THÈSE

pour obtenir le grade de

DOCTEUR de l'Université de Paris XII-Val de Marne

Spécialité : **MATHÉMATIQUES**

présentée et soutenue par
Boris HASPOT

le 29 Novembre 2007

Etude d'équations liées à la mécanique des fluides compressibles capillaires

Directeur de Thèse : **Raphaël DANCHIN**

Jury composé de

Mme.	**Sylvie BENZONI**	**Examinateur**
Mr.	**Didier BRESCH**	**Rapporteur**
Mr.	**Frédéric COQUEL**	**Examinateur**
Mr.	**Antonín NOVOTNY**	**Rapporteur**
Mr.	**Marius PAICU**	**Examinateur**

Remerciements

Je voudrais en premier lieu remercier Raphaël Danchin d' avoir dirigé ma thèse, et de m' avoir initié à l' étude des équations aux dérivées partielles, ainsi qu' à de passionnants sujets de recherche. Ses conseils, sa disponibilité, et sa clairvoyance mathématique m' ont été très stimulants.
Je suis très reconnaissant à Didier Bresch et Antonín Novotný, qui ont bien voulu établir un rapport sur ma thèse. Je les remercie pour l' intérêt qu' ils ont porté à mon travail. Leurs commentaires m'ont été très utiles et m'ont permis d' envisager plusieurs prolongements supplémentaires à mes travaux.
J' aimerais aussi exprimer ma gratitude à Sylvie Benzoni, Frédéric Coquel et Marius Paicu qui me font l' honneur d' être membres du jury de soutenance.
Je remercie le Laboratoire d'Analyse et Système Aléatoire de Paris XII, son Directeur Franck Pacard et tous ses membres qui m'ont acceuilli, et qui m' ont donné des conditions de travail très agréables.
Je voudrais remercier mes camarades thésards de Paris 12 et d'ailleurs, qui m' ont accompagné pendant ces années : Anna, Abdellatif, Arnaud, Aurélia, Fethi, Habib, Hassen, Marianne, Saida, Zaynab et les autres, merci à eux pour leur gentillesse et pour toutes ces discussions partagées.
Mes pensées vont à mes amis, pour tout ce qu' ils m' apportent, mais surtout pour leur amitié.
Enfin, je tiens à exprimer ma profonde affection à ma famille et à mes parents qui m' ont toujours soutenu.

Table des matières

Chapitre 1

Introduction

Cette introduction se décompose en deux grandes parties : la première section est destinée à une présentation des équations de Navier-Stokes compressible avec un terme de capillarité de type Korteweg pour des interfaces diffuses. Nous commençons par décrire les phénomènes de changements de phase pour des fluides multi-constituants, en exposant quelques méthodes de simulations numériques qui permettent de tenir en compte des différents paramètres physiques influençant le comportement du mélange aux interfaces. On s'intéresse plus particulièrement au cas d'un fluide pur se présentant sous forme de deux phases liquide et vapeur, on rappelle ainsi la méthode dite du second gradient. Nous reprenons ici dans une large mesure les grandes lignes des travaux de [45], [44] et [33]. Nous écrivons ensuite les équations de bilan en expliquant les relations de fermeture ainsi que le terme de capillarité introduit par Korteweg. Puis nous donnons un rapide historique des résultats mathématiques portant sur le sujet. Pour conclure cette section, nous présentons les résultats obtenus dans cette thèse sur le système de Korteweg à interfaces diffuses. Dans le premier chapitre on obtient l'existence et l'unicité de solutions pour le système non isotherme avec des données initiales critiques du point de vue du scaling des équations. On distingue le cas de données initiales proches d'un état stable et celles dites globales. Dans le second chapitre on s'intérresse à l'existence globale de solutions faibles en dimension un et deux. Dans le cas de la dimension deux on remarque qu'une hypothèse sur le contrôle du vide est nécessaire.

Dans la seconde section, on s'intéresse à nouveau au cas d'un mélange liquide vapeur, à la différence que le modèle est étudié sous le point de vue de méthodes dites à interfaces discontinues. On reprend ainsi le système de Coquel, Rohde et leurs collaborateurs (voir [19]) qui introduisent un nouveau terme de capillarité et où le modèle considéré est isotherme. On rappelle dans un premier temps les travaux antérieurs sur le sujet. Nous présentons ensuite les résultats obtenus dans la thèse sur ce système. On montre ainsi dans le troisième chapitre l'existence globale de solutions faibles pour des dimensions $N \geq 2$ avec des données initiales appartenant aux espaces d'énergie. Enfin dans le dernier chapitre, on obtient l'existence et l'unicité de solutions dans des espaces critiques pour le scaling des équations. On distingue à nouveau le cas des données initiales proche d'un équilibre et celui avec des données grandes.

1 Equations générales

1.1 Présentation des méthodes d'étude du modèle

Nous allons nous intéresser au cas des écoulements multi-constituants qui mettent en jeu des phénomènes de changement de phase. Ceux-ci interviennent dans de très nombreux processus industriels. Ainsi, dans le domaine de l'*industrie chimique*, les processus de distillation sont utilisés afin de séparer les composants d'un mélange par évaporations et recondensations successives.
Dans l'*industrie nucléaire*, les phénomènes de changement de phase se révèlent tout autant primordiaux, aussi bien en fonctionnement nominal qu'en situation incidentelle ou accidentelle et occupent ainsi une place prépondérante dans les études de sûreté. En particulier, les situations incidentelles ou accidentelles provoquant le renoyage du coeur d'un réacteur font intervenir des phénomènes de changement de phase multi-constituants importants comme lors de la mise en contact d'eau froide avec un gaz chaud, typiquement de l'azote ou de l'hydrogène.
Les nombreuses études sur les phénomènes de changement de phase de fluide multi-constituants ont ainsi permis la mise en évidence du comportement souvent complexe des mélanges. *Le comportement d'un mélange diffère en effet bien souvent de celui d'un fluide pur possédant les mêmes propriétés physiques*, notamment en ce qui concerne le coefficient d'échange d'un mélange aussi bien en ébullition qu'en condensation. L'étude des coefficients d'échange de mélanges s'avère être un problème délicat et impose une bonne compréhension des *phénomènes locaux* intervenant dans les processus de changement de phases des mélanges.
En outre l'étude des phénomènes locaux permet une meilleure compréhension des phénomènes à plus grande échelle. Cependant, les expérimentations présentent certaines limites en particulier dans le domaine des écoulements diphasiques où elles sont complexes à réaliser ; on privilégie ainsi la **Simulation Numérique Directe** (**SND**) consistant en la résolution des équations du mouvement *locales et instantanées*. La SND apparaît ainsi comme un complément aux études expérimentales et elle permet d'avoir accès à l'ensemble des grandeurs instantanées, comme par exemple la température aux interfaces dont les valeurs influencent les coefficients d'échange, alors que la mesure de celle-ci via des expérimentations s'avère compliquée.
La simulation numérique directe des écoulements diphasiques a eu un essor important vers la fin des années 80. Il existe actuellement de très nombreuses méthodes de SND, certaines reposant sur la dynamique moléculaire et les autres sur la mécanique des milieux continus.
Nous allons à présent nous concentrer sur les méthodes de SND basées sur la mécanique des milieux continus, en distinguant deux type de méthodes, les *méthodes à interfaces discontinues* et les *méthodes à interfaces diffuses*. Leurs différences reposent sur la description des interfaces.

Méthodes à interfaces discontinues

Ainsi dans le cas des méthodes à interfaces discontinues, l'interface séparant deux phases est considérée comme une *surface de discontinuité*. Une telle description repose sur la théorie de Gibbs dans laquelle une interface est d'épaisseur nulle et dotée de propriétés physiques, en particulier d'une énergie surfacique qui correspond à la tension surfacique du fluide (voir [53]). Dans ces méthodes, les équations du mouvement du fluide, à savoir les équations de bilan de masse, de quantité de mouvement et d'énergie, sont résolues de façon séparée dans chacune des phases. Des bilans et des conditions aux limites classiques aux interfaces (voir [24] et [42]) sont appliquées afin de raccorder entre elles les solutions obtenues dans chacune des phases. On étudiera plus précisément leur comportement dans la section 4 en s'intéressant à un modèle d'interfaces discontinues introduit par Coquel, Rohde et leurs collaborateurs dans [19].

Méthodes à interfaces diffuses

Nous allons maintenant nous concentrer sur les méthodes dites à *interfaces diffuses*. Elles sont liées à une modélisation des interfaces séparant deux phases comme des *zones volumiques d'épaisseur non nulle à travers lesquelles les grandeurs physiques d'un fluide varient de façon continue*. Une telle modélisation est basée sur une formulation thermodynamique issue de la *théorie de la capillarité de Van der Waals*, qui a pour origine le modèle des interfaces liquide-vapeur et **qui suppose que l'énergie d'un fluide dépend du gradient de sa masse volumique**. On peut montrer en conséquence qu'elle implique une valeur non nulle de l'épaisseur des interfaces et que les interfaces sont alors munies d'une énergie en excès non nulle qui est précisément la tension interfaciale. Nous allons à présent étudier la méthode dite du *second gradient* qui est destinée à la simulation de fluides diphasiques liquide-vapeur (voir [43] et [44]) et correspond en fait à une dérivation moderne des travaux de Van der Waals. C'est cette modélisation qui sera développée dans la suite pour décrire les équations de Korteweg.

Méthode du second gradient

Dans le cas des méthodes du second gradient, c'est **la masse volumique** ou **densité** du fluide qui sert de **paramètre d'ordre** conformément à la formulation initiale de la théorie de la capillarité introduite par Van der Waals. L'énergie interne volumique du fluide est alors supposée être de la forme :

$$U(S,\rho,\nabla\rho) = U^{cl}(S,\rho) + \frac{\kappa}{2}|\nabla\rho|^2 \tag{1.1}$$

où ρ définit la densité, S est l'entropie volumique, U^{cl} représente l'énergie interne volumique *classique*, c'est à dire indépendante du gradient de masse volumique, et κ désigne le coefficient de capillarité interne. Une dépendance en $|\nabla\rho|^2$ plutôt qu'en $\nabla\rho$ s'explique par le fait que l'énergie est une grandeur indépendante du système de coordonnées choisi et ne

peut donc dépendre que du produit scalaire de ses variables vectorielles. Cette modélisation permet de rendre compte des fortes variations de masse volumique à travers les interfaces liquide-vapeur supposées d'épaisseur non nulle, et donc de *considérer la masse volumique comme une fonction continue de l'espace, même à travers les interfaces.* En fait l'ensemble des grandeurs physiques relatives au fluide peuvent être considérées comme continue.
L'intérêt de cette méthode repose sur *la gestion des interfaces et de leur déplacement qui est intrinsèque au modèle* puisqu'un seul système d'équations aux dérivées partielles suffit pour décrire l'ensemble d'un système liquide-vapeur, y compris les interfaces, permettant ainsi de prendre en compte directement le changement de phase ainsi que les changements de topologie des interfaces. De plus le *modèle complet reste thermodynamiquement cohérent*, ce qui permet en particulier que la prise en compte de la tension interfaciale soit intrinsèque au modèle.
En revanche, la méthode du second gradient présente une difficulté importante liée à l'épaisseur des interfaces : physiquement, les interfaces liquide-vapeur ont certes une épaisseur non nulle, mais, sauf au voisinage immédiat du point critique, cette épaisseur est trop faible (de l'ordre d'une dizaine d'Angströms) pour que les interfaces puissent être discrétisées par un nombre suffisant de points tout en gardant un nombre raisonnable de mailles pour le système simulé. Il est donc nécessaire, pour *des raisons numériques*, *d'augmenter artificiellement les interfaces.*

1.2 Présentation des systèmes d'équations

Nous allons à présent rappeler le système général d'équations de bilan régissant le mouvement d'un fluide liquide- vapeur. Nous insisterons aussi grandement sur les différences notables qu'engendre chaque méthode de simulations numériques que ce soit celle à interfaces discontinues ou à interfaces diffuses. Ces variations se situent essentiellement dans l'appréciation du tenseur de capillarité. Ainsi les deux systèmes que l'on étudiera se distingueront par leur comportement aux interfaces et donc implicitement par la régularité des solutions.
Nous rappelons maintenant le système général d'un fluide lors d'un mélange fluide-vapeur et par la même nous montrerons certaines similitudes qu'il partage avec le système de Navier-Stokes classique.
Nous considérons ainsi un fluide de densité $\rho \geq 0$, de vitesse $u \in \mathbb{R}^N$, d'entropie s, d'énergie de densité e et de température $\theta = (\frac{\partial e}{\partial s})_\rho$. Nous notons $w = \nabla\rho$ et nous supposons que l'énergie interne spécifique e dépend de la densité, de l'entropie spécifique s et de w. En terme dénergie libre, le principe de la thermodynamique prend la forme d'une relation de Gibbs généralisée :

$$de = \widetilde{T}ds + \frac{P}{\rho^2}d\rho + \frac{1}{\rho}\phi^* \cdot dw$$

où $\widetilde{T}$ est la température, P la pression et ϕ un vecteur colonne de $\mathbb{R}^N$ et ϕ^* le vecteur adjoint. Dans la suite nous préciserons la forme que prend ϕ, celui ci modélisant la partie capillaire du système.

La conservation de masse, du moment et d'énergie s'écrivent :

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u + pI) = \mathrm{div}(K + D) + \rho f, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \mathrm{div}\big(u(\rho e + \frac{1}{2}\rho|u|^2 + p)\big) \\ \qquad\qquad = \mathrm{div}\big((D + K)\cdot u - Q + W\big) + \rho f \cdot u, \end{cases} \tag{1.2}$$

où :

$$D = (\lambda \mathrm{div} u)I + \mu(du + \nabla u), \text{ est le tenseur de diffusion,}$$

$$Q = -\eta \nabla \widetilde{T}, \text{ est le flux de chaleur.}$$

Le terme

$$W = (\partial_t \rho + u^* \cdot \nabla \rho)\phi = -(\rho \mathrm{div} u)\phi$$

correspond au travail intersticiel qui a pour but d'assurer la balance entropique, il a été introduit pour la première fois par Dunn and Serrin dans [28]. K est le tenseur de capillarité et nous préciserons ultérieurement sa forme selon les méthodes SDN employées.

Concernant les coefficients (λ, μ) ils représentent la viscosité du fluide et dépendent à la fois de la densité ρ et de la température $\widetilde{T}$. Le coefficient thermal η est une fonction positive dépendant de la température $\widetilde{T}$ et de la densité ρ.

liquide-vapeur.

Cas du système de Korteweg

Nous allons ici succintement rappeler la forme du système de Korteweg et plus précisément la forme du tenseur de capillarité K. En suivant le modèle de Dunn et Serrin dans [28] (pour plus de précision on réfère le lecteur aux sections 3 et 4), il existe trois fonctions Π_0, Π_1 et φ telles que la pression et l'énergie interne s'écrivent sous la forme :

$$P(\rho, \widetilde{T}) = \widetilde{T} P_1(v) + P_0(v),$$

$$e_0 = -\Pi_0(v) + \varphi(\widetilde{T}) - \widetilde{T}\varphi'(\widetilde{T}),$$

avec $v = \frac{1}{\rho}$ et où $P_1 = \Pi_1'$ et $P_0 = \Pi_0'$.

De plus en supposant que l'énergie interne soit une fonction croissante de la température $\widetilde{T}$, on pose :

$$\Psi(\widetilde{T}) = \varphi(\widetilde{T}) - \widetilde{T}\varphi'(\widetilde{T}) \quad \text{avec} \quad \Psi'(\widetilde{T}) > 0\,.$$

Enfin pour simplifier les notations, on pose $\theta = \Psi(\widetilde{T})$. On peut alors réécrire le système (1.2) sous la forme suivante :

$$
NHV \qquad \begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\[2ex] \partial_t u + u\cdot\nabla u - \dfrac{\mathrm{div}(D)}{\rho} + \dfrac{\nabla(P_0(\rho) + \Psi^{-1}(\theta)P_1(\rho))}{\rho} = \mathrm{div}K, \\[2ex] \partial_t \theta + u\cdot\nabla\theta - \dfrac{\mathrm{div}(\chi\nabla\theta)}{\rho} + \Psi^{-1}(\theta)\dfrac{P_1(\rho)}{\rho}\mathrm{div}(u) = \dfrac{D:\nabla u}{\rho}. \end{cases}
$$

avec χ le coefficient thermal. K représente le tenseur de cappilarité et celui-ci s'écrit sous la forme suivante :

$$K = (\rho\mathrm{div}\phi)I - \phi w^*,$$

avec $\phi = \kappa w$ où κ est le coefficient de capillarité.

Cas du système de Rohde

Nous allons ici seulement rappeler le système de Rohde en insistant sur la forme du tenseur de capillarité, on précisant ici que ce dernier emploie une méthode à interfaces diffuses. De plus on s'intéresse au cas du système isotherme. Pour plus de précision sur la physique du système on renvoie à la section 4 de cette introduction.
Le système s'écrit sous la forme suivante :

$$
(NSK) \qquad \begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho\, u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div} u + \nabla(P(\rho)) = \kappa\rho\nabla D[\rho] \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0) \end{cases}
$$

avec : $D[\rho] = \kappa(\phi * \rho - \rho)$, on précisera ultérieurement les conditions sur la fonction ϕ et κ coefficient de capillarité.
Nous pouvons à présent faire les remarques suivantes sur chacun des deux systèmes.

- Le premier système (NHV) découle de la théorie du second gradient via des méthodes thermodynamiques et implique par la même un tenseur de capillarité très régulier, effectivement la modélisation physique suppose que les interfaces sont d'épaisseur non nulle. Nous présenterons par la suite une méthode permettant d'établir ces équations. Nous rappellerons ainsi dans la section 3 la forme de la fermeture thermodynamique proposée par C. Fouillet dans [33].
- Le second système (NSK) est obtenu via une méthode d'interfaces discontinues où l'on cherche à minimiser l'énergie libre volumique et elle a pour conséquence un tenseur de capillarité fortement discontinu d'un point de vue mathématique. Elle est introduite dans [19] par Coquel, Rohde et leurs collaborateurs. Nous présenterons dans la section 4 la description physique précise du modèle.

Enfin nous pouvons constater que lorsque dans chaque système le tenseur de capillarité est nul, on retrouve le système de Navier-Stokes classique. On va ainsi voir par la suite que le comportement du système (NSK) est proche de celui de Navier-Stokes alors que le comportement du système (NHV) diffère considérablement par la régularité qu'implique le tenseur de capillarité.Nous allons maintenant rappeler dans la section suivante certains résultats majeurs sur le système de Navier-Stokes compressible et par la suite mettre en lumière les différences fondamentales ou les similitudes que possèdent dans chaque cas les systèmes capillaires avec le système de Navier-Stokes.

2 Système de Navier-Stokes standard

2.1 Présentation du modèle physique

Nous allons maintenant rappeler le système de Navier-Stokes compressible non isotherme qui est en fait un cas particulier du système (NSK) lorsque $\kappa = 0$. Ensuite nous passerons en revue certains travaux antérieurs concernant ce système.
Les équations de Navier-Stokes compressibles non isothermes décrivent l'évolution d'un gaz compressible, notamment dans l'atmosphère pour des altitudes suffisamment faibles. Les équations complètes de Navier-Stokes modélisant les fluides compressibles s'écrivent sous la forme suivante dans $(0,T)\times\mathbb{R}^N$, avec $T>0$ fixé :

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \operatorname{div}\sigma = \rho f, \\ \partial_t(\rho E) + \operatorname{div}(\rho u H) + P\operatorname{div}u = \operatorname{div}((\sigma + PI)\cdot u) + \operatorname{div}(k\nabla\widetilde{T}) + \rho f\cdot u, \\ E = e + \dfrac{|u|^2}{2},\; H = h + \dfrac{|u|^2}{2},\; h = e + \dfrac{P(\rho)}{\rho}, \end{cases} \tag{2.3}$$

où ρ représente la densité du fluide, $u\in\mathbb{R}^N$ sa vitesse, $\widetilde{T}$ la température, k le coefficient de conductivité thermique, σ le tenseur de tension, P la pression, e l'énergie spécifique interne et h l'enthalpie spécifique. On note E l'énergie spécifique totale et H l'enthalpie spécifique associée. Les forces de masse sont décrites par le champ de vecteur f.
Les équations de (2.3) représentent respectivement l'équation de la masse, l'équation du moment et celle de l'énergie. De plus pour obtenir la fermeture du système on suppose le fluide Newtonien, il existe alors deux coefficients de viscosité μ et λ tels que :

$$\sigma = 2\mu D(u) + (\lambda\operatorname{div}u - P)I$$

où $D(u)$ est le symétrisé du gradient de vitesse ∇u. Enfin pour obtenir une clôture thermodynamique des équations, la pression ansi que l'énergie interne dépendent de la densité ρ et de la température $\widetilde{T}$:

$$P = \mathcal{P}(\rho,\widetilde{T}) \;\text{ et }\; e = \mathcal{E}(\rho,\widetilde{T}).$$

D'après le second principe de la thermodynamique, on impose des conditions de compatibilité dites *équations de Maxwell* :

$$\mathcal{P}(\rho,\widetilde{T}) = \rho^2\Big(\frac{\partial\mathcal{E}}{\partial\rho}\Big)_{\widetilde{T}} + \widetilde{T}\Big(\frac{\partial\mathcal{P}}{\partial\widetilde{T}}\Big)_{\rho}.$$

L'entropie est définie à une constante près sous la forme :

$$(\frac{\partial \mathcal{S}}{\partial e})_\rho = \frac{1}{\widetilde{T}} \ \text{ et } \ (\frac{\partial \mathcal{S}}{\partial \rho})_{\widetilde{T}} = \frac{1}{\widetilde{T}} = -\frac{P}{\rho^2 \widetilde{T}}.$$

D'autre part l'entropie doit être une fonction concave de $(\frac{1}{\rho}, e)$.

2.2 Historique des résultats

Nous allons ici plus précisément nous concentrer sur les solutions faibles de Navier-Stokes et ainsi faire référence à certains résultats majeurs notamment ceux de P-L Lions.

2.3 Solutions faibles

- **Cas du système isotherme**

Nous considérons ici le cas isentropique où l'on postule que la pression P ne dépend que de la densité ρ. Pour fixer les idées, on prend :

$$P(\rho) = a\rho^\gamma, \ \ a > 0, \ \ \gamma > 1.$$

Cette restriction consiste essentiellement à considérer une évolution adiabatique du fluide en négligeant le flux de chaleur visqueux. Le système correspondant devient :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu d + \lambda(\operatorname{div} u) I) + \nabla P = \rho f, \end{cases} \tag{2.4}$$

avec les conditions initiales :

$$\begin{cases} \rho_{/t=0} = \rho_0 \in L^1(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N), \\ (\rho u)_{/t=0} = m_0 \ \text{ avec } \ \dfrac{|m_0|^2}{\rho_0} \in L^1(\mathbb{R}^N), \end{cases}$$

où par convention $\frac{|m_0|^2}{\rho_0}(x) = 0$ lorsque $\rho_0(x) = 0$.

L'existence de solutions faibles globales en temps pour le système complet (2.3) incluant la température est encore ouvert sauf dans le cas monodimensionnel. Les résultats d'existence et d'unicité en dimension 1 sont dûs à A.V. Kazhikov et V.V. Shelukin [48], [46], [47], [67], [66], D. Serre [65] et D. Hoff [38], [39], [40].

Nous allons maintenant rappeler un résultat fondamental de ces dernières années dû à P.-L. Lions dans [52] qui montre l'existence de solutions faibles globales pour le système (2.4). Ce résultat représente réellement le pendant du théorème d'existence globale de solutions faibles de J. leray pour le système de Navier-Stokes incompressible. P.-L. Lions obtient ainsi le théorème suivant :

Théorème 1. *Si* $\gamma \geq \frac{3}{2}$ *et* $N = 2$, $\gamma \geq \frac{9}{5}$ *et* $N = 3$, *ou* $\gamma > \frac{N}{2}$ *et* $N \geq 4$, *alors, il existe une solution globale* (ρ, u) *dans* $L^{\infty}(0,\infty; L^{\gamma}(\mathbb{R}^N)) \cap L^2(0,\infty;\dot{H}^1(\mathbb{R}^N))$, *telle que* $\rho \in C([0,\infty); L^p(\mathbb{R}^N))$ *si* $1 \leq p < \gamma$, $\rho|u|^2 \in L^{\infty}(0,\infty; L^1(\mathbb{R}^N))$, *et* $\rho \in L^q((0,T)\times\mathbb{R}^N)$ *pour tout* $q \in [1, \gamma - 1 + \frac{2\gamma}{N}]$ *si* $N \geq 3$, *et* $q \in [1, 2\gamma - 1)$ *si* $N = 2$. *En outre, pour tout* $t \geq 0$, *on a l'inégalité d'énergie :*

$$\begin{aligned}\int_{\mathbb{R}^N}(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma-1}\rho^{\gamma})(t,x)dx + \int_0^t\int_{\mathbb{R}^N}(\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2)(s,x)dxds \\ \leq \int_{\mathbb{R}^N}(\frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{a}{\gamma-1}\rho_0^{\gamma})(x)dx.\end{aligned} \tag{2.5}$$

Nous allons maintenant expliquer succintement les idées de la démonstration de ce théorème. Lions commence par construire des solutions approchées $(\rho_n, u_n)_{n\in\mathbb{N}}$ pour (2.4) vérifiant les inégalités d'énergie du système, ces solutions sont construites après plusieurs approximations successives du système (2.4) par régularisation en introduisant notamment des effets de viscosité sur la densité. La partie délicate consiste ensuite dans la stabilité de ces solutions. Toute la difficulté repose ainsi dans le passage à la limite dans le terme de pression. Un des phénomènes primordiaux qui y contribue grandement est la notion de pression efficace $\widetilde{P} = P(\rho) - (2\mu + \lambda)\mathrm{div}u$ introduite par Hoff dans [39]. Lions montre ainsi que cette expression multipliée par ρ^{ε} avec $\varepsilon > 0$ est plus régulière au sens des distributions que les inégalités d'énergie initiales ne pouvaient le laissent présager. L'utilisation astucieuse de cette expression dans l'équation du moment permet ainsi d'avoir des résultats de convergence forte sur la suite ρ_n dans $L^{\gamma}_{loc}((0,T)\times\mathbb{R}^N)$ vers une limite ρ. Un des éléments essentiels pour cela est la notion de solutions renormalisées introduite par Lions et Di Perna dans [26], [27] et qui permet en effet de tester la convergence forte de la suite approchée $(\rho_n)_{n\in\mathbb{N}}$ vers ρ via l'utilisation de fonctions concaves.
En outre un autre point crucial de cette démonstration est un théorème de commutateur de type Coiffman, Meyer, Rochberg (voir dans [18]).
On peut cependant remarquer que pour $N = 2, 3$, on ne peut atteindre le même seuil limite, c'est à dire $\gamma > \frac{N}{2}$. En fait la difficulté majeure correspond à pouvoir renormaliser l'équation de masse sans supposer nécessairement que ρ appartient à $L^2_{loc}(L^2)$ (ce qui est le cas dans les travaux de P.-L. Lions qui a besoin de choisir γ assez grand pour obtenir $\rho \in L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^N)$, ceci en utilisant un gain classique d'intégrabilité sur la densité ρ qui dépend du coefficient γ).

Le théorème 1 de Lions a ainsi été récemment amélioré par E. Feireisl, A. Novotný et H. Petzeltová dans [29], [30], [32] en ce qui concerne le coefficient γ de la pression dans les cas spécifiques $N = 2$ et $N = 3$. Effectivement pour $N = 2, 3$ Feireisl, A. Novotný et H. Petzeltová dans [32] a étendu le seuil critique de γ à $\gamma > \frac{N}{2}$ comme c'est le cas chez Lions pour les dimensions supérieures. Pour de plus nombreux détails nous faisons aussi référence au livre de Novotný et Straškraba [60].
Pour contourner la difficulté concernant la possibilité de renormaliser l'équation de masse,

E. Feireisl dans [29] introduit une nouvelle notion appelée *oscillations defect measure* et notée $\mathrm{osc}_p[\rho_n \to \rho]$.

Définition 1. *Soit Ω un ouvert de $\mathbb{R}^N$ et une suite $(\rho_n)_{n\in\mathbb{N}}$ telle que :*

$$\rho_n \to \rho \ \textit{faiblement dans} \ L^1(\Omega).$$

Nous définissons l'expression $osc_p[\rho_n \to \rho]$ comme suit :

$$osc_p[\rho_n \to \rho](\Omega) = \sup_{k\geq 1}\big(\lim \sup_{n\to+\infty}\int_\Omega |T_k(\rho_n) - T_k(\rho)|^p dxdt\big),$$

où T_k est une fonction troncature.

E. Feireisl montre ainsi que si l'on contrôle ces *oscillations defect measures* alors on peut renormaliser l'équation de masse.

Proposition 1. *Soit $\Omega \subset \mathbb{R}^N$ un domaine arbitraire, $N \geq 2$ et $(\rho_n)_{n\in\mathbb{N}}$ une suite de fonctions positives telle que :*

$$\rho_n \to \rho \ \textit{faiblement}(-^*) \ \textit{dans} \ L^\infty(0,T;L^\gamma(\Omega)), \ \ \gamma > \frac{2N}{N+2}$$

et

$$osc_p[\rho_n \to \rho](O) < c(O) \ \textit{pour un certain} \ p > 2,$$

et pour tout ouvert borné $O \subset (0,T)\times\Omega$.
De plus si :

$$u_n \to u \ \textit{faiblement dans} \ L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^N)),$$

où ρ_n, u_n résolvent l'équation de la masse au sens des solutions renormalisées sur $(0,T)\times\Omega$. Alors (ρ,u) sont des solutions renormalisées de l'équation de la masse sur $(0,T)\times\Omega$.

La subtilité de ces *oscillations defect measures* $\mathrm{osc}_p[\rho_n \to \rho]$ s'explique par le fait que leur contrôle permet d'être dans l'une des situations suivantes :

- ρ_n converge vers ρ fortement dans $L^1(O)$, $O \subset (0,T)\times\Omega$ un ouvert.
- ρ_n est borné dans $L^2\big(((0,T)\times\Omega)\setminus O\big)$.

Chacun de ces deux phénomènes considérés individuellement permet alors aisément en s'appuyant sur la démonstration de Lions de conclure à la convergence forte de ρ_n vers ρ. Enfin on remarque que la proposition 1 peut s'appliquer pour $\gamma > \frac{N}{2}$ car on a $\frac{N}{2} \geq \frac{2N}{N+2}$ pour $N \geq 2$.

Nous allons maintenant considérer le cas le plus général du système de Navier-Stokes isotherme où les coefficients de viscosité dépendent de la densité ρ. Le système s'écrit alors :

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div}u)) + \nabla P(\rho) = \rho f, \end{cases} \tag{2.6}$$

On peut remarquer que lorsque $\mu(\rho)=\rho$, $\lambda(\rho)=0$, $P(\rho)=\rho^2$ et $N=2$ le système (2.6) correspond au fameux modèle de Saint-Venant.
Ici dans le cas de la recherche de solutions faibles du système (2.6), on ne peut appliquer aisément les techniques de P-L Lions. Effectivement il s'avère délicat de pouvoir extirper la pression efficace en appliquant un opérateur différentiel du type $(\Delta)^{-1}\mathrm{div}$. Il devient donc difficile d'acquérir simplement un gain d'intégrabilité sur la densité ρ.
De ce fait, le premier résultat véritablement marquant et faisant apparaître un gain de régularité sur la densité ρ est dû à D. Bresch, B. Desjardins et C.-K Lin dans [13] où ils s'intéressent à des coefficients de viscosité spécifiques de type Saint-Venant $\mu(\rho)=\rho$ et $\lambda(\rho)=0$. Ils montrent ainsi pour le système de Korteweg et avec ce choix de coefficients de viscosité une nouvelle inégalité d'énergie permettant un gain de régularité sur la densité ρ, avec $\rho\in L^2(H^2)$.
A la suite de cela, ils étendent dans [8] ce type d'inégalité d'énergie à des coefficients de viscosité plus généraux que ceux de Saint-Venant. Il est à noter cependant que cette condition (2.3) sur les coefficients de viscosité n'englobe pas le cas des coefficients constants étudiés par P.-L. Lions. Ainsi sous la condition algébrique suivante :

$$\lambda(s)=2(s\mu^{'}(s)-\mu(s)),$$

ils démontrent alors la proposition fondamentale suivante qui permet un meilleur contrôle de la densité ρ que ne le permet les inégalités d'énergie classiques.

Proposition 2.

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega\rho|u|^2+\int_\Omega 2\mu(\rho)D(u):D(u)+\int_\Omega\lambda(\rho)|\mathrm{div}u|^2+\frac{\chi}{2}\frac{\partial}{\partial t}\int_\Omega|\nabla\mu(\rho)|^2\\=\int_\Omega P(\rho,\theta)\mathrm{div}u.\end{aligned}\tag{2.7}$$

où χ correspond à un coefficient de tension surfacique.

