdot).$$

Let us take for example $\kappa(x)=(2\pi)^{\frac{-N}{2}}e^{\frac{-|x|^2}{2}}$. By using the theorem 1.1 and by setting $\rho_0^n=\rho_0+\frac{f_0}{n}$ with f_0 continuous and belonging in $L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap W^{1,3}(\mathbb{R}^N)\cap W^{1,1}(\mathbb{R}^N)$ and strictly positive and ρ_0 continuous we know that there exists a global regular solution of the system (1.24) with initial data $(\rho_0^n,-\nabla\varphi(\rho_0^n))$. Indeed ρ_0^n verifies the hypothesis of theorem 1.2 since ρ_0^n is continuous in $L^1(\mathbb{R}^N)$ and strictly positive. We also observe that ρ_0^n converges strongly to ρ_0 in $L^1(\mathbb{R}^N)$. Let us deal now with a more simple case when we assume that ρ_0 belongs also in $L^\infty(\mathbb{R}^N)\cap W^{1,3}(\mathbb{R}^N)\cap W^{1,1}(\mathbb{R}^N)$. Next we have:

$$\begin{aligned}\sqrt{\rho_0^n}\varphi'(\rho_0^n)\nabla\rho_0^n&=\alpha(\rho_0^n)^{\alpha-\frac{3}{2}}\nabla\rho_0^n,\\&=\alpha(\rho_0+\frac{f_0}{n})^{\alpha-\frac{3}{2}}\nabla\rho_0+\frac{\alpha}{n}(\rho_0+\frac{f_0}{n})^{\alpha-\frac{3}{2}}\nabla f_0.\end{aligned}\tag{5.98}$$

Let us distinguish two cases $\alpha\geq\frac{3}{2}$ and $1-\frac{1}{N}\alpha\leq\frac{3}{2}$.

• $\alpha\geq\frac{3}{2}$. In this case since ρ_0 belongs in $H^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ we deduce that $\nabla f(\rho_0^n)$ and $\sqrt{\rho_0^n}u_0^n$ are uniformly bounded in $L^\infty(L^2(\mathbb{R}^N))$. Indeed we have:

$$|\sqrt{\rho_0^n}\varphi'(\rho_0^n)\nabla\rho_0^n|\leq(\alpha+1)(\|\rho_0\|_{L^\infty(\mathbb{R}^N)}+\|f_0\|_{L^\infty(\mathbb{R}^N)})^{\alpha-\frac{3}{2}}\big(|\nabla\rho_0|+|\nabla f_0|\big),$$

which implies the previous statement.
Let us prove that $\sqrt{\rho_0^n}(u_0^n)^{1+\frac{1}{p}}$ is uniformly bounded in $L^2(\mathbb{R}^N)$ for a $p>1$ large enough. It will be then sufficient in order to show that $\sqrt{\rho_0^n}u_0^n\sqrt{\ln(1+|u_0^n|^2)}$ is uniformly bounded in $L^2(\mathbb{R}^N)$. We have then for p large enough $(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}\geq0$ and:

$$\begin{aligned}&|\sqrt{\rho_0^n}(u_0^n)^{1+\frac{1}{p}}|\leq C(\rho_0^n)^{(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}}|\nabla\rho_0^n|^{1+\frac{1}{p}},\\&\leq C(\|\rho_0\|_{L^\infty(\mathbb{R}^N)}+\|f_0\|_{L^\infty(\mathbb{R}^N)})^{(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}}(|\nabla\rho_0|^{1+\frac{1}{p}}+|\nabla f_0|^{1+\frac{1}{p}}).\end{aligned}\tag{5.99}$$

Since we have assumed that ρ_0 and f_0 are in $W^{1,1}(\mathbb{R}^N)\cap W^{1,3}(\mathbb{R}^N)$ it shows that for p large enough, $\sqrt{\rho_0^n}(u_0^n)^{1+\frac{1}{p}}$ is bounded in $L^2(\mathbb{R}^N)$ which implies the desired result.
Let us finished by proving the strong convergence of $\rho_0^n u_0^n$ to $\rho_0 u_0$. We have then:

$$|\rho_0^n u_0^n-\rho_0 u_0|=|\alpha[(\rho_0+\frac{f_0}{n})^{\alpha-1}-\rho_0^{\alpha-1}]\nabla\rho_0+\frac{\alpha}{n}(\rho_0+\frac{f_0}{n})^{\alpha-1}\nabla f_0|.\tag{5.100}$$

Since ρ_0 and f_0 are bounded in $L^\infty(\mathbb{R}^N)$ we deduce that:

$$\|(\rho_0+\frac{f_0}{n})^{\alpha-1}-\rho_0^{\alpha-1}\|_{L^\infty(\mathbb{R}^N)}\rightarrow_{n\rightarrow+\infty}0.$$

This last inequality and Hölder's inequality show in particular that $[(\rho_0+\frac{f_0}{n})^{\alpha-1}-(\rho_0)^{\alpha-1}]\nabla\rho_0$ converges strongly in $L^1(\mathbb{R}^N)$ to 0. The second term of (5.100) is easy to treat. It gives that $\rho_0^n u_0^n$ converges strongly to $\rho_0 u_0$.
In particular via the first part of the theorem 1.2, it implies that (ρ_n,u_n) converge in distribution sense to a global weak solution (ρ,u).

- $1-\frac{1}{N}<\alpha<\frac{3}{2}$. In this case we shall assume moreover that $\sqrt{\rho_0}|\nabla\varphi(\rho_0)|^{1+\frac{1}{p}}$ and $\sqrt{f_0}|\nabla\varphi(f_0)|^{1+\frac{1}{p}}$ are bounded in $L^2(\mathbb{R}^N)$ for p large enough. We have via (5.98):

$$|\sqrt{\rho_0^n}\varphi^{'}(\rho_0^n)\nabla\rho_0^n|\leq\alpha\rho_0^{\alpha-\frac{3}{2}}|\nabla\rho_0|+\alpha n^{\frac{1}{2}-\alpha}f_0^{\alpha-\frac{3}{2}}|\nabla f_0|. \tag{5.101}$$