L'égalité (2.7) se distingue ainsi des égalités d'énergie classiques par le fait qu'elle nous incite à considérer de nouveaux espaces d'énergie avec un renseignement sur le gradient de la densité ρ sous la forme suivante $\nabla\mu(\rho)\in L^\infty(L^2)$.
On arrive ainsi à avoir quasiment des renseignements en norme $L^\infty(H^1)$ sur ρ ce qui est totalement nouveau et permet par la même de passer facilement à la compacité dans le terme de pression (alors que cela consistait la difficulté essentielle pour le système de Navier-Stokes à coefficients constants).
Cependant selon le choix des coefficients de viscosité, une nouvelle difficulté entre en jeu lorsque du vide apparaît, effectivement la vitesse u ne peut être définie lorsque les coefficients de viscosité $\lambda(\rho)$ et $\mu(\rho)$ s'annulent. On perd alors des renseignements sur ∇u. Cette difficulté rend alors délicat le passage à la compacité sur le terme $\rho u\otimes u$ où l'on ne peut utiliser les résultats classiques de type Lions.
D. Bresch et B. Desjardins évitent cette difficulté en introduisant une pression qui empêche

d'une certaine manière l'apparition du vide, celle-ci étant de plus choisie en corrélation avec les coefficients de viscosité et leur comportement autour du vide. Ils imposent ainsi les relations suivantes entre la pression P et les coefficients de viscosité lorsque ρ est petit :

- Pour tout $s < A$, $\mu(s) \geq c_0 s^n$ avec A, c_0 des constantes et $\frac{N-1}{N} < n < 1$ et $c_1 s^m \leq \mu(s)$ pour $s \geq A$ avec $m > 1$.
- $P_c(\rho) \sim -\rho^{-l}$ pour $\rho \leq \rho_*$ avec ρ_* une constante. D'autre part l dépend notamment de n.

Cette pression P explose ainsi au voisinage du vide, ce qui physiquement a pour but de montrer la contraction extrême des élément lorsque le vide apparaît, ce phénomène explique physiquement l'impossibilité d'une densité nulle.
Pour plus de détails sur le choix des coefficients voir [9]. Il est à noter que loin du vide on est proche d'une pression type gaz parfait.
En fait ce lien entre la pression froide et les termes de viscosité est essentiel et permet ainsi de contrôler div$u$ en norme $L^2(L^2)$ lorsque la viscosité s'annule. Ceci permet ensuite de passer à la compacité le terme $\rho u \otimes u$. Via des inégailtés de Hölder la pression froide absorbe les difficultés liées au vide sur les termes $\mu(\rho)|\nabla u|^2$ et $\lambda(\rho)|\text{div}u|^2 \in L^1(L^1)$. Nous rappelons ici le résultat principal de Bresch et Desjardins dans [9].

Théorème 2. *Supposons que la viscosité vérifie (2.3) ainsi que d'autres hypothèses plus techniques et que le gaz considéré soit parfait de type polytropique. Les données initiales* (ρ_0, m_0, G_0) *avec* $G_0 = \rho_0 e_0$ *sont prises telles que :*

$$\mathcal{H}(0) = \int_\Omega (\rho_0 e_0 + \frac{|m_0|^2}{2\rho_0}) dx < +\infty,$$

la densité initiale ρ_0 *vérifie :*

$$\rho_0 \in L^1(\Omega), \quad \textit{et } \frac{\mu(\rho_0)}{\sqrt{\rho_0}} \in L^2(\Omega)^N$$

et l'entropie initiale de densité $s_0 = C_v \log(\frac{\theta_0}{\rho_0})$ *satisfait :*

$$\rho_0 s_0 \in L^1(\Omega).$$

Alors il existe une solution globale en temps de (2.3).

Il est à noter que le théorème précédent est un résultat de stabilité, car il admet l'existence d'une suite de solutions approchées au problème vérifiant les inégalités d'entropie requises. Il reste alors à construire ces suites approchées vérifiant les nouvelles inégalités d'énergie introduites par Bresch, Desjardins.
Ainsi dans [10], Bresch et Desjardins construisent des solutions approchées du système (2.6) vérifiant le choix précédent sur les coefficients de viscosité ainsi que sur la pression (qui est somme d'un terme de type pression froide et d'un terme barotropique). De plus ces solutions approchées vérifient uniformément les inégalités d'énergie introduite dans [8]. La construction de ces solutions repose sur la régularisation du système (2.6) par un terme

de type capillaire qui permet le contrôle du vide et donc l'existence globale de solutions. Finalement Bresch et Desjardins obtiennent l'existence globale de solutions faibles pour le système de Navier-Stokes isotherme avec leur choix spécifique concernant les coefficients de viscosité ainsi que la pression.

Enfin dans le même registre, il est à signaler cet article de Bresch, Desjardins et Varet dans [12] qui considèrent toujours le système de Navier-Stokes compressible avec le même choix pour les coefficients de viscosité mais avec en plus un terme de traînée de la forme $r_0\rho|u|u$. Ils obtiennent alors l'existence globale de solutions faibles avec cette fois-ci des pressions barotropiques de la forme $a\rho^\gamma$ avec $\gamma > 1$. Il est à noter que ce résultat répond à l'existence globale de solutions faibles pour le système de Saint-Venant avec terme de traînée modélisant l'aspect onduleux des fonds marins. La clef de ce résultat est le gain naturel d'intégrabilité sur la vitesse u qu'apporte le terme de traînée, il permet ainsi de passer à la compacité dans le terme $\rho u \otimes u$ sans évoquer le rajout d'une pression froide.

Pour pallier à cette restriction qu'impose une pression froide dans le cas du système (2.6), récemment dans un article extrêmement intéressant A. Mellet et A. Vasseur dans [55] étendent le résultat de Bresch et Desjardins dans le cas isotherme en autorisant le choix d'une pression isentropique. Cela permet ainsi de traiter le cas très important du modèle de Saint-Venant :

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\rho \nabla u) + \nabla \rho^2 = 0 \end{cases}$$

La preuve repose sur une nouvelle inégalité d'énergie des plus astucieuses qui permet d'obtenir un gain d'intégrabilité sur la vitesse u, élément essentiel pour ensuite passer à la compacité dans le terme $\rho u \otimes u$.

Ils obtiennent ainsi la proposition suivante :

Proposition 3. *Supposons que :*

$$2\mu(\rho) + N\lambda(\rho) \geq \nu\mu(\rho)$$

pour un certain $\nu \in (0,1)$ et soit $\delta \in (0, \frac{\nu}{4})$. Alors nous avons l'inégalité suivante :

$$\frac{d}{dt}\int_{\mathbb{R}^N} \rho\frac{1+|u|^2}{2}\ln(1+|u|^2)dx + \frac{\nu}{2}\int \mu(\rho)[1+\ln(1+|u|^2)]|D(u)|^2dx \leq$$
$$C\bigg(\int_{\mathbb{R}^N}(\frac{\rho^{2\gamma-\frac{\delta}{2}}}{\mu(\rho)})^{\frac{2}{2-\delta}}dx\bigg)^{\frac{2}{2-\delta}}\bigg(\int \rho[2+\ln(1+|u|^2)]^{\frac{2}{\delta}}dx\bigg)^{\frac{\delta}{2}} + C\int_{\mathbb{R}^N}\mu(\rho)|\nabla u|^2dx.$$

Finalement A.Mellet et A. Vasseur obtiennent le théorème de stabilité suivant.

Théorème 3. *Soit $\Omega = \mathbb{R}^N$ ou $\mathbb{T}^N$. Supposons que $\gamma > 1$ et que $\mu(\rho)$ et $\lambda(\rho)$ sont deux fonctions C^2 de ρ vérifiant les conditions de Bresch, Desjardins exceptés celle sur la minoration de μ et λ. Soit $(\rho_n, u_n)_{n\in\mathbb{N}}$ une suite de solutions de satisfaisant les inégalités avec des données initiales :*

$$\rho_n/_{t=0} = \rho_0^n \ \ et \ \ \rho_n u_n/_{t=0} = \rho_0^n u_0^n$$

où ρ_0^n *et* u_0^n *sont tels que :*

$$\rho_0^n \geq 0, \quad \rho_0^n \to \rho_0 \ \ dans \ \ L^1(\mathbb{R}^N), \quad \rho_0^n u_0^n \to \rho_0 u_0 \ \ dans \ \ L^1(\mathbb{R}^N),$$

et satisfont les inégalités suivantes :

$$\int_\Omega \rho_0^n \frac{|u_0^n|^2}{2} + \frac{\gamma}{\gamma-1}(\rho_0^n)^\gamma < C, \qquad \int_\Omega \frac{1}{\rho_0^n}|\nabla\mu(\rho_0^n)|^2 dx < C,$$

et

$$\int_\Omega \rho_0^n \frac{1+|u_0^n|^2}{2}\ln(1+|u_0^n|^2) < C,$$

pour un $\delta > 0$ *petit.*
Alors à extraction près, $(\rho_n, \sqrt{\rho}_n u_n)$ *converge fortement vers une solution faible de (2.4) satisfaisant les inégalités d'entropie.*
La densité ρ_n *converge fortement dans* $C^0(([0,T]; L^{\frac{3}{2}}_{loc}(\Omega))$, $\sqrt{\rho}_n u_n$ *converge fortement dans* $L^2(0,T; L^2_{loc}(\Omega))$ *et le moment* $m_n = \rho_n u_n$ *converge fortement dans* $L^1(0,T; L^1_{loc}(\Omega))$ *pour tout* $T > 0$*.*

Ce théorème est un résultat de stabilité. Pour garantir un résultat final de solutions faibles, comme dans le cas de Bresch et Desjardins il est nécessaire de réussir à construire une suite de solutions approchées $(\rho_n, u_n)_{n\in\mathbb{N}}$ vérifiant les différentes inégalités d'énergie, et ceci reste compliqué étant donné la complexité des deux inégalités d'entropie et notamment celle introduite sur la vitesse u. Il semble difficile de régulariser le système (2.6) tout en conservant chacune des deux inégalités d'énergie. Ainsi il est à noter que les solutions approchées introduites par Bresch et Desjardins dans [10] ne sont pas adaptées au cas du résultat de Mellet et Vasseur, car elles ne préservent pas le gain d'énergie sur la vitesse u. Le problème concernant l'existence de solutions faibles du système de Navier-Stokes isotherme avec des pressions barotropes pour ce type de viscosité reste donc à l'heure actuelle complètement ouvert pour les dimensions supérieurs à 2. Cependant dans le cas spécifique de la dimension une Mellet et Vasseur dans [56] obtiennent l'existence de solutions fortes globales avec un choix précis sur la viscosité $\mu(\rho)$ pour les faibles densités avec un comportement de type ρ^α où $\alpha \in (0, \frac{1}{2})$. Effectivement ceci permet de contrôler le vide et d'obtenir ainsi des solutions fortes, il est à préciser que ce qui rend possible cette démonstration est qu'en dimension une, la structure du tenseur de viscosité n'introduit qu'un seul coefficient μ et donc on est soulagé de la contrainte du type $2\mu + N\lambda \geq 0$.
Enfin on rappelle aussi des travaux concernant le comportement asymptotique des solutions faibles de Feireisl sur le système (2.4) avec $f = \nabla\phi$ de A. Novotný et I. Straškraba dans [60], [61], [62]. En fait ils montrent que les solutions ont des trajectoires compactes. Ainsi en travaillant sur Ω un domaine borné et en choisissant certaines conditions sur γ le coefficient de puissance dans la pression, ils montrent que pour toute suite $(t_n)_{n\in\mathbb{N}}$ on peut extraire une suite $(s_n)_{n\in\mathbb{N}}$ telle que :

$$\lim_{n\to+\infty} \|\rho(s_n) - \rho_\infty\|_{L^q(\Omega)} = 0,$$

$$\lim_{n\to+\infty} \|\rho u(s_n)\|_{L^r(\Omega)} = 0.$$

De plus la fonction ρ_∞ est une densité d'équilibre qui conserve la masse de ρ_0 et qui vérifie les conditions suivantes :

$$\nabla P(\rho_\infty) = \rho_\infty \nabla\phi,$$
$$\int_\Omega \rho_\infty dx = \int_\Omega \rho_0 dx, \quad \rho_\infty \geq 0.$$

Novotný et Straškraba montrent donc que les solutions se stabilisent donc asymptotiquement vers un état d'équilibre $(\rho, \rho u) = (\rho_\infty, 0)$. Il peut être ensuite intéressant de mieux connaitre ρ_∞ pour savoir si celui-ci admet du vide en fonction du choix de ρ_0.

Cas du système non isotherme

Dans cette section nous nous intéressons à présent aux résultats concernant Navier-Stokes non isotherme, et nous allons rendre compte des différences fondamentales qu'implique l'équation de la chaleur. Nous rappelons ici l'équation de Navier-Stokes non isotherme :

$$\begin{aligned}
&\partial_t\rho + \mathrm{div}(\rho u) = 0,\\
&\partial_t(\rho\, u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla(P(\rho)) = 0,\\
&\partial_t(\rho Q(\widetilde{T})) + \mathrm{div}(\rho Q(\widetilde{T})\, u) - \mathrm{div}(K(\widetilde{T})\nabla\widetilde{T}) = (2\mu D(u) + \lambda \mathrm{div} u\, I) : \nabla u \\
&\qquad\qquad - \widetilde{T} P \mathrm{div} u,\\
&(\rho_{t=0}, u_{t=0}) = (\rho_0, u_0, \widetilde{T}_0),
\end{aligned} \tag{2.8}$$

où K est le coefficient du flux de chaleur et Q une fonction $C^2(0,\infty)$ représentant la contribution d'énergie thermale et dépendant de la chaleur.
Les premiers résultats d'existence globale de solutions faibles sont ceux de P-L Lions dans [52] pour le cas de pression isentropique avec des coefficients de viscosité constants, cependant une nuance importante est à noter sur la définition de solutions faibles. Effectivement si concernant l'équation de masse et l'équation du moment notre solution (ρ, u, θ) vérifie le système au sens des destributions, il en est tout autrement en ce qui concerne l'équation sur la température. Cette dernière n'est satisfaite que sous forme d'une inégalité au sens des distributions :

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \rho Q_h(\widetilde{T})\partial_t\phi + \rho\psi(\widetilde{T}) u\cdot\nabla\phi + K_h(\widetilde{T})\Delta\phi dxdt \\
&\qquad \leq \int_0^T \int_{\mathbb{R}^N} h(\widetilde{T})(\widetilde{T} P_0 \mathrm{div} u - D : \nabla u)\phi\, dxdt \\
&\qquad\quad + \int_0^T \int_{\mathbb{R}^N} h'(\widetilde{T})\eta(\widetilde{T})|\nabla\widetilde{T}|^2\phi\, dxdt - \int_{\mathbb{R}^N} \rho_0 Q_h(\widetilde{T}_0)\phi(0)dx,
\end{aligned}$$

pour toutes fonctions h telle que :

- $h \in C^2[0,\infty)$, $h(0) = 1$, h non décroissante sur $[0,\infty)$, $\lim_{z\to\infty} h(z) = 0$,
- $h''(z)h(z) \geq 2(h'(z)^2$ pour tout $z \geq 0$.

et nous avons :

$K_h(\widetilde{T}) = \int_0^{\widetilde{T}} -\eta(\rho,\widetilde{T})h(z)dz$, et $Q_h(\widetilde{T}) = \int_0^{\widetilde{T}} s_0 h(z)dz$

– et pour toute fonction ϕ satisfaisant :
  $\phi \geq 0$, $\phi \in W^{2,\infty}((0,T) \times \mathbb{R}^N)$.

Il est à noter que physiquement cette inégalité au sens des distributions a un sens car elle respecte les inégalités thermodynamiques. Le problème reste cependant entièrement ouvert concernant l'obtention de solutions faibles vérifiant véritablement au sens des distributions le système complet (2.8).

Les premiers à avoir comblé (avec des conditions spécifiques sur les coefficients de viscosité et avec l'ajout d'une pression froide comme on l'a vu précédemment) cette lacune en obtenant des solutions faibles sur le système de Navier-Stokes non isotherme sont D. Bresch et B. Desjardins dans [9] avec de véritables égalités au sens des distributions sur l'équation thermique. La difficulté principale pour passer à la limite dans l'équation thermique relève du manque de régularité sur la densité ρ et l'impossibilité alors d'employer des méthodes de compacité. Seulement dans le cas de Bresch et Desjardins la découverte de cette nouvelle inégalité d'énergie apporte de nouveau renseignement sur la densité et suffisament de régularité pour permettre des méthodes de compacité dans l'équation thermique. Dernièrement Feireisl dans [31] étend encore les résultats précédents au cas où les coefficients de viscosité dépendent de la température, pour cela il extirpe à nouveau la pression efficace $\widetilde{P}(\rho) = P(\rho) - (2\mu(\rho) + \lambda(\rho))\text{div}u$ et montre des propriétés similaires sur $\widetilde{P}$ que Lions en utilisant astucieusement un commutateur sur les coefficients de viscosité.

2.4 Solutions fortes

– **Cas du système isotherme**

L'existence de solutions régulières locales ou globales pour des données petites est connue depuis J. Nash et V. Solonnikov [68],[69], qui a obtenu les premiers résultats d'existence et de régularité en temps petit. Partant de données initiales $(\rho_0, u_0) \in L^\infty(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \times W^{1,2-\frac{2}{q}}(\mathbb{R}^N)$ pour un $q > N$ telles qu'il existe $m > 0$ et $M > 0$ vérifiant $0 < m \leq \rho_0(x) \leq M$ presque partout dans $\mathbb{R}^N$, il obtient l'existence de solutions fortes (ρ, u) sur un intervalle de temps $(0,T)$ appartenant à $L^\infty((0,T)\times\mathbb{R}^N) \cap L^2(0,T;W^{1,q}(\mathbb{R}^N))$ telles que $0 < m(t) \leq \rho(t,x) \leq M(t) < +\infty$, $\partial_t\rho$ et $\partial_t u \in L^q((0,T) \times \mathbb{R}^N)$.

Les travaux sur les solutions fortes consistent ainsi essentiellement à construire des solutions dont les données initiales soient le plus proche possible des espaces d'énergie ou du moins critiques du point de vue du scaling des équations.

Ainsi nous pouvons citer parmi les très nombreux travaux répertoriés des résultats de Solonnikov dans [68], [69] ou encore Hoff dans [38], [39], [41]. Ces solutions ont toutes la même particularité à savoir que la densité est minorée par une constante strictement positive, effectivement on cherche à éviter les phénomènes délicats engendrés par le vide, à savoir une perte de parabolicité sur l'équation du moment et donc une perte d'effet régularisant.

Enfin comme c'est le cas pour Navier-Stokes incompressible atteindre des espaces de données critiques pour le sacling des équations est un enjeu et s'avère souvent délicat via des difficultés liées à des problèmes d'analyse harmonique, les derniers résultats marquants à ce niveau sont l'oeuvre de R. Danchin dans [23], [20] et [22]. Il montre l'existence et l'unicité de solutions globales avec des données initiales proches de l'équilibre qui appartiennent à des espaces de Besov critiques au sens du scaling. De plus il obtient dans ces mêmes espaces de Besov l'existence et l'unicité de solutions locales en temps avec des données initiales grandes, en imposant cependant une condition de petitesse sur la densité initiale associée à une condition de stricte positivité sur la densité. Dans ces théorèmes, un point essentiel est donc le contrôle du vide ainsi que celui de div$u$ dans $L^1_T(B^{\frac{N}{2}}) \hookrightarrow L^1(L^\infty)$ qui s'avère essentiel pour contrôler l'équation de masse et donc propager la régularité sur la densité ρ.
Enfin concernant la résolution du problème de shallow-water, peu de travaux ont été réalisés concernant les solutions fortes ; on peut citer ceux de W. Wang et C.-J. Xu dans [73] où ils obtiennent le résultat suivant :

Théorème 4. *Soit $s > 0$. On prend les conditions initiales suivantes : $u_0 \in H^{2+s}(\mathbb{R}^2)$, $h_0 - \bar{h} \in H^{2+s}(\mathbb{R}^2)$ et on impose la condition de petitesse suivante :*

$$\|h_0 - \bar{h}\|_{H^{2+s}} << \bar{h}.$$

Alors il existe un temps T, et une unique solution (u, h) au problème telle que :

$$u,\, h - \bar{h}_0 \in L^\infty([0,T]; H^{2+s}), \ \ \nabla u \in L^2([0,T], H^{2+s}).$$

De plus, il existe une constante c telle que si $\|h_0 - \bar{h}\|_{H^{2+s}} + \|u_0\|_{H^{2+s}} \leq c$ alors on a $T = +\infty$.

Comme on l'a vu, la plupart des travaux sur les solutions fortes considérent donc des données initiales telles que la densité initiale soit minorée par un nombre strictement positif, ceci afin de contrôler le vide et bénéficier des effets régularisants de l'équation du moment, cependant peu de travaux sur les solutions fortes concernent des résultats avec une densité initiale pouvant admettre l'état de vide. Ceci est d'autant plus intéressant que les solutions faibles de P.-L. Lions n'imposent aucune contrainte sur le vide, effectivement la densité ρ_0 appartient à $L^1(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$. Cependant il est vrai que Lions obtient aussi des résultats dans des espaces d'Orlicz où la densité est proche d'un état stable ce qui est le cas des travaux sur les solutions fortes préalablement cités.
L'un des premiers à avoir répondu à ce problème est B. Desjardins dans [25] qui considère le problème de Navier-Stokes compressible isotherme sur un tore. Il obtient alors dans le cas de la dimension $N = 2, 3$ un résultat d'existence de solutions faibles en temps fini en prenant $\rho_0 \geq 0$, $\rho_0 \in L^\infty(\mathbb{T}^N)$ et $u_0 \in H^1(\mathbb{T}^N)^N$. De plus il aboutit à un résultat de solution fort-faible à partir de ses solutions qui sont plus régulières que celles de Lions, elles conservent ainsi la norme L^∞ sur la densité. On rappelle que le problème des solutions

forts-faibles sur les solutions de Lions reste complètement ouvert même en dimension 2.
Enfin dernièrement Cho, Choe et Kim dans [16] (voir aussi [17]) obtiennent de véritables solutions fortes avec des données initales qui autorisent le vide. Effectivement ils prolongent les résultats de Desjardins, ainsi dans le cas $N = 3$ ils obtiennent l'existence et l'unicité de solutions en temps fini dans un ouvert Ω en choisissant des données initiales telles que $0 \leq \rho_0 \in H^1 \cap W^{1,q}$ avec $3 < q < +\infty$ et $u_0 \in D_0^1 \cap D^2$ avec la condition de compatibilité suivante :

$$Lu_0 + \nabla P(\rho_0) = \sqrt{\rho_0} g \quad \text{avec} \quad g \in L^2.$$

Concernant le comportement asymptotique des solutions fortes avec une donnée initiale proche d'un état stable, on renvoie aux travaux de Kobayashi et Shibata dans [49].

– **Cas du système non isotherme**

Dans le cas non isotherme d'autres nombreux résultats d'existence de solutions fortes sont à signaler dans la littérature à savoir notamment ceux de Hoff dans [40], [39].
Enfin dans le cas d'une densité initiale minoré par une constante strictement positive, R. Danchin dans [21] obtient des résultats similaires au cas isotherme. Il assure ainsi l'existence et l'unicité de solutions en temps fini pour des données initiales grandes citiques pour le scaling des équations avec $(q_0, u_0, \theta_0) \in B^{\frac{N}{2}} \times B^{\frac{N}{2}-1} \times B^{\frac{N}{2}-2}$. Cependant il est à préciser que dans ces travaux il travaille avec une énergie interne bien spécifique, c'est à dire linéaire par rapport à la température ce qui lui permet d'atteindre le cas des espaces critiques. En outre il impose une condition de petitesse sur la donnée initiale. Enfin il obtient l'existence de solutions globales et leur unicité pour des données initiales proche d'un état physique stable. Pour conclure nous citerons aussi les travaux de Cho et Kim qui étendent leur résultats au cas non isotherme (voir [15]), où l'on rappelle que les densités initiales peuvent être astreintes à du vide.

3 Navier-Stokes capillaire de type Korteweg

3.1 Présentation physique

Etablissement des équations de bilan

Nous allons maintenant rappeler le système d'équations de bilan régissant le mouvement d'un fluide décrit par la théorie du second gradient après avoir présenté une méthode permettant d'établir ces équations. Ce système d'équations nécessite d'être fermé par l'expression de l'énergie du fluide en fonction des variables thermodynamiques du système.
Nous présenteront ainsi la forme de la fermeture thermodynamique proposée par C. Fouillet dans [33].

Dans [43], les équations de bilan régissant le mouvement d'un fluide pur décrit par la théorie de la capillarité de Van der Waals sont établies à partir du principe des puissances virtuelles, ici une méthode plus simple et plus directe va être utilisée.
Les équations de bilan pour un fluide pur peuvent s'écrire sous la forme générale suivante :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \\ \rho \dfrac{du}{dt} = \nabla \cdot \mathbf{T} + \rho g, \\ \rho \dfrac{de}{dt} + \nabla \cdot Q = \nabla \cdot (\nabla \cdot \mathbf{T}) + \rho g \cdot u, \\ \rho \dfrac{ds}{dt} + \nabla \cdot Q_s = \Delta_s, \end{cases} \tag{3.9}$$

où $\frac{d}{dt}$ représente la dérivée particulaire définie par $\frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla$. La première équation de (3.9) désigne le bilan de masse où ρ définit la densité du fluide, u sa vélocité ; la seconde équation est le bilan de quantité de mouvement, $\mathbf{T}$ représentant le tenseur des contraintes et g le vecteur gravité (la seule force extérieure est la force de gravité) ; la troisième équation est le bilan d'énergie totale, l'énergie totale massique e étant définie à partir de l'énergie interne massique $u^{'}$ par :

$$e = u^{'} + \frac{1}{2}|u|^2, \tag{3.10}$$

et Q caractérisant le flux d'énergie.
Enfin la dernière équation de (3.9) est le bilan d'entropie où s définit l'entropie spécifique, Q_s est le flux d'entropie et Δ_s la création d'entropie.

Afin de déterminer les expressions de $\mathbf{T}$ et Q, une équation de bilan d'entropie est écrite à partir des trois premières équations de (3.9) ainsi que (3.10), et celle-ci est identifiée à la dernière équation de (3.9) pour déterminer Q_s et Δ_s. L'application du second principe de la thermodynamique selon lequel la création d'entropie Δ_s doit être positive quel que soit le mouvement considéré nous permet alors de déterminer la forme du tenseur des contraintes et du flux d'énergie.
Si l'on fait l'hypothèse que l'on se situe dans des conditions d'équilibre thermodynamique local, l'expression (1.1) de l'énergie interne conduit à la relation suivante :

$$\frac{du^{'}}{dt} = \widetilde{T}\frac{ds}{dt} + \frac{\widetilde{P}}{\rho^2}\frac{d\rho}{dt} + \frac{\kappa}{\rho}\nabla\rho \cdot \frac{d\nabla\rho}{dt} \tag{3.11}$$

où $\widetilde{T}$ représente la température, κ le coefficient de cappilarité interne et $\widetilde{P}$ la pression thermodynamique totale définie par :

$$\widetilde{P} = \rho^2 \left(\frac{\partial u^{'}}{\partial \rho}\right)_{s,\nabla\rho} \tag{3.12}$$

La température $\widetilde{T}$ du système peut être définie par :

$$\widetilde{T} = \left(\frac{\partial u^{'}}{\partial s}\right)_{\rho,\nabla\rho} \tag{3.13}$$

En utilisant la définition (3.10) de l'énergie e ainsi que l'expression (3.11) de $\frac{du'}{dt}$ et la première équation du système (3.9), puis en multipliant l'équation du moment par u, le bilan d'énergie assure l'expression suivante :

$$\rho\widetilde{T}\frac{ds}{dt} = -\nabla\cdot q + \tau^D : \nabla u \tag{3.14}$$

où q représente le flux d'énergie et τ^D le tenseur des contraintes dissipatives, de plus on a les expressions suivantes :

$$\begin{aligned} q &= Q - \kappa\rho\nabla\rho\nabla\cdot u, \\ \tau^D &= \mathbf{T} + \widetilde{P}I + \kappa\nabla\rho\otimes\nabla\rho - \rho\nabla\cdot(\kappa\rho)I. \end{aligned} \tag{3.15}$$

En identifiant la forme générale du bilan d'entropie de (3.9) et l'équation d'entropie sous la forme (3.14), on déduit les expressions suivantes pour la création d'entropie Δ_s et le flux d'entropie Q_s :

$$\begin{aligned} \Delta_s &= -\frac{q}{\widetilde{T}^2}\cdot\nabla\widetilde{T} + \frac{1}{\widetilde{T}}\tau^D : \nabla u, \\ Q_s &= \frac{q}{\widetilde{T}}. \end{aligned} \tag{3.16}$$

Ces expressions de la création d'entropie et de flux d'entropie sont telles que les équations d'énergie et d'entropie du système d'équation de bilan de (3.9) sont redondantes : les équations de masse, de quantité de mouvement et de quantité d'énergie ainsi que l'équation (3.11) permettent en effet d'écrire l'équation d'entropie (3.14) dont la forme est strictement équivalente à l'équation d'entropie de (3.9) du fait des relations (3.16).
Le système (3.9) se réduit alors à un système de trois équations qui s'écrivent en utilisant (3.15) sous la forme :

$$\begin{aligned} &\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0, \\ &\rho\frac{du}{dt} = -\nabla\big(\widetilde{P} - \rho\nabla\cdot(\kappa\nabla\rho)\big) - \nabla\cdot(\kappa\rho\otimes\nabla\rho) + \nabla\cdot\tau^D + \rho g, \\ &\rho\widetilde{T}\frac{ds}{dt} = -\nabla\cdot q + \tau^D : \nabla u. \end{aligned} \tag{3.17}$$

Il est à préciser que l'équation de bilan d'entropie de (3.17) est équivalente à l'équation de bilan d'énergie totale de (3.10) si l'on tient compte des relations (3.11) et (3.15). Il reste maintenant à préciser les formes générales du flux d'énergie q et du tenseur des contraintes dissipatives τ^D.

Relation de fermeture

En utilisant *le second principe de la thermodynamique* qui affirme que la création d'entropie doit être positive, i.e $\Delta_s \geq 0$, si l'on fait l'hypothèse que les dissipations thermiques et mécaniques sont découplées, on en déduit les expression de q et τ^D sous la forme suivante :

$$\begin{aligned} q &= -k\nabla\widetilde{T}, \\ \tau^D &= 2\mu D + (\kappa - \frac{2}{3}\mu)\mathrm{tr}(D)I, \end{aligned} \tag{3.18}$$

(où par simplification on a supposé la conductivité thermique q et le tenseur de contraintes visqueuses τ^D isotropes)
La première équation est la relation de Fourier classique, où k désigne la conductivité thermique, et la seconde représente le tenseur des contraintes où μ désigne la viscosité dynamique et D la partie symétrique du tenseur $\nabla \cdot u$.
Pour que le système des équations du mouvement (3.17) soit complètement fermé, il est également nécessaire de se donner une forme pour l'énergie du système afin de pouvoir exprimer $\widetilde{P}$ et s en fonction des données du système. Du fait que la densité et la température sont les variables thermodynamiques les plus aisées à contrôler physiquement, l'utilisation de l'énergie libre semble particulièrement appropriée.
L'énergie libre volumique F du système est définie à partir de l'énergie interne volumique U suivant la relation :

$$F = U - S(\frac{\partial U}{\partial S})_{\rho,\nabla\rho} \tag{3.19}$$

En supposant que le coefficient de capillarité interne κ est indépendant de l'entropie volumique S, on peut montrer que F a la même dépendance que l'énergie interne dans (1.1) selon $\nabla\rho$:

$$F(\widetilde{T}, \rho, \nabla\rho) = F^{cl}(\widetilde{T}, \rho) + \frac{\kappa}{2}|\nabla\rho|^2. \tag{3.20}$$

Si l'on se donne une forme pour $F^{cl}(\widetilde{T}, \rho)$, les expressions de $\widetilde{P}$ et s peuvent alors être déduites de (3.12), (3.13) et (3.19) :

$$\begin{aligned} \widetilde{P} &= -F + \rho(\frac{\partial F}{\partial \rho})_{\widetilde{T},\nabla\rho}, \\ s &= -\frac{1}{\rho}(\frac{\partial F}{\partial \widetilde{T}})_{\rho,\nabla\rho}. \end{aligned} \tag{3.21}$$

Dans la suite pour simplifier la présentation, on suppose κ constant, on expliquera dans le chapitre 2 les modifications à apporter lorsque κ varie.

Forme finale du système

En définissant la *pression thermodynamique classique* P par la relation :

$$P = \rho^2(\frac{\partial (u')^{cl}}{\partial \rho})_s = -F^{cl} + \rho(\frac{\partial F^{cl}}{\partial \rho})_{\widetilde{T}}, \tag{3.22}$$

en utilisant (1.1) et (3.12), et l'hypothèse κ constant, la pression thermodynamique totale du système $\widetilde{P}$ s'exprime en fonction de P par :

$$\widetilde{P} = P - \frac{\kappa}{2}|\nabla\rho|^2. \tag{3.23}$$

En utilisant de plus la relation vectorielle suivante :

$$\nabla \cdot (\kappa \nabla \rho \otimes \nabla \rho) = \nabla (\frac{\kappa}{2} |\nabla \rho|^2) + \nabla \big(\rho \nabla \cdot (\kappa \nabla \rho)\big) - \rho \nabla \big(\nabla \cdot (\kappa \nabla \rho)\big), \tag{3.24}$$

le bilan de quantité de mouvement (3.17) peut alors être écrit sous la forme :

$$\rho \frac{du}{dt} = -\nabla P + \rho \nabla \big(\nabla \cdot (\kappa \nabla \rho)\big) + \nabla \cdot \tau^D + \rho g. \tag{3.25}$$

De plus en introduisant le *potentiel chimique du fluide* (ou enthalpie libre massique) défini par :

$$\mu^{'} = (\frac{\partial U^{cl}}{\partial \rho})_S = (\frac{\partial F^{cl}}{\partial \rho})_{\widetilde{T}} \tag{3.26}$$

la relation de Gibbs-Duhem s'écrit :

$$dP = \rho d\mu^{'} + \rho s d\widetilde{T} \tag{3.27}$$

on peut alors réécrir le **bilan de quantité de mouvement** (3.25) sous la forme :

$$\rho \frac{du}{dt} = -\rho \nabla \mu^g - \rho s \nabla \widetilde{T} + \nabla \cdot \tau^D + \rho g. \tag{3.28}$$

où μ^g est un potentiel chimique, appelé potentiel chimique généralisé, défini par la relation :

$$\mu^g = \mu^{'} - \nabla \cdot (\kappa \nabla \rho). \tag{3.29}$$

Enfin intéressons-nous à l'équation d'entropie. En définissant la *capacité calorifique spécifique à volume constant* c_v par :

$$c_v = (\frac{\partial u^{'}}{\partial \widetilde{T}})_{\rho, \nabla \rho}, \tag{3.30}$$

et en utilisant (3.13), on montre la relation suivante :

$$c_v = \widetilde{T} (\frac{\partial s}{\partial \widetilde{T}})_\rho. \tag{3.31}$$

L'équation d'entropie dans (3.17) peut alors s'écrire sous la forme d'une équation d'évolution de la température :

$$\rho c_v \frac{d\widetilde{T}}{dt} = -\nabla \cdot q + \tau^D : \nabla u - \rho \widetilde{T} (\frac{\partial s}{\partial \rho})_{\widetilde{T}} \frac{d\rho}{dt}. \tag{3.32}$$

Enfin en utilisant la relation de Maxwell suivante :

$$(\frac{\partial s}{\partial \rho})_{\widetilde{T}} = -\frac{1}{\rho^2} (\frac{\partial P}{\partial \widetilde{T}})_\rho, \tag{3.33}$$

l'équation (3.32) peut alors s'écrire sous la forme :

$$\rho c_v \frac{d\widetilde{T}}{dt} = -\nabla \cdot q + \tau^D : \nabla u - \frac{\rho}{\widetilde{T}} (\frac{\partial P}{\partial \widetilde{T}})_\rho \frac{d\rho}{dt}. \tag{3.34}$$

Cette forme de l'équation en température consiste en fait à une simple réécriture du terme de *changement de phase* $\rho \widetilde{T} (\frac{\partial s}{\partial \rho})_{\widetilde{T}} \frac{d\rho}{dt}$ où $\frac{d\rho}{dt}$ prend des valeurs substantielles à l'interface

seulement si celle-ci est traversée par un flux de masse dû à un phénomène de changement de phase.
On obtient finalement le système d'équations du mouvement suivant :

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \\ &\rho \frac{du}{dt} = -\rho \nabla \mu^g - \rho s \nabla \widetilde{T} + \nabla \cdot \tau^D + \rho g, \\ &\rho c_v \frac{d\widetilde{T}}{dt} = -\nabla \cdot q + \tau^D : \nabla u + \frac{\widetilde{T}}{\rho} (\frac{\partial P}{\partial \widetilde{T}})_\rho \frac{d\rho}{dt}, \end{aligned} \tag{3.35}$$

où les expressions de q et τ^D sont données par les relations (3.18) et les expressions de μ^g, s, c_v et P sont déduites de l'expression de l'énergie libre volumique grâce aux relations (3.29), (3.26), (3.21), (3.31) et (3.22).
On peut alors remarquer que, du fait que les variables ρ, u et $\widetilde{T}$ sont continues dans tout le système, ce système décrit **le mouvement de l'ensemble du fluide, y compris celui des interfaces**. Les interfaces sont localisées dans les zones de fortes variations de densité ρ, leurs déplacements par convection ou par changement de phase sont donc complètement intégrés dans la solution des équations du mouvement.
Pour rappel ce modèle physique remonte donc aux premiers travaux de Korteweg et Van der Waals dans [50], par contre son étude mathématique n'est que très récente et fait suite aux nouveaux travaux de Jamet dans [44] ou Truedell dans [70] qui utilisèrent comme expliqué précédemment la théorie du second gradient. Nous allons donc ici décrire certains travaux récents en commençant par ceux sur les solutions faibles puis ceux traitant des solutions fortes. Le modèle (3.35) du point de vue mathématique sera étudié sous la forme du système (NHV) précédemment cité dans la section 1.

3.2 Solutions faibles

Cas du système isotherme

On s'intéresse ici au système (NHV) sous sa forme isotherme que l'on réécrit comme suit :

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho) D u) - \nabla(\lambda(\rho)\text{div} u) + \nabla P(\rho) \\ \qquad\qquad = \nabla(\kappa \Delta \rho + \frac{\kappa_\rho^{'}}{2} |\nabla \rho|^2). \end{cases} \tag{3.36}$$

Avant de rappeler les principaux résultats concernant les solutions faibles, nous allons commencer par décrire les inégalités d'énergie liées au système lorsque les coefficients de viscosité et de capillarité sont des constantes (ceci pour simplifier les écritures). En effet ceci nous donnera alors une idée du comportement mathématique du système et surtout des termes éventuellement délicats à contrôler pour obtenir des solutions faibles. Effectivement les inégalités d'énergie vont tout de suite mettre en lumière les termes présentant un manque de compacité. On suppose (ρ, u) une solution approchée du système (3.36), et l'on multiplie l'équation du moment par u, on obtient alors l'inégalité suivante :

$$\int_{\mathbb{R}^N}(\frac{1}{2}\rho|u|^2+(\Pi(\rho)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho|^2)(t)dx+2\int_0^t\int_{\mathbb{R}^N}(2\mu D(u):D(u)$$
$$+(\lambda+\mu)|\mathrm{div}u|^2)dx\leq\int_{\mathbb{R}^N}(\frac{|m_0|^2}{2\rho}+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho_0|^2)dx,$$

où Π est défini comme suit :

$$\Pi(s)=s\Big(\int_{\bar{\rho}}^s\frac{P_0(z)}{z^2}dz-\frac{P_0(\bar{\rho})}{\bar{\rho}}\Big).$$

En imposant l'inégalité suivante sur les données initiales :

$$\epsilon_0=\int_{\mathbb{R}^N}(\frac{|m_0|^2}{2\rho}+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho_0|^2)dx<+\infty\,, \tag{3.37}$$

on obtient les estimations à priori suivantes :

$$\Pi(\rho)-\Pi(\bar{\rho}),\ \ \text{et}\ \ \rho|u|^2\in L^\infty(0,\infty,L^1(\mathbb{R}^N))$$
$$\nabla\rho\in L^\infty(0,\infty,L^2(\mathbb{R}^N))^N,\ \ \text{et}\ \ \nabla u\in L^2(0,\infty,\mathbb{R}^N)^{N^2}.$$

Afin d'obtenir des solutions faibles, R. Danchin et B. Desjardins dans [23] considèrent une suite de solutions approchées $(\rho_n,u_n)_{n\in\mathbb{N}}$ du système (3.36) avec des coefficients physiques constants et ils cherchent à obtenir des propriétés de stabilité sur le système. On se rend alors rapidement compte que le terme délicat pour l'obtention de solutions faibles correspond au terme quadratique $\nabla\rho_n\otimes\nabla\rho_n$ provenant du tenseur de capillarité. En effet d'après nos estimations à priori venant des inégalités d'énergie, $\nabla\rho_n\otimes\nabla\rho_n$ est uniformément borné dans $L^\infty(L^1)$ si l'on choisit des données initiales vérifiant uniformément (3.37), on obtient donc qu'une convergence au sens de la mesure à extraction près vers une mesure ν. Toute la difficulté repose sur le fait de savoir si ν correspond réellement à $\nabla\rho\otimes\nabla\rho$ avec ρ défini comme limite de ρ_n. Il est à préciser que par rapport à Navier-Stokes isotherme le terme de pression $P(\rho)$ est aisé à traiter étant donné la régularité $L^\infty(L^2)$ que l'on a sur $\nabla\rho_n$.

R. Danchin et B. Desjardins répondent partiellement au problème dans le cas de la dimension $N=2$ en imposant certaines conditions sur la densité ρ_n, ils obtiennent ainsi le résultat suivant :

Proposition 4. *Il existe $\eta>0$, $T>0$ tel que tant que :*

$$(\mathcal{E}_0)_n+\sup_{t\in[0,T]}\delta_n(t)\leq\eta\quad\forall n.$$

avec :

$$\delta_n(t)=\|\frac{\rho_n-\bar{\rho}}{\bar{\rho}}\|_{L^\infty}$$

alors il existe une solution faible (ρ,u) sur $(0,T)$ du système (NHV) isotherme telle que $\rho-\bar{\rho}\in L^2(0,T;\dot{H}^1(\mathbb{R}^2)\cap\dot{H}^2(\mathbb{R}^2))$, $u\in L^2(0,T;\dot{H}^1(\mathbb{R}^2))\cap L^\infty(0,T;L^2(\mathbb{R}^2))^2$.

La preuve repose sur des inégalités d'énergie qui permettent un gain de dérivabilité sur la densité ρ, ceci en extirpant astucieusement un effet régularisant du tenseur de capillarité. Le passage à la limite par compacité de (ρ_n, u_n) vers une solution est alors aisé.

Le problème des solutions faibles pour le système isotherme avec des coefficients de viscosité constants restant complètement ouvert, D. Bresch, B. Desjardins et C-K. Lin dans [13] se sont intéressés au cas où les coefficients de viscosité étaient variables. Ils se sont concentrés tout particulièrement sur le cas des coefficients intervenant dans le modèle de Saint-Venant $\mu(\rho) = C\rho$ et $\lambda(\rho) = 0$ avec $C > 0$. Le système s'écrit alors :

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - C\mathrm{div}(\rho D(u)) + \nabla(P(\rho)) = \kappa\rho\nabla\Delta\rho. \end{cases} \tag{3.38}$$

On rappelle ici leur définition de solutions faibles.

Définition 2. *Nous disons que (ρ, u) est une solution faible sur $(0, T)$ du système précédent si et seulement si les propriétés suivantes sont vérifiées :*

– *Nous supposons que :*

$$\mathcal{E}_0 = \int_\Omega \big(\bar{\kappa}\frac{|\nabla\rho_0|^2}{2} + \Pi(\rho_0) + \rho_0\frac{|u_0|^2}{2}\big) < +\infty$$

et :

$$\mathcal{F}_0 = 2\nu^2 \int_\Omega |\nabla\sqrt{\rho_0}|^2 = \frac{1}{2}\int_\Omega \rho_0|\nu\nabla\log\rho_0|^2 < +\infty$$

– *en plus nous avons :*

$$\begin{cases} \rho \in L^2(0, T; H^2(\Omega)), \\ \nabla\rho \;\; et \;\; \nabla\sqrt{\rho} \in L^\infty(0, T; (L^2(\Omega))^N), \\ \sqrt{\rho}u \in L^\infty(0, T; (L^2(\Omega))^N), \\ \sqrt{\rho}D(u) \in L^2(0, T; (L^2(\Omega))^{N\times N}), \end{cases}$$

avec $\rho \geq 0$ presque partout, et :

$$\partial_t\rho + \mathrm{div}(\rho u) = 0 \quad dans \;\; \mathcal{D}'((0, T) \times \Omega), \quad \rho_{t=0} = \rho_0 \;\; dans \;\; \mathcal{D}'(\Omega)$$

et pour tout $\varphi \in C^\infty([0, T] \times \Omega)^N$ tel que $\varphi(T, \cdot) = 0$, on a :

$$\int_\Omega \rho_0 u_0 \cdot \rho_0\varphi(0, \cdot) + \int_0^T \int_\Omega (\rho^2 u \cdot \partial_t\varphi + \rho u \otimes \rho u : D(\varphi) - \rho^2(u \cdot \varphi)\mathrm{div}u - \nu\rho D(u) : \rho D(\varphi)$$
$$- \nu\rho D(u) : \varphi \otimes \nabla\rho + \Pi(\rho)\mathrm{div}\varphi - \bar{\kappa}\rho^2\Delta\rho\mathrm{div}\varphi - 2\bar{\kappa}\rho(\varphi \cdot \nabla\rho)\Delta\rho = 0.$$

Ils obtiennent alors le résultat d'existence suivant.

Théorème 5. *Soit $N = 2, 3$. Il existe alors pour le cas périodique $\Omega = \mathbb{T}^N$ une solution globale (ρ, u) du système (3.38) qui vérifie la définition 2.*

On peut remarquer que ces solutions faibles sont très particulières dans la mesure où les fonctions tests dépendent elles-mêmes de la solution et s'écrivent sous la forme $\rho\varphi$ avec $\varphi \in C^\infty([0,T]\times\Omega)^N$ (ce que l'on va expliquer). La démonstration de ce résultat repose sur un gain de dérivée par rapport à la densité ρ en utilisant des inégalités d'énergie, qui proviennent essentiellement de la structure très spécifique des coefficients de viscosité (on a vu que ces gains de dérivabilité sur la densité ρ ont été étendus par ces mêmes auteurs à des coefficients de viscosité plus généraux dans le cas du système de Navier-Stokes). En contrepartie du choix de ces coefficients de viscosité une nouvelle difficulté apparaît, consistant en une perte d'information sur la vitesse u lorsque le vide intervient. C'est pourquoi D. Bresch, B. Desjardins et C-K. Lin ont besoin d'utiliser ces fonctions tests $\rho\varphi$ pour pouvoir passer à la compacité notamment dans le terme $\rho u\otimes u$.
On rappelle ici le gain de dérivée par rapport à la densité ρ obtenu :

Lemme 1. *Nous avons l'inégalité suivante :*

$$4C\int_\Omega P'(\rho)|\nabla\sqrt{\rho}|^2 + C\kappa\int_\Omega |\nabla\nabla\rho|^2 + \frac{\partial}{\partial t}\int_\Omega \big(\frac{1}{2}\rho|u + C\nabla\log\rho|^2 + \Pi(\rho) + \frac{\kappa}{2}|\nabla\rho|^2\big) \leq C\int_\Omega \rho\, D(u) : D(u).$$

Enfin concernant le cas *non isotherme* aucun travaux à ma connaissance traitent du sujet, on pourrait cependant sans difficultés aucune transcrire les résultats de Bresch et Desjardins avec un choix de coefficients de viscosité de type Saint-Venant au cas non isotherme.

3.3 Solutions fortes

Les premiers à avoir vérifié si le modèle complet *non isotherme* était bien posé sont Li et Hattori dans [36] et [37] en 1996. Ils obtiennent des résultats d'existence globale et d'unicité avec des données initiales proches d'un état stable dans les espaces de Sobolev $H^s\times H^{s-1}\times H^{s-2}$ avec $s\geq \frac{N}{2}+4$ (où N représente la dimension) ainsi que des résultats d'existence en temps fini et unicité avec des données initiales grandes. De plus ils considèrent des pressions convexes, n'intégrant pas le cas des équations de Van der Waals. Plus récemment un travail de Raphaël Danchin et Benoît Desjardins concernant le cas *isotherme* dans [6] améliore considérablement les résultats précédents, effectivement ils obtiennent des solutions avec des données initiales dans des espaces critiques du point de vue du scaling des équations. Expliquons brièvement ce que nous entendons par scaling du système. La notion de scaling fut introduite par Fujita et Kato dans [34] pour les équations de Navier-Stokes incompressible, et son usage est depuis lors devenu classique pour différents types de système d'équations aux dérivées partielles.
On peut ainsi vérifier aisément que si (ρ,u,θ) est une solution du système (NHV), alors $(\rho_\lambda,u_\lambda,\theta_\lambda)$ est aussi solution du système où on a posé :

$$\rho_\lambda(t,x)=\rho(\lambda^2t,\lambda x)\,,\ u_\lambda(t,x)=\lambda u(\lambda^2t,\lambda x)\ \text{ et }\ \theta_\lambda(t,x)=\lambda^2\theta(\lambda^2t,\lambda x)$$

et où les pressions P_0, P_1 ont été changées en $\lambda^2 P_0, \lambda^2 P_1$.

Definition 3.1. *Nous dirons qu'un espace est critique au sens du scaling si sa norme associée est invariante par la transformation suivante :*

$$(\rho, u, \theta) \longrightarrow (\rho_\lambda, u_\lambda, \theta_\lambda)$$

(ceci pour n'importe quel $\lambda > 0$).

Ceci suggère de choisir les données initiales (ρ_0, u_0, θ_0) dans un espace dont la norme est invariante pour tout $\lambda > 0$ par la transformation :

$$(\rho_0, u_0, \theta_0) \longrightarrow (\rho_0(\lambda \cdot), \lambda u_0(\lambda \cdot), \lambda^2 \theta_0(\lambda \cdot)).$$

Un candidat naturel vérifiant cette propriété est le produit suivant d'espaces de Sobolev homogènes $E = \dot{H}^{N/2} \times (\dot{H}^{N/2-1})^N \times \dot{H}^{N/2-2}$. Cependant une difficulté majeure apparaît après utilisation de cet espace, celle du contrôle du vide et donc du comportement en norme L^∞ de la densité ρ. Effectivement le vide comme dans de nombreux problèmes liés à la mécanique des fluides engendre de nombreuses contraintes techniques souvent infranchissables et dans le cas présent des solutions fortes, la principale correspond à une perte de parabolicité sur l'équation du moment.
L'idée est donc de trouver un espace invariant selon le scaling des équations qui permet en plus d'absorber le vide.
Danchin et Desjardins choisissent alors dans le cas du système isotherme de prendre leurs données initiales dans l'espace $F = B_{2,1}^{\frac{N}{2}} \times (B_{2,1}^{\frac{N}{2}-1})^N$. Effectivement cet espace F est critique pour le scaling des équations et est en réalité un raffinement du produit d'espaces de Sobolev homogènes correspondant, on a ainsi $B_{2,1}^{\frac{N}{2}} \hookrightarrow \dot{H}^{\frac{N}{2}}$. Mais le choix de F permet surtout de contrôler le vide car on a l'injection suivante $B_{2,1}^{\frac{N}{2}} \hookrightarrow L^\infty$.

Notation 1. *Dans la suite on utilisera la notation :*

$$B_{2,1}^s = B^s.$$

Avant de préciser le résultat de Danchin et Desjardins, pour la compréhension des théorèmes donnons la définition suivante :

Définition 3. *Soit $\bar{\rho} > 0$, $\bar{\theta} > 0$. Nous noterons dans la suite :*

$$q = \frac{\rho - \bar{\rho}}{\bar{\rho}} \;\; et \;\; \mathcal{T} = \theta - \bar{\theta}.$$

Ils obtiennent ainsi un résultat d'existence globale pour des données proche de l'équilibre dans [23].

Théorème 6. *Soit $\bar{\rho} > 0$ tel que $P'(\bar{\rho}) > 0$. Supposons que la fluctuation de la densité initiale q_0 appartienne à $B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}$, que le moment initial m_0 soit dans $B^{\frac{N}{2}-1}$ et le* *terme de force f est dans $L^1(\mathbb{R}, B^{\frac{N}{2}-1})^N$. Il existe alors une constante $\varepsilon_0 > 0$ dépendant seulement des coefficients physiques ainsi que de la dimension telle que si :*

$$\|q_0\|_{B^{\frac{N}{2}}\cap B^{\frac{N}{2}-1}} + \|m_0\|_{B^{\frac{N}{2}}} \leq \varepsilon_0$$

alors le système a une unique solution (q, m) dans :

$$E = \big(\widetilde{C}(B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}) \cap L^1(B^{\frac{N}{2}+2} \cap B^{\frac{N}{2}+1})\big) \times \big(\widetilde{C}(B^{\frac{N}{2}-1}) \cap L^1(B^{\frac{N}{2}+1})\big)^N.$$

Il est à noter que la densité initiale q_0 appartient à $B^{\frac{N}{2}} \cap B^{\frac{N}{2}-1}$ et non pas seulement à $B^{\frac{N}{2}}$, ceci est dû au fait que sur un théorème d'existence globale, on doit aussi de manière conséquente tenir compte du comportement des basses fréquences.
Danchin et Desjardins obtiennent en plus un théorème d'existence et d'unicité avec des données grandes sur un intervalle de temps fini.

Théorème 7. *Supposons que le terme de force extérieure appartienne à $(L^1_T(B^{\frac{N}{2}-1})^N$, que le moment initial m_0 soit dans $(B^{\frac{N}{2}-1})^N$ et la fluctuation de densité initiale q_0 dans $B^{\frac{N}{2}}$. De plus on impose que la densité initiale soit strictement positive soit $\rho_0 \geq c > 0$.*
Alors il existe un temps $T > 0$ tel que le système avec pour données initiales (q_0, m_0) ait une unique solution (q, m) dans :

$$F_T = \big(\widetilde{C}([0,T]; B^{\frac{N}{2}}) \cap L^1(0,T; B^{\frac{N}{2}+2})\big) \times \big(\widetilde{C}([0,T]; B^{\frac{N}{2}-1}) \cap L^1(0,T; B^{\frac{N}{2}+1})\big)^N.$$

La démonstration de ces théorèmes repose essentiellement sur le contrôle du vide où notamment pour le second théorème on vérifie que la condition de stricte positivité de la donnée initiale est conservée à temps petit, ceci permet ainsi d'utiliser un effet la parabolicité de l'équation du moment. Enfin on remarque qu'on obtient également un effet régularisant sur la densité ρ, qui provient évidemment du terme de capillarité. Ce phénomène radicalement nouveau par rapport à des résultats similaires de solutions fortes sur le système de Navier-Stokes compressible permet ainsi de ne pas imposer de conditions de petitesses sur la densité q_0, ceci dû au fait que l'on évite certains cas critiques lors de l'utilisation du paraproduit de Bony.
On peut aussi rappeler les résultats de Benzoni, Danchin et Descombes dans [4], [5] qui se sont intéressés au cas du système de Korteweg isotherme non visqueux. Comme pour le cas d'Euler compressible, une difficulté majeure se révèle avec la perte de parabolicité sur l'équation du moment. On perd ainsi le gain de dérivabilité sur la vélocité u, la difficulté est alors d'obtenir de bonnes égalités d'énergie sur ρ et u sans pertes de dérivées avec des données initiales assez régulières, la démonstration repose notamment sur des théorèmes de commutateurs. Ainsi ils obtiennent des résultats de solutions fortes en temps fini avec les données initiales (q_0, u_0) dans $H^{\frac{N}{2}+2+\varepsilon} \times H^{\frac{N}{2}+1+\varepsilon}$ avec $\varepsilon > 0$. D'autres travaux sur la stabilité à l'infini du système sont à noter, on renvoie aux travaux [2], [3].

4 Navier-Stokes de type capillaire avec méthode à interfaces discontinues

4.1 Présentation physique

On s'intéresse maintenant au cas du système de Navier-Stokes compressible isotherme avec un terme de capillarité non local récemment mis à l'honneur par Rohde dans [63] et [19].

Avant de passer en revue certains travaux sur le sujet nous rappelons ce nouveau modèle concernant toujours les changements de phases liquide-vapeur où F. Coquel, C. Rohde et leurs collaborateurs dans [19] dérivent le modèle de transition de phase en considérant un nouveau terme de capillarité. Ils se concentrent donc sur le problème d'un fluide compressible pouvant évoluer à la fois sous forme liquide ou vapeur, la densité déterminera l'état du fluide et celle-ci variera dans l'intervalle $(0, b)$ avec $b > 0$. Nous écrivons ensuite l'énergie libre sous la forme :

$$W(\rho) = \rho w(\frac{1}{\rho}) \quad \text{et} \quad W \in C^2((0, b))$$

où $w(\rho)$ est l'énergie interne. On va noter les différents états du liquide comme suit :

- vapeur si $\rho \in (0, \alpha_1)$
- spinodal si $\rho \in (\alpha_1, \alpha_2)$
- liquide si $\rho \in (\alpha_2, b)$

avec $\alpha_1, \alpha_2 \in (0, b)$.

On va ici utiliser une méthode basée sur les *modèles à interfaces discontinues* pour déterminer un nouveau tenseur de capillarité. Dans ce type de méthodes on rappelle que les équations du mouvement sont généralement résolues de façon séparée dans chacune des phases, ce qui entraîne nécessairement des discontinuités aux interfaces qui sont supposées d'épaisseur nulle.

Ainsi si l'on considère le problème à interfaces discontinues, chercher l'équilibre statistique du système revient alors à minimiser la fonctionnelle suivante :

$$F_0[\rho] = \inf_{\rho \in A_0} \int_\Omega W(\rho(x))dx$$

où l'on définit l'ensemble admissible A_0 pour la densité ρ avec $m > 0$ comme suit :

$$A_0 = \{\rho \in L^1(\Omega)/W(\rho) \in L^1(\Omega),\ \int_\Omega \rho(x)dx = m\}.$$

où Ω est un ouvert de $\mathbb{R}^N$

On impose donc une conservation de la masse avec la condition $\int_\Omega \rho(x)dx = m$, et on cherche à minimiser l'énergie libre pour atteindre l'équilibre statistique. Le calcul variationnel nous indique alors que l'on obtient un état constant par morceaux sur la densité ne prenant que deux états physiques, c'est à dire ceux de Maxwell β_1 et β_2 (voir [19] pour plus de précision sur les états de Maxwell).

Cependant il existe de nombreux minimiseurs de la fonctionnelle F_0 prenant seulement la valeur β_1 et β_2, ainsi il apparaît nécessaire de sélectionner les minimiseurs en choisissant la bonne solution physique, de ce fait on décide d' imposer une condition sur la largeur des interfaces dans le cas des modèles à interfaces diffuses.

Cependant Van der Waals fut le premier à proposer une autre approche au problème en constatant cette non unicité des minimiseurs. Il décida plutôt de pénaliser le comportement aux interfaces en introduisant un terme de capillarité, il considéra ainsi le problème de minimisation suivant :

$$F^{\varepsilon}_{local}[\rho_\varepsilon] = \inf_{\rho \in A_{local}} \int_\Omega \big(W(\rho_\varepsilon(x)) + \frac{\varepsilon^2 \kappa}{2} |\nabla \rho_\varepsilon(x)|^2\big) dx.$$

avec :

$$A_{local} = H^1(\Omega) \cap A_0$$

et κ représente le coefficient de capillarité interne.

Modulo de bonnes estimations sur W on peut vérifier que F^{ε}_{local} possède bien des minimiseurs ρ^ε et si la suite de ces minimiseurs $(\rho^\varepsilon)_{\varepsilon>0}$ converge dans $L^1(\Omega)$ alors sa limite ρ répond bien aux deux seuls états physiques β_1 et β_2. De plus cette solution minimise l'épaisseur des interfaces (voir [58]).

Coquel, Rohde et leurs collaborateurs dans [19] présentent un autre choix de minimiseurs afin d'éviter de supposer les interfaces d'épaisseur non nulles et ainsi obtenir des solutions régulières comme c'est le cas pour les méthodes d'interfaces diffuses initiées par Van der Waals. Ils se ramènent ainsi au cas original des interfaces discontinues avec la fonctionnelle F_0. Ils choisissent alors un domaine d'admissibilité proche de A_0 :

$$A_{global} = L^2(\Omega) \cap A_0.$$

On choisit ensuite ϕ une fonction paire telle que :

$$\phi \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi(x) dx = 1, \ \text{ et } \ \phi \geq 0.$$

On pose ensuite :

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^N} \phi(\frac{x}{\varepsilon}).$$

Maintenant pour $\kappa > 0$, nous cherchons un minimiseur $\rho^\varepsilon \in A_{global}$ pour la fonctionnelle suivante :

$$F^{\varepsilon}_{global}[\rho^\varepsilon] = \inf_{\rho \in A_{local}} \int_\Omega \left(W(\rho^\varepsilon(x)) + \frac{\kappa}{4} \int_\Omega \phi_\varepsilon(x-y)(\rho^\varepsilon(y) - \rho^\varepsilon(x))^2 dy \right) dx.$$

Coquel, Rohde et leurs collaborateurs ont ainsi introduit un nouveau terme de capillarité. Ce terme non local pénalise les fortes variations de densité aux interfaces sur une échelle de largeur de l'ordre de ε. L'ensemble A_{global} peut à présent contenir des fonctions avec des sauts, ce qui signifie que les interfaces peuvent être considérées comme discontinues.

Ensuite ils dérivent l'équation en transformant l'étude variationnelle en un système équivalent d'équations. Dans la suite on fixera $\varepsilon = 1$ et on considèrera le système obtenu. Celui-ci s'écrit sous la forme :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D(u)) - \nabla((\lambda(\rho) + \mu(\rho))\operatorname{div} u) + \nabla(P(\rho)) \\ \hfill = \kappa \rho \nabla D[\rho], \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0). \end{cases} \tag{4.39}$$

où $\kappa \nabla D[\rho]$ correspond au terme de capillarité et l'on écrit $D[\rho]$ ainsi :

$$D[\rho] = \phi * \rho - \rho.$$

Les résultats mathématiques recensés sur ce système sont ceux de Rohde dans [63], ce dernier montre dans le cas de la dimension deux que le modèle est bien posé et obtient un résultat d'existence et d'unicité en temps fini pour des données initiales grandes (q_0, u_0) appartenant au produit d'espace de Sobolev suivant $H^4 \times H^4$.

5 Présentation des résultats sur Korteweg

Je vais ici présenter mes résultats concernant le système de Korteweg en commençant par traiter du cas des solutions fortes pour le système non isotherme et ensuite de celui des solutions faibles.

5.1 Solutions fortes

Dans le chapitre 2, je me suis ainsi intéressé au système (NHV) non isotherme avec des coefficients dépendant de la densité ainsi que de la température. Effectivement il est très important de considérer toutes les dépendances du système, ceci car au voisinage des interfaces, les fluctuations de densité sont très importantes et jouent considérablement sur la variation de la capillarité. D'autre part ces travaux font la connexion avec les travaux de Bresch, Desjardins et Lin dans [13]. On peut ainsi relier ces résultats avec coefficients de viscosité variable et les leurs ; effectivement en considérant le comportement asymptotique en temps des solutions de ces derniers on tombe alors dans le cadre de théorème d'existence à données petites. Ainsi les résultats suivants permettent de définir leurs solutions faibles comme de vraies solutions au sens des distributions avec des fonctions test $\varphi \in C^\infty([0,T] \times \Omega)^N$ pour des temps suffisamment grand.

Théorème 8. *Soit $N \geq 3$. Supposons que la fonction Ψ satisfasse $\Psi(\widetilde{T}) = A\widetilde{T}$ avec $A > 0$ et que tous les coefficients physiques soient des fonctions régulières dépendant seulement de la densité. Soit $\bar{\rho} > 0$ tel que :*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}) > 0, \quad \lambda(\bar{\rho}) + 2\mu(\bar{\rho}) > 0, \quad \eta(\bar{\rho}) > 0 \;\; \text{et} \;\; \partial_\rho P_0(\bar{\rho}) > 0.$$

Supposons de plus que :

$$q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}},\ u_0 \in B^{\frac{N}{2}-1},\ \mathcal{T}_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}.$$

Il existe un ε_0 dépendant seulement des coefficients physiques tel que si :

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1}} + \|\mathcal{T}_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}} \leq \varepsilon_0$$

alors (NHV) a une unique solution globale $(\rho, u, \mathcal{T})$ dans $E^{N/2}$:

$$\begin{aligned} E^{\frac{N}{2}} =& [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})] \times [C_b(\mathbb{R}_+, B^{\frac{N}{2}-1})^N \cap L^1(\mathbb{R}_+, B^{\frac{N}{2}+1})^N] \\ & \times [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})]. \end{aligned}$$

Remarque 1. *La notation $\widetilde{B}^{s,t}$ dans le précédent théorème définit un espace de Besov avec différents niveaux de régularité, s en basses fréquences et t en hautes fréquences.*

On peut généraliser le résultat précédent au cas $N = 2$, cependant il est nécessaire d'exiger plus de régularité sur les données initiales. Effectivement on est soumis à quelques difficultés techniques liées à des cas critiques concernant le paraproduit de J.-M. Bony.

Nous souhaitons ensuite montrer l'existence globale de solutions pour le système de Korteweg pour des données petites avec les conditions les plus générales possibles en ce qui concerne les coefficients physiques. Afin de contrôler les termes non linéaires de l'équation thermique, plus de régularité est exigée. Nous avons notamment besoin de contrôler la norme L^∞ de la température.

Théorème 9. *Soit $N \geq 2$. Supposons que Ψ soit une fonction dépendant de θ. Nous supposons en plus que tous les coefficients physiques sont réguliers et dépendent de ρ et θ excepté de κ qui dépend seulement de la densité. Nous choisissons $(\bar{\rho}, \bar{T})$ tels que :*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}, \bar{T}) > 0, \quad \lambda(\bar{\rho}, \bar{T}) + 2\mu(\bar{\rho}, \bar{T}) > 0, \quad \eta(\bar{\rho}, \bar{T}) > 0 \;\; et \;\; \partial_\rho P_0(\bar{\rho}, \bar{T}) > 0.$$

De plus nous supposons que :

$$q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1},\ u_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}},\ \mathcal{T}_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}.$$

Il existe un ε_1 dépendant seulement des coefficients physiques tel que si :

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\mathcal{T}_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \leq \varepsilon$$

alors (NHV) a une unique solution globale $(\rho, u, \mathcal{T})$ dans :

$$\begin{aligned} F^{\frac{N}{2}} =& [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+3})] \times [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})^N \\ & \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})^N] \times [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})]. \end{aligned}$$

Dans le théorème suivant j'établis l'existence en temps fini de solutions pour le système de Korteweg avec des données grandes. De plus on suppose la densité initiale minorée par une constante strictement positive. On montrera comme dans tous ces théorèmes la persistence en temps petit du contrôle du vide.