Since we have $\alpha>\frac{1}{2}$ it implies that $\sqrt{\rho_0^n}\varphi^{'}(\rho_0^n)\nabla\rho_0^n$ is uniformly bounded in $L^2(\mathbb{R}^N)$ by using the fact that $\sqrt{\rho_0}\nabla\varphi(\rho_0)$ and $\sqrt{f_0}\nabla\varphi(f_0)$ are bounded in $L^2(\mathbb{R}^N)$. It implies that $\nabla f(\rho_0^n)$ and $\sqrt{\rho_0^n}u_0^n$ are uniformly bounded in $L^\infty(L^2(\mathbb{R}^N))$. Next by (5.99) we have:

$$\begin{aligned}|\sqrt{\rho_0^n}(u_0^n)^{1+\frac{1}{p}}|&\leq C(\rho_0^n)^{(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}}|\nabla\rho_0^n|^{1+\frac{1}{p}},\\&\leq C(\rho_0^n)^{(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}}(|\nabla\rho_0|^{1+\frac{1}{p}}+\frac{1}{n^{1+\frac{1}{p}}}|\nabla f_0|^{1+\frac{1}{p}}),\\&\leq C(\rho_0^{(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}}|\nabla\rho_0|^{1+\frac{1}{p}}+n^{(1+\frac{1}{p})(2-\alpha)-\frac{3}{2}-\frac{1}{p}}f_0^{\frac{1}{2}+(1+\frac{1}{p})(\alpha-2)}|\nabla f_0|^{1+\frac{1}{p}},\\&\leq C(\rho_0^{(1+\frac{1}{p})(\alpha-2)+\frac{1}{2}}|\nabla\rho_0|^{1+\frac{1}{p}}+n^{(\frac{1}{2}-\alpha+\frac{1}{p}(1-\alpha)}f_0^{\frac{1}{2}+(1+\frac{1}{p})(\alpha-2)}|\nabla f_0|^{1+\frac{1}{p}}.\end{aligned} \tag{5.102}$$

Since $\alpha>\frac{1}{2}$ by choosing p large enough we obtain that $n^{(\frac{1}{2}-\alpha+\frac{1}{p}(1-\alpha)}$ which is uniformly bounded. By the fact that $\sqrt{\rho_0}|\nabla\varphi(\rho_0)|^{1+\frac{1}{p}}$ and $\sqrt{f_0}|\nabla\varphi(f_0)|^{1+\frac{1}{p}}$ are bounded in $L^2(\mathbb{R}^N)$, it implies that $\sqrt{\rho_0^n}(u_0^n)^{1+\frac{1}{p}}$ is bounded in $L^2(\mathbb{R}^N)$ and so that $\rho_0^n|u_0^n|^2\ln(1+|u_0^n|^2)$ is uniformly bounded in $L^2(\mathbb{R}^N)$.
Finally by (5.100) we are going to prove that $\rho_0^n u_0^n$ converges strongly to $\rho_0 u_0$. We start by remarking that when $\alpha\leq 1$:

$$|[(\rho_0+\frac{f_0}{n})^{\alpha-1}-\rho_0^{\alpha-1}]\nabla\rho_0|\leq 2\rho_0^{\alpha-1}|\nabla\rho_0|.$$

We deduce by the theorem of dominated convergence that $[(\rho_0+\frac{f_0}{n})^{\alpha-1}-\rho_0^{\alpha-1}]\nabla\rho_0$ converges strongly to 0 in $L^1(\mathbb{R}^N)$. The second term on the right hand side of (5.98) goes also trivially to 0. It the case where $1\leq\alpha<\frac{3}{2}$ it suffices to observe that:

$$\||[(\rho_0+\frac{f_0}{n})^{\alpha-1}-\rho_0^{\alpha-1}]\|_{L^\infty(\mathbb{R}^N)}\rightarrow_{n\rightarrow+\infty}0.$$

It implies in particular since $\nabla\rho_0$ belongs in $W^{1,1}(\mathbb{R}^N)$ that $[(\rho_0+\frac{f_0}{n})^{\alpha-1}-\rho_0^{\alpha-1}]\nabla\rho_0$ converges strongly to 0 in $L^1(\mathbb{R}^N)$. It achieves the proof of the strong convergence of

$\rho_0^n u_0^n$ to $\rho_0 u_0$. Finally it implies that (ρ_n, u_n) converge also in the case $1 - \frac{1}{N} < \alpha < \frac{3}{2}$ to a global weak solution (ρ, u) of the system (1.24).

We have previously proved the existence of a global weak solution (ρ, u) of the system (1.24) by assuming extra conditions on the initial density ρ_0, typically ρ_0 belonging in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$, ρ_0 continuous and $\sqrt{\rho_0}|\nabla\varphi(\rho_0)|^{1+\frac{1}{p}}$ bounded in $L^2(\mathbb{R}^N)$ for p large enough. Let us deal with the general case, it suffices by a second approximation on the initial data $(\rho_0^n, \sqrt{\rho_n}(u_0^n)^{1+\frac{1}{p}})$ with p large enough to pass to the limit by using the first part of the theorem 1.2. More precisely by a convolution approximation we choose $\rho_0^n = \rho_0 * \kappa_n$ which belongs in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$ and is continuous and we set $\sqrt{\rho_n}(u_0^n)^{1+\frac{1}{p}} = (\sqrt{\rho} u_0^{1+\frac{1}{p}}) * \kappa_n$.

Let us now describe the form of the solution (ρ, u), in particular we are interested in proving that ρ is also the unique global strong solution of the system 1.12) (for a such result, we refer to the theorem 2.4). To do this we recall that our first approximation (ρ_n, u_n) is solution of (1.12) with a initial data strictly positive in $L^1(\mathbb{R}^N)$ and continuous. Let us recall that from theorem 2.4 the porous media equation verifies a crucial property which ensures the uniqueness the L^1 contraction principle. Let us apply this property to the sequence $(\rho_n)_{n\in\mathbb{N}}$, we have then for all $n, m \in \mathbb{N}$:

$$\|\rho_n(t, \cdot) - \rho_m(t, \cdot)\|_{L^1(\mathbb{R}^N)} \le \|\rho_0^n - \rho_0^m\|_{L^1(\mathbb{R}^N)}.$$