Théorème 10. *Soit $N \geq 3$, Ψ et les coefficients physiques sont comme dans le théorème 8. Nous supposons que $(q_0, u_0, \mathcal{T}_0) \in B^{\frac{N}{2}} \times (B^{\frac{N}{2}-1})^N \times B^{\frac{N}{2}-2}$ et que $\rho_0 \geq c$ pour un certain $c > 0$.*

Alors il existe un temps T tel que le système (NHV) a une unique solution dans F_T

$$\begin{aligned} F_T =& [\widetilde{C}_T(B^{\frac{N}{2}}) \cap L^1_T(B^{\frac{N}{2}+2})] \times [\widetilde{C}_T(B^{\frac{N}{2}-1})^N \cap L^1_T(B^{\frac{N}{2}+1})^N] \\ &\times [\widetilde{C}_T(B^{\frac{N}{2}-2}) \cap L^1_T(B^{\frac{N}{2}})] \,. \end{aligned}$$

Enfin on peut obtenir le même type de résultat dans le cas $N = 2$ modulo le choix de données initiales plus régulières et ceci pour les mêmes raisons que dans le cas du théorème 8. D'autre part le théorème 10 sera étendu au cas général concernant le choix des coefficients physiques. Comme dans le théorème 9 plus de régularité sur les données initiales sera exigé.

5.2 Solutions faibles

Dans le chapitre 3 je me suis intéressé à la recherche de solutions faibles du système de Korteweg isotherme, afin de prolonger les résultats de R. Danchin et B. Desjardins dans [23].

Dans le cas isotherme, on rappelle que le système s'écrit sous la forme suivante :

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho\, u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u(\lambda + \mu)\mathrm{div} u + \nabla(P(\rho)) = \kappa\rho\nabla\Delta\rho. \end{cases} \tag{5.40}$$

On rappelle ici la forme du tenseur de capillarité :

$$\kappa\nabla\Delta\rho = \mathrm{div}K,$$

$$\text{avec } \; K_{i,j} = \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\delta_{i,j} - \kappa\partial_i\rho\partial_j\rho.$$

Dans le théorème suivant nous généralisons un résultat de R. Danchin et B. Desjardins dans [23], en montrant que pour avoir des solutions faibles globales en dimension $N = 2$ il n'est pas nécessaire de contrôler la norme L^∞ de la densité ρ. Effectivement de la même façon nous avons cherché un gain de dérivée sur la densité ρ et plus particulièrement sur ρ^2, afin de pouvoir contrôler par compacité le terme quadratique en $\nabla\rho \otimes \nabla\rho$. Dans ce but nous nous sommes intéressés à des dérivées fractionnaires permettant ainsi d'être plus précis dans les inégalités d'énergie.

De plus il est à noter que la structure régularisante du terme de capillarité ne permet qu'un gain de dérivabilité sur $\nabla\rho^2$ alors que l'on en souhaiterait un sur $\nabla\rho$, ainsi le contrôle du vide semble primordial.

Nous considérons alors une suite approchée $(\rho_n, u_n)_{n\in\mathbb{N}}$ du système (5.40) avec des données initiales uniformément bornées dans les espaces d'énergie.

Théorème 11. *Soit $N = 2$. Nous supposons qu'il existe un $\beta > 0$ tel que :*

$$\rho_n(t,x) \geq \beta \quad \textit{pour presque tout } (t,x) \in (0,+\infty) \times \mathbb{R}^2 \quad \textit{et } \forall\, n.$$

Alors il existe $\eta > 0$ tel que si les données initiales vérifient :

$$\|\nabla \rho_0\|_{L^2} + \|\sqrt{\rho_0} u_0\|_{L^2} \leq \eta$$

il existe une solution globale (ρ, u) du système (5.40) avec $\frac{1}{\rho} \in L^\infty(L^\infty)$ et $\rho^2 \in L^2(H^{1+\frac{s}{2}})$ pour $0 < s < 2$.

Dans le théorème suivant, je me suis intéressé au cas de la dimension un, effectivement on remarque que dans nos inégalités d'énergie apparaît un terme en $\Lambda^s \rho^2$ avec $s < \frac{1}{2}$ que l'on peut contrôler en norme L^∞ en dimension un, on peut donc avoir un gain de dérivabilité sur la densité ρ sans d'hypothèses de petitesse sur les données initiales et de contrôle du vide.
On obtient ainsi l'existence de solutions faibles en temps fini en dimension un avec des données initiales grandes dans l'espace d'énergie. Cependant on impose une condition de stricte positivité sur ρ_0 et on montre qu'on peut la conserver sur un temps petit.

Théorème 12. *Soit $N = 1$. Soient (ρ_0, u_0) les données initiales du système (5.40) dans l'espace d'énergie avec $\rho_0 \geq c > 0$. Alors il existe un temps T tel que nous avons une solution (ρ, u)sur $(0,T) \times \mathbb{R}$ au sens des distributions.*

Enfin on montre aussi en dimension un, l'existence de solutions faibles globales avec des données initiales proches d'un état stable, effectivement notre gain de dérivabilité étant sur ρ^2 on a besoin de contrôler $\frac{1}{\rho}$ pour retranscrire celui-ci sur $\nabla \rho$ et ainsi traiter par compacité le terme $\nabla \rho \otimes \nabla \rho$.
La condition de petitesse permet ainsi de maîtriser le vide.

Théorème 13. *Soit $N = 1$. Soient (ρ_0, u_0) les données initiales du système (5.40) dans l'espace d'énergie et vérifiant la condition suivante :*

$$\|\nabla \rho_0\|_{L^2} + \|\sqrt{\rho_0} u_0\|_{L^2} \leq \eta$$

pour un $\eta > 0$ assez petit.
Il esiste alors une solution (ρ, u) du système (5.40) en temps global.

6 Présentation des résultats sur Rohde

Nous allons nous concentrer ici sur des résultats relatives au système de Rohde en commençant par l'existence de solutions faibles puis par celle de solutions fortes.

6.1 Solutions faibles

Dans le chapitre 4, j'étudie donc le système (4.39). Dans le cas présent, on considère les coefficients de viscosité constants et on considère le système suivant :

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div}u + \nabla(P(\rho)) = \kappa\rho\nabla D[\rho], \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0). \end{cases} \tag{6.41}$$

Je me suis d'abord concentré sur le problème de l'existence de solutions faibles avec données initiales dans l'espace d'énergie pour des pressions de la forme $P(\rho) = a\rho^\gamma$. On utilise des techniques proches de celles de P-L Lions pour le cas de Navier-Stokes compressible, cependant la présence du terme de capillarité permet de choisir $\gamma \geq 1$ pour la dimension deux et trois car il nous fournit facilement un contrôle L^2 sur la densité. De ce fait, on peut renormaliser les solutions sans contraintes sur le coefficient γ. Pour simplifier les notations nous écrivons le théorème pour le cas des dimensions $N \leq 3$ et nous supposons construite une suite (ρ_n, u_n) de solutions approchées du système (6.41) conservant les inégalités classiques d'entropie. Nous obtenons alors le résultat suivant :

Théorème 14. *Soit $N = 2, 3$ et $P(\rho) = a\rho^\gamma$ avec $\gamma \geq 1$. De plus les données initiales appartiennent à l'espace d'énergie et vérifient les conditions suivantes :*

- *$\rho_0 \geq 0$ presque partout dans $\mathbb{R}^N$, $\rho_0 \in L^1 \cap L^s$ avec $s = \max(\gamma, 2)$,*
- *$m_0 = 0$ presque partout sur $\{\rho_0 = 0\}$ et $\frac{|m_0|^2}{\rho_0} \in L^1$.*

Si en plus nous supposons que $(\rho_0)_n$ converge dans $L^1(\mathbb{R}^N)$ vers ρ_0 alors il existe une solution faible (ρ, u) du système (6.41) et nous avons en plus :

- *$\rho_n \to_n \rho$ dans $C([0,T], L^p(\mathbb{R}^N)) \cap L^r((0,T)\times\mathbb{R}^N)$ pour tout $1 \leq p < s$, $1 \leq r < q$, avec $q = s + \frac{N\gamma}{2} - 1$ si $N = 3$,*
- *$\rho_n \to_n \rho$ dans $C([0,T], L^p(\mathbb{R}^2)) \cap L^r((0,T)\times K)$ pour tout $1 \leq p < s$, $1 \leq r < q$, avec K un compact arbitraire dans $\mathbb{R}^2$ si $N = 2$.*

De plus nous avons :

- *$\rho_n u_n \to \rho u$ dans $L^p(0,T; L^r(\mathbb{R}^N))$ pour tout $1 \leq p < +\infty$ et $1 \leq r < \frac{2s}{2s+1}$,*
- *$\rho_n(u_i)_n(u_j)_n \to \rho_n u_i u_j$ dans $L^p(0,T; L^1(\mathbb{R}^N))$ pour tout $1 \leq p < +\infty$, $1 \leq i,j \leq N$ si $N = 3$.*
- *$\rho_n(u_i)_n(u_j)_n \to \rho_n u_i u_j$ dans $L^p(0,T; L^1(\Omega))$ pour tout $1 \leq p < +\infty$, $1 \leq i,j \leq N$ avec Ω un ouvert arbitraire de $\mathbb{R}^2$ si $N = 2$.*
- *$\rho_n \to \rho$ dans $C([0,T], L^p(\mathbb{R}^N)) \cap L^r((0,T)\times\mathbb{R}^N)$ pour tout $1 \leq p < s$, $1 \leq r < q$, avec $q = s + \frac{N\gamma}{2} - 1$,*

Ce théorème étant simplement un résultat de stabilité, on s'attache dans le chapitre 6 à la construction de ces solutions approchées (ρ_n, u_n), ceci requiert plusieurs niveau d'approximation du système (6.41), notamment en introduisant un terme de viscosité sur la densité. On peut ensuite étendre le résultat à des pressions plus générales et notamment au cas de lois de pression croissantes mais non convexes.

Théorème 15. *Le théorème 14 a encore lieu et de plus, $P(\rho_n)$ converge vers $P(\rho)$ dans $L^1(K\times(0,T))$ pour tout compact K.*

De plus comme le modèle est destiné à décrire le phénomène de transition de phase (où les lois de Van der Waals sont particulièrement adaptées), on considère une pression approchée de celle de Van der Waals, on définit ainsi la pression P_θ sous la forme :

$$P_\theta(\rho)=\frac{RT_*\rho}{b-\rho}-a\rho^2 \;\text{ for }\; \rho\leq b-\theta \quad\text{pour un }\; \theta>0 \;\text{ assez petit sur }\; \rho\geq b-\theta,$$

et nous étendrons la fonction P_θ à une fonction croissante sur $\rho\geq b-\theta$. On montre alors que le théorème 14 a encore lieu pour P_θ.
Enfin on finit ce chapitre par l'étude de solutions faibles proche d'un état d'équilibre, ceci dans le but d'être dans les conditions des résultats de solutions fortes précédemment rappelés, c'est à dire éviter le vide. Dans ce cas en travaillant dans des espaces d'Orlicz avec $(\rho_0-\bar{\rho})\in L^s_2$, on obtient à nouveau l'existence globale de solutions faibles.
Enfin dans le dernier chapitre on conclura cette thèse par des perspectives sur les différents sujets étudiés, en donnant quelques directions pour des travaux ultérieurs.

6.2 Solutions fortes

Dans cette section on va se concentrer sur les résultats concernant les solutions fortes. Ainsi dans le chapitre 5 je me suis intéressé au cas du système (4.39) de Navier-Stokes compressible isotherme avec terme de capillarité non local récemment remis à l'honneur par Rohde dans [63] et [19]. Je cherche à améliorer les résultats de Rohde dans [63], Weike et Xu dans [72] et j'explique le lien de ces solutions fortes avec les nouvelles inégalités d'énergie introduites par Bresch, Desjardins et Mellet, Vasseur.
Nous allons maintenant donner les principaux résultats obtenus concernant l'existence globale et l'unicité de solutions avec des données initiales proches d'un état d'équilibre et l'existence et l'unicité de solutions avec des données initiales grandes.
Dans le théorème suivant on montre l'existence et l'unicité d'une solution globale en temps pour des pressions générales avec des données initiales proches d'un état stable. De plus l'on travaille dans des espaces de Besov construits sur la norme L^2 qui sont critiques pour le scaling des équations.

Théorème 16. *Nous supposons $N\geq 3$. Soit $\bar{\rho}>0$ tel que $P^{'}(\bar{\rho})>0$. Nous supposons que les coefficients μ et λ sont infiniment réguliers et vérifient :*

$$\mu(\bar{\rho})>0 \;\; et \;\; 2\mu(\bar{\rho})+\lambda(\bar{\rho})>0.$$

Il existe deux constantes ε_0 et M telles que si $q_0\in\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}$, $u_0\in B^{\frac{N}{2}-1}$ et

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_0\|_{B^{\frac{N}{2}-1}}\leq\varepsilon_0$$

alors (4.39) a une unique solution globale (q,u) dans $E^{\frac{N}{2}}$ qui satisfait :

$$\|(q,u)\|_{E^{\frac{N}{2}}} \leq M\big(\,\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}-1}}\big).$$

où :

$$E^{\frac{N}{2}} = [C_b(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}) \cap L^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})] \times [C_b(\mathbb{R}^+,B^{\frac{N}{2}-1})^N \cap L^1(\mathbb{R}^+,B^{\frac{N}{2}+1})^N].$$

On remarque que l'on impose plus de régularité sur les données initiales en supposant $q_0 \in B^{\frac{N}{2}-1}$ afin de contrôler le comportement en basses fréquences des solutions.
Il est à noter que le théorème suivant prend en compte le fameux cas du système de Saint-Venant lorsque $\kappa = 0$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ et $P(\rho) = \rho^2$ et améliore les travaux de Weike Wang et Chao-Jiang Xu (voir [72]).
Dans le cas de la dimension deux, on montre aussi l'existence globale et l'unicité de solutions avec des données initiales proches d'un état stable. Cependant pour des raisons techniques liées au paraproduit de J-M Bony, on exige un peu plus de régularité sur les données initiales que l'on prend dans $\widetilde{B}^{0,1+\varepsilon} \times \widetilde{B}^{0,\varepsilon}$ avec $\varepsilon > 0$.

Dans le théorème suivant, on montre l'existence et l'unicité de solutions en temps fini avec des données initiales grandes. Cependant on remarque que pour pouvoir travailler dans des espaces critiques du point de vue du scaling, on est obligé d'imposer une condition de petitesse sur q_0. On s'intéresse aussi aux espaces de Besov construits à partir des espaces L^p avec $p \in [1,+\infty[$, l'intérêt étant de pouvoir avoir des données initiales appartenant à des espaces de Besov avec une régularité négative.
Effectivement dans le cas où $\kappa = 0$, cela nous permet d'être proche des nouvelles inégalités d'énergie introduites par D. Bresch et B. Desjardins dans [7] ainsi que Mellet et Vasseur dans [55]. En particulier Mellet et Vasseur dans [55] obtiennent un gain d'intégrabilité sur la vélocité $\sqrt{\rho}u$ dans l'espace $L^\infty(L^{2+\alpha})$ avec $\alpha > 0$, donc modulo le contrôle du vide on peut obtenir un gain d'intégrabilité sur u. Pour pouvoir utiliser les résultats de Mellet et Vasseur (c'est à dire contrôler la norme $B^{\frac{N}{p}-1}$ de u_0 avec $p > N$ par $\|u_0\|_{L^p}$, plus des renseignements en basses féquences sur u_0) on aurait besoin d'avoir leurs inégalités d'énergie dans des espaces d'Orlicz avec une condition à la Lions sur les données initiales $(\rho_0 - \bar{\rho}) \in L^\gamma_2$. Ainsi on ne peut appliquer tels quels leurs résultats à nos théorèmes car nous travaillons dans des espaces qui n'autorisent pas le vide. Dans le dernier chapitre sur les perspectives, j'explique l'intérêt d'essayer d'obtenir les inégalités d'énergie de Bresch, Desjardins et Mellet, Vasseur dans des espaces d'Orlicz pour éviter le vide et ainsi faire le lien avec les solutions fortes dans le cas de la dimension deux.

Théorème 17. *Soit $p \in [1,+\infty[$, $N \geq 2$ et $\bar{\rho} > 0$ telle que $P'(\bar{\rho}) > 0$. Nous supposons que les coefficients μ et λ sont infiniment réguliers et que :*

$$\mu(\bar{\rho}) > 0 \;\; et \;\; 2\mu(\bar{\rho}) + \lambda(\bar{\rho}) > 0.$$

Soit $q_0 \in B_p^{\frac{N}{p}}$ *et* $u_0 \in B_p^{\frac{N}{p}-1}$*. Supposons alors que :*

$$\|q_0\|_{B_p^{\frac{N}{p}}} \leq \varepsilon$$

pour une constante ε *suffisamment petite.*
Alors il existe un temps $T > 0$ *telle que les résultats suivants ont lieu :*

- *i)Existence : Si* $p \in [1, 2N[$ *alors le système (4.39) a une solution* (ρ, u) *dans* $F_p^{\frac{N}{p}}$ *avec :*

$$F_p^{\frac{N}{p}} = \widetilde{C}_T(B_p^{\frac{N}{p}}) \times \big(L_T^1(B_p^{\frac{N}{p}+1} \cap \widetilde{C}_T(B_p^{\frac{N}{p}-1})\big)$$

- *ii)Unicité : Si de plus* $1 \leq p \leq N$ *alors l'unicité a lieu dans* $F_p^{\frac{N}{p}}$*.*

Il est à noter que dans le cas de l'existence on peut faire varier p dans l'intervalle $[1, 2N[$ alors que dans le cas des résultats d'unicité p ne peut appartenir qu'à l'intervalle $[1, N]$ avec N qui est un cas limite.
De plus on peut minorer le temps T en fonction des données initiales, nous le préciserons dans le chapitre 5.
Dans le théorème suivant, nous considérons le cas où la densité initiale appartient à l'espace $\bar{\rho} + \widetilde{B}_p^{\frac{N}{p}, \frac{N}{p}+\varepsilon}$ et satisfait $0 < \underline{\rho} < \rho_0$. Ceci permet l'existence et l'unicité de solutions en temps fini sans imposer de conditions de petitesse sur $\|q_0\|_{B^{\frac{N}{p}}}$. En contrepartie on est obligé d'imposer plus de régularité à q_0. Ce théorème améliore les résultas de Rohde [63] dans le cas de la dimension deux ainsi que ceux de Wang et Xu sur shallow-water dans [72].

Théorème 18. *Soit* $\varepsilon > 0$*,* $N \geq 2$ *et* $p \in [1, \frac{N}{1-\varepsilon}]$*. Soit* $\bar{\rho} > 0$ *une constante telle que* $P'(\bar{\rho}) > 0$*. Nous supposons que les coefficients* μ *et* λ *sont infiniment réguliers et que :*

$$\mu(\bar{\rho}) > 0 \;\; et \;\; 2\mu(\bar{\rho}) + \lambda(\bar{\rho}) > 0.$$

Soit $q_0 \in B_p^{\frac{N}{p}, \frac{N}{p}+\varepsilon}$ *pour une constante* $\bar{\rho} > 0$ *et* $u_0 \in B^{\frac{N}{p}+\varepsilon-1}$*. Supposons qu'il existe une constante* $\underline{\rho}$ *telle que nous avons la condition de stricte positivité :*

$$0 < \underline{\rho} \leq \rho_0.$$

Alors il existe un temps $T > 0$ *tel que le système (4.39) a une unique solution* (ρ, u) *dans* $(\bar{\rho}, 0) + F^{\frac{N}{p}+\varepsilon}$*.*

De plus nous obtenons à nouveau un contrôle sur le temps T qui peut être minoré en fonction des données initiales.

Bibliographie

[1] D.M. Anderson, G.B McFadden and A.A. Wheller. Diffuse-interface methods in fluid mech. In Annal review of fluid mechanics, Vol. 30, p 139-165. Annual Reviews, Palo Alto, CA, 1998.

[2] S. Benzoni-Gavage. Linear stability of propagating phase boundaries in capillary fluids. Phys. D, 155(3-4) : 235-273, 2001.

[3] S. Benzoni-Gavage, R. Danchin, S. Descombes, and D. Jamet. Structure of Korteweg models and stability of diffuse interfaces, Interfaces Free Boundaries, 7, 371-414 (2005).

[4] S. Benzoni-Gavage, R. Danchin and S. Descombes. Well-posedness of one-dimensional Korteweg models. Electronic Journal of Differential Equations, Vol 2006(2006), N 59, pp. 1-35.

[5] S. Benzoni-Gavage, R. Danchin and S. Descombes,Well-posedness of multi-dimensional Korteweg models. à paraître dans Indiana University Mathematics Journal.

[6] D. Bresch and B. Desjardins, Sur un modèle de Saint-Venant visqueux et sa limite quasi-géostrophique. C. R. Math. Acad. Sci. Paris, 335(12) : 1079-1084, 2002.

[7] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D Viscous shallow water equations and convergence to the quasi-geostrophic model. Comm. Math. Phys., 238(1-2) : 211-223, 2003.

[8] D. Bresch and B. Desjardins. Some diffusive capillary models of Korteweg type. C. R. Math. Acad. Sci. Paris, Section Mécanique , 332(11) : 881-886, 2004.

[9] D. Bresch and B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, Journal de Mathématiques Pures et Appliqués, Volume 87, Issue 1, January 2007, Pages 57-90.

[10] D. Bresch and B. Desjardins, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models,Journal de Mathématiques Pures et Appliqués Volume 86, Issue 4, October 2006, Pages 362-368.

[11] D. Bresch and B. Desjardins, Some diffusive capillary models of Koretweg type. C. R. Math. Acad. Sci. Paris, Section Mécanique, 332(11) : 881-886, 2004.

[12] D. Bresch, B. Desjardins and D. Gérard-Varet, On compressible Navier-Stokes equations with density dependent viscosities in bouded domains, Preprint DMA 2006.

[13] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models : Korteweg,lubrication and shallow water systems. Comm. Partial Differential Equations, 28(3-4) : 843-868, 2003.

[14] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I. Interfacial free energy, J. Chem. Phys. 28 (1998) 258-267.

[15] H. J. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, preprint 2005.

[16] Y. Cho, H.J Choe and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, Journal de Mathématiques pures. 83(2), (2004), 243-275(33).

[17] H. J. Choe and H. Kim, Strong solution of the Navier-Stokes equations for isentropic compressible fluids, J. Differential Equations 190 (2003), 504-523.

[18] R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes, Compensated-compactness and Hardy spaces, J. Math. Pures Appl., 72 (1993), p 247-286.

[19] F. Coquel, D. Diehl, C. Merkle and C. Rohde, Sharp and diffuse interface methods for phase transition problems in liquid-vapour flows. Numerical Methods for Hyperbolic and Kinetic Problems, 239-270, IRMA Lect. Math. Theor. Phys,7,Eur. Math. Soc, Zürich, 2005.

[20] R. Danchin, Global Existence in Critical Spaces for compressible Navier-Stokes equations. Invent. math. 141, 579-614(2000)

[21] R. Danchin, Global Existence in Critical Spaces for Flows of Compressible Viscous and Heat-Conductive Gases, Arch.Rational Mech.Anal. 160 (2001) 1-39

[22] Danchin.R, Local Theory in critical Spaces for Compressible Viscous and Heat-Conductive Gases,Communication in Partial Differential Equations 26 (78), 1183-1233 (2001)

[23] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP,Analyse non linéaire 18, 97-133 (2001)

[24] J.M Delhaye. Jump conditions and entropy sources in two-phase systems. Local instant frmulation. Int. J. Multiphase Flow, 1, 395-409, (1974).

[25] B. Desjardins, Regularity of weak solutions of the compressible isentropic Navier-Stokes equations, Comm. P.D.E., no 5-6 (22) p. 977-1008, 1997.

[26] R. J. Diperna and P.L. Lions, On the global existence for Boltzmann equations : global existence and weak stability. Ann. Math., 130 (1989), p. 321-366.

[27] R. J. Diperna and P.L. Lions, Equations différentielles ordinaires et équations de transport avec des coefficients irréguliers. In Séminaire EDP 1988-1989, Ecole Polytechnique, Palaiseau, 1989.

[28] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working ,Arch. Rational Mech. Anal. 88(2) (1985) 95-133.

[29] E. Feireisl, Dynmamics of Viscous Compressible Fluids-Oxford Lecture Series in Mathematics and its Applications-26 (2004).

[30] E. Feireisl. Compressible Navier-Stokes equations with a non-monotone pressure law. J. Differential Equations, 184(1) : 97-108, 2002.

[31] E. Feireisl. On the motion of a viscous, compressible, and heat conducting equation. Indiana Univ. Math. J., 53(6) : 1705-1738, 2004.

[32] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. J. Math. Fluid Mech., 3 : 358-392, 2001.

[33] C. Fouillet. Généralisation à des méalnges binaires de la méthode du second gradient et application à la simulation numérique directe de l'ébullition nuclée. Thèse de doctorat, Ecole Centrale Paris. (2003).

[34] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, Archive for Rational Mechanics and Analysis 16, 269-315 (1964).

[35] M.E. Gurtin, D. Poligone and J. Vinals, Two-phases binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci.. 6(6) (1996) 815–831.

[36] H. Hattori and D.Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differential Equations 9(4) (1996) 323-342.

[37] H. Hattori and D. Li, Global Solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198(1) (1996) 84-97.

[38] David Hoff. Construction of solutions for compressible, isentropic Navier-Stokes equations with large initial data, Trans. Amer. Math. Soc. 303 (1987), p. 310-315.

[39] David Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans. Amer. Math. Soc, 303(1) : 169-181, 1987.

[40] David Hoff. Discontinuous solutions of the Navier-Stokes equations for compressible flow, Arch. Rat. Mech. Anal. 114 (1991), p. 15-46.

[41] Gui-Qiang Chen, David Hoff, and Konstantina Trivisa, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, Communications in Partial Differnetial Equations, vol. 25(11 et 12), p 2233-2257 (2000)

[42] Ishii M. (1975). Thermo-fluid dynamic theory of two-phase flow. Eyrolles, Paris.

[43] D. Jamet, Etude des potentialités de la théorie du second gradient pour la simulation numérique directe des écoulements liquide-vapeur avec changement de phase. Thèse de doctorat, Ecole Centrale Paris. (1998).

[44] D. Jamet, O. Lebaigue, N. Coutris and J.M. Delhaye, The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. J. Comput. Phys, 169(2) : 624–651, (2001).

[45] Song Jiang and Ping Zhang. Axisymetrics solutions of the 3D Navier-Stokes equations for compressible isentropic fluids. J. Math. Pures Appl. (9), 82(8) : 949-973, 2003.

[46] A. V. Kazhikov. The equation of potential flows of a compressible viscous fluid for small Reynolds numbers : existence, uniqueness and stabilization of solutions. Sibirsk. Mat. Zh., 34 (1993), no. 3, p. 70-80.

[47] A. V. Kazhikov , On the Cauchy problem for the equation of a viscous gas. (Russian), Sibirsk. Mat. Phys. 82 (1981/1982), no. 2, p ; 37-62.

[48] A. V. Kazhikov and V. V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one- dimensional equations of a viscous gas. Prikl. Mat. Meh., 41(2) : 282-291, 1977.

[49] T. Kobayashi and Y. Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbb{R}^3$. Communications in Mathematical Physics, 1999, 200, 621-660.

[50] D.J. Korteweg. Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité. Arch. Néer. Sci. Exactes Sér. II, 6 : 1-24, 1901.

[51] L. D. Landau and E. M. Lifshitz. Fluid mechanics. Translated from the Russian by J. B. Sykes and W. H. Reid. Course of Theoretical Physics, Vol. 6. Pergamon Press, London, 1959.

[52] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Compressible models, Oxford lecture series in mathematics and its application. 10 (1998).

[53] Lupis C. H. P. (1983). Chemical Thermodynamics of Materials. North-Holland, New-York.

[54] Akitaka Matsumura and Takaaki Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. Proc. Japan Acad. Ser. A Math. Sci, 55(9) : 337-342, 1979.

[55] A.Mellet and A.Vasseur, On the barotropic compressible Navier-Stokes equation, Comm. Partial Differential Equations 32 (2007), no. 1-3, 431–452.

[56] A.Mellet and A.Vasseur, Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations, SIAM J. Math. Anal. 39 (2007/08), no. 4, 1344–1365.

[57] Y.Meyer, Ondelettes et opérateurs, tome 3, Hermann, Paris, 1991

[58] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. Arch. Ration. Mech. Anal., 98 : 123-142, 1987.

[59] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, Bulletin de la Société Mathématique de France, 1962, 90, 487-497.

[60] A. Novotný and I. Straškraba. Introduction to the mathematical theory of compressible flow, Oxford lecture series in mathematics and its application. 27 (2004)

[61] A. Novotný and I. Straškraba. (2000). Stabilization of solutions to compressible Navier-Stokes equations. J. Math. Kyoto Univ., 40(2), 217-245.

[62] A. Novotný and I. Straškraba. (2001).Convergence to equilibria for compressible Navier-Stokes equations with large data. Ann. di Mat. Pura et Appl, CLXXIX(IV), 263-287.

[63] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. ZAMMZ. Angew. Math. Mech. 85(2005), no. 12, 839-857.

[64] J.S. Rowlinson, Translation of J.D Van der Waals "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". J.Statist. Phys., 20(2) : 197-244, 1979.

[65] Denis Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible., 303(13) :639-642, 1986

[66] V.V. Shelukhin, Evolution of a contact discontinuity in the baratropic flow of a viscous gas, J. Appl. Math. Mech. 47 (1983), no. 5, p. 698-700.

[67] V.V. Shelukhin, On the structure of generalized solutions of the one dimensional equations of a polytropic viscous gas, Prikl. Matem. Mekhan. 48 (1984), p. 912-920.

[68] V.A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes systems. Zap. Nauchn. Sem. LOMI, 38, (1973), p.153-231 ; J. Soviet Math. 8, (1977), p. 467-529.

[69] V.A. Solonnikov. Solvability of the initial boundary value problem for the equation of a viscous compressible fluid, J. Soviet Math., 14 (1980), p. 1120-1133.

[70] C. Truedelland W. Noll. The nonlinear field theories of mechanics. Springer-Verlag, Berlin, second edition, 1992.

[71] A. Valli, W. Zajaczkowski. Navier-Stokes equations for compressible fluids : global existence and qualitative properties of the silutions in the general case. Commun. Math. Phys., 103 (1986) no 2., p. 259-296.

[72] Weike Wang and Chao-Jiang Xu. The Cauchy problem for viscous shallow water equations. Rev. Mat. Iberoamericana 21, no. 1 (2005), 1-24.

[73] Z. Xin, Blow up of smooth solutions to the compressible Navier-Stokes equation with compact density, Communications in Pure and Applied Mathematics, 1998, 51, 229-240.

Chapitre 2

Existence of solutions for compressible fluid models of Korteweg type

Abstract

This chapter is devoted to the study of the initial boundary value problem for a general non isothermal model of capillary fluids derived by J.E Dunn and J.Serrin (1985) in [15, 25], which can be used as a phase transition model.
We distinguish two cases, when the physical coefficients depend only on the density, and the general case. In the first case we can work in critical scaling spaces, and we prove global existence of solution and uniqueness for data close to a stable equilibrium. For general data, existence and uniqueness is stated on a short time interval.
In the general case with physical coefficients depending on density and on temperature, additional regularity is required to control the temperature in L^∞ norm. We prove global existence of solution close to a stable equilibrium and local in time existence of solution with more general data. Uniqueness is also obtained.

1 Introduction

1.1 Derivation of the Korteweg model

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by van der Waals and Korteweg [21] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [20, 26].
Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. Let us consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^N$ ($N \geq 2$), entropy density e, and temperature $\theta = (\frac{\partial e}{\partial s})_\rho$. We note $w = \nabla\rho$, and we

suppose that the intern specific energy, e depends on the density ρ, on the entropy specific s, and on w. In terms of the free energy, this principle takes the form of a generalized Gibbs relation :

$$de = \widetilde{T}ds + \frac{p}{\rho^2}d\rho + \frac{1}{\rho}\phi^* \cdot dw$$

where $\widetilde{T}$ is the temperature, p the pressure, ϕ a vector column of $\mathbb{R}^N$ and ϕ^* the adjoint vector.
In the same way we can write a differential equation for the intern energy per unit volume, $E = \rho e$,

$$dE = \widetilde{T}dS + gd\rho + \phi^* \cdot dw$$

where $S = \rho s$ is the entropy per unit volume and $g = e - s\widetilde{T} + \frac{p}{\rho}$ is the chemical potential. In terms of the free energy, the Gibbs principle gives us :

$$dF = -Sd\widetilde{T} + gd\rho + \phi^* \cdot dw.$$

In the present chapter, we shall make the hypothesis that :

$$\phi = \kappa w.$$

The nonnegative coefficient κ is called the capillarity and may depend on both ρ and $\widetilde{T}$. All the thermodynamic quantities are sum of their classic version (it means independent of w) and of one term in $|w|^2$.
In this case the free energy F decomposes into a standard part F_0 and an additional term due to gradients of density :

$$F = F_0 + \frac{1}{2}\kappa|w|^2.$$

We denote $v = \frac{1}{\rho}$ the specific volume and $k = v\kappa$. Similar decompositions hold for S, p and g :

$$p = p_0 - \frac{1}{2}K_p|w|^2 \quad \text{where : } K_p = k_v^{'} \text{ and } p_0 = -(f_0)_v^{'}$$
$$g = g_0 + \frac{1}{2}K_g|w|^2 \quad \text{where : } K_e = k - \widetilde{T}k_{\widetilde{T}}^{'} \text{ and } e_0 = f_0 - \widetilde{T}(f_0)_{\widetilde{T}}^{'}.$$

The model deriving from a Cahn-Hilliard like free energy (see the pioneering work by J.E.Dunn and J.Serrin in [15] and also in [1, 8, 17]), the conservation of mass, momentum and energy read :

$$\begin{cases} \partial_t\rho + \text{div}(\rho u) = 0, \\ \partial_t(\rho u) + \text{div}(\rho u \otimes u + pI) = \text{div}(K + D) + \rho f, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \text{div}(u(\rho e + \frac{1}{2}\rho|u|^2 + p)) = \text{div}((D + K)\cdot u - Q + W) + \rho f \cdot u, \end{cases}$$

with :

$$D = (\lambda\text{div}u)I + \mu(du + \nabla u), \text{ is the diffusion tensor}$$

$$K = (\rho\,\text{div}\phi)I - \phi w^*, \text{ is the Korteweg tensor}$$

$$Q = -\eta\nabla\widetilde{T}, \text{ is the heat flux.}$$

The term

$$W = (\partial_t \rho + u^* \cdot \nabla \rho)\phi = -(\rho \mathrm{div} u)\phi$$

is the intersticial work which is needed in order to ensure the entropy balance and was first introduced by Dunn and Serrin in [15].