Since we know that ρ_0^n converge to ρ_0 in $L^1(\mathbb{R}^N)$ it implies that $(\rho_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T], L^1(\mathbb{R}^N))$ for any $T > 0$ which implies that $(\rho_n)_{n\in\mathbb{N}}$ converges strongly to ρ_1 in $C([0,T], L^1(\mathbb{R}^N))$. But since via the first part of the theorem 1.2 we know that ρ_n converges also strongly in $C([0,T], L^1_{loc}(\mathbb{R}^N))$ to ρ, it implies that $\rho = \rho_1$. Furthermore by the definition of the L^1 solution of the porous media equation (indeed these last one are defined as limit of energy global weak solution after a regularization of the initial data ρ_0 where we use the fundamental L^1 contraction principle which ensures the uniqueness of a such process, we called this the limit solutions in the L^1 setting. We refer to the proof of the theorem 2.4 where we explained precisely how are defined the L^1 solutions. For more details on the theory the reader can also consult the chapter 6 and 9 of the excellent book of Vázquez [33]), we know that ρ_1 is the unique solution of porous media equation with $\rho_0^1 = \rho_0$. It proves that $\rho = \rho_1$ is the global unique solution of (1.12) with initial data ρ_0 which belongs in $L^1(\mathbb{R}^N)$. We proceed similarly for the second approximation (ρ_n, u_n) since each time the approximated sequel verify the L^1 contraction principle of the porous media equation. It concludes the proof of the theorem 1.2. □

5.1 Proof of Theorem 1.3 and corollary 1

5.2 Proof of theorem 1.3

We are now going to prove that if we have some global weak solution $(\rho_\epsilon, u_\epsilon)$ for the system (1.23) in the sense of the definition 1.3 (or see [29]), then these global weak solution converge in distribution sense to a quasi-solution (ρ, u) with initial data (ρ_0, u_0). To prove this, it will suffice to use the same compactness argument than in the previous section; except that we shall deal with the pressure term and that the entropy (3.43) is quite more complicated since there is a reminder term to deal with. Via the entropy (3.41), (3.42) we have:

$$\begin{aligned}&\int_{\mathbb{R}^N}\big[\rho_\epsilon|u_\epsilon(t,x)|^2(t,x)+\frac{\epsilon}{\gamma-1}\rho_\epsilon^\gamma\big]\,dx+\int_0^t\int_{\mathbb{R}^N}\mu(\rho_\epsilon)|Du_\epsilon|^2dxdt\\&\qquad+\int_0^t\int_{\mathbb{R}^N}\lambda(\rho_\epsilon)|\mathrm{div}u_\epsilon|^2dxdt\leq\int_{\mathbb{R}^N}\big[\rho_0|u_0|^2(x)+\frac{\epsilon}{\gamma-1}\rho_0^\gamma\big]dx.\\&\int_{\mathbb{R}^N}\big[\rho_\epsilon|u_\epsilon(t,x)|^2+\rho_\epsilon|\nabla\varphi(\rho_\epsilon)|^2(t,x)\big]\,dx+\epsilon\int_0^t\int_{\mathbb{R}^N}\nabla\varphi(\rho_\epsilon)\cdot\nabla\rho_\epsilon^\gamma dxdt\\&\qquad\leq C(\int_{\mathbb{R}^N}\big(\rho_0|u_0|^2(x)+\rho_0|\nabla\varphi(\rho_0)|^2(x)+\frac{\epsilon}{\gamma-1}\rho_0^\gamma(x)\big)\,dx).\end{aligned}\tag{5.103}$$

Lemma 6 *We are going to distinguish two cases.*

- $\nu_1\geq 2$ $\sqrt{\epsilon}\rho_\epsilon^{\frac{\gamma}{2}}$ *is uniformly bounded in* $L^2_T(L^6(\mathbb{R}^N))$ *for any* $T>0$ *when* $N=3$ *and in* $L^2_T(L^q(\mathbb{R}^N))$ *for any* $T>0$ *and* $q\geq 2$ *when* $N=2$.
- $0<\nu_1<2$
  1. $\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ *is uniformly bounded in* $L^1_T(L^3(\mathbb{R}^N))$ *for any* $T>0$ *when* $N=3$.
  2. $\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ *is bounded in any* $L^1_T(L^p(\mathbb{R}^N))$ *with* $p\in[2,+\infty[$ *when* $N=2$.

Proof: The inequality (5.103) insures a uniform bound of $\sqrt{\epsilon}\sqrt{\varphi'(\rho_\epsilon)\rho_\epsilon^{\gamma-1}}\nabla\rho_\epsilon$ in $L^2((0,T),L^2(\mathbb{R}^N))$ for any $T>0$. Let us evaluate $\sqrt{\varphi'(\rho_\epsilon)\rho_\epsilon^{\gamma-1}}\nabla\rho_\epsilon$, we have using (1.6) and (1.5):

$$\begin{aligned}\sqrt{\varphi'(\rho_\epsilon)\rho_\epsilon^{\gamma-1}}\nabla\rho_\epsilon&=\sqrt{\frac{N\lambda(\rho_\epsilon)+2\mu(\rho_\epsilon)}{N}\rho_\epsilon^{\gamma-3}+2(1-\frac{1}{N})\mu(\rho_\epsilon)\rho_\epsilon^{\gamma-3}}\;\;\nabla\rho_\epsilon,\\&\geq\sqrt{2(1-\frac{1}{N}+\frac{\nu_1}{2})\mu(\rho_\epsilon)\rho_\epsilon^{\gamma-3}}\;\;\nabla\rho_\epsilon,\end{aligned}$$

Via (1.9) we deduce that:

$$\begin{aligned}&\rho_\epsilon^{\frac{\gamma}{2}-1-\frac{1}{2N}+\frac{\nu_1}{4N}}|\nabla\rho_\epsilon|\leq C|\sqrt{\varphi'(\rho_\epsilon)\rho_\epsilon^{\gamma-1}}\nabla\rho_\epsilon|\;\;\forall\rho_\epsilon>1,\\&\rho_\epsilon^{\frac{\gamma}{2}-1-\frac{1}{2N}+\frac{\nu_2}{4N}}|\nabla\rho_\epsilon|\leq C|\sqrt{\varphi'(\rho_\epsilon)\rho_\epsilon^{\gamma-1}}\nabla\rho_\epsilon|\;\;\forall\rho_\epsilon\leq 1.\end{aligned}$$

Let set ψ a C_0^∞ function such that $\psi=1$ on $B(0,1)$ and $\psi=0$ on ${}^cB(0,2)$. It implies that since $\operatorname{supp}\psi'$ is included in the shell $C(0,1,2)$:

- $\sqrt{\epsilon}\psi(\rho_\epsilon)\nabla\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_2}{4N}}$ is uniformly bounded in $L^2_T(L^2(\mathbb{R}^N))$ for any $T>0$.
- $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\nabla\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is uniformly bounded in $L^2_T(L^2(\mathbb{R}^N))$ for any $T>0$.

Next by (5.103) we know that $\epsilon^{\frac{1}{\gamma}}\rho_\epsilon$ is uniformly bounded in $L^\infty_T(L^\gamma(\mathbb{R}^N))$ for any $T>0$ which implies that $\sqrt{\epsilon}\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for any $T>0$. Let us deal with two different cases.

$\nu_1 \geq 2$

In this case we have : $-\frac{1}{2N}+\frac{\nu_1}{4N}\geq 0$ it implies that $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\nabla\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^2_T(L^2(\mathbb{R}^N))$ for any $T>0$. And since $\epsilon^{\frac{1}{2}}\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for any $T>0$, we deduce that $\epsilon^{\frac{1}{2}}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for any $T>0$. We deduce that $\epsilon^{\frac{1}{2}}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^2_T(H^1(\mathbb{R}^N))$ for any $T>0$. Easily since $\epsilon^{\frac{1}{2}}\psi(\rho_\epsilon)\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))\cap L^\infty_T(L^\infty(\mathbb{R}^N))$, we deduce by Sobolev embedding that $\epsilon^{\frac{1}{2}}\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^2_T(L^6(\mathbb{R}^N))$ when $N=3$ and in $L^2_T(L^q(\mathbb{R}^N))$ for any $q\geq 2$.

$0<\nu_1<2$

Let us distinguish the case $N=2$ and $N=3$.

$\bullet N=3$
We know that $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\nabla\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is uniformly bounded in $L^2_T(L^2(\mathbb{R}^N))$ for any $T>0$. Furthermore since $\sqrt{\epsilon}\rho_\epsilon^{\frac{\gamma}{2}}$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$ we deduce that $\epsilon^{\frac{1}{p}}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is uniformly bounded in $L^\infty_T(L^p(\mathbb{R}^N))$ with $p(\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N})=2$ with $p>2$ (indeed it is possible because $\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}>0$) . Indeed we have:

$$\|\epsilon^{\frac{1}{p}}\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}\|_{L^\infty_T(L^p(\mathbb{R}^N))}=\|\epsilon^{\frac{1}{\gamma}}\rho_\epsilon\|^{\frac{\gamma}{p}}_{L^\infty_T(L^\gamma(\mathbb{R}^N))}.$$

Since $p>2$ it implies that $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is uniformly bounded in $L^\infty_T(L^p(\mathbb{R}^N))$ and also in $L^\infty_T(L^2(\mathbb{R}^N))$ because $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is strictly positive only a set of finite measure (it is a direct consequence of the Tchebytchev lemma). We have shown that $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is uniformly bounded in $L^2_T(H^1(\mathbb{R}^N))$ for any $T>0$. By Sobolev embedding we deduce that $\epsilon(1-\psi(\rho_\epsilon))\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^1_T(L^3(\mathbb{R}^N))$ for any $T>0$.
Since we know that $\psi(\rho_\epsilon)\rho_\epsilon$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))\cap L^\infty_T(L^\infty(\mathbb{R}^N))$ we deduce that $\epsilon\psi(\rho_\epsilon)\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^1_T(L^3(\mathbb{R}^N))$ for any $T>0$ because if $\gamma-\frac{1}{N}+\frac{\nu_1}{2N}\geq 1$ this is obvious by interpolation. In the other case $\epsilon\psi(\rho_\epsilon)\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^\infty_T(L^p(\mathbb{R}^N))$ with $p(\gamma-\frac{1}{N}+\frac{\nu_1}{2N})=1$. And we have $p=\frac{1}{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}\leq 2$, we conclude also by interpolation in order to prove that $\epsilon\psi(\rho_\epsilon)\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^\infty_T(L^3(\mathbb{R}^N))$.
Finally we obtain that $\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^1_T(L^3(\mathbb{R}^N))$ for any $T>0$.

$\bullet\ N=2$
Similarly we obtain that $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ is uniformly bounded in $L^2_T(H^1(\mathbb{R}^N))$ for any $T>0$. It provides a uniform bound for $\sqrt{\epsilon}(1-\psi(\rho_\epsilon))\rho_\epsilon^{\frac{\gamma}{2}-\frac{1}{2N}+\frac{\nu_1}{4N}}$ in any $L^2_T(L^q(\mathbb{R}^N))$ with $q\in[2,+\infty[$. It means that $\epsilon(1-\psi(\rho_\epsilon))\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is in any $L^1_T(L^p(\mathbb{R}^N))$

with $p\in[1,+\infty[$. Since $\psi(\rho_\epsilon)\rho_\epsilon$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))\cap L^\infty_T(L^\infty(\mathbb{R}^N))$ we deduce that $\epsilon\psi(\rho_\epsilon)\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is bounded in any $L^1_T(L^p(\mathbb{R}^N))$ with $p\in[1,+\infty[$ if $\gamma-\frac{1}{N}+\frac{\nu_1}{2N}\geq 1$ and in any $L^1_T(L^p(\mathbb{R}^N))$ with $[p_1,+\infty[$ where $p_1(\gamma-\frac{1}{N}+\frac{\nu_1}{2N})=1$ in the other case where $p_1\leq 2$. It implies that:

- $\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is bounded in any $L^1_T(L^p(\mathbb{R}^N))$ with $p\in[1,+\infty[$ if $\gamma-\frac{1}{N}+\frac{\nu_1}{2N}\geq 1$.
- $\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is bounded in any $L^1_T(L^p(\mathbb{R}^N))$ with $p\in[p_1,+\infty[$ with $p_1(\gamma-\frac{1}{N}+\frac{\nu_1}{2N})=1$ in other case.

It achieves the proof of the lemma. □

In the following lemma we are going to get uniform estimate on the pressure ρ_ϵ^γ.