The coefficients (λ, μ) represent the viscosity of the fluid and may depend on both the density ρ and the temperature $\widetilde{T}$. The thermal coefficient η is a given non negative function of the temperature $\widetilde{T}$ and of the density ρ.

Differentiating formally the equation of conservation of the mass, we obtain a law of conservation for w :

$$\partial_t w + \mathrm{div}(u w^* + \rho d u) = 0 \; .$$

One may obtain an equation for e by using the mass and momentum conservation laws and the relations :

$$\mathrm{div}((-pI + K + D)u) = (\mathrm{div}(-pI + K + D)) \cdot u - p\mathrm{div}(u) + (K + D) : \nabla u \; .$$

Multiplying the momentum equation by u yields :

$$\big(\mathrm{div}(-pI + K + D)\big) \cdot u = \big(\partial_t(\rho u) + \mathrm{div}(\rho u u^*)\big) \cdot u = \partial_t(\frac{\rho |u|^2}{2}) + \mathrm{div}(\frac{\rho |u|^2}{2} u) \; .$$

We obtain then :

$$\rho(\partial_t e + u^* \cdot \nabla e) + p \, \mathrm{div} u = (K + D) : \nabla u + \mathrm{div}(W - Q) \; .$$

In substituting K, we have (with the summation convention over repeated indices) :

$$K : \nabla u = \rho \, \mathrm{div}\phi \, \mathrm{div} u - \phi_i w_j \partial_i u_j \; ,$$

while :

$$-\mathrm{div} W = \mathrm{div}((\rho \mathrm{div} u)\phi) = \rho(\mathrm{div}\phi)(\mathrm{div} u) + (w^* \cdot \phi)\mathrm{div} u + \phi_i \, \rho \, \partial^2_{j,i} u_j .$$

In using $w_j = \partial_j \rho$, we obtain :

$$\begin{aligned} K : \nabla u - \mathrm{div} W &= -(w^* \cdot \phi)\mathrm{div} u - \phi_i \partial_j(\rho \partial_i u_j) \\ &= -(w^* \cdot \phi)\mathrm{div} u - (\mathrm{div}(\rho d u)) \cdot \phi . \end{aligned}$$

Finally, the equation for e rewrites :

$$\rho(\partial_t e + u^* \cdot \nabla e) + (p + w^* \cdot \phi)\mathrm{div} u = D : \nabla u - (\mathrm{div}(\rho d u)) \cdot \phi - \mathrm{div} Q \; .$$

From now on, we shall denote : $d_t = \partial_t + u^* \cdot \nabla$.

1.2 The case of a generalized Van der Waals law

From now on, we assume that there exist two functions Π_0 and Π_1 such that :

$$p_0 = \widetilde{T}\Pi_1^{'}(v) + \Pi_0^{'}(v),$$

$$e_0 = -\Pi_0(v) + \varphi(\widetilde{T}) - \widetilde{T}\varphi^{'}(\widetilde{T}).$$

We now suppose that the coefficients λ, μ depend on the density and on the temperature, and in all the sequel the capillarity κ doesn't depend on the temperature.
Moreover we suppose that the intern specific energy is an increasing function of $\widetilde{T}$:

$$(A) \qquad \Psi^{'}(\widetilde{T}) > 0 \text{ with } \Psi(\widetilde{T}) = \varphi(\widetilde{T}) - \widetilde{T}\varphi^{'}(\widetilde{T}).$$

We then set $\theta = \Psi(\widetilde{T})$ and we search to obtain an equation on θ. In what follows, we assume that κ depends only on the specific volume.

Obtaining an equation for θ :

As :

$$e = -\Pi_0(v) + \theta + \frac{1}{2}\kappa|w|^2,$$

we thus have :

$$d_t e = -\Pi_0^{'}(v) d_t v + d_t\theta + \frac{1}{2}\kappa_v^{'}|w|^2 d_t v + \kappa w^* \cdot d_t w\ .$$

By a direct calculus we find :

$$d_t v = v\text{div}u \quad \text{and} \quad w^* \cdot d_t w = -|w|^2\text{div}u - \text{div}(\rho\, du)\cdot w\ .$$

Then we have :

$$d_t\theta = d_t e + v(p - \widetilde{T}\Pi_1^{'}(v))\text{div}u + \kappa|w|^2\text{div}u + \kappa\text{div}(\rho du)\cdot w.$$

And in using the third equation of the system, we get an equation on θ :

$$d_t\theta + v\text{div}Q + v\widetilde{T}\Pi_1^{'}(v)\text{div}u = vD : \nabla u + \text{div}(\rho du)\cdot(\kappa w - v\phi) + (\kappa|w|^2 - vw^*\cdot\phi)\text{div}u\ .$$

But as we have $\phi = \kappa w$ and $k = v\kappa$ we conclude that :

$$d_t\theta - v\text{div}(\chi\nabla\theta) + v\Psi^{-1}(\theta)\Pi_1^{'}(v)\text{div}u = vD : \nabla u$$

with : $\chi(\rho,\theta) = \eta(\rho,\widetilde{T})(\Psi^{-1})^{'}(\theta)$.

Obtaining a system for ρ, u, θ :

We obtain then for the momentum equation :

$$d_t u - \frac{\text{div}D}{\rho} + \frac{\nabla p_0}{\rho} = \frac{\text{divK}}{\rho} + \frac{1}{2}\frac{\nabla(K_p|w|^2)}{\rho}$$

where $K_p = \kappa - \rho\kappa_\rho^{'}$.
And by a calculus we check that :

$$\mathrm{div}K + \frac{1}{2}\nabla(K_p|w|^2) = \rho\nabla(\kappa\Delta\rho) + \frac{\rho}{2}\nabla(\kappa_\rho^{'}|\nabla\rho|^2).$$

Indeed we have :

$$\begin{aligned}
I =&\nabla(\rho\mathrm{div}(\kappa\nabla\rho)) - \mathrm{div}(\kappa w w^*) + \frac{1}{2}\nabla(K_p|w|^2)\\
=&[\rho\nabla(\kappa\Delta\rho) + \kappa\nabla\rho\Delta\rho + \rho\nabla(\kappa_\rho^{'}|\rho|^2) + \kappa_\rho^{'}|\nabla\rho|^2\nabla\rho] - [\kappa\mathrm{div}(ww^*) + \kappa_\rho^{'}\, w\cdot\nabla\rho\, w],\\
&+[\frac{\kappa}{2}\nabla|w|^2 + \frac{\kappa_\rho^{'}}{2}\nabla|w|^2\nabla\rho - \frac{1}{2}\nabla(\rho\kappa_\rho^{'}|w|^2)],\\
=&[\rho\nabla(\kappa\Delta\rho) + \kappa\nabla\rho\Delta\rho + \frac{\rho}{2}\nabla(\kappa_\rho^{'}|\rho|^2)] - [\kappa\mathrm{div}(ww^*)] + [\frac{\kappa}{2}\nabla|w|^2],\\
=&[\rho\nabla(\kappa\Delta\rho) + \kappa\nabla\rho\Delta\rho + \frac{\rho}{2}\nabla(\kappa_\rho^{'}|\rho|^2)] - [\kappa w\mathrm{div}w + \frac{\kappa}{2}\nabla|w|^2] + [\frac{\kappa}{2}\nabla|w|^2],\\
=&\rho\nabla(\kappa\Delta\rho) + \frac{\rho}{2}\nabla(\kappa_\rho^{'}|\nabla\rho|^2)\ .
\end{aligned}$$

Finally we have obtained the following system :

$$(NHV)\qquad \begin{cases}
\partial_t\rho + \mathrm{div}(\rho u) = 0,\\
\partial_t u + u\cdot\nabla u - \dfrac{\mathrm{div}D}{\rho} - \nabla(\kappa\Delta\rho) + \dfrac{\nabla(P_0(\rho) + \Psi^{-1}(\theta)P_1(\rho))}{\rho}\\
\qquad\qquad\qquad\qquad = \nabla(\dfrac{\kappa_\rho^{'}}{2}|\nabla\rho|^2),\\
\partial_t\theta + u\cdot\nabla\theta - \dfrac{\mathrm{div}(\chi\nabla\theta)}{\rho} + \Psi^{-1}(\theta)\dfrac{P_1(\rho)}{\rho}\mathrm{div}(u) = \dfrac{D:\nabla u}{\rho},
\end{cases}$$

where : $P_0 = \Pi_0^{'}$, $P_1 = \Pi_1^{'}$ and $\widetilde{T} = \Psi^{-1}(\theta)$.
We supplement (NHV) with initial conditions :

$$\rho_{t=0} = \rho_0 \geq 0\ \ u_{t=0} = u_0,\ \text{and}\ \ \theta_{t=0} = \theta_0.$$

1.3 Classical a priori-estimates

Before getting into the heart of mathematical results, let us derive the physical energy bounds of the (NHV) system when κ is a constant and where the pressure just depends on the density to simplify. Let $\bar{\rho} > 0$ be a constant reference density, and let Π be defined by :

$$\Pi(s) = s\Big(\int_{\bar{\rho}}^{s}\frac{P_0(z)}{z^2}dz - \frac{P_0(\bar{\rho})}{\bar{\rho}}\Big)$$

so that $P_0(s) = s\Pi^{'}(s) - \Pi(s)$, $\Pi^{'}(\bar{\rho}) = 0$ and :

$$\partial_t\Pi(\rho) + \mathrm{div}(u\Pi(\rho)) + P_0(\rho)\mathrm{div}(u) = 0\ \ \text{in}\ \ \mathcal{D}^{'}((0,T)\times\mathbb{R}^N).$$

Notice that Π is convex as far as P is non decreasing (since $P_0'(s) = s\Pi''(s)$), which is the case for γ-type pressure laws or for Van der Waals law above the critical temperature.
Multiplying the equation of momentum conservation in the system (NHV) by ρu and integrating by parts over $\mathbb{R}^N$, we obtain the following estimate :

$$\begin{aligned}\int_{\mathbb{R}^N}\big(\frac{1}{2}\rho|u|^2+\rho\theta+(\Pi(\rho)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho|^2\big)(t)dx+2\int_0^t\int_{\mathbb{R}^N}\big(2\mu D(u):D(u)\\ +(\lambda+\mu)|{\rm div}u|^2\big)dx\leq\int_{\mathbb{R}^N}\big(\frac{|m_0|^2}{2\rho}+\rho_0\theta_0+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho_0|^2\big)dx.\end{aligned}$$

It follows that assuming that the initial total energy is finite :

$$\epsilon_0=\int_{\mathbb{R}^N}\big(\frac{|m_0|^2}{2\rho}+\rho_0\theta_0+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho_0|^2\big)dx<+\infty\,,$$

then we have the a priori bounds :

$$\Pi(\rho)-\Pi(\bar{\rho}),\;\;\rho|u|^2,\;\;\text{and}\;\;\rho\,\theta\in L^\infty(0,\infty,L^1(\mathbb{R}^N))$$

$$\nabla\rho\in L^\infty(0,\infty,L^2(\mathbb{R}^N))^N,\;\;\text{and}\;\;\nabla u\in L^2(0,\infty,\mathbb{R}^N)^{N^2}.$$

2 Mathematical results

We wish to prove existence and uniqueness results for (NHV) in functions spaces very close to energy spaces. In the non isothermal non capillary case and $P(\rho) = a\rho^\gamma$, with $a > 0$ and $\gamma > 1$, P-L. Lions in [22] proved the global existence of variational solutions (ρ, u, θ) to (NHV) with $\kappa = 0$ for $\gamma > \frac{N}{2}$ if $N \geq 4$, $\gamma \geq \frac{3N}{N+2}$ if $N = 2, 3$ and initial data (ρ_0, m_0) such that :

$$\Pi(\rho_0)-\Pi(\bar{\rho}),\;\;\frac{|m_0|^2}{\rho_0}\in L^1(\mathbb{R}^N),\;\;\text{and}\;\;\rho_0\theta_0\in L^1(\mathbb{R}^N).$$

These solutions are weak solutions in the classical sense for the equation of mass conservation and for the equation of the momentum.
On the other hand, the weak solution satisfies only an inequality for the thermal energy equation.
Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties of the density ρ in L^p_{loc} spaces required to pass to the limit in the pressure term $P(\rho) = a\rho^\gamma$.
Let us mention that Feireisl in [16] generalized the result to $\gamma > \frac{N}{2}$ in establishing that we can obtain renormalized solution without imposing that $\rho \in L^2_{loc}$, for this he introduces the concept of oscillation defect measure evaluating the lost of compactness.
We can finally cite a very interesting result from Bresch-Desjardins in [5],[6] where they show the existence of global weak solution for (NHV) with $\kappa = 0$ in choosing specific type of viscosity where μ and λ are linked. It allows them to get good estimate on the density in using energy inequality and to can treat by compactness all the delicate terms. This result is very new because the energy equation is verified really in distribution sense. In

[23], Mellet and Vasseur improve the results of Bresch,Desjardins in generalize to some coefficient μ and λ admitting the vacuum in the case of Navier-Stokes isothermal, they use essentially a gain of integrability on the velocity.

In the case $\kappa > 0$, we remark then that the density belongs to $L^\infty(0,\infty,\dot{H}^1(\mathbb{R}^N))$. Hence, in contrast to the non capillary case one can easily pass to the limit in the pressure term. However let us emphasize at this point that the above a priori bounds do not provide any L^∞ control on the density from below or from above. Indeed, even in dimension $N = 2$, H^1 functions are not necessarily locally bounded. Thus, vacuum patches are likely to form in the fluid in spite of the presence of capillary forces, which are expected to smooth out the density. Danchin and Desjardins show in [14] that the isothermal model has weak solutions if there exists c_1 and M_1 such that :

$$c_1 \leq |\rho| \leq M_1 \;\text{ and }\; |\rho - 1| << 1.$$

The vacuum is one of the main difficulties to get weak solutions, and the problem remains open.
In the isothermal capillary case with specific type of viscosity and capillarity $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$, Bresch, Desjardins and Lin in [7] obtain the global existence of weak solutions without smallness assumption on the data. We can precise the space of test functions used depends on the solution itself which are on the form $\rho\phi$ with $\phi \in C_0^\infty(\mathbb{R}^N)$. The specificity of the viscosity allows to get a gain of one derivative on the density : $\rho \in L^2(H^2)$.

Existence of strong solution with κ, μ and λ constant is known since the work by Hattori an Li in [18], [19] in the whole space $\mathbb{R}^N$. In [14], Danchin and Desjardins study the well-posedness of the problem for the isothermal case with constant coefficients in critical Besov spaces.
Here we want to investigate the well-posedness of the full non isothermal problem in critical spaces, that is, in spaces which are invariant by the scaling of Korteweg's system. Recall that such an approach is now classical for incompressible Navier-Stokes equation and yields local well-posedness (or global well-posedness for small data) in spaces with minimal regularity.
Let us explain precisely the scaling of Korteweg's system. We can easily check that, if (ρ, u, θ) solves (NHV), so does $(\rho_\lambda, u_\lambda, \theta_\lambda)$, where :

$$\rho_\lambda(t,x) = \rho(\lambda^2 t, \lambda x)\,,\; u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x) \;\text{ and }\; \theta_\lambda(t,x) = \lambda^2\theta(\lambda^2 t, \lambda x)$$

provided the pressure laws P_0, P_1 have been changed into $\lambda^2 P_0, \lambda^2 P_1$.

Definition 2.2. *We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation :*

$$(\rho, u, \theta) \longrightarrow (\rho_\lambda, u_\lambda, \theta_\lambda)$$

(up to a constant independent of λ).

This suggests us to choose initial data (ρ_0, u_0, θ_0) in spaces whose norm is invariant for all $\lambda > 0$ by $(\rho_0, u_0, \theta_0) \longrightarrow (\rho_0(\lambda\cdot), \lambda u_0(\lambda\cdot), \lambda^2\theta_0(\lambda\cdot))$.

A natural candidate is the homogeneous Sobolev space $\dot{H}^{N/2} \times (\dot{H}^{N/2-1})^N \times \dot{H}^{N/2-2}$, but since $\dot{H}^{N/2}$ is not included in L^∞, we cannot expect to get L^∞ control on the density when $\rho_0 \in \dot{H}^{N/2}$. The same problem occurs in the equation for the temperature when dealing with the non linear term involving $\Psi^{-1}(\theta)$.
This is the reason why, instead of the classical homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$, we will consider homogeneous Besov spaces with the same derivative index $B^s = \dot{B}^s_{2,1}(\mathbb{R}^N)$ (for the corresponding definition we refer to section 4).
One of the nice property of B^s spaces for critical exponent s is that $B^{N/2}$ is an algebra embedded in L^∞. This allows to control the density from below and from above, without requiring more regularity on derivatives of ρ. For similar reasons, we shall take θ_0 in $B^{\frac{N}{2}}$ in the general case where appear non-linear terms in function of the temperature.
Since a global in time approach does not seem to be accessible for general data, we will mainly consider the global well-posedness problem for initial data close enough to stable equilibria (Section 5). This motivates the following definition :

Definition 2.3. *Let $\bar{\rho} > 0$, $\bar{\theta} > 0$. We will note in the sequel :*

$$q = \frac{\rho - \bar{\rho}}{\bar{\rho}} \quad \text{and} \quad \mathcal{T} = \theta - \bar{\theta}.$$

One can now state the main results of the paper.
The first three theorems concern the global existence and uniqueness of solution to the Korteweg's system with *small* initial data. In particulary the first two results concern Korteweg's system with coefficients depending only on the density and where the intern specific energy is a linear function of the temperature.

Theorem 2.1. *Let $N \geq 3$. Assume that the function Ψ defined in (A) satisfies $\Psi(\widetilde{T}) = A\widetilde{T}$ with $A > 0$ and that all the physical coefficients are smooth functions depending only on the density. Let $\bar{\rho} > 0$ be such that :*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}) > 0, \quad \lambda(\bar{\rho}) + 2\mu(\bar{\rho}) > 0, \quad \eta(\bar{\rho}) > 0 \ \text{ and } \ \partial_\rho P_0(\bar{\rho}) > 0.$$

Moreover suppose that :

$$q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}, \ u_0 \in B^{\frac{N}{2}-1}, \ \mathcal{T}_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}.$$

There exists an ε_0 depending only on the physical coefficients (that we will precise later) such that if :

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1}} + \|\mathcal{T}_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2}} \leq \varepsilon_0$$

then (NHV) has a unique global solution $(\rho, u, \mathcal{T})$ *in* $E^{N/2}$ *where* E^s *is defined by :*

$$E^s = [C_b(\mathbb{R}_+, \widetilde{B}^{s-1,s}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{s+1,s+2})] \times [C_b(\mathbb{R}_+, B^{s-1})^N \cap L^1(\mathbb{R}_+, B^{s+1})^N] \times [C_b(\mathbb{R}_+, \widetilde{B}^{s-1,s-2}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{s+1,s})].$$

Remark 1. *Above,* $\widetilde{B}^{s,t}$ *stands for a Besov space with regularity* B^s *in low frequencies and* B^t *in high frequencies (see definition 3.4).*

The case $N = 2$ requires more regular initial data because of technical problems involving some nonlinear terms in the temperature equation.

Theorem 2.2. *Let* $N = 2$. *Under the assumption of the theorem 2.1 for* Ψ *and the physical coefficients, let* $\varepsilon' > 0$ *and suppose that :*

$$q_0 \in \widetilde{B}^{0,1+\varepsilon'},\ u_0 \in \widetilde{B}^{0,\varepsilon'},\ \mathcal{T}_0 \in \widetilde{B}^{0,-1+\varepsilon'}.$$

There exists an ε_0 *depending only on the physical coefficients such that if :*

$$\|q_0\|_{\widetilde{B}^{0,1+\varepsilon'}} + \|u_0\|_{\widetilde{B}^{0,\varepsilon'}} + \|\mathcal{T}_0\|_{\widetilde{B}^{0,-1+\varepsilon'}} \leq \varepsilon_0$$

then (NHV) has a unique global solution $(\rho, u, \mathcal{T})$ *in the space :*

$$E' = [C_b(\mathbb{R}_+, \widetilde{B}^{0,1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,3+\varepsilon'})] \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,\varepsilon'})^2 \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,2+\varepsilon'})^2] \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,-1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,1+\varepsilon'})].$$

In the following theorem we are interested by showing the global existence of solution for Korteweg's system with general conditions and small initial data. In order to control the non linear terms in temperature more regularity is required. That's why we want control the temperature in norm L^∞.

Theorem 2.3. *Let* $N \geq 2$. *Assume that* Ψ *be a regular function depending on* θ. *Assume that all the coefficients are smooth functions of* ρ *and* θ *except* κ *which depends only on the density. Take* $(\bar{\rho}, \bar{T})$ *such that :*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}, \bar{T}) > 0, \quad \lambda(\bar{\rho}, \bar{T}) + 2\mu(\bar{\rho}, \bar{T}) > 0, \quad \eta(\bar{\rho}, \bar{T}) > 0 \ \text{ and } \ \partial_\rho P_0(\bar{\rho}, \bar{T}) > 0.$$

Moreover suppose that :

$$q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1},\ u_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}},\ \mathcal{T}_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}.$$

There exists an ε_1 *depending only on the physical coefficients such that if :*

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1}} + \|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|\mathcal{T}_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} \leq \varepsilon$$

then (NHV) has a unique global solution $(\rho, u, \mathcal{T})$ *in :*

$$F^{\frac{N}{2}} = [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+3})] \times [C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})^N \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})^N][C_b(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})].$$

In the previous theorem we can observe for the case $N=2$ that the initial data are very close from the energy space of Bresch, Desjardins and Lin in [7].
In the following three theorems we are interested by the existence and uniqueness of solution in finite time for *large* data. We distinguish always the differents cases $N\geq 3$ and $N=2$ if the coefficients depend only on ρ, and the case where the coefficients depend also on $\tilde{T}$.

Theorem 2.4. *Let $N\geq 3$, and Ψ and the physical coefficients be as in theorem 2.1. We suppose that $(q_0,u_0,\mathcal{T}_0)\in B^{\frac{N}{2}}\times(B^{\frac{N}{2}-1})^N\times B^{\frac{N}{2}-2}$ and that $\rho_0\geq c$ for some $c>0$.*
Then there exists a time T such that system (NHV) has a unique solution in F_T

$$
\begin{aligned}
F_T=&[\widetilde{C}_T(B^{\frac{N}{2}})\cap L^1_T(B^{\frac{N}{2}+2})]\times[\widetilde{C}_T(B^{\frac{N}{2}-1})^N\cap L^1_T(B^{\frac{N}{2}+1})^N]\\
&\times[\widetilde{C}_T(B^{\frac{N}{2}-2})\cap L^1_T(B^{\frac{N}{2}})]\,.
\end{aligned}
$$

For the same reasons as previously in the case $N=2$ we can not reach the critical level of regularity.

Theorem 2.5. *Let $N=2$ and $\varepsilon'>0$. Under the assumptions of theorem 2.1 for Ψ and the physical coefficients we suppose that $(q_0,u_0,\mathcal{T}_0)\in\widetilde{B}^{1,1+\varepsilon'}\times(\widetilde{B}^{0,\varepsilon'})^2\times\widetilde{B}^{-1,-1+\varepsilon'}$ and $\rho_0\geq c$ for some $c>0$.*
Then there exists a time T such that the system has a unique solution in $F_T(2)$ with :

$$
\begin{aligned}
F_T(2)=&[\widetilde{C}_T(\widetilde{B}^{1,1+\varepsilon'})\cap L^1_T(\widetilde{B}^{3,3+\varepsilon'})]\times[\widetilde{C}_T(\widetilde{B}^{0,\varepsilon'})^2\cap L^1_T(\widetilde{B}^{2,2+\varepsilon'})^2]\\
&\times[\widetilde{C}_T(\widetilde{B}^{-1,-1+\varepsilon'})\cap L^1_T(\widetilde{B}^{1,1+\varepsilon'})]\,.
\end{aligned}
$$

In the last theorem we see the general system without conditions, and like previously we need more regular initial data.

Theorem 2.6. *Under the hypotheses of theorem 2.3 we suppose that :*

$$(q_0,u_0,\mathcal{T}_0)\in\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1}\times(B^{\frac{N}{2}})^N\times B^{\frac{N}{2}}\quad\textit{and}\ \ \rho_0\geq c\ \ \textit{for some}\ \ c>0.$$

Then there exists a time T such that the system has a unique solution in :

$$
\begin{aligned}
F'_T=&[\widetilde{C}_T(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})\cap L^1_T(\widetilde{B}^{\frac{N}{2}+2,\frac{N}{2}+3})]\times[\widetilde{C}_T(B^{\frac{N}{2}})^N\cap L^1_T(B^{\frac{N}{2}+2})^N]\\
&\times[\widetilde{C}_T(B^{\frac{N}{2}})\cap L^1_T(B^{\frac{N}{2}+2})]\,.
\end{aligned}
$$

This chapter is structured in the following way, first of all we recall in the section 3 some definitions on Besov spaces and some useful theorem concerning Besov spaces. Next we will concentrate in the section 4 on the global existence and uniqueness of solution for our system (NHV) with small initial data. In subsection 4.1 we will give some necessary conditions to get the stability of the linear part associated to the system (NHV). In subsection 4.2 we will study the case where the specific intern energy is linear and where the physical coefficients are independent of the temperature. In our proof we will distinguish the case $N\geq 3$ and the case $N=2$ for some technical reasons. In the section 5 we will examine the local existence and uniqueness of solution with general initial data. For the same reasons as section 4 we will distinguish the cases in function of the behavior of the coefficients and of the intern specific energy.

3 Littlewood-Paley theory and Besov spaces

3.1 Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables.
We can use for instance any $\varphi \in C^\infty(\mathbb{R}^N)$, supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that :

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \ \text{ if } \ \xi \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by :

$$\Delta_l u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy \ \text{ and } \ S_l u = \sum_{k \leq l-1} \Delta_k u\,.$$

Formally, one can write that :

$$u = \sum_{k \in \mathbb{Z}} \Delta_k u\,.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^N)$ for two reasons :

1. The right hand-side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^N)$.
2. Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^N)$ (consider the case of the polynomials).

However, this equality holds true modulo polynomials hence homogeneous Besov spaces will be defined modulo the polynomials, according to [4].

3.2 Homogeneous Besov spaces and first properties

Definition 3.4. *For $s \in \mathbb{R}$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set :*

$$\|u\|_{B^s} = (\sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^2}).$$

A difficulty due to the choice of homogeneous spaces arises at this point. Indeed, $\|.\|_{B^s}$ cannot be a norm on $\{u \in \mathcal{S}'(\mathbb{R}^N), \|u\|_{B^s} < +\infty\}$ because $\|u\|_{B^s} = 0$ means that u is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces, see [4].

Definition 3.5. *Let $s \in \mathbb{R}$.*
Denote $m = [s - \frac{N}{2}]$ if $s - \frac{N}{2} \notin \mathbb{Z}$ and $m = s - \frac{N}{2} - 1$ otherwise.

- *If $m < 0$, then we define B^s as :*

$$B^s = \Big\{ u \in \mathcal{S}'(\mathbb{R}^N) \,/ \|u\|_{B^s} < \infty \ \text{and} \ u = \sum_{l \in \mathbb{Z}} \Delta_l u \ \text{in} \ \mathcal{S}'(\mathbb{R}^N) \Big\}.$$

– *If $m \geq 0$, we denote by $\mathcal{P}_m[\mathbb{R}^N]$ the set of polynomials of degree less than or equal to m and we set :*

$$B^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \, / \|u\|_{B^s} < \infty \text{ and } u = \sum_{l \in \mathbb{Z}} \Delta_l u \text{ in } \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \right\}.$$

Proposition 3.1. *The following properties hold :*

1. *Density : If $|s| \leq \frac{N}{2}$, then C_0^∞ is dense in B^s.*
2. *Derivatives : There exists a universal constant C such that :*

$$C^{-1}\|u\|_{B^s} \leq \|\nabla u\|_{B^{s-1}} \leq C\|u\|_{B^s}.$$

3. *Algebraic properties : For $s > 0$, $B^s \cap L^\infty$ is an algebra.*
4. *Interpolation : $(B^{s_1}, B^{s_2})_{\theta,1} = B^{\theta s_2 + (1-\theta)s_1}$.*

3.3 Hybrid Besov spaces and Chemin-Lerner spaces

Hybrid Besov spaces are functional spaces where regularity assumptions are different in low frequency and high frequency, see [12].
They may be defined as follows :

Definition 3.6. *Let $s,\, t \in \mathbb{R}$. We set :*

$$\|u\|_{\widetilde{B}^{s,t}} = \sum_{q \leq 0} 2^{qs}\|\Delta_q u\|_{L^2} + \sum_{q > 0} 2^{qt}\|\Delta_q u\|_{L^2}\,.$$

Let $m = -[\frac{N}{2} + 1 - s]$, we then define :
– *$\widetilde{B}^{s,t} = \{u \in \mathcal{S}'(\mathbb{R}^N) \, / \|u\|_{\widetilde{B}^{s,t}} < +\infty\}$, if $m < 0$*
– *$\widetilde{B}^{s,t} = \{u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \, / \|u\|_{\widetilde{B}^{s,t}} < +\infty\}$ if $m \geq 0$.*

Let us now give some properties of these hybrid spaces and some results on how they behave with respect to the product. The following results come directly from the paradifferential calculus.

Proposition 3.2. *We recall some inclusion :*
– *We have $\widetilde{B}^{s,s} = B^s$.*
– *If $s \leq t$ then $\widetilde{B}^{s,t} = B^s \cap B^t$, if $s > t$ then $\widetilde{B}^{s,t} = B^s + B^t$.*
– *If $s_1 \leq s_2$ and $t_1 \geq t_2$ then $\widetilde{B}^{s_1,t_1} \hookrightarrow \widetilde{B}^{s_2,t_2}$.*

Proposition 3.3. *For all $s,\, t > 0$, we have :*

$$\|uv\|_{\widetilde{B}^{s,t}} \leq C(\|u\|_{L^\infty}\|v\|_{\widetilde{B}^{s,t}} + \|v\|_{L^\infty}\|u\|_{\widetilde{B}^{s,t}})\,.$$

For all $s_1,\, s_2,\, t_1,\, t_2 \leq \frac{N}{2}$ such that $\min(s_1+s_2,\, t_1+t_2) > 0$ we have :

$$\|uv\|_{\widetilde{B}^{s_1+t_1-\frac{N}{2},s_2+t_2-\frac{N}{2}}} \leq C\|u\|_{\widetilde{B}^{s_1,t_1}}\|v\|_{\widetilde{B}^{s_2,t_2}}\,.$$

For a proof of this proposition see [12]. We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces :

$$L^\rho_T(B^s) := L^\rho(0,T,B^s).$$

Definition 3.7. *Let $\rho \in [1,+\infty[$, $T \in [1,+\infty]$ and $s \in \mathbb{R}$. We then denote :*

$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} = \sum_{l\leq 0} 2^{ls_1}\big(\int_0^T \|\Delta_l u(t)\|^\rho_{L^2}dt\big)^{1/\rho} + \sum_{l>0} 2^{ls_2}\big(\int_0^T \|\Delta_l u(t)\|^\rho_{L^2}dt\big)^{1/\rho}.$$

And we have in the case $\rho = \infty$:

$$\|u\|_{\widetilde{L}^\infty_T(\widetilde{B}^{s_1,s_2})} = \sum_{l\leq 0} 2^{ls_1}\|\Delta_l u\|_{L^\infty(L^2)} + \sum_{l>0} 2^{ls_2}\|\Delta_l u\|_{L^\infty(L^2)}.$$

We note that thanks to Minkowsky inequality we have :

$$\|u\|_{L^\rho_T(\widetilde{B}^{s_1,s_2})} \leq \|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} \quad \text{and} \quad \|u\|_{L^1_T(\widetilde{B}^{s_1,s_2})} = \|u\|_{\widetilde{L}^1_T(\widetilde{B}^{s_1,s_2})}.$$

From now on, we will denote :

$$\|u\|^-_{\widetilde{L}^\rho_T(B^{s_1})} = \sum_{l\leq 0} 2^{ls_1}\big(\int_0^T \|\Delta_l u(t)\|^\rho_{L^p}dt\big)^{1/\rho}$$

$$\|u\|^+_{\widetilde{L}^\rho_T(B^{s_2})} = \sum_{l>0} 2^{ls_2}\big(\int_0^T \|\Delta_l u(t)\|^\rho_{L^p}dt\big)^{1/\rho}.$$

Hence :

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1})} = \|u\|^-_{\widetilde{L}^\rho_T(B^{s_1})} + \|u\|^+_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}$$

We then define the space :

$$\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}) = \{u \in L^\rho_T(\widetilde{B}^{s_1,s_2}) / \|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})} < \infty\}.$$

We denote moreover by $\widetilde{C}_T(\widetilde{B}^{s_1,s_2})$ the set of those functions of $\widetilde{L}^\infty_T(\widetilde{B}^{s_1,s_2})$ which are continuous from $[0,T]$ to $\widetilde{B}^{s_1,s_2}$. In the sequel we are going to give some properties of this spaces concerning the interpolation and their relationship with the heat equation.