Lemma 4 *Let us distinguish two cases:*

- $\nu_1\geq 2$. *The pressure $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^{\frac{5}{3}}((0,T)\times\mathbb{R}^N)$ when $N=3$ and $L^r((0,T)\times\mathbb{R}^N)$ for all $r\in[1,2[$ when $N=2$.*
- $0<\nu_1<2$. *The pressure $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^{r_1}_T(L^{r_1}(\mathbb{R}^N))$ with $r_1=2-\frac{2-\nu_1}{6(1+\nu_1)}$ when $N=3$.*

Proof:
• $\nu_1\geq 2$:
We have seen in the lemma 6 that when $N=2$, $\sqrt{\epsilon}\rho_\epsilon^{\frac{\gamma}{2}}$ is bounded in $L^2(0,T;L^q(\mathbb{R}^N))$ for any $q\geq 2$. We deduce that $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^1(0,T;L^p(\mathbb{R}^N))\cap L^\infty(L^1(\mathbb{R}^N))$ for all $p\in[1,+\infty[$, hence by interpolation $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^r((0,T)\times\mathbb{R}^N)$ for all $r\in[1,2[$.
When $N=3$, by Sobolev embedding we only obtain that $\sqrt{\epsilon}\rho_\epsilon^{\frac{\gamma}{2}}$ bounded in $L^2(0,T;L^6(\mathbb{R}^N))$ which gives that $\epsilon\rho_\epsilon^\gamma$ is uniformly bounded in $L^1(0,T;L^3(\mathbb{R}^N))$. By Hölder inequality we have:

$$\|\epsilon\rho_\epsilon^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\mathbb{R}^N}\leq\|\epsilon\rho_\epsilon^\gamma\|^{\frac{2}{5}}_{L^\infty(0,T;L^1(\mathbb{R}^N))}\|\epsilon\rho_\epsilon^\gamma\|^{\frac{3}{5}}_{L^1(0,T;L^3(\mathbb{R}^N))}$$

Hence $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^{\frac{5}{3}}((0,T)\times\mathbb{R}^N)$.

• $0<\nu_1<2$:

When $N=3$ we know via the lemma 6 that $\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}$ is bounded in $L^1_T(L^3(\mathbb{R}^N))$. We have in particular that:

$$\epsilon\rho_\epsilon^\gamma=(\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}})\rho_\epsilon^{\frac{1}{N}-\frac{\nu_1}{2N}}.$$

Via lemma 3 we have seen that ρ_ϵ is uniformly bounded in $L^\infty_T(L^{1+\nu_1}(\mathbb{R}^N))$ when $N=3$. We define p such that:

$$p(\frac{1}{N}-\frac{\nu_1}{2N})=p(\frac{1}{3}-\frac{\nu_1}{6})=1+\nu_1\;\;\Leftrightarrow\;\;p=\frac{6(1+\nu_1)}{2-\nu_1}\;\;\text{with}\;\;\nu_1<2.$$

By Hölder's inequality we obtain with $\frac{1}{p_1}=\frac{1}{3}+\frac{1}{p}$ that:

$$\|\epsilon\rho_\epsilon^\gamma\|_{L^1_T(L^{p_1}(\mathbb{R}^N))}=\|\epsilon\rho_\epsilon^{\gamma-\frac{1}{N}+\frac{\nu_1}{2N}}\|_{L^1_T(L^3(\mathbb{R}^N))}\|\rho_\epsilon^{\frac{1}{N}-\frac{\nu_1}{2N}}\|_{L^\infty_T(L^p(\mathbb{R}^N))}.$$

Now since $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^1_T(L^{p_1}(\mathbb{R}^N))\cap L^\infty_T(L^1(\mathbb{R}^N))$ we have by interpolation that $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^r_T(L^q(\mathbb{R}^N))$ with:

$$\begin{cases}\dfrac{1}{r}=\theta,\\ \dfrac{1}{q}=\dfrac{\theta}{p_1}+1-\theta,\end{cases}$$

It implies the following relation $\frac{1}{q}+\frac{1}{r}(1-\frac{1}{p_1})=1$. In particular we obtain that $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^{r_1}_T(L^{r_1}(\mathbb{R}^N))$ with $r_1=2-\frac{2-\nu_1}{6(1+\nu_1)}$. □

We are going finally to prove that $\sqrt{\rho_\epsilon}|u_\epsilon|(\ln(1+|u_\epsilon|^2)^{\frac{1}{2}}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$.

Lemma 5 *$\sqrt{\rho_\epsilon}|u_\epsilon|(\ln(1+|u_\epsilon|^2)^{\frac{1}{2}}$ is uniformly bounded in $L^\infty_T(L^2(\mathbb{R}^N))$ for the following situations:*

- $\nu_1\geq 2$ *and:*

$$\begin{aligned}&\frac{5}{6}+\frac{\nu_2}{12}<\gamma<2+\frac{\nu_1}{2}\quad \textit{if}\;\;N=3,\\ &\frac{5}{6}+\frac{\nu_2}{12}<\gamma<\frac{5}{6}+\frac{7}{12}\nu_1\quad \textit{if}\;\;N=3,\\ &\frac{1}{4}+\frac{\nu_2}{8}<\gamma\quad \textit{if}\;\;N=2.\end{aligned}$$

- $0<\nu_1<2$ *and:*

$$\begin{aligned}&\frac{5}{6}+\frac{\nu_2}{12}<\gamma<\frac{(4-\nu_1)(1+\nu_1)}{2-\nu_1}\quad \textit{if}\;\;N=3,\\ &\frac{5}{6}+\frac{\nu_2}{12}<\gamma<\frac{5}{6}+\frac{7}{12}\nu_1\quad \textit{if}\;\;N=3,\\ &\frac{1}{4}+\frac{\nu_2}{8}<\gamma\quad \textit{if}\;\;N=2.\end{aligned}$$

Proof: Let us come back to the inequality (3.43), we have $\forall\delta\in(0,2)$:

$$\begin{aligned}&\int_{\mathbb{R}^N}\rho_\epsilon\frac{1+|u_\epsilon|^2}{2}\ln(1+|u_\epsilon|^2)(t,x)dx+\nu\int_0^t\int_{\mathbb{R}^N}\mu(\rho_\epsilon)(1+\ln(1+|u_\epsilon|^2))|Du_\epsilon|^2(t,x)dxdt\\ &\leq C\int_0^t\int_{\mathbb{R}^N}\mu(\rho_\epsilon)|\nabla u_\epsilon|^2(t,x)dxdt+C_\delta\epsilon^2\int_0^t\big(\int_{\mathbb{R}^N}(\frac{\rho_\epsilon^{2\gamma-\frac{\delta}{2}}}{\mu(\rho_\epsilon)})^{\frac{2}{2-\delta}}dx\big)dt.\end{aligned}\tag{5.104}$$

Let us deal with this inequality in order to prove the uniform bound of $\rho_\epsilon\frac{1+|u_\epsilon|^2}{2}\ln(1+|u_\epsilon|^2)$ in $L^\infty_T(L^1(\mathbb{R}^N))$ for any $T>0$. To do this we shall estimate the right hand side of

(5.104). Using the inequality (1.9) we obtain that:

$$\begin{aligned}&\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)}\leq\frac{1}{C}\rho^{2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N}}\quad\forall\rho>1,\\&\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)}\leq\frac{1}{C}\rho^{2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_2}{2N}}\quad\forall\rho\leq 1.\end{aligned}\tag{5.105}$$

Let us distinguish two cases when $\nu_1\geq 2$ and when $0<\nu_1<2$.

• $\nu_1\geq 2$ and $N=3$
From the lemma 4, we know that $\epsilon\rho_\epsilon^\gamma$ is uniformly bounded in $L^{\frac{5}{3}}((0,T)\times\mathbb{R}^N)\cap L^\infty_T(L^1(\mathbb{R}^N))$ for $N=3$ which implies that it exists $C>0$ such that:

$$\epsilon^{\frac{5}{3}}\int_0^t\int_{\mathbb{R}^N}\rho_\epsilon^{\frac{5}{3}\gamma}dxdt\leq C.\tag{5.106}$$

In particular it implies that for δ small enough $\epsilon^2(1_{\{\rho_\epsilon\geq 1\}}\frac{\rho_\epsilon^{2\gamma-\frac{\delta}{2}}}{\mu(\rho_\epsilon)})^{\frac{2}{2-\delta}}$ is uniformly bounded in $L^1((0,T)\times\mathbb{R}^N)$ under the conditions that:

$$2\gamma-1+\frac{1}{N}-\frac{\nu_1}{2N}<\frac{5}{3}\gamma\ \ \Leftrightarrow\ \ \gamma<2+\frac{\nu_1}{2}.\tag{5.107}$$

Indeed by Tchebytchev lemma we have:

$$|\{x,\ |\rho_\epsilon(t,x)|\geq 1\}|\leq\|\rho_0\|_{L^1(\mathbb{R}^N)}.$$

We choose p such that $p(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}=\frac{5}{3}\gamma$ with δ small enough. By Hölder's inequality we have:

$$\begin{aligned}\epsilon^2\int_0^T\int_{\mathbb{R}^N}1_{\{\rho_\epsilon\geq 1\}}\rho^{(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}}dxdt&\leq\epsilon^2\int_0^T(\int_{\mathbb{R}^N}\rho^{\frac{5}{3}\gamma}dx)^{\frac{1}{p}}\|\rho_0\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p'}}dt,\\&\leq\epsilon^{2-\frac{5}{3p}}(T\|\rho_0\|_{L^1(\mathbb{R}^N)})^{\frac{1}{p'}}(\int_0^T\int_{\mathbb{R}^N}\epsilon^{\frac{5}{3}}\rho^{\frac{5}{3}\gamma}dxdt)^{\frac{1}{p}}.\end{aligned}$$

with $\frac{1}{p'}=1-\frac{1}{p}$ and $p>1$. It implies by (5.106) that $\epsilon^2 1_{\{\rho_\epsilon\geq 1\}}\rho^{(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}}$ is uniformly bounded in $L^1((0,T)\times\mathbb{R}^N)$ under the hypothesis (5.107).
Let us deal now with the term $\epsilon^2 1_{\{\rho_\epsilon\leq 1\}}\rho^{(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}}$, it suffices to assume that:

$$2\gamma-1+\frac{1}{N}-\frac{\nu_2}{2N}>1\ \ \Leftrightarrow\ \ \gamma>\frac{5}{6}+\frac{\nu_2}{12}.\tag{5.108}$$

Indeed we know that $1_{\{\rho_\epsilon\leq 1\}}\rho_\epsilon$ is bounded in $L^\infty_T(L^1(\mathbb{R}^N))\cap L^\infty_T(L^\infty(\mathbb{R}^N))$ which insures the uniform bound of $1_{\{\rho_\epsilon\leq 1\}}\rho^{(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}}$ in $L^\infty_T(L^1(\mathbb{R}^N))$ by interpolation. This achieves the proof of the case $N=3$.

The last situation consists when $N=3$ in using the lemma 3 when $2\gamma-1+\frac{1}{N}-\frac{\nu_1}{2N}\leq 1+\frac{3\nu_1}{N}$ which is equivalent to $\gamma\leq\frac{5}{6}+\frac{7}{12}\nu_1$. Indeed via the lemma 3 we know that $\rho_\epsilon^{1+\frac{\nu_1}{2N}}$ is uniformly bounded in $L^\infty_T(L^6(\mathbb{R}^N))$. It implies that $\epsilon^2 1_{\{\rho_\epsilon\geq 1\}}\rho^{(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}}$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$ when $2\gamma-1+\frac{1}{N}-\frac{\nu_1}{2N}\leq 1+\frac{3\nu_1}{N}$. In order to bounde $\epsilon^2 1_{\{\rho_\epsilon\leq 1\}}\rho^{(2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N})\frac{2}{2-\delta}}$ we use the same hypothesis than in the previous case.