Proposition 3.4. *Let s, t, s_1, $s_2 \in \mathbb{R}$, ρ, ρ_1, $\rho_2 \in [1,+\infty]$. We have :*

1. *Interpolation :*

$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s,t})} \leq \|u\|^\theta_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s_1,t_1})}\|u\|^{1-\theta}_{\widetilde{L}^{\rho_2}_T(\widetilde{B}^{s_2,t_2})} \quad \text{with } \theta \in [0,1] \text{ and } \frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2},$$

$$s = \theta s_1 + (1-\theta)s_2,\ t = \theta t_1 + (1-\theta)t_2.$$

2. *Embedding :*

$$\widetilde{L}^\rho_T(\widetilde{B}^{s,t}) \hookrightarrow L^\rho_T(C_0) \text{ and } \widetilde{C}_T(B^{\frac{N}{2}}) \hookrightarrow C([0,T]\times\mathbb{R}^N).$$

The $\widetilde{L}^{\rho}_T(B^s_p)$ spaces suit particulary well to the study of smoothing properties of the heat equation. In [9], J-Y. Chemin proved the following proposition :

Proposition 3.5. *Let $p \in [1,+\infty]$ and $1 \leq \rho_2 \leq \rho_1 \leq +\infty$. Let u be a solution of :*

$$\begin{cases} \partial_t u - \mu \Delta u = f \\ u_{t=0} = u_0 \, . \end{cases}$$

Then there exists $C > 0$ depending only on N, μ, ρ_1 and ρ_2 such that :

$$\|u\|_{\widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1})} \leq C\|u_0\|_{B^s} + C\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2})} \, .$$

To finish with, we explain how the product of functions behaves in the spaces of Chemin-Lerner. We have the following properties :

Proposition 3.6.
Let $s > 0$, $t > 0$, $1/\rho_2 + 1/\rho_3 = 1/\rho_1 + 1/\rho_4 = 1/\rho \leq 1$, $u \in \widetilde{L}^{\rho_3}_T(\widetilde{B}^{s,t}) \cap \widetilde{L}^{\rho_1}_T(L^\infty)$ and $v \in \widetilde{L}^{\rho_4}_T(\widetilde{B}^{s,t}) \cap \widetilde{L}^{\rho_2}_T(L^\infty)$.
Then $uv \in \widetilde{L}^{\rho}_T(\widetilde{B}^{s,t})$ and we have :

$$\|uv\|_{\widetilde{L}^{\rho}_T(\widetilde{B}^{s,t})} \leq C\|u\|_{\widetilde{L}^{\rho_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{\rho_4}_T(\widetilde{B}^{s,t})} + \|v\|_{\widetilde{L}^{\rho_2}_T(L^\infty)}\|u\|_{\widetilde{L}^{\rho_3}_T(\widetilde{B}^{s,t})}.$$

If $s_1, s_2, t_1, t_2 \leq \frac{N}{2}$, $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho} \leq 1$, $u \in \widetilde{L}^{\rho_1}_T(\widetilde{B}^{s_1,t_1})$ and $v \in \widetilde{L}^{\rho_2}_T(\widetilde{B}^{s_2,t_2})$ then $uv \in \widetilde{L}^{\rho}_T(\widetilde{B}^{s_1+s_2-\frac{N}{2},t_1+t_2-\frac{N}{2}})$ and :

$$\|uv\|_{\widetilde{L}^{\rho}_T(\widetilde{B}_2^{s_1+s_2-\frac{N}{2},t_1+t_2-\frac{N}{2}}),} \leq C\|u\|_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s_1,t_1})}\|v\|_{\widetilde{L}^{\rho_2}_T(\widetilde{B}^{s_2,t_2})} \, .$$

For a proof of this proposition see [12]. Finally we need an estimate on the composition of functions in the spaces $\widetilde{L}^{\rho}_T(\widetilde{B}^s_p)$ (see the proof in the appendix).

Proposition 3.7. *Let $s > 0$, $p \in [1,+\infty]$ and $u_1, u_2 \cdots , u_d \in \widetilde{L}^{\rho}_T(B^s_p) \cap L^\infty_T(L^\infty)$.*

(i) Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0) = 0$. Then $F(u_1, u_2, \cdots , u_d) \in \widetilde{L}^{\rho}_T(B^s_p)$.
More precisely, there exists a constant C depending only on s, p, N and F such that :

$$\|F(u_1, u_2, \cdots , u_d)\|_{\widetilde{L}^{\rho}_T(B^s_p)} \leq C(\|u_1\|_{L^\infty_T(L^\infty)}, \|u_2\|_{L^\infty_T(L^\infty)}, \cdots , \|u_d\|_{L^\infty_T(L^\infty)})$$
$$(\|u_1\|_{\widetilde{L}^{\rho}_T(B^s_p)} + \cdots + \|u_d\|_{\widetilde{L}^{\rho}_T(B^s_p)}).$$

(ii) Let $u \in \widetilde{L}^{\rho}_T(\widetilde{B}^{s_1,s_2})$, s_1, $s_2 > 0$ then we have $F(u) \in \widetilde{L}^{\rho}_T(\widetilde{B}^{s_1,s_2})$ and

$$\|F(u)\|_{\widetilde{L}^{\rho}_T(\widetilde{B}^{s_1,s_2})} \leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^{\rho}_T(\widetilde{B}^{s_1,s_2})}.$$

(iii) If $v,\, u\in \widetilde{L}^{\rho}_{T}(B^{s}_{p})\cap L^{\infty}_{T}(L^{\infty})$ and $G\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^{N})$ then $G(u)-G(v)$ belongs to $\widetilde{L}^{\rho}_{T}(B^{s}_{p})$ and there exists a constant C depending only of s,p,N and G such that :

$$\begin{aligned}\|G(u)-G(v)\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p})}\leq C(\|u\|_{L^{\infty}_{T}(L^{\infty})},\|v\|_{L^{\infty}_{T}(L^{\infty})})\big(\|v-u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p})}(1+\|u\|_{L^{\infty}_{T}(L^{\infty})}\\ +\|v\|_{L^{\infty}_{T}(L^{\infty})})+\|v-u\|_{L^{\infty}_{T}(L^{\infty})}(\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p})}+\|v\|_{\widetilde{L}^{\rho}_{T}(B^{s}_{p})})\big).\end{aligned}$$

(iv) If $v,\, u\in \widetilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p})\cap L^{\infty}_{T}(L^{\infty})$ and $G\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^{N})$ then $G(u)-G(v)$ belongs to $\widetilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p})$ and it exists a constant C depending only of s,p,N and G such that :

$$\begin{aligned}\|G(u)-G(v)\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})}\leq C(\|u\|_{L^{\infty}_{T}(L^{\infty})},\|v\|_{L^{\infty}_{T}(L^{\infty})})\big(\|v-u\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})}(1+\|u\|_{L^{\infty}_{T}(L^{\infty})}\\ +\|v\|_{L^{\infty}_{T}(L^{\infty})})+\|v-u\|_{L^{\infty}_{T}(L^{\infty})}(\|u\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})}+\|v\|_{\widetilde{L}^{\rho}_{T}(\widetilde{B}^{s_1,s_2}_{p})})\big).\end{aligned}$$

The proof is an adaptation of a theorem by J.Y. Chemin and H. Bahouri in [2], see the proof in the Appendix.

4 Existence of solutions for small initial data

4.1 Study of the linear part

This section is devoted to the study of the linearization of system (NHV) in order to get conditions for the existence of solution. We recall the system (NHV) in the case where κ depends only on the density ρ :

$$(NHV)\qquad \begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=0,\\ \partial_t(\rho\, u)+\operatorname{div}(\rho u\otimes u)-\operatorname{div}D-\rho\nabla(\kappa\Delta\rho)+(\nabla(P_0(\rho)+TP_1(\rho))\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =\rho\nabla(\frac{\kappa'_{\rho}}{2}|\nabla\rho|^2),\\ \partial_t\theta-\frac{\operatorname{div}(\chi\nabla\theta)}{\rho}+T\frac{P_1(\rho)}{\rho}\operatorname{div}(u)=\frac{D:\nabla u}{\rho}.\end{cases}$$

Moreover we have :

$$\begin{aligned}\operatorname{div}(D)&=(\lambda+\mu)\nabla\operatorname{div}u+\mu\Delta u+\nabla(\lambda)\operatorname{div}u+(du+\nabla u)\nabla\mu,\\ &=(2\lambda+\mu)\nabla\operatorname{div}u+\mu\Delta u+\partial_1\lambda(\rho,\theta)\nabla\rho\operatorname{div}u+\partial_2\lambda(\rho,\theta)\nabla\theta\operatorname{div}u\\ &\quad+(du+\nabla u)\partial_1\mu(\rho,\theta)\nabla\rho+(du+\nabla u)\partial_2\mu(\rho,\theta)\nabla\theta.\end{aligned}$$

We transform the system to study it in the neighborhood of $(\bar{\rho}, 0, \bar{\theta})$. Using the notation of definition 2.3, we obtain the following system where F, G, H contain the non linear part :

$$(M)\qquad \begin{cases} \partial_t q + \mathrm{div} u = F\,, \\ \partial_t u - \dfrac{\bar{\mu}}{\bar{\rho}}\Delta u - \dfrac{(\bar{\lambda}+\bar{\mu})}{\bar{\rho}}\nabla \mathrm{div}\, u - \bar{\rho}\bar{\kappa}\nabla\Delta q + (P_0'(\bar{\rho}) + \bar{T}P_1'(\bar{\rho}))\nabla q \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{P_1(\bar{\rho})}{\bar{\rho}\psi'(\bar{T})}\nabla\mathcal{T} = G, \\ \partial_t \mathcal{T} - \dfrac{\bar{\chi}}{\bar{\rho}}\Delta\mathcal{T} + \dfrac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}}\mathrm{div} u\S = \S H. \end{cases}$$

This induces us to study the following linear system :

$$(M')\qquad \begin{cases} \partial_t q + \mathrm{div} u = F \\ \partial_t u - \widetilde{\mu}\Delta u - (\widetilde{\mu} + \widetilde{\lambda}) - \varepsilon\nabla\Delta q - \beta\nabla q - \gamma\nabla\mathcal{T} = G \\ \partial_t\mathcal{T} - \alpha\Delta\mathcal{T} + \delta\mathrm{div} u = H \\ \partial_t u - \widetilde{\mu}\Delta u = PG \end{cases}$$

where ν, ε, α, β, γ, δ and $\widetilde{\mu}$ are given real parameters. Note that system (M) with right hand side considered as source terms enters in the class of models (M'), it is only a matter of setting :

$$\widetilde{\mu} = \frac{\bar{\mu}}{\bar{\rho}},\ \widetilde{\lambda} = \frac{\bar{\lambda}}{\bar{\rho}},\ \varepsilon = \bar{\rho}\,\bar{\kappa},\ \beta = P_0'(\bar{\rho}) + \bar{T}P_1'(\bar{\rho}),\ \gamma = \frac{P_1(\bar{\rho})}{\bar{\rho}\psi'(\bar{T})},\ \alpha = \frac{\bar{\chi}}{\bar{\rho}},\ \delta = \frac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}}\ .$$

We transform the system in setting :

$$d = \Lambda^{-1}\mathrm{div}\, u \ \text{ and } \ \Omega = \Lambda^{-1}\mathrm{curl}\, u$$

where we set : $\Lambda^s h = \mathcal{F}^{-1}(|\xi|^s\hat{h})$ (the curl is defined in the appendix).
We finally obtain the following system in projecting on divergence free vector fields and on potential vector fields :

$$(M_1')\qquad \begin{cases} \partial_t q + \Lambda d = F, \\ \partial_t d - \nu\Delta d - \varepsilon\Lambda^3 q - \beta\Lambda q - \gamma\Lambda\mathcal{T} = \Lambda^{-1}\mathrm{div}\, G, \\ \partial_t\mathcal{T} - \alpha\Delta\mathcal{T} + \delta\Lambda d = H, \\ \partial_t\Omega - \widetilde{\mu}\Delta\Omega = \Lambda^{-1}\mathrm{curl}\, G, \\ u = -\Lambda^{-1}\nabla d - \Lambda^{-1}\mathrm{div}\,\Omega. \end{cases}$$

The last equation is just a heat equation. Hence we are going to focus on the first three equations. However the last equation gives us an idea of which spaces we can work with. The first three equation can be read as follows :

$$(M_2')\qquad \partial_t \begin{pmatrix} \hat{q}(t,\xi) \\ \hat{d}(t,\xi) \\ \hat{\mathcal{T}}(t,\xi) \end{pmatrix} + A(\xi)\begin{pmatrix} \hat{q}(t,\xi) \\ \hat{d}(t,\xi) \\ \hat{\mathcal{T}}(t,\xi) \end{pmatrix} = \begin{pmatrix} \hat{F}(t,\xi) \\ \Lambda^{-1}\mathrm{div}\,\hat{G}(t,\xi) \\ \hat{H}(t,\xi) \end{pmatrix}$$

where we have :

$$A(\xi) = \begin{pmatrix} 0 & |\xi| & 0 \\ -\varepsilon|\xi|^3 - \beta|\xi| & \nu|\xi|^2 & -\gamma|\xi| \\ 0 & \delta|\xi| & \alpha|\xi|^2 \end{pmatrix}.$$

The eigenvalues of the matrix $-A(\xi)$ are of the form $|\xi|^2\lambda_\xi$ with λ_ξ being the roots of the following polynomial :

$$P_\xi(X) = X^3 + (\nu+\alpha)X^2 + \left(\varepsilon + \nu\alpha + \frac{\gamma\delta+\beta}{|\xi|^2}\right)X + \left(\alpha\varepsilon + \frac{\alpha\beta}{|\xi|^2}\right).$$

For very large ξ, the roots tend to those of the following polynomial (by virtue of continuity of the roots in function of the coefficients) :

$$X^3 + (\nu+\alpha)X^2 + (\varepsilon+\nu\alpha)X + \alpha\varepsilon.$$

The roots are $-\alpha$ and $-\frac{\nu}{2}(1 \pm \sqrt{1-\frac{4\varepsilon}{\nu^2}})$.
The system $(M_1^{'})$ is well-posed if and only if for $|\xi|$ tending to $+\infty$ the real part of the eigenvalues associated to $A(\xi)$ stay non positive. Hence, we must have :

$$\varepsilon,\ \nu,\ \alpha \geq 0.$$

Let us now state a necessary and sufficient condition for the global stability of $(M^{'})$.

Proposition 4.8. *The linear system $(M^{'})$ is globally stable if and only if the following conditions are verified :*

$$(*) \qquad \nu,\ \varepsilon,\ \alpha \geq 0,\ \ \alpha\beta \geq 0,\ \ \gamma\delta(\nu+\alpha)+\nu\beta \geq 0,\ \ \gamma\delta+\beta \geq 0.$$

If all the inequalities are strict, the solutions tend to 0 in the sense of distributions and the three eigenvalues $\lambda_1(\xi)$, $\lambda_+(\xi)$, $\lambda_-(\xi)$ have the following asymptotic behavior when ξ tends to 0 :

$$\lambda_1(\xi) \sim -(\frac{\alpha\beta}{\beta+\gamma\delta})|\xi|^2,\ \ \lambda_\pm(\xi) \sim -(\frac{\gamma\delta(\nu+\alpha)+\nu\beta}{2(\gamma\delta+\beta)})|\xi|^2 \pm i|\xi|\sqrt{\gamma\delta+\beta}.$$

Proof :

We already know that the system is well-posed if and only if $\nu, \alpha \geq 0$. We want that all the eigenvalues have a negative real part for all ξ.
We have to distinguish two cases : either all the eigenvalues are real or there are two complex conjugated eigenvalues.

First case :

The eigenvalues are real. A necessary condition for negativity of the eigenvalues is that $P(X) \geq 0$ for $X \geq 0$. We must have in particular :

$$P_\xi(0) = \alpha\varepsilon + \frac{\alpha\beta}{|\xi|^2} \geq 0 \quad \forall \xi \neq 0.$$

This imply that $\alpha\beta \geq 0$ and $\alpha\varepsilon \geq 0$. Hence, given that $\alpha \geq 0$, we must have $\beta \geq 0$ and $\varepsilon \geq 0$. For ξ tending to 0, we have :

$$P_\xi(\lambda) \sim \frac{\lambda(\gamma\delta + \beta) + \alpha\beta}{|\xi|^2}.$$

Making λ tend to infinity, we must have $P_\xi(\lambda) \geq 0$ and so $\gamma\delta + \beta \geq 0$.
The converse is trivial.

Second case :

P_ξ has two complex roots $z_\pm = a \pm ib$ and one real root λ, we have :

$$P_\xi(X) = (X - \lambda)(X^2 - 2aX + |z_\pm|^2).$$

A necessary condition to have the real parts negative is in the same way that $P_\xi(X) \geq 0$ for all $X \geq 0$.
If $\gamma\delta + \beta > 0$, we are in the case where ξ tends to 0 (and we see that P_ξ is increasing).
We can observe the terms of degree 2 and we get : $\lambda + 2a = -\alpha - \nu$ then λ and α are non positive if and only if $P_\xi(-\alpha - \nu) \leq 0$ (for this it suffices to rewrite P_ξ like $P_\xi(X) = (X - \lambda)(X^2 - 2aX + |z_\pm|^2)$). Calculate :

$$P_\xi(-\alpha - \nu) = -\nu\varepsilon - \nu^2\alpha - \frac{\nu\beta + \nu\gamma\delta + \alpha\gamma\delta}{|\xi|^2}.$$

With the hypothesis that we have made, we deduce that $P_\xi(-\alpha - \nu) \leq 0$ for ξ tending to 0 if and only if $\nu\beta + \nu\gamma\delta + \alpha\gamma\delta \geq 0$.

Behavior of the eigenvalues in low frequencies :

Let us now study the asymptotic behavior of the eigenvalues when ξ tends to 0 and all the inequalities in (A) are strict.
We remark straight away that the condition $\gamma\delta + \beta > 0$ ensures the strict monotonicity of the function : $\lambda \rightarrow P_\xi(\lambda)$ for ξ small. Then there's only one real eigenvalue $\lambda_1(\xi)$ and two complex eigenvalues $\lambda_\pm(\xi) = a(\xi) \pm ib(\xi)$.

Let $\varepsilon^- < -\frac{\alpha\beta}{\gamma\delta+\beta} < \varepsilon^+ < 0$. When ξ tends to 0, we have :

$$P_\xi(\lambda) \sim |\xi|^{-2}(\lambda(\gamma\delta + \beta) + \alpha\beta)).$$

Then $P_\xi(\varepsilon^-) < 0$ and $P_\xi(\varepsilon^+) > 0$ and P_ξ has a unique real root included between ε^- and ε^+. These considerations give the asymptotic value of $\lambda_1(\xi)$.
Finally, we have :

$$\lambda_1(\xi) + 2a(\xi) = -\alpha - \nu \text{ and } -(a(\xi)^2 + b(\xi)^2)\lambda(\xi) = \alpha\xi + \frac{\alpha\beta}{|\xi|^2} \sim \frac{\alpha\beta}{|\xi|^2},$$

whence the result. □

We summarize this results in the following remark.

Remark 2. *According to the analysis made in proposition 4.8, we expect the system (M) to be locally well-posed close to the equilibrium $(\bar{\rho}, 0, \bar{T})$ if and only if we have :*

$$(C) \qquad \mu(\bar{\rho},\bar{\theta}) \geq 0,\ \lambda(\bar{\rho},\bar{\theta}) + 2\mu(\bar{\rho},\bar{\theta}) \geq 0,\ \kappa(\bar{\rho}) \geq 0, \quad \textit{and}\ \ \chi(\bar{\rho},\bar{T}) \geq 0.$$

By the calculus we have :

$$\beta = \partial_\rho p_0(\bar{\rho},\bar{T}),\ \gamma = \frac{\partial_T p_0(\bar{\rho},\bar{T})}{\bar{\rho}\,\partial_T e_0(\bar{\rho},\bar{T})},\ \delta = \frac{\bar{T}\,\partial_T p_0(\bar{\rho},\bar{T})}{\bar{\rho}}.$$

We remark that $\gamma\delta \geq 0$ if $\partial_T e_0(\bar{\rho},\bar{T}) \geq 0$. In the case where η verifies $\eta(\bar{\rho},\bar{T}) > 0$, the supplementary condition giving the global stability reduces to :

$$(D) \qquad \partial_\rho p_0(\bar{\rho},\bar{T}) \geq 0.$$

Now that we know the stability conditions on the coefficients of the system (M'), we aim at proving estimates in the space $E^{\frac{N}{2}}$.
We add a condition in this following proposition compared with the proposition 4.8 which is : $\gamma\delta > 0$, but it's not so important because in the system (NHV) we are interested in, we have effectively $\gamma\delta = \frac{1}{\bar{T}\Psi'(\bar{T})} > 0$.

Proposition 4.9. *: Under the conditions of proposition 4.8 with strict inequalities and with the condition $\gamma\delta > 0$, let $(q, d, \mathcal{T})$ be a solution of the system (M') on $[0, T)$ with initial data $(q_0, u_0, \mathcal{T}_0)$ such that :*

$$q_0 \in \widetilde{B}^{s-1,s}, d_0 \in B^{s-1}, \mathcal{T}_0 \in \widetilde{B}^{s-1,s-2} \ \textit{for some}\ s \in \mathbb{R}\,.$$

Moreover we suppose that for some $1 \leq r_1 \leq +\infty$, we have :

$$F \in \widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}}),\ \ G \in \widetilde{L}^{r_1}_T(B^{s-3+\frac{2}{r_1}}),\ \ H \in \widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-4+\frac{2}{r_1}}).$$

We then have the following estimate for all $r_1 \leq r \leq +\infty$:

$$\|q\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})} + \|\mathcal{T}\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s-2+\frac{2}{r}})} + \|u\|_{\widetilde{L}^r_T(B^{s-1+\frac{2}{r}})} \lesssim \|q_0\|_{\widetilde{B}^{s-1,s}} + \|u_0\|_{B^{s-1}}$$

$$+ \|\mathcal{T}_0\|_{\widetilde{B}^{s-1,s-2}} + \|F\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})} + \|G\|_{L^{r_1}_T(B^{s-3+\frac{2}{r_1}})} + \|H\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-4+\frac{2}{r_1}})}\,.$$

Proof :

We are going to separate the case of the low, medium and high frequencies, particulary the low and high frequencies which have a different behavior, and depend on the indice of Besov space.

1) Case of low frequencies :

Let us focus on just the first three equation because the last one is a heat equation that we can treat independently. Applying operator Δ_l to the system $(M_1^{'})$, we obtain then in setting :

$$q_l = \Delta_l q,\ d_l = \Delta_l d,\ \mathcal{T}_l = \Delta_l \mathcal{T}$$

the following system :

$$\partial_t q_l + \Lambda d_l = F_l, \tag{4.1}$$

$$\partial_t d_l - \nu\Delta d_l - \varepsilon\Lambda^3 q_l - \beta\Lambda q_l - \gamma\Lambda\mathcal{T}_l = \Lambda^{-1}\mathrm{div}G_l, \tag{4.2}$$

$$\partial_t \mathcal{T}_l - \alpha\Delta\mathcal{T}_l + \delta\Lambda d_l = H_l. \tag{4.3}$$

Throughout the proof, we assume that $\delta \neq 0$: if not we have just a heat equation on (4.3) and we can use the proposition 3.5 to have the estimate on $\mathcal{T}$ and we have just to deal with the first two equations. Denoting by $W(t)$ the semi-group associated to $(4.1-4.3)$ we have :

$$\begin{pmatrix} q(t) \\ u(t) \\ \theta(t) \end{pmatrix} = W(t)\begin{pmatrix} q_0 \\ u_0 \\ \theta_0 \end{pmatrix} + \int_0^t W(t-s)\begin{pmatrix} F(s) \\ G(s) \\ H(s) \end{pmatrix} ds\ .$$

We set :

$$f_l^2 = \beta\|q_l\|_{L^2}^2 + \|d_l\|_{L^2}^2 + \frac{\gamma}{\delta}\|\mathcal{T}_l\|_{L^2}^2 - 2K\langle\Lambda q_l, d_l\rangle$$

for some $K \geq 0$ to be fixed hereafter and $\langle\cdot,\cdot\rangle$ noting the L^2 inner product.
To begin with, we consider the case where $F = G = H = 0$.
Then we take the inner product of (4.2) with d_l, of (4.1) with βq_l and of (4.3) with $\gamma\mathcal{T}_l$. We get :

$$\frac{1}{2}\frac{d}{dt}\big(\|d_l\|_{L^2}^2 + \beta\|q_l\|_{L^2}^2 + \frac{\gamma}{\delta}\|\mathcal{T}_l\|_{L^2}^2\big) + \nu\|\nabla d_l\|_{L^2}^2 - \varepsilon\langle\Lambda^3 q_l, d_l\rangle + \frac{\gamma\alpha}{\delta}\|\nabla\mathcal{T}_l\|_{L^2}^2 = 0. \tag{4.4}$$

Next, we apply the operator Λ to (4.2) and take the inner product with q_l, and we take the scalar product of (4.1) with Λd_l to control the term $\frac{d}{dt}\langle\Lambda q_l, d_l\rangle$. Summing the two resulting equalities, we get :

$$\frac{d}{dt}\langle\Lambda q_l, d_l\rangle + \|\Lambda d_l\|_{L^2}^2 - \nu\langle\Delta d_l, \Lambda q_l\rangle - \varepsilon\|\Lambda^2 q_l\|_{L^2}^2 - \beta\|\Lambda q_l\|_{L^2}^2 - \gamma\langle\Lambda\mathcal{T}_l, \Lambda q_l\rangle = 0. \tag{4.5}$$

We obtain then in summing (4.4) and (4.5) :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}f_l^2 + (\nu\|\nabla d_l\|_{L^2}^2 - K\|\Lambda d_l\|_{L^2}^2) + (K\beta\|\Lambda q_l\|_{L^2}^2 + K\varepsilon\|\Lambda^2 q_l\|_{L^2}^2) + \frac{\gamma\alpha}{\delta}\|\nabla\mathcal{T}_l\|_{L^2}^2 \\ + K\nu\langle\Delta d_l, \Lambda q_l\rangle + K\gamma\langle\Lambda\mathcal{T}_l, \Lambda q_l\rangle - \varepsilon\langle\Lambda^3 q_l, d_l\rangle = 0\,.\end{aligned} \tag{4.6}$$

Like indicated, we are going to focus on low frequencies so assume that $l \leq l_0$ for some l_0 to be fixed hereafter. We have then $\forall c,\ b,\ d > 0$:

$$
\begin{aligned}
|\langle \Delta d_l, \Lambda q_l\rangle| &\leq \frac{b}{2}\|\Lambda q_l\|_{L^2}^2 + \frac{1}{2b}\| \Delta d_l\|_{L^2}^2 \\
&\leq \frac{b}{2}\|\Lambda q_l\|_{L^2}^2 + \frac{C2^{2l_0}}{2b}\|\Lambda d_l\|_{L^2}^2 \,, \\
|\langle \Lambda^3 q_l, d_l\rangle| = |\langle \Lambda^2 q_l, \Lambda d_l\rangle| &\leq \frac{C2^{2l_0}}{2c}\|\Lambda q_l\|_{L^2}^2 + \frac{c}{2}\|\Lambda d_l\|_{L^2}^2 \,.
\end{aligned}
\tag{4.7}
$$

Moreover we have : $\|\nabla d_l\|_{L^2}^2 = \|\Lambda d_l\|_{L^2}^2$. Finally we obtain :

$$
\frac{1}{2}\frac{d}{dt}f_l^2 + \big[\nu - (K + \frac{C2^{2l_0}}{2b}K\nu + \frac{c\varepsilon}{2})\big]\|\Lambda d_l\|_{L^2}^2 + \big[\frac{\gamma\alpha}{\delta} - \frac{K\gamma}{2d}\big]\|\Lambda \mathcal{T}_l\ \|_{L^2}^2
$$
$$
+ K\big[\beta + \varepsilon C2^{2l_0} - \nu\frac{b}{2} - \varepsilon\frac{C2^{2l_0}}{2c} - \gamma\frac{d}{2}\big]\|\Lambda q_l\|_{L^2}^2 \leq 0.
$$

Then we choose (b, c, d) such that :

$$
b = \frac{\beta}{2\nu}\ ,\ c = \frac{\nu}{\varepsilon}\ ,\ d = \frac{\beta}{2\gamma}\ ,
$$

which is possible if $\gamma > 0$ as $\nu > 0,\ \varepsilon > 0$. In the case where $\gamma \leq 0$, we recall that γ and δ have the same sign, we have then no problem because with our choice the first and third following inequalities will be satisfied and if $\gamma \leq 0$ in the second equation the term $\gamma\frac{d}{2}$ is positive in taking $d > 0$. So we assume from now on that $\gamma > 0$ and so with this choice, we want that :

$$
\begin{aligned}
&\frac{\nu}{2} - K(1 + C2^{2l_0}\frac{\nu^2}{\beta}) > 0, \\
&\frac{\beta}{2} + \varepsilon C2^{2l_0} - C2^{2l_0}\frac{\varepsilon^2}{2\nu} > 0, \\
&\frac{\gamma\alpha}{\delta} - K\frac{\gamma}{2} > 0.
\end{aligned}
$$

We recall that in your case $\nu > 0,\ \beta > 0,\ \alpha > 0$ and $\gamma > 0,\ \delta > 0$. So it suffices to choose K and l_0 such that :

$$
K < \min\Bigg(\frac{\nu}{2(1 + C2^{2l_0}\frac{\nu^2}{2\beta})}, \frac{2\alpha}{\delta}\Bigg) \quad \text{and} \quad 2^{2l_0} < \min\Bigg(\frac{\beta\nu}{6C\varepsilon^2}, \frac{1}{6\varepsilon C}\Bigg).
$$

Finally we conclude in using Proposition 3.1 part (ii) with a c' small enough. We get :

$$
\frac{1}{2}\frac{d}{dt}f_l^2 + c'2^{2l}f_l^2 \leq 0 \ \text{ for } l \leq l_0 \,. \tag{4.8}
$$

2) Case of high frequencies :

We are going to work with $l \geq l_1$ where we will determine l_1 hereafter. We set then :

$$
f_l^2 = \varepsilon B\|\Lambda q_l\|_{L^2}^2 + B\|d_l\|_{L^2}^2 + \|\Lambda^{-1}\mathcal{T}_l\|_{L^2}^2 - 2K\langle \Lambda q_l, d_l\rangle,
$$

and we choose B and K later on.