• $\nu_1\geq 2$, $N=2$
In this case, the situation is quite simple, indeed we know via the lemma 3 that ρ_ϵ is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $q\geq 1$. In particular it implies that $1_{\{\rho_\epsilon\geq 1\}}\rho^{2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N}}$ is bounded in $L^\infty_T(L^1(\mathbb{R}^N))$ without any specific condition. However we shall require a hypothesis for dealing with the term $1_{\{\rho_\epsilon\leq 1\}}\rho^{2\gamma-\frac{\delta}{2}-1+\frac{1}{N}-\frac{\nu_1}{2N}}$ which is similar to the previous section:

$$2\gamma-1+\frac{1}{N}-\frac{\nu_2}{2N}>1 \;\;\Leftrightarrow\;\; \gamma>\frac{1}{4}+\frac{\nu_2}{8}. \tag{5.109}$$

• $0<\nu_1<2$, $N=2$
The proof in this situation is exactly the same than in the previous case by using the lemma 3. We need only:

$$2\gamma-1+\frac{1}{N}-\frac{\nu_2}{2N}>1 \;\;\Leftrightarrow\;\; \gamma>\frac{1}{4}+\frac{\nu_2}{8}. \tag{5.110}$$

• $0<\nu_1<2$, $N=3$

Via the lemma 4 we have seen that $\epsilon\rho_\epsilon^\gamma$ is bounded in $L^{r_1}_T(L^{r_1}(\mathbb{R}^N))$ with $r_1=2-\frac{2-\nu_1}{6(1+\nu_1)}$. By using exactly the same argument than in the previous case we have two possibility to bound the last term on the right hand side of (5.104):

$$\begin{aligned}&2\gamma-\frac{2}{3}-\frac{\nu_1}{6}<(2-\frac{2-\nu_1}{6(1+\nu_1)})\gamma \;\;\Leftrightarrow\;\; \gamma<2+\frac{\nu_1}{2}.\\ &2\gamma-1+\frac{1}{N}-\frac{\nu_2}{2N}>1\end{aligned} \tag{5.111}$$

or

$$\frac{5}{6}+\frac{\nu_2}{12}<\gamma\frac{5}{6}+\frac{7\nu_1}{12}. \tag{5.112}$$

It achieves the proof of the lemma 5. □

We have now proved that $\rho_\epsilon\frac{1+|u_\epsilon|^2}{2}\ln(1+|u_\epsilon|^2)$ is uniformly bounded in $L^\infty_{loc}(L^1(\mathbb{R}^N))$. We can then pass to the limit when ϵ goes to 0, more precisely by using lemmas 1, 2 we show that ρ_ϵ, $\rho_\epsilon u_\epsilon$ and $\sqrt{\rho_{ep}}u_\epsilon\otimes\sqrt{\rho_{ep}}u_\epsilon$ converges in distribution sense to ρ, ρu and $\sqrt{\rho}u\otimes\sqrt{\rho}u$ and lemma 3 give us the convergence in distribution sense of the diffusion term. Furthermore the lemmas 1 and 2 give us the following desired strong convergence:

- ρ_ϵ converges strongly to ρ in $C([0,T],L^{1+\alpha}_{loc}(\mathbb{R}^N))$ for α small enough when $N=3$.
- ρ_ϵ converges strongly to ρ in $C([0,T],L^p_{loc}(\mathbb{R}^N))$ for any $p\geq 1$ when $N=2$.
- $\sqrt{\rho_\epsilon}u_\epsilon$ converges strongly to $\sqrt{\rho}u$ in $L^2_{loc}((0,T)\times\mathbb{R}^N))$ for any $T>0$.

It remains only to deal with the term $\epsilon\nabla\rho_\epsilon^\gamma$ and to prove that it converges in distribution sense to 0.

Lemma 6 *Let us distinguish two cases:*

- *When $N = 3$*
  $\epsilon^\alpha \rho_\epsilon^\gamma$ *converges strongly to* 0 *in* $L^\infty_T(L^1_{loc}(\mathbb{R}^N))$ *for any* $\alpha > 0$ *when* ϵ *goes to* 0 *for:*

$$\gamma < 1 + \nu_1.$$

  $\epsilon^\alpha \rho_\epsilon^\gamma$ *converges strongly to* 0 *in* $L^{\frac{5}{3}-\alpha}((0,T) \times \mathbb{R}^N))$ *for* $\nu_1 \geq 2$ *and for any* $\alpha > 0$ *small enough when* ϵ *goes to* 0*.*
  $\epsilon^\alpha \rho_\epsilon^\gamma$ *converges strongly to* 0 *in* $L^{1+\alpha}_T(L^{r_1-\alpha}(\mathbb{R}^N))$ *for* $0 < \nu_1 < 2$ *and for any* $\alpha > 0$ *small enough when* ϵ *goes to* 0*.*

- *When $N = 2$*
  $\epsilon^\alpha \rho_\epsilon^\gamma$ *converges strongly to* 0 *in* $L^\infty_T(L^1_{loc}(\mathbb{R}^N))$ *for any* $\alpha > 0$ *when* ϵ *goes to* 0*.*

Proof: When $N = 2$ we know via the lemma 3 that ρ_ϵ is bounded in $L^\infty_T(L^q(\mathbb{R}^N))$ for any $q \geq 1$. It implies trivially that $\epsilon^\alpha \rho_\epsilon^\gamma$ converges strongly to 0 in $L^\infty_T(L^1_{loc}(\mathbb{R}^N))$ for any $\alpha > 0$ when ϵ goes to 0.
When $N = 3$ we are in a similar situation when $\gamma < 1 + \nu_1$ via the lemma 3. If $\nu_1 \geq 2$ we have seen in the lemma 4 that $\epsilon \rho_\epsilon^\gamma$ is uniformly bounded in $L^{\frac{5}{3}}((0,T) \times \mathbb{R}^N))$, combining this result with the fact that $\epsilon \rho_\epsilon^\gamma$ is uniformly bounded in $L^\infty_T(L^1(\mathbb{R}^N))$ and an interpolation argument we obtain the result that we wish. In the case $0 < \nu_1 < 2$ we apply a similar argument with r_1 by using the lemma 4. It concludes the proof of the lemma. □

Furthermore in the same spirit by using exactly the same arguments than in the proof of theorem 1.2 and the previous estimate on the pressure we prove also the stability of the global weak solutions for the system (1.1).

5.3 Proof of the corollary 1

From the previous theorem we know that $(\rho_\epsilon, u_\epsilon)$ converges to a quasi solution (ρ, u) and that ρ_ϵ converges strongly to ρ in $C([0,T], L^1_{loc}(\mathbb{R}^N))$. In particular when $\mu(\rho) = \mu\rho^\alpha$ if we assume that there exists a unique global quasi solution we know via the theorem 1.2 that this quasi solution verifies the porous media or the fast diffusion equation in function of α inasmuch as ρ is solution of (1.12).
In particular when we assume that the initial density ρ_0 has a compact support, it implies that when $\alpha > 1$ the support of the density ρ remains bounded along the time. Indeed it consists merely of not ing that we can find a delayed Barrenblatt solution centered for instance at 0 that lies on the top of ρ_0, it means:

$$0 \leq \rho_0(x) \leq U_m(\tau, x) \quad \forall x \in \mathbb{R}^N,$$

with m large enough and $\tau > 0$. By theorem 2.4 and the maximum principle we know that:

$$0 \leq \rho(t,x) \leq (t+\tau)^{-\gamma_1} F(\frac{x}{(t+\tau)^\beta}) \quad \forall x \in \mathbb{R}^N,$$

with $F(x) = (C - \frac{(\alpha-1)\gamma_1}{2\alpha}|x|^2)_+^{\frac{1}{\alpha-1}}$. In particular it implies an information on the expansion of the support of the solution since we observe that the support of $\rho(t, \cdot)$ is included

in a set $E(t)=CB(0,M(t+\tau)^{\frac{\beta}{2}}$ with $C>0$, $M>0$ independent of t.
Let us prove now that ρ_ϵ converges strongly to ρ in $C([0,T],L^1(\mathbb{R}^N))$. We know for the moment that ρ_ϵ converges strongly to ρ in $C([0,T],L^1_{loc}(\mathbb{R}^N))$, it suffices to consider K a compact set large enough such that for any $t\in[0,T]$ we have supp$\rho(t,\cdot)\subset K$, we have then:

$$\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(K)}\rightarrow_{\epsilon\rightarrow 0} 0. \tag{5.113}$$

Now we have by conservation of the mass and the fact that $\rho(t,\cdot)=0$ in K^c for any $t\in[0,T]$:

$$\begin{aligned}\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(K^c)}&=\|\rho_\epsilon(t,\cdot)\|_{L^1(K^c)},\\&=\|\rho_0\|_{L^1(\mathbb{R}^N)}-\|\rho_\epsilon(t,\cdot)\|_{L^1(K)}.\end{aligned} \tag{5.114}$$

In particular since ρ_ϵ converges strongly to ρ in $C([0,T],L^1(K))$, (5.114) implies that $\|\rho_\epsilon(t,\cdot)\|_{L^1(K)}$ converges uniformly on $[0,T]$ to $\|\rho(t,\cdot)\|_{L^1(K)}$ when ϵ goes to 0. But since $\|\rho(t,\cdot)\|_{L^1(K)}=\|\rho_0\|_{L^1(\mathbb{R}^N)}$ for any $t\in[0,T]$ (indeed the support of $\rho(t,\cdot)$ is completely included in K), it induces that $\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(K^c)}$ converges uniformly on $[0,T]$ to 0 when ϵ goes to 0.
Finally we have by using (5.114):

$$\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}\leq\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(K)}+\|\rho_0\|_{L^1(\mathbb{R}^N)}-\|\rho_\epsilon(t,\cdot)\|_{L^1(K)}.$$

It implies that $\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}$ converges uniformly on $[0,T]$ to 0 when ϵ goes to 0 and we have shown that ρ_ϵ converges strongly to ρ in $C([0,T],L^1(\mathbb{R}^N))$. In particular it implies that for ϵ small enough ρ_ϵ is the sum of a solution with compact support on $[0,T]$ and of a term of small L^1 norm. In this sense we can claim that the propagation speed of the free boundary of ρ_ϵ is not so far to be finite at a small L^1 perturbation. This implies in particular by interpolation that ρ_ϵ converges strongly to ρ in $C([0,T],L^p(\mathbb{R}^N))$ for any $T>$, any $p\leq 1+\alpha$ with α small enough if $N=3$ and with $p\geq 1$ if $N=2$.
In particular since via the theorem 2.6, we have:

$$\|\rho(t)\|_{L^p(\mathbb{R}^N)}\leq Ct^{-\sigma_p}\|\rho_0\|^{\alpha_p}_{L^1(\mathbb{R}^N)},$$

with $\sigma_p=\frac{N(\alpha-1)+2p}{(N(\alpha-1)+2)p}$ and $\alpha_p=\frac{N(p-1)}{(N(\alpha-1)+2)p}$. It shows that up a remainder term of small norm in L^p, the L^p norm of ρ_ϵ decrease in time for small ϵ. It means that in some sense the density is subjected to a damping effect in time for the L^p norm which is very surprising since this effect seems purely non linear.

Let us deal now with the time asymptotic behavior of ρ_ϵ. We expect that $\rho_\epsilon(t,\cdot)$ goes asymptotically in time to the Barrenblatt solution U_m of (1.12) of mass $\|\rho_0\|_{L^1(\mathbb{R}^N)}=m$. We have then:

$$\|U_m(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}\leq\|U_m(t,\cdot)-\rho(t,\cdot)\|_{L^1(\mathbb{R}^N)}+\|\rho(t,\cdot)-\rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}. \tag{5.115}$$

Via the theorem 2.7 we know that $\|U_m(t,\cdot)-\rho(t,\cdot)\|_{L^1(\mathbb{R}^N)}$ converges asymptotically to 0 when t goes to $+\infty$. The second term converges also to 0 when ϵ goes to 0 since we have shown that ρ_ϵ converges strongly to ρ in $C([0,T],L^1(\mathbb{R}^N))$ for any $T>0$. In particular it implies that for any $\alpha>0$ it exists $T>0$ such that:

$$\|U_m(t,\cdot)-\rho(t,\cdot)\|_{L^1(\mathbb{R}^N)}\leq\alpha\quad\forall t>T.$$

Furthermore it exists $\epsilon_0 > 0$ such that for all $\epsilon \le \epsilon_0$ we have:

$$\|\rho(t,\cdot) - \rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le \alpha \quad \forall t \in [0, nT], \quad \text{with } n \in \mathbb{N}.$$

It implies that for all $\alpha > 0$ it exits $T > 0$ such that for all $n \in \mathbb{N}$ it exits $\epsilon_0 > 0$ such that for all $0 < \epsilon \le \epsilon_0$ we have:

$$\|U_m(t,\cdot) - \rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le 2\alpha \quad \forall t \in [T, nT].$$

In this sense we observe that for ϵ small enough the solution ρ_ϵ tends to converge asymptotically to a Barrenblatt solution of mass $\|\rho_0\|_{L^1(\mathbb{R}^N)}$. □

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