Then we take the inner product of (4.2) with d_l :

$$\frac{1}{2}\frac{d}{dt}\|d_l\|_{L^2}^2+\nu\|\nabla d_l\|_{L^2}^2-\varepsilon\langle\Lambda^3 q_l,d_l\rangle-\beta\langle\Lambda q_l,d_l\rangle-\gamma\langle\Lambda\mathcal{T}_l,d_l\rangle=0. \tag{4.9}$$

Moreover we have in taking the scalar product of (4.1) with $\Lambda^2 q_l$:

$$\frac{1}{2}\frac{d}{dt}\|\Lambda q_l\|_{L^2}^2+\langle\Lambda^2 d_l,\Lambda q_l\rangle=0. \tag{4.10}$$

And in the same way with (4.3), we have :

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-1}\mathcal{T}_l\|_{L^2}^2+\alpha\|\mathcal{T}_l\|_{L^2}^2+\delta\langle d_l,\Lambda^{-1}\mathcal{T}_l\rangle=0. \tag{4.11}$$

After we sum (4.9), (4.10) and (4.11) to get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(B\|d_l\|_{L^2}^2+\varepsilon B\|\Lambda q_l\|_{L^2}^2+\|\Lambda^{-1}\mathcal{T}_l\|_{L^2}^2\big)+B\nu\|\nabla d_l\|_{L^2}^2+\alpha\|\mathcal{T}_l\|_{L^2}^2\\ -B\beta\langle\Lambda q_l,d_l\rangle-B\gamma\langle\Lambda\mathcal{T}_l,d_l\rangle+\delta\langle d_l,\Lambda^{-1}\mathcal{T}_l\rangle=0.\end{aligned} \tag{4.12}$$

Then like previously we can play with $\langle\Lambda q_l,d_l\rangle$ to obtain a term in $\|\Lambda q_l\|_{L^2}^2$. We have then again the following equation :

$$\frac{d}{dt}\langle\Lambda q_l,d_l\rangle+\|\Lambda d_l\|_{L^2}^2-\nu\langle\Delta d_l,\Lambda q_l\rangle-\varepsilon\|\Lambda^2 q_l\|_{L^2}^2-\beta\|\Lambda q_l\|_{L^2}^2-\gamma\langle\Lambda\mathcal{T}_l,\Lambda q_l\rangle=0. \tag{4.13}$$

We sum all these expressions and get :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}f_l^2+\big[B\nu\|\nabla d_l\|_{L^2}^2-K\|\Lambda d_l\|_{L^2}^2\big]+\alpha\|\mathcal{T}_l\|_{L^2}^2+K\big[\beta\|\Lambda q_l\|_{L^2}^2+\varepsilon\|\Lambda^2 q_l\|_{L^2}^2\big]\\ &-B\beta\langle\Lambda q_l,d_l\rangle-B\gamma\langle\Lambda\mathcal{T}_l,d_l\rangle+\delta\langle d_l,\Lambda^{-1}\mathcal{T}_l\rangle+K\nu\langle\Delta d_l,\Lambda q_l\rangle+\gamma K\langle\Lambda\mathcal{T}_l,\Lambda q_l\rangle=0.\end{aligned} \tag{4.14}$$

The main term in high frequencies will be : $\|\Lambda^2 q_l\|_{L^2}^2$. The other terms may be treated by mean of Young's inequality :

$$\begin{aligned}|\langle\Lambda q_l,d_l\rangle|&\leq\frac{1}{2a}\|\Lambda q_l\|_{L^2}^2+\frac{a}{2}\|\Lambda d_l\|_{L^2}^2,\\ &\leq\frac{1}{2a\,c2^{2l_1}}\|\Lambda^2 q_l\|_{L^2}^2+\frac{a}{2}\|\Lambda d_l\|_{L^2}^2.\end{aligned}$$

We do as before with the others terms in the second line of (4.14) and we obtain :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}f_l^2+(B\nu-K)\|\Lambda d_l\|_{L^2}^2]+\alpha\|\mathcal{T}_l\|_{L^2}^2+K(\frac{\beta}{c2^{2l_1}}+\varepsilon)\|\Lambda^2 q_l\|_{L^2}^2\leq\\ &B\gamma\big[\frac{1}{2a}\|\mathcal{T}_l\|_{L^2}^2+\frac{a}{2}\|\Lambda d_l\|_{L^2}^2\big]+K\big[\frac{\nu b}{2}\|\Lambda^2 q_l\|_{L^2}^2+\frac{2\nu}{b}\|\Lambda d_l\|_{L^2}^2+\frac{\gamma}{2c'}\|\mathcal{T}_l\|_{L^2}^2+\frac{\gamma c'}{2}\|\Lambda^2 q_l\|_{L^2}^2\big]\\ &+B\beta\big[\frac{1}{2d}\frac{1}{c2^{2l_1}}\|\Lambda^2 q_l\|_{L^2}^2+\frac{d}{2}\frac{1}{c2^{2l_1}}\|\Lambda d_l\|_{L^2}^2\big]+\delta\big[\frac{1}{2e}\frac{1}{c2^{2l_1}}\|\mathcal{T}_l\|_{L^2}^2+\frac{e}{2}\frac{1}{c2^{2l_1}}\|\Lambda d_l\|_{L^2}^2\big].\end{aligned}$$

We obtain then for some $a,\ b,\ c^{'},\ d,\ e$ to be chosen :

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}f_l^2+\big[B\nu-(K+B\gamma\frac{a}{2}+K\nu\frac{2}{b}+B\beta\frac{d}{2c2^{2l_1}}+\delta\frac{e}{2}\frac{1}{c2^{2l_1}})\big]\|\Lambda d_l\|_{L^2}^2\\
&+\big[\alpha-(B\gamma\frac{1}{2a}+\gamma K\frac{1}{2c^{'}}+\delta\frac{1}{2e}\frac{1}{c2^{2l_1}})\big]\|\mathcal{T}_l\|_{L^2}^2 \qquad (4.15)\\
&+\big[\frac{\beta K}{c2^{2l_1}}+\varepsilon K-K\nu\frac{b}{2}-\gamma K\frac{c^{'}}{2}-B\beta\frac{1}{2d}\frac{1}{2^{2l_1}}\big]\|\Lambda^2 q_l\|_{L^2}^2\leq 0\ .
\end{aligned}$$

We claim that $a,\ b,\ c^{'},\ d,\ e,\ l_1,\ K$ may be chosen so that :

$$B\nu-(K+B\gamma\frac{a}{2}+K\nu\frac{2}{b}+B\beta\frac{d}{2}+\delta\frac{e}{2}\frac{1}{2^{2l_1}})>0, \tag{4.16}$$

$$\alpha-(B\gamma\frac{1}{2a}+\gamma K\frac{1}{2c^{'}}+\delta\frac{1}{2e}\frac{1}{c2^{2l_1}})>0, \tag{4.17}$$

$$\frac{\beta K}{c2^{2l_1}}+\varepsilon K-K\nu\frac{b}{2}-\gamma K\frac{c^{'}}{2}-B\beta\frac{1}{2d}\frac{1}{2^{2l_1}}>0. \tag{4.18}$$

We want at once that for (4.16) and (4.18) :

$$\nu-\gamma\frac{a}{2}-\beta\frac{d}{2}>0\ , \tag{4.19}$$

$$\varepsilon-\nu\frac{b}{2}-\gamma\frac{c^{'}}{2}>0\ . \tag{4.20}$$

So we take :

$$\begin{aligned}
&e=1,\ a=2h\delta,\ d=2h,\ h=\frac{\nu}{2(\gamma\delta+\beta)},\ b=2\beta h^{'},\\
&c^{'}=2\delta(\nu+\alpha)h^{'}\quad\text{and}\quad h^{'}=\frac{\varepsilon}{2(\gamma\delta(\nu+\alpha)+\nu\beta)}\ .
\end{aligned}$$

With this choice, we get (4.19) and (4.20). In what follows it suffices to choice B,K small enough and l_1 large enough. We have then :

$$f_l\simeq \text{Max}(1,2^l)\|q_l\|_{L^2}+\|d_l\|_{L^2}+\text{Min}(1,2^l)\|\mathcal{T}_l\|_{L^2}$$

We have so obtain for $l\leq l_0$, $l\geq l_1$ and for a $c^{'}$ small enough :

$$\frac{1}{2}\frac{d}{dt}f_l^2+c^{'}2^{2l}f_l^2\leq 0.$$

3) Case of Medium frequencies :

For $l_0\leq l\leq l_1$, there is only a finite number of terms to treat. So it suffices to find a C such that for all these terms :

$$(B)\qquad \|q_l\|_{L^r_T(L^2)}\leq C,\ \|d_l\|_{L^r_T(L^2)}\leq C,\ \|\mathcal{T}_l\|_{L^r_T(L^2)}\leq C\quad\text{for all}\quad T\in[0,+\infty]\ \text{and}\ \ r\in[1,+\infty]$$

with C large enough independent of T.

And this is true because the system is globally stable : indeed according to proposition 4.8, we have :

$$\left\|W(t)\begin{pmatrix} a\\ b\\ c\end{pmatrix}\right\|_{L^2}\lesssim e^{-c_1(\xi)t}\left\|\begin{pmatrix} a\\ b\\ c\end{pmatrix}\right\|_{L^2}\quad\forall a,b,c\in L^2$$

with $c_1(\xi)=\min_{2^{l_0}\leq|\xi|\leq 2^{l_1}}(\mathrm{Re}(\lambda_1(\xi)),\ \mathrm{Re}(\lambda_2(\xi)),\ \mathrm{Re}(\lambda_3(\xi)))$ where the $\lambda_i(\xi)$ correspond to the eigenvalues of the system. We have then in using the estimate in low and high frequencies in part 4.1 and the continuity of $c_1(\xi)$ the fact that there exits c_1 such that :

$$c_1(\xi)\geq c_1>0.$$

So that we have :

$$\left(\int_0^T\begin{pmatrix}\|q_l(t)\|_{L^2}^r\\ \|u_l(t)\|_{L^2}^r\\ \|\mathcal{T}_l(t)\|_{L^2}^r\end{pmatrix}dt\right)^{\frac{1}{r}}\lesssim\left(\int_0^T e^{-c_1 rs}ds\right)^{\frac{1}{r}}\begin{pmatrix}\|(q_0)_l\|_{L^2}\\ \|(u_0)_l\|_{L^2}\\ \|(\mathcal{T}_0)_l\|_{L^2}\end{pmatrix}\quad\text{for }\ l_0\leq l\leq l_1.$$

And so we have the result (B).

4) Conclusion :

In using Duhamel formula for W and in taking C large enough we have for all l :

$$\begin{aligned}\max(1,2^l)\|q_l(t)\|_{L^2}+\|d_l(t)\|_{L^2}+\min(1,2^{-l})\|\mathcal{T}_l(t)\|_{L^2}\leq Ce^{-c2^{2l}t}\big(\max(1,2^l)\|(q_0)_l\|_{L^2}\\ +\|(d_0)_l\|_{L^2}+\min(1,2^{-l})\|(\mathcal{T}_0)_l\|_{L^2}\big)+C\int_0^t e^{-c2^{2l}(t-s)}\big(\max(1,2^{2l})\|F_l\|_{L^2}+\|G_l\|_{L^2}\\ +\min(1,2^{-l})\|H_l\|_{L^2})ds\,.\end{aligned}$$

Now we take the L^r norm in time and we sum in multiplying by $2^{l(s-1+\frac{2}{r})}$ for the low frequencies and we sum in multiplying by $2^{l(s+\frac{2}{r})}$ for the high frequencies.
This yields :

$$\begin{aligned}&\|q\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})}+\|\mathcal{T}\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s-2+\frac{2}{r}})}+\|d\|_{\widetilde{L}^r_T(B^{s-1+\frac{2}{r}})}\leq\|q_0\|_{\widetilde{B}^{s-1,s}}+\|\mathcal{T}_0\|_{\widetilde{B}^{s-1,s-2}}\\ &+\|d_0\|_{B^{s-1}}+\sum_{l\leq 0}2^{l(s-1+\frac{2}{r})}\int_0^T\left(\int_0^T e^{c(t-\tau)}\big(\|F_l(\tau)\|_{L^2}+\|G_l(\tau)\|_{L^2}+\|H_l(\tau)\|_{L^2}\big)d\tau\right)^r dt\Big)^{\frac{1}{r}}\\ &+\sum_{l\geq 0}2^{l(s+\frac{2}{r})}\left(\int_0^T\left(\int_0^T e^{c(t-\tau)}\big(\|\nabla F_l(\tau)\|_{L^2}+\|G_l(\tau)\|_{L^2}+\|\Lambda^{-1}H_l(\tau)\|_{L^2}\big)d\tau\right)^r dt\right)^{\frac{1}{r}}.\end{aligned}$$

Bounding the right hand-side may be done by taking advantage of convolution inequalities. To complete the proof of proposition 4.9, it suffices to use that $u=-\Lambda^{-1}\nabla d-\Lambda^{-1}\mathrm{div}\Omega$ and to apply proposition 3.5. □

4.2 Global existence for temperature independent coefficients

This section is devoted to the proof of theorem 2.1 and 2.3. Let us first recall the spaces in which we work with for the theorem 2.1 :

$$\begin{aligned} E^s =&[C_b(\mathbb{R}_+,\widetilde{B}^{s-1,s})\cap L^1(\mathbb{R}_+\widetilde{B}^{s+1,s+2})]\times[C_b(\mathbb{R}_+,B^{s-1})^N\cap L^1(\mathbb{R}_+,B^{s+1})^N]\\ &\times[C_b(\mathbb{R}_+,\widetilde{B}^{s-1,s-2})\cap L^1(\mathbb{R}_+,\widetilde{B}^{s+1,s})]. \end{aligned}$$

In what follows, we assume that $N\geq 3$.

Proof of theorem 2.1 :

We shall use a contracting mapping argument for the function ψ defined as follows :

$$\psi(q,u,\mathcal{T})=W(t,\cdot)*\begin{pmatrix} q_0\\ u_0\\ \mathcal{T}_0\end{pmatrix}+\int_0^t W(t-s)\begin{pmatrix} F(q,u,\mathcal{T})\\ G(q,u,\mathcal{T})\\ H(q,u,\mathcal{T})\end{pmatrix}ds\,. \tag{4.21}$$

In what follows we set :

$$\rho=\bar{\rho}(1+q)\ ,\ \theta=\bar{\theta}+\mathcal{T}\ ,\ \widetilde{T}=\Psi^{-1}(\theta).$$

The non linear terms F,G,H are defined as follows :

$$\begin{aligned} F=&-\operatorname{div}(qu),\\ G=&-u.\nabla u+\nabla(\frac{K^{'}_\rho}{2}|\nabla\rho|^2\ )+\Big[\frac{\mu(\rho)}{\rho}-\frac{\mu(\bar{\rho})}{\bar{\rho}}\Big]\Delta u+\Big[\frac{\zeta(\rho)}{\rho}-\frac{\zeta(\bar{\rho})}{\bar{\rho}}\Big]\nabla\operatorname{div}u\\ &+(\nabla((K(\rho)-K(\bar{\rho}))\Delta\rho)+\Big[\frac{P_0^{'}(\rho)+\widetilde{T}P_1^{'}(\rho)}{\rho}-\frac{P_0^{'}(\bar{\rho})+\bar{T}P_1^{'}(\bar{\rho})}{\bar{\rho}}\Big]\nabla\rho\\ &+\Big[\frac{P_1(\rho)}{\rho\Psi'(\mathcal{T})}-\frac{P_1(\bar{\rho})}{\bar{\rho}\Psi'(\bar{T})}\Big]\nabla\theta+\frac{\lambda^{'}(\rho)\nabla\rho\operatorname{div}u}{\rho}+\frac{(du+\nabla u)\mu^{'}(\rho)\nabla\rho}{\rho}\,, \end{aligned} \tag{4.22}$$

where we note : $\zeta=\lambda+\mu$, and :

$$H=\Big(\frac{\operatorname{div}(\chi(\rho)\nabla\theta)}{\rho}-\frac{\bar{\chi}}{\bar{\rho}}\Delta\theta\Big)+\Big[\frac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}}-\frac{\widetilde{T}P_1(\rho)}{\rho}\Big]\operatorname{div}u-u^*.\nabla\theta+\frac{D:\nabla u}{\rho}\,. \tag{4.23}$$

1) First step, uniform bounds :

Let :

$$\eta=\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_0\|_{B^{\frac{N}{2}-1}}+\|\mathcal{T}_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\,.$$

We are going to show that ψ maps the ball $B(0,R)$ into itself if R is small enough. According to proposition 4.9, we have :

$$\|W(t,\cdot)*\begin{pmatrix} q_0\\ u_0\\ \mathcal{T}_0\end{pmatrix}\|_{E^{\frac{N}{2}}}\leq C\eta\,. \tag{4.24}$$

We have then according (4.21), proposition 4.9 and 4.24 :

$$\|\psi(q,u,\mathcal{T})\|_{E^{\frac{N}{2}}} \leq C\eta + \|F(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} + \|G(q,u,\mathcal{T})\|_{L^1(B^{\frac{N}{2}-1})} + \|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\,. \tag{4.25}$$

Moreover we suppose for the moment that :

$$(\mathcal{H}) \qquad \|q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)} \leq 1/2\,.$$

We will use the different theorems on the paradifferential calculus to obtain estimates on

$$\|F(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},\ \|G(q,u,\mathcal{T})\|_{L^1(B^{\frac{N}{2}-1})} \text{ and } \|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}.$$

1) Let us first estimate $\|F(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$. According to proposition 3.6, we have :

$$\|\mathrm{div}(qu)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} \leq \|qu\|_{L^1(B^{\frac{N}{2}})} + \|qu\|_{L^1(B^{\frac{N}{2}+1})}$$

and :

$$\|qu\|_{L^1(B^{\frac{N}{2}})} \leq \|q\|_{L^2(B^{\frac{N}{2}})}\|u\|_{L^2(B^{\frac{N}{2}})}$$

$$\|qu\|_{L^1(B^{\frac{N}{2}+1})} \leq \|q\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})} + \|q\|_{L^2(B^{\frac{N}{2}+1})}\|u\|_{L^2(B^{\frac{N}{2}})}.$$

Because $\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1} \hookrightarrow B^{\frac{N}{2}}$ and $\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1} \hookrightarrow B^{\frac{N}{2}+1}$ (from proposition 3.2), we get :

$$\|\mathrm{div}(qu)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} \leq \|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|u\|_{L^1(B^{\frac{N}{2}+1})} + \|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|u\|_{L^2(B^{\frac{N}{2}})}.$$

2) We have to estimate $\|G(q,u,\mathcal{T})\|_{L^1(B^{\frac{N}{2}-1})}$. We see straight away that :

$$[\frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}}]\Delta u = K(q)\Delta u$$

for some smooth function K such that $K(0) = 0$. Hence by propositions 3.7, 3.6 and 3.2 yield :

$$\begin{aligned}\Big\|[\frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}}]\Delta u\Big\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|K(q)\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})},\\ &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})},\\ &\lesssim \|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})}\,.\end{aligned}$$

In the same way we have :

$$\|[\frac{\zeta(\rho)}{\rho} - \frac{\zeta(\bar{\rho})}{\bar{\rho}}]\nabla\mathrm{div}u\|_{L^1(B^{\frac{N}{2}-1})} \lesssim \|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})},$$

$$\|\nabla(K(\rho) - K(\bar{\rho}))\Delta q)\|_{L^1(B^{\frac{N}{2}-1})} \lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+2})},$$

$$\begin{aligned}\|[\frac{P_0'(\rho)}{\rho} - \frac{P_0'(\bar{\rho})}{\bar{\rho}}]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}-1})}\|q\|_{L^1(B^{\frac{N}{2}+1})},\\ &\lesssim \|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}\,.\end{aligned}$$

After it remains two terms to treat :

$$\|\Big[\frac{\widetilde{T}P_1'(\rho)}{\rho}-\frac{\bar{T}P_1'(\bar{\rho})}{\bar{\rho}}\Big]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\ \|\big[(\frac{P_1'(\rho)}{\rho}-\frac{P_1'(\bar{\rho})}{\bar{\rho}})\,\bar{\theta}\big]\nabla q\|_{L^1(B^{\frac{N}{2}-1})}$$

$$+\|\frac{P_1'(\bar{\rho})}{\bar{\rho}}\mathcal{T}\nabla q\|_{L^1(B^{\frac{N}{2}-1})}+\|\mathcal{T}\big(\frac{P_1'(\rho)}{\rho}-\frac{P_1'(\bar{\rho})}{\bar{\rho}}\big)\nabla q\|_{L^1(B^{\frac{N}{2}-1})},$$

$$\|\Big[\frac{\widetilde{T}P_1'(\rho)}{\rho}-\frac{\bar{T}P_1'(\bar{\rho})}{\bar{\rho}}\Big]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\|q\|_{L^\infty(B^{\frac{N}{2}-1})}\|q\|_{L^1(B^{\frac{N}{2}+1})}+\|\mathcal{T}\nabla q\|_{L^1(B^{\frac{N}{2}-1})}$$

$$+\|K_1(q)\mathcal{T}\nabla q\|_{L^1(B^{\frac{N}{2}-1})},$$

According to proposition 3.7, we have :

$$\|\mathcal{T}\nabla q\|_{L^1(B^{\frac{N}{2}-1})}\leq\|\mathcal{T}\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}-1})}\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\ ,$$

$$\|K_1(q)\mathcal{T}\nabla q\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\|\mathcal{T}\nabla q\|_{L^1(B^{\frac{N}{2}-1})}\ .$$

Therefore :

$$\|\big[\frac{\widetilde{T}P_1'(\rho)}{\rho}-\frac{\bar{T}P_1'(\bar{\rho})}{\bar{\rho}}\big]\nabla\rho\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}$$

$$+(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})})\|\mathcal{T}\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}-1})}\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\,.$$

In the same spirit :

$$\|(\frac{P_1(\rho)}{\rho}-\frac{P_1(\bar{\rho})}{\bar{\rho}})\nabla\theta\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}\ ,$$

$$\|(\frac{P_1(\rho)}{A\rho}-\frac{P_1(\bar{\rho})}{A\bar{\rho}})\nabla\theta\|_{L^1(B^{\frac{N}{2}-1})}\leq\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|\mathcal{T}_{BF}\|_{L^2(B^{\frac{N}{2}})}$$

$$+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\mathcal{T}_{HF}\|_{L^1(B^{\frac{N}{2}})}\ ,$$

where we have :

$$\mathcal{T}_{BF}=\sum_{l\leq 0}\Delta_l\mathcal{T}\ \text{ and }\ \mathcal{T}_{HF}=\sum_{l>0}\Delta_l\mathcal{T}\ .$$

Next we have the following term :

$$\|u^*.\nabla u\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\|u\|_{L^\infty(B^{\frac{N}{2}-1})}\|u\|_{L^1(B^{\frac{N}{2}+1})}.$$

And finally we have the terms coming from $\operatorname{div}(D)$ which are of the form :

$$\|\frac{\lambda'(\rho)\nabla\rho\,\operatorname{div}u}{\rho}\|_{L^1(B^{\frac{N}{2}-1})}\leq\|L(q)\nabla\rho\,\operatorname{div}u\|_{L^1(B^{\frac{N}{2}-1})}+\|\frac{\lambda'(\bar{\rho})}{\bar{\rho}}\nabla\rho\,\operatorname{div}u\|_{L^1(B^{\frac{N}{2}-1})}$$

where we have set :

$$L(x_1)=\frac{\lambda'(\bar{\rho}(1+x_1))}{\bar{\rho}(1+x_1)}-\frac{\lambda'(\bar{\rho})}{\bar{\rho}}.$$

Afterwards we can apply proposition 3.7 to get :

$$\|\nabla\rho\,\mathrm{div}u\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(B^{\frac{N}{2}})}.$$

$$\|L(q)\nabla\rho\mathrm{div}u\|_{L^1(B^{\frac{N}{2}-1})}\leq\|L(q)\|_{L^\infty(B^{\frac{N}{2}})}\|\nabla\rho\,\mathrm{div}u\|_{L^1(B^{\frac{N}{2}-1})}.$$

As we assumed that $(\mathcal{H})$ is satisfied, we have in using proposition 3.7 :

$$\|L(q)\|_{L^\infty(B^{\frac{N}{2}})}\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\,.$$

So we have :

$$\|\frac{\lambda^{'}(\rho)\nabla\rho\,\mathrm{div}u}{\rho}\|_{L^1(B^{\frac{N}{2}-1})}\lesssim\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}).$$

In the same way we have in using 3.6, 3.7 and 3.2 :

$$\|\frac{(du+\nabla u)\nabla\rho\,\mu^{'}(\rho)}{\rho}\|_{L^1(B^{\frac{N}{2}-1})}\leq C\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(B^{\frac{N}{2}})}(1+\|q\|_{L^\infty(B^{\frac{N}{2}})}).$$

$$\begin{aligned}\|\nabla(\frac{K^{'}_{\rho}}{2}|\nabla\rho|^2)\|_{L^1(B^{\frac{N}{2}-1})}&\lesssim\|(\frac{K^{'}_{\rho}}{2}-\frac{K^{'}_{\bar\rho}}{2})|\nabla\rho|^2)\|_{L^1(B^{\frac{N}{2}})}+\|\frac{K^{'}_{\bar\rho}}{2}|\nabla\rho|^2\|_{L^1(B^{\frac{N}{2}})},\\&\lesssim\|L(q)\|_{L^\infty(B^{\frac{N}{2}})}\||\nabla\rho|^2\|_{L^1(B^{\frac{N}{2}})}+\||\nabla\rho|^2\|_{L^1(B^{\frac{N}{2}})},\\&\lesssim\|q\|_{L^\infty(B^{\frac{N}{2}})}\|\nabla\rho\|^2_{L^2(B^{\frac{N}{2}})}+\|\nabla\rho\|^2_{L^2(B^{\frac{N}{2}})},\\&\lesssim\|q\|_{L^\infty(B^{\frac{N}{2}})}\|q\|^2_{L^2(B^{\frac{N}{2}+1})}+\|q\|^2_{L^2(B^{\frac{N}{2}+1})}.\end{aligned}$$

where $L(q)=\frac{K^{'}_{\rho}}{2}-\frac{K^{'}_{\bar\rho}}{2}$.

3) Let us finally estimate $\|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$:

$$\begin{aligned}\|\frac{\mathrm{div}(\chi(\rho)\nabla\theta)}{\rho}-\frac{\bar\chi}{\bar\rho}\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}&\leq\|K(q)\mathrm{div}(K_1(q)\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\\&+\|\mathrm{div}(K_1(q)\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}+\|K(q)\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})},\end{aligned}$$

and we have :

$$\|\mathrm{div}(K_1(q)\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}.$$

So finally :

$$\begin{aligned}\|\frac{\mathrm{div}(\chi(\rho)\nabla\theta)}{\rho}-\frac{\bar\chi}{\bar\rho}\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}&\lesssim\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}\\&\times(2+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})})\,.\end{aligned}$$

Next we have :

$$\|(\frac{\theta P_1(\rho)}{A\rho}-\frac{\bar{\theta}P_1(\bar{\rho})}{A\bar{\rho}})\mathrm{div}\,u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\ \|\mathcal{T}\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$$
$$+\|\mathcal{T}L_1(q)\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}+\|L_1(q)\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

where we denote :

$$L_1(x)=\frac{P_1(\bar{\rho}(1+x))}{\bar{\rho}(1+x)}-\frac{P_1(\bar{\rho})}{\bar{\rho}}\ .$$

On one hand,

$$\|\mathcal{T}\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\|u\|_{L^1(B^{\frac{N}{2}+1})},$$
$$\|L_1(q)\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\|L_1(q)\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\|u\|_{L^1(B^{\frac{N}{2}+1})},$$

whence the desired result :

$$\|(\frac{\theta P_1(\rho)}{A\rho}-\frac{\bar{\theta}P_1(\bar{\rho})}{A\bar{\rho}})\mathrm{div}\,u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\ \|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\|u\|_{L^1(B^{\frac{N}{2}+1})}$$
$$+\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\|u\|_{L^1(B^{\frac{N}{2}+1})}(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}).$$

We proceed in the same way for the others terms which are similar, and we finish with the last two following terms :

$$\|u^*.\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}\|u\|_{L^\infty(B^{\frac{N}{2}-1})},$$
$$\|\frac{D:\nabla u}{\rho}\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\|K(q)\nabla u:\nabla u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}+\|\nabla u:\nabla u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}.$$

and :

$$\|K(q)\nabla u:\nabla u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim\|q\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^2(B^{\frac{N}{2}})}.$$

so the result :

$$\|\frac{D:\nabla u}{\rho}\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}\lesssim(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})})\|u\|_{L^2(B^{\frac{N}{2}})}.$$

Finally in using (4.24), (4.25) and all the previous bound, we get :

$$\|\psi(q,u,\mathcal{T})\|_{E^{\frac{N}{2}}}\leq C((C+1)\eta+R)^2. \tag{4.26}$$

Let c be such that $\|\cdot\|_{B^{\frac{N}{2}}}\leq c$ implies that : $\|\cdot\|_{L^\infty}\leq 1/3$. Then we choose R and η such that :

$$R\leq\inf((3C)^{-1},c,1),\ \text{and}\ \ \eta\leq\frac{\inf(R,c)}{C+1}\ .$$

So $(\mathcal{H})$ is verified and we have :

$$\psi(B(0,R))\subset B(0,R)\ .$$

2) Second step : Property of contraction

We consider $(q_1^{'},u_1^{'},\mathcal{T}_1^{'})$, $(q_2^{'},u_2^{'},\mathcal{T}_2^{'})$ in $B(0,R)$ where we note :

$$\theta_i=\mathcal{T}_i+\bar{\theta}\ ,\ \widetilde{T}_i=\Psi^{-1}(\theta_i)$$

and we set :

$$(\delta q=q_2^{'}-q_1^{'},\ \delta u=u_2^{'}-u_1^{'},\ \delta\mathcal{T}=\mathcal{T}_2^{'}-\mathcal{T}_1^{'})\ .$$

We have according to proposition 4.9 and (4.21) :

$$\begin{aligned}\|\psi_{(q_L,u_L,\mathcal{T}_L)}(q_2^{'},u_2^{'},\mathcal{T}_2^{'})-\psi_{(q_L,u_L,\mathcal{T}_L)}(q_1^{'},u_1^{'},\mathcal{T}_1^{'})\|_{E^{\frac{N}{2}}}\lesssim&\\ \|F(q_2,u_2,\mathcal{T}_2)-F(q_1,u_1,\mathcal{T}_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}&\\ +\|G(q_2,u_2,\mathcal{T}_2)-G(q_1,u_1,\mathcal{T}_1)\|_{L^1(B^{\frac{N}{2}-1})}&\\ +\|H(q_2,u_2,\mathcal{T}_2)-H(q_1,u_1,\mathcal{T}_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}.&\end{aligned}\tag{4.27}$$

where we have :

$$F(q_2,u_2,\mathcal{T}_2)-F(q_1,u_1,\mathcal{T}_1)=-\mathrm{div}(q_2u_2)+\mathrm{div}(q_1u_1)$$

$$\begin{aligned}&G(q_2,u_2,\mathcal{T}_2)-G(q_1,u_1,\mathcal{T}_1)=\delta u^*.\nabla u_2+u_1^*.\nabla\delta u+\nabla(\frac{1}{2}(K_{\rho_2}^{'}-K_{\rho_1}^{'})|\nabla\rho_2|^2)\\
&+\nabla(\frac{1}{2}K_{\rho_1}^{'}(|\nabla\rho_2|^2-|\nabla\rho_1|^2)-\mu(\bar{\rho})\Delta\delta u+\frac{\mu(\rho_1)}{\rho_1}\Delta\delta u-\zeta(\bar{\rho})\,\nabla\mathrm{div}(\delta u)\\
&+\nabla(K(\rho_1)\Delta(\bar{\rho}\delta q))+\nabla((K(\rho_2)-K(\rho_1))\Delta\rho_2)+[\frac{P_0^{'}(\rho_2)}{\rho_2}-\frac{P_0^{'}(\rho_1)}{\rho_1}]\bar{\rho}\,\nabla\delta q\\
&+(\frac{P_0^{'}(\rho_1)}{\rho_1}-\frac{P_0^{'}(\bar{\rho})}{\bar{\rho}})\nabla(\delta q)+\frac{P_0^{'}(\bar{\rho})}{\bar{\rho}}\nabla(\delta q)+\bar{T}P_1^{'}(\bar{\rho})\nabla\delta q+\bar{\rho}A\,\delta\mathcal{T}\,P_1^{'}(\rho_2)\\
&+\bar{\rho}\,A\theta_1(P_1^{'}(\rho_2)-P_1^{'}(\rho_1))\nabla q_2+\frac{1}{A}[P_1(\rho_2)-P_1(\rho_1)]\nabla\theta_2+\frac{1}{A}[P_1(\rho_1)-P_1(\bar{\rho})]\nabla\delta\mathcal{T}\\
&+\bar{\rho}\,A\theta_1P_1^{'}(\rho_1)\nabla\delta q+(\frac{\lambda^{'}(\rho_2)}{\rho_2}-\frac{\lambda^{'}(\rho_1)}{\rho_1})\nabla\rho_2\,\mathrm{div}u_2+\frac{\lambda^{'}(\rho_1)}{\rho_1}\,(\bar{\rho}\,\nabla\delta\,q\mathrm{div}u_2\\
&+\nabla\rho_1\mathrm{div}\,\delta u)+(\frac{\mu^{'}(\rho_2)}{\rho_2}-\frac{\mu^{'}(\rho_1)}{\rho_1})\,(du_2+\nabla u_2)\nabla\rho_2+\frac{\mu^{'}(\rho_1)}{\rho_1}(d(\delta u)+\nabla\delta u\nabla\rho_1\\
&+\bar{\rho}\,\nabla(\delta q)(du_2+\nabla u_2)).\end{aligned}$$

And we have for the part pertaining to H :

$$H(q_2,u_2,\mathcal{T}_2)-H(q_1,u_1,\mathcal{T}_1)=(\frac{1}{\rho_2}-\frac{1}{\rho_1})\,\text{div}(\chi(\rho_2)\nabla\theta_2)+\frac{1}{\rho_1}\text{div}((\chi(\rho_2)-\chi(\rho_1))\nabla\theta_2)$$

$$+\frac{1}{\rho_1}\,\text{div}(\chi(\rho_1)\nabla\delta\mathcal{T})+\left[\frac{P_1(\rho_1)}{\rho_1}-\frac{P_1(\rho_2)}{\rho_2}\right]\frac{\theta_1}{A}\text{div}u_1+\frac{P_1(\rho_2)}{\rho_2}\frac{\delta\mathcal{T}}{A}\text{div}u_1-\delta u^*.\nabla\theta_2$$

$$+\frac{P_1(\rho_2)}{\rho_2}\frac{\theta_2}{A}\text{div}\delta u-u_1^*\nabla\delta\mathcal{T}+(\frac{1}{\rho_2}-\frac{1}{\rho_1})D_2:\nabla u_2-\frac{1}{\rho_1}D_1:\nabla\delta u-\frac{1}{\rho_1}(D_2-D_1):\nabla u_2$$

Let us first estimate $\|F(q_2,u_2,\mathcal{T}_2)-F(q_1,u_1,\mathcal{T}_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$. We have :

$$\|F(q_2,u_2,\mathcal{T}_2)-F(q_1,u_1,\mathcal{T}_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|\text{div}((q_2-q_1)u_2)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|\text{div}(q_1(u_2-u_1))\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

$$\lesssim\|\delta q\|_{L^2(B^{\frac{N}{2}})}\|u_2\|_{L^2(B^{\frac{N}{2}})}+\|\delta q\|_{L^\infty(B^{\frac{N}{2}})}\|u_2\|_{L^1(B^{\frac{N}{2}+1})}+\|\delta q\|_{L^2(B^{\frac{N}{2}+1})}\|u_2\|_{L^2(B^{\frac{N}{2}})}$$
$$+\|q_1\|_{L^2(B^{\frac{N}{2}})}\|\delta u\|_{L^2(B^{\frac{N}{2}})}+\|q_1\|_{L^\infty(B^{\frac{N}{2}})}\|\delta u\|_{L^1(B^{\frac{N}{2}+1})}+\|q_1\|_{L^2(B^{\frac{N}{2}+1})}\|\delta u\|_{L^2(B^{\frac{N}{2}})}.$$

Next, we have to bound $\|G(q_2,u_2,\mathcal{T}_2)-G(q_1,u_1,\mathcal{T}_1)\|_{L^1(B^{\frac{N}{2}-1})}$. We treat only one typical term, the others are of the same form.
We use essentially the proposition 3.7 to treat the product and the composition, so we get :

$$\|\frac{\mu(\rho_1)}{\rho_1}\Delta(u_2-u_1)\|_{L^1(B^{\frac{N}{2}-1})}\lesssim(1+\|q_1\|_{L^\infty(B^{\frac{N}{2}})})\|\delta u\|_{L^1(B^{\frac{N}{2}+1})}.$$

Bounding $\|H(q_2,u_2,\mathcal{T}_2)-H(q_1,u_1,\mathcal{T}_1)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}-2})}$ is left to the reader. So we get in using the proposition 4.9 :

$$\|\Psi(q_2^{'},u_2^{'},\mathcal{T}_2^{'})-\Psi(q_1^{'},u_1^{'},\mathcal{T}_1^{'})\|_{E^{\frac{N}{2}}}\leq C\,\|(\delta q,\delta u,\delta\mathcal{T})\|_{E^{\frac{N}{2}}}\Big(\|(q_1^{'},u_1^{'},\mathcal{T}_1^{'})\|_{E^{\frac{N}{2}}}$$
$$+\|(q_2^{'},u_2^{'},\mathcal{T}_2^{'})\|_{E^{\frac{N}{2}}}+2\|(q_L,u_L,\mathcal{T}_L)\|_{E^{\frac{N}{2}}}\Big).$$

If one chooses R small enough, we end up with in using (4.27) and the previous estimates :

$$\|\Psi(q_2^{'},u_2^{'},\mathcal{T}_2^{'})-\Psi(q_1^{'},u_1^{'},\mathcal{T}_1^{'})\|_{E^{\frac{N}{2}}}\leq\frac{3}{4}\,\|(\delta q,\delta u,\delta\mathcal{T})\|_{E^{\frac{N}{2}}}.$$

We thus have the property of contraction and so by the fixed point theorem, we have existence of a solution to (NHV). Indeed we can see easily that $E^{\frac{N}{2}}$ is a Banach space.

3)Uniqueness of the solution :

The proof is similar to the proof of contraction, hence we will have the same type of estimates. So consider two solutions in $\widetilde{E}^{\frac{N}{2}}$: $(q_1,u_1,\mathcal{T}_1)$ and $(q_2,u_2,\mathcal{T}_2)$ of the system (NHV)

with the same initial data. With no loss of generality, one can assume that $(q_1, u_1, \mathcal{T}_1)$ is the solution found in the previous section.
We thus have :

$$(\mathcal{H}) \qquad \|q_1\|_{L^\infty([0,T]\times\mathbb{R}^N)} \leq \frac{1}{2}.$$

Let $\bar{T}$ be the largest time such that q_2 verifies $(\mathcal{H})$. By continuity, we have $0 < \bar{T} \leq T$.
Next we see that :

$$\delta q = q_2 - q_1,\ \delta u = u_2 - u_1,\ \delta\mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1$$

verifies the system :

$$\begin{cases} \partial_t \delta q + \mathrm{div}\delta u = F(q_2, u_2, \mathcal{T}_2) - F(q_1, u_1, \mathcal{T}_1), \\ \partial_t \delta u - \dfrac{\bar{\mu}}{\bar{\rho}}\Delta\delta u - \dfrac{\bar{\zeta}}{\bar{\rho}}\nabla\mathrm{div}\delta u - \bar{\rho}\bar{K}\nabla\Delta\delta q + (P_0^{'}(\bar{\rho}) + \bar{T}P_1^{'}(\bar{\rho}))\nabla\delta q + \dfrac{P_1(\bar{\rho})}{\bar{\rho}\psi^{'}(\bar{T})}\nabla\delta\mathcal{T} \\ \qquad\qquad = G(q_2, u_2, \mathcal{T}_2) - G(q_1, u_1, \mathcal{T}_1), \\ \partial_t \delta\mathcal{T} - \dfrac{\bar{\chi}}{\bar{\rho}}\Delta\delta\mathcal{T} + \dfrac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}}\mathrm{div}\delta u\S = \S H(q_2, u_2, \mathcal{T}_2) - H(q_1, u_1, \mathcal{T}_1). \end{cases}$$

We apply the proposition 4.9 on $[0, T_1]$ with $0 < T_1 \leq \bar{T}$ and we have :

$$\|(\delta q, \delta u, \delta\mathcal{T})\|_{\widetilde{E}^{\frac{N}{2}}} \leq A(T_1)\|(\delta q, \delta u, \delta\mathcal{T})\|_{\widetilde{E}^{\frac{N}{2}}}$$

where we have for T_1 enough small $A(T_1) \leq \frac{1}{2}$.

And we thus have : $\delta q = 0$, $\delta u = 0$, $\delta\mathcal{T} = 0$ on $[0, T_1]$ for T_1 small enough and we conclude after by connectivity. □

We treat now the specific case of $N = 2$, where we need more regularity for the initial data because we cannot use the proposition 3.6 in the case $N = 2$ with the previous initial data. Indeed we cannot treat some non-linear terms such as $\|\mathcal{T}\mathrm{div}u\|_{L^1(\widetilde{B}^{0,-1})}$ or $\|u^*.\nabla\theta\|_{L^1(\widetilde{B}^{0,-1})}$ because if we want to use proposition 3.6, we are in the case $s_1 + s_2 = 0$. This is the reason why more regularity is required.

We recall the space in which we are working :

$$\begin{aligned} E^{'} =& [C_b(\mathbb{R}_+, \widetilde{B}^{0,1+\varepsilon^{'}}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,3+\varepsilon^{'}})] \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,\varepsilon^{'}})^N \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,2+\varepsilon^{'}})^N] \\ & \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,-1+\varepsilon^{'}}) \cap L^1(\mathbb{R}_+, \widetilde{B}^{2,1+\varepsilon^{'}})] \end{aligned}$$

with $\varepsilon^{'} > 0$, $E^{'}$ being the space in which we have a solution . And $\widetilde{E}^{'}$ corresponds to the space where we show the uniqueness of solution.

$$\begin{aligned} \widetilde{E}^{'} =& [C_b(\mathbb{R}_+, \widetilde{B}^{0,1+\varepsilon^{'}}) \cap L^2(\mathbb{R}_+, \widetilde{B}^{1,2+\varepsilon^{'}})] \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,\varepsilon^{'}})^N \cap L^2(\mathbb{R}_+, \widetilde{B}^{1,1+\varepsilon^{'}})^N] \\ & \times [C_b(\mathbb{R}_+, \widetilde{B}^{0,-1+\varepsilon^{'}}) \cap L^2(\mathbb{R}_+, \widetilde{B}^{1,\varepsilon^{'}})]. \end{aligned}$$

Proof of theorem 2.2

The proof is similar to the previous one except that we have changed the functional space, in which the fixed point theorem is applied. So we want verify that the function ψ is contracting to apply the fixed point. We denote by $(q_L, u_L, \mathcal{T}_L)$ the solution of the linear system $(M^{'})$ with $F = G = H = 0$ and with initial data $(q_0, u_0, \mathcal{T}_0)$
Arguing as before, we get :

$$\begin{aligned} \|\psi(q,u,\mathcal{T})\|_{E'} \leq C\eta + \|F(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,1+\varepsilon'})} \\ + \|G(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,\varepsilon'})} + \|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})} . \end{aligned} \tag{4.28}$$

if :

$$\|q_0\|_{\widetilde{B}^{0,1+\varepsilon'}} + \|u_0\|_{\widetilde{B}^{0,\varepsilon'}} + \|\mathcal{T}_0\|_{\widetilde{B}^{0,-1+\varepsilon'}} \leq \eta.$$

Let us estimate $\|F(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,1+\varepsilon'})}$, $\|G(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,\varepsilon'})}$ and $\|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})}$, we just give two examples of estimates in the space $E^{'}$,the other estimates are left to the reader.

$$\|\mathrm{div}(qu)\|_{L^1(\widetilde{B}^{0,1+\varepsilon'})} \leq \|qu\|_{L^1(B^1)} + \|qu\|_{L^1(B^{2+\varepsilon'})},$$

and :

$$\|qu\|_{L^1(B^1)} \lesssim \|q\|_{L^2(B^1)}\|u\|_{L^2(B^1)},$$

$$\|qu\|_{L^1(B^{2+\varepsilon'})} \lesssim \|q\|_{L^\infty(B^1)}\|u\|_{L^1(B^{2+\varepsilon'})} + \|q\|_{L^2(B^{2+\varepsilon'})}\|u\|_{L^2(B^1)}.$$

We do similarly for $\|G(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,\varepsilon'})}$. The new difficulty appears on the last term $\|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})}$. In fact it's only for this term that that additional regularity is needed. Proposition 3.6 enables us to write :

$$\|\mathcal{T}\mathrm{div}u\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})} \lesssim \|\mathcal{T}\|_{L^\infty(\widetilde{B}^{0,-1+\varepsilon'})}\|u\|_{L^1(B^2)}.$$

$$\|u^*.\nabla\theta\|_{L^1(\widetilde{B}^{0,-1+\varepsilon'})} \lesssim \|\mathcal{T}\|_{L^1(\widetilde{B}^{1,1+\varepsilon'})}\|u\|_{L^\infty(B^0)}.$$

To conclude we follow the previous proof. Uniqueness in $\widetilde{E}'$ goes along the lines of the proof of uniqueness in dimension $N \geq 3$. □

4.3 Existence of a solution in the general case with small initial data

In this section we are interested by the general case where all the coefficients depend on the density and the temperature except κ. In this case to control the non-linear terms we need that θ be bounded, that's why we need to take more regular initial data to preserve the L^∞ bound.
As the initial data are more regular, we need to obtain new estimates in Besov spaces on the linear system $(M^{'})$.

Proposition 4.10. *Under conditions of proposition 4.8 with strict inequality, let* $(q,u,\mathcal{T})$ *be a solution of the system* (M') *on* $[0,T)$ *with initial conditions* $(q_0,u_0,\mathcal{T}_0)$ *such that :*

$$q_0\in\widetilde{B}^{s-1,s+1},u_0\in\widetilde{B}^{s-1,s},\mathcal{T}_0\in\widetilde{B}^{s-1,s}.$$

Moreover we suppose $1\leq r_1\leq+\infty$ *and :*

$$F\in\widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-1+\frac{2}{r_1}}),\ \ G\in\widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}}),\ \ H\in\widetilde{L}^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}}).$$

We then have the following estimate for all $r\in[r_1,+\infty]$ *:*

$$\|q\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+1+\frac{2}{r}})}+\|u\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})}+\|\mathcal{T}\|_{\widetilde{L}^r_T(\widetilde{B}^{s-1+\frac{2}{r},s+\frac{2}{r}})}\lesssim\|q_0\|_{\widetilde{B}^{s-1,s+1}}+\|u_0\|_{\widetilde{B}^{s-1,s}}$$

$$+\|\mathcal{T}_0\|_{\widetilde{B}^{s-1,s}}+\|F\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-1+\frac{2}{r_1}})}+\|G\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})}+\|H\|_{L^{r_1}_T(\widetilde{B}^{s-3+\frac{2}{r_1},s-2+\frac{2}{r_1}})}.$$

Proof :

The proof is similar to that of proposition 4.9. Low frequencies are treated as in proposition 4.9 because we don't change the regularity index for the low frequencies. On the other hand in the case of high frequencies the regularity index has changed so that we have to see what is new. For the medium frequencies we can proceed as in proposition 4.9.

Case of high frequencies :

We are going to work with $l\geq l_1$ where we will determine l_1 hereafter. We set :

$$f_l^2=\varepsilon B\|\Lambda q_l\|^2_{L^2}+B\|d_l\|^2_{L^2}+\|\mathcal{T}_l\|^2_{L^2}-2K\langle\Lambda q_l,d_l\rangle$$

where B and K will be chosen later on.
Then we take the scalar product of (4.3) with $\mathcal{T}_l$, we get :

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{T}_l\|^2_{L^2}+\alpha\|\nabla\mathcal{T}_l\|^2_{L^2}+\delta\langle\Lambda d_l,\mathcal{T}_l\rangle=0. \tag{4.29}$$

After we sum (4.9), (4.10) and (4.29) to get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(B\|d_l\|^2_{L^2}+\varepsilon B\|\Lambda q_l\|^2_{L^2}+\|\mathcal{T}_l\|^2_{L^2})+B\nu\|\nabla d_l\|^2_{L^2}+\alpha\|\nabla\mathcal{T}_l\|^2_{L^2}\\ -B\beta\langle\Lambda q_l,d_l\rangle-B\gamma\langle\Lambda\mathcal{T}_l,d_l\rangle+\delta\langle\Lambda d_l,\mathcal{T}_l\rangle=0.\end{aligned} \tag{4.30}$$

We sum (4.30) and (4.13) and we get :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}f_l^2+[B\nu\|\nabla d_l\|^2_{L^2}-K\|\Lambda d_l\|^2_{L^2}]+\alpha\|\nabla\mathcal{T}_l\|^2_{L^2}+[\beta K\|\Lambda q_l\|^2_{L^2}+\varepsilon K\|\Lambda^2 q_l\|^2_{L^2}]\\ &-B\beta\langle\Lambda q_l,d_l\rangle-B\gamma\langle\Lambda\mathcal{T}_l,d_l\rangle+\delta\langle\Lambda d_l,\mathcal{T}_l\rangle+K\nu\langle\Delta d_l,\Lambda q_l\rangle+\gamma K\langle\Lambda\mathcal{T}_l,\Lambda q_l\rangle=0.\end{aligned} \tag{4.31}$$

We interest us after only to the terms of high frequencies, so arguing as in proposition 4.9

we get :

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}f_l^2+\big[B\nu-(K+B\gamma\frac{a}{2c2^{2l_1}}+K\nu\frac{2}{b}+B\beta\frac{d}{2c2^{2l_1}}+\delta\frac{e}{2})\big]\|\Lambda d_l\|_{L^2}^2\\&+[\alpha-(B\gamma\frac{1}{2a}+\gamma K\frac{1}{2c'}+\delta\frac{1}{2e}\frac{1}{c2^{2l_1}})]\|\Lambda\mathcal{T}_l\|_{L^2}^2\\&+(\frac{\beta K}{c2^{2l_1}}+\varepsilon K-K\nu\frac{b}{2}-\gamma K\frac{c'}{2c2^{2l_1}}-B\beta\frac{1}{2d}\frac{1}{2^{2l_1}})\|\Lambda^2 q_l\|_{L^2}^2\leq 0\,.\end{aligned}\tag{4.32}$$

Let us assume that :

$$(1)\qquad\begin{aligned}&B\nu-(K+B\gamma\frac{a}{2c2^{2l_1}}+K\nu\frac{2}{b}+B\beta\frac{d}{2c2^{2l_1}}+\delta\frac{e}{2})>0,\\&\alpha-(B\gamma\frac{1}{2a}+\gamma K\frac{1}{2c'}+\delta\frac{1}{2e}\frac{1}{2^{2l_1}})>0,\\&\frac{\beta K}{c2^{2l_1}}+\varepsilon K-K\nu\frac{b}{2}-\gamma K\frac{c'}{2c2^{2l_1}}-B\beta\frac{1}{2d}\frac{1}{2^{2l_1}}>0.\end{aligned}$$

We recall that $\nu>0$, and $\alpha>0$. Next we want to have :

$$\varepsilon-\nu\frac{b}{2}>0.$$

So we take : $b=\frac{\nu}{\varepsilon}$ (we recall that $\varepsilon>0$). So with this choice we get (1) in taking $B,\ K$ small enough and l_1 big enough in following the same type of estimate as in the proof of the proposition 4.9. We have then for $l\leq l_0$, $l\geq l_1$ and c' small enough :

$$\frac{1}{2}\frac{d}{dt}f_l^2+c'2^{2l}f_l^2\leq 0.$$

and :

$$f_l\simeq\max(1,2^l)\|q_l\|_{L^2}+\|d_l\|_{L^2}+\|\mathcal{T}_l\|_{L^2}$$

Next we conclude in a similar way as in proposition 4.9. □

In the general case the coefficients depend on the temperature and we have to control the norm L^∞ in order to apply the theorems of composition. This motivates us to work in the following spaces :

$$\begin{aligned}F^{\frac{N}{2}}=&[C_b(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})\cap L^1(\mathbb{R}_+\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+3})]\times[C_b(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})\cap\\&L^1(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})]^N\times[C_b(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})\cap L^1(\mathbb{R}_+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})].\end{aligned}$$

Proof of theorem 2.3 :

The principle of the proof is similar to the previous one and we use the same notation. We define the map ψ as before with the same F, G and H except that our coefficients depends on the density and the temperature. We will verify only that ψ maps a ball $B(0,R)$ into itself, the end is left to the reader.

1) First step, uniform Bounds :

We set :

$$\alpha_0=\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1}}+\|u_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|\mathcal{T}_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}\,.$$

We denote $(q_L,u_L,\mathcal{T}_L)$ the solution of $(M^{'})$ with initial data $(q_0,u_0,\mathcal{T}_0)$. We have so in accordance with proposition 4.10 the following estimates :

$$\|(q_L,u_L,\mathcal{T}_L)\|_{F^{\frac{N}{2}}}\leq C\alpha_0\,,\tag{4.33}$$

$$\begin{aligned}\|\psi(q,u,\mathcal{T})\|_{F^{\frac{N}{2}}}\leq\;& C\alpha_0+\|F(q,u,\mathcal{T})\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\\&+\|G(q,u,\mathcal{T})\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|H(q,u,\mathcal{T})\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\,.\end{aligned}\tag{4.34}$$

Moreover we suppose for the moment that :

$$(\mathcal{H})\qquad\qquad \|q\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}\leq 1/2\;\;\text{and}\;\;\|\mathcal{T}\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}\leq 1/2\,.$$

We will now treat each term : $\|F(q,u,\mathcal{T})\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}$, $\|G(q,u,\mathcal{T})\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$ and $\|H(q,u,\mathcal{T})\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$.

1) We notice that :

$$\begin{aligned}\|\text{div}(qu)\|_{L^1(B^{\frac{N}{2}-1})}&\leq\|q\|_{L^2(B^{\frac{N}{2}})}\|u\|_{L^2(B^{\frac{N}{2}})},\\&\leq\|q\|_{L^2(B^{\frac{N}{2}+2})}\|u\|_{L^2(B^{\frac{N}{2}})}+\|u\|_{L^1(B^{\frac{N}{2}+2})}\|q\|_{L^\infty(B^{\frac{N}{2}})}.\end{aligned}$$

2) After we focus on $\|G(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$. We have according to proposition 3.7 :

$$\begin{aligned}\Big\|[\frac{\mu(\rho,\theta)}{\rho}-\frac{\mu(\bar{\rho},\bar{\theta})}{\bar{\rho}}]\Delta u\Big\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}&\leq\|K(q,\mathcal{T})\|_{L^\infty(B^{\frac{N}{2}})}\|u\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}\\&\lesssim(\|q\|_{L^\infty(B^{\frac{N}{2}})}+\|\mathcal{T}\|_{L^\infty(B^{\frac{N}{2}})})\|u\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}\,.\end{aligned}$$

We proceed in a similar way for the term :

$$(\zeta(\rho)-\zeta(\bar{\rho}))\nabla\text{div}u.$$

Next we have in using propositions 3.6 and 3.2 :

$$\begin{aligned}\|\rho\nabla\big(\kappa(\rho)-\kappa(\bar{\rho})\big)\Delta q)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\Big(&\|q\|_{L^1(\widetilde{B}^{\frac{N}{2}+2,\frac{N}{2}+3})}\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\\&+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\|q\|_{L^1(B^{\frac{N}{2}+2})}\Big)(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})})\end{aligned}$$

$$\|\,[\,\frac{\bar{\rho}\,P_0^{'}(\rho)}{\rho}-P_0^{'}(\bar{\rho})\,]\,\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}.$$

Next, we have to treat the following terms :

$$\|\big[\frac{\widetilde{T}\bar{\rho}\,P_1'(\rho)}{\rho}-\bar{T}P_1'(\bar{\rho})\big]\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\ \|L_1(q)L_2(\mathcal{T})\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|L_1(q)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|L_2(\mathcal{T})\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

where L_1 and L_2 are regular function in the sense of proposition 3.7. And we have :

$$\|L_1(q)\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})},$$
$$\|L_2(\mathcal{T})\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}.$$

Finally :

$$\|\big[\frac{\widetilde{T}\bar{\rho}\,P_1'(\rho)}{\rho}-\bar{T}P_1'(\bar{\rho})\big]\nabla q\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\ \|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q\|_{L^1(B^{\frac{N}{2}+1})}$$
$$+\big(\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}+\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\big)\|q\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+3})}$$

and :

$$\|(\frac{P_1(\rho)}{\rho\Psi'(\widetilde{T})}-\frac{P_1(\bar{\rho})}{\bar{\rho}\,\Psi'(\bar{T})})\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|L_1(q)\nabla\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|L_2(\mathcal{T})\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|L_1(q)L_2(\mathcal{T})\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

$$\|L_2(\mathcal{T})\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|\mathcal{T}\|_{L^1(B^{\frac{N}{2}+1})},$$
$$\|L_1(q)\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}.$$

Finally :

$$\|(\frac{P_1(\rho)}{\rho\Psi'(T)}-\frac{P_1(\bar{\rho})}{\bar{\rho}\,\Psi'(\bar{T})})\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\ (\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})})^2\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}$$
$$+\big(\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\big)\|\mathcal{T}\|_{L^1(B^{\frac{N}{2}+1,\frac{N}{2}+2})}.$$

After we have the following terms :

$$\|u^*.\nabla u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|u\|^2_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}.$$

And we have the terms coming from $\mathrm{div}(D)$. We will treat this one :

$$\|\,\frac{\lambda_1'(\rho,\theta)\nabla\rho\ \mathrm{div}u}{\rho}\,\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|L(q,\mathcal{T})\nabla\rho\,\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|\nabla\rho\,\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$\lesssim(1+\|q\|_{L^\infty(B^{\frac{N}{2}-1,\frac{N}{2}+1})}+\|\mathcal{T}\|_{L^\infty(B^{\frac{N}{2}-1,\frac{N}{2}})})\,\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})}\|u\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}.$$

Afterwards in the same way we can treat the terms of the type :

$$\frac{(du+\nabla u)\nabla\rho\ \mu_1'(\rho,\theta)}{\rho},\ \frac{(du+\nabla u)\nabla\theta\ \mu_2'(\rho,\theta)}{\rho}\ \text{ and }\ \frac{\lambda_2'(\rho,\theta)\nabla\theta\,\mathrm{div}u}{\rho}.$$

Finally, we have :

$$\|\nabla(K'_\rho|\nabla\rho|^2)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} \lesssim \|(K'_\rho-K'_{\bar\rho})|\nabla\rho|^2\|_{L^1(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}+\||\nabla\rho|^2\|_{L^1(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}$$
$$\lesssim (1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}+1})})\|q\|^2_{L^2(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}.$$

3) Let us finally estimate $\|H(q,u,\mathcal{T})\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$:

$$\|\frac{\mathrm{div}(\chi(\rho,\theta)\nabla\theta)}{\rho}-\frac{\chi(\bar\rho,\bar\theta)}{\bar\rho}\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq \|K(q)\mathrm{div}(K_1(q,\mathcal{T})\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|\mathrm{div}(K_1(q,\mathcal{T})\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|K(q)\Delta\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$

and in using the propositions 3.7 and 3.6 we get :

$$\|\mathrm{div}(K_1(q,\mathcal{T})\nabla\theta)\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim(\|q\|_{L^\infty(B^{\frac{N}{2}})}+\|\mathcal{T}\|_{L^\infty(B^{\frac{N}{2}})})\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}$$
$$+(\|q\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}+\|\mathcal{T}\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})})\|\mathcal{T}\|_{L^2(B^{\frac{N}{2}+1})}.$$

Next we have :

$$\|(\frac{TP_1(\rho)}{\rho}-\frac{\bar TP_1(\bar\rho)}{\bar\rho})\mathrm{div}\,u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|L_1(q)\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\|L_1(q)L_2(\mathcal{T})\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|L_2(\mathcal{T})\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

where :

$$\|L_1(q)\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})}$$
$$\|L_2(\mathcal{T})\mathrm{div}u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})}$$

so we get :

$$\|(\frac{TP_1(\rho)}{\rho}-\frac{\bar TP_1(\bar\rho)}{\bar\rho})\mathrm{div}\,u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\lesssim\ \|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|u\|_{L^1(B^{\frac{N}{2}+1})}\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$+\big(\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|\mathcal{T}\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\big)\|u\|_{L^1(B^{\frac{N}{2}+1})}\ .$$

To end with, we have the last two terms :

$$\|u^*.\nabla\theta\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|\mathcal{T}\|_{L^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}\|u\|_{L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$
$$\|\frac{D:\nabla u}{\rho}\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\leq\|K(q)\nabla u:\nabla u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}+\|\nabla u:\nabla u\|_{L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$$
$$\lesssim(1+\|q\|_{L^\infty(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})})\|u\|_{L^2(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|u\|_{L^2(B^{\frac{N}{2}+1})}.$$

Finally we have in using (4.33), (4.34) and the previous bounds :

$$\|\psi(q,u',\mathcal{T}')\|_{E^{N/2}}\leq C((C+1)\eta+R)^2 \tag{4.35}$$

Let c such that $\|\cdot\|_{B^{N/2}} \leq c$ implies that : $\|\cdot\|_{L^\infty} \leq 1/3$ then we choose R and α_0 such that :

$$R \leq \inf((3C)^{-1}, c, 1),\ \alpha_0 \leq \inf \frac{(R,c)}{C+1}\ .$$

So $(\mathcal{H})$ is verified and we have then :

$$\psi(B(0,R)) \subset B(0,R)\ .$$

Next one can proceed as in the proof of the theorem 2.1, we have to show the contraction of the application ψ to use the theorem of the fixed point.
The uniqueness of the solution in the space $F^{\frac{N}{2}}$ follows the same lines as in theorem 2.1. The details are left to the reader.

5 Local theory for large data

In this part we are interested in results of existence in finite time for general initial data with density bounded away from zero. We focus on the case where the coefficients depend only on the density with linear specific energy, and next we will treat the general case. As a first step, we shall study the linear part of the system (NHV) about non constant reference density and temperature, that is :

$$(N)\qquad \begin{cases} \partial_t q + \text{div} u = F, \\ \partial_t u - \text{div}(a\nabla u) - \nabla(b\text{div} u) - \nabla(c\Delta q) = G, \\ \partial_t \mathcal{T} - \text{div}(d\nabla \mathcal{T}) = H, \end{cases}$$

5.1 Study of the linearized equation

We want to prove a priori estimates for system (N) with the following hypotheses on a, b, c, d :

$$0 < c_1 \leq a < M_1 < \infty,\ 0 < c_2 \leq a + b < M_2 < \infty,\ 0 < c_3 \leq c < M_3 < \infty,$$
$$0 < c_4 \leq d < M_4 < \infty.$$

We remark that the last equation is just a heat equation with variable coefficients so that one can apply the following proposition proved in [13].

Proposition 5.11. *Let $\mathcal{T}$ solution of the heat equation :*

$$\partial_t \mathcal{T} - \text{div}(d\nabla \mathcal{T}) = H,$$

we have so for all index τ such that $-\frac{N}{2} - 1 < \tau \leq \frac{N}{2} - 1$ the following estimate for all $\alpha \in [1, +\infty]$:

$$\|\mathcal{T}\|_{\widetilde{L}^\alpha_T(B^{\tau+\frac{2}{\alpha}})} \leq \|\mathcal{T}_0\|_{B^\tau} + \|H\|_{\widetilde{L}^1_T(B^\tau)} + \|\nabla d\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})} \|\nabla \mathcal{T}\|_{\widetilde{L}^1_T(B^{\tau+1})}.$$

We are now interested by the first two equations of the system (N).

$$(N') \qquad \begin{cases} \partial_t q + \mathrm{div} u = F \\ \partial_t u - \mathrm{div}(a\nabla u) - \nabla(b\mathrm{div} u) - \nabla(c\Delta q) = G \end{cases}$$

where we keep the same hypothesis on a, b and c. We have then the following estimate of the solution in the spaces of Chemin-Lerner :

Proposition 5.12. *Let $1 \leq r_1 \leq r \leq +\infty$, $0 \leq s \leq 1$, $(q_0, u_0) \in B^{\frac{N}{2}+s} \times (B^{\frac{N}{2}-1+s})^N$, and $(F, G) \in \widetilde{L}^{r_1}_T(B^{\frac{N}{2}-2+s+2/r_1}) \times (\widetilde{L}^{r_1}_T(B^{\frac{N}{2}-3+s+2/r_1}))^N$.*

Suppose that ∇a , ∇b , ∇c belong to $\widetilde{L}^2_T(B^{\frac{N}{2}})$ and that $\partial_t c \in L^1_T(L^\infty)$.

Let $(q, u) \in (\widetilde{L}^r_T(B^{\frac{N}{2}+s+2/r}) \cap \widetilde{L}^2_T(B^{\frac{N}{2}+s+1})) \times ((\widetilde{L}^r_T(B^{\frac{N}{2}+s-1+2/r}))^N \cap (\widetilde{L}^2_T(B^{\frac{N}{2}+s})^N)$ be a solution of the system (N').

Then there exists a constant C depending only on r, r_1, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\kappa}$, c_1, c_2, M_1 and M_2 such that :

$$\begin{aligned} \|(\nabla q, u)\|_{\widetilde{L}^r_T(B^{\frac{N}{2}-1+s+2/r})}(1 - C\|\nabla c\|_{L^2_T(L^\infty)}) \leq \|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}}} + \|(\nabla F, G)\|_{\widetilde{L}^{r_1}_T(B^{\frac{N}{2}-3+s+2/r_1})} \\ + \|\nabla q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s})}\|\partial_t c\|_{L^1_T(L^\infty)} + \|(\nabla q, u)\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+s})}(\|\nabla a\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})} + \|\nabla b\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})} \\ + \|\nabla c\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}) . \end{aligned}$$

Proof :

Like previously we are going to show estimates on q_l and u_l. So we apply to the system the operator Δ_l , and we have then :

$$\partial_t q_l + \mathrm{div} u_l = F_l \tag{5.36}$$

$$\partial_t u_l - \mathrm{div}(a\nabla u_l) - \nabla(b\,\mathrm{div} u_l) - \nabla(c\Delta q_l) = G_l + R_l \tag{5.37}$$

where we denote :

$$R_l = \mathrm{div}([a, \Delta_l]\nabla u) - \nabla([b, \Delta_l]\mathrm{div} u_l) - \nabla([c, \Delta_l]\Delta q).$$

Performing integrations by parts and usinf (5.36) we have :

$$\begin{aligned} -\int_{\mathbb{R}^N} u_l \nabla(c\Delta q_l)dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} c|\nabla q_l|^2 dx - \int_{\mathbb{R}^N} \big(\mathrm{div} u_l\,(\nabla q_l.\nabla c) + \frac{|\nabla q|^2}{2}\partial_t c \\ + c.\nabla q_l.\nabla F_l\big)\, dx. \end{aligned}$$

Next, we take the inner product of (5.37) with u_l and we use the previous equality, we have then :

$$\frac{1}{2}\frac{d}{dt}(\|u_l\|_{L^2}^2+\int_{\mathbb{R}^N}c|\nabla q_l|^2dx)+\int_{\mathbb{R}^N}(a|\nabla u_l|^2+b|\mathrm{div}u_l|^2)dx=\int_{\mathbb{R}^N}((G_l+R_l).u_l\,dx$$
$$+\int_{\mathbb{R}^N}\big((\mathrm{div}u_l(\nabla c.\nabla q_l)+\frac{|\nabla q_l^2|}{2}\partial_t c+c\nabla q_l.\nabla F_l)\big)\,dx\,.$$

In order to recover some terms in Δq_l we take the inner product of the gradient of (5.36) with u_l, the inner product scalar of (5.37) with ∇q_l and we sum, we obtain then :

$$\frac{d}{dt}\int_{\mathbb{R}^N}\nabla q_l.u_l dx+\int_{\mathbb{R}^N}c(\Delta q_l)^2dx=\int_{\mathbb{R}^N}((G_l+R_l).\nabla q_l+|\mathrm{div}u_l|^2+u_l.\nabla F_l$$
$$-a\nabla u_l:\nabla^2 q_l-b\Delta q_l\mathrm{div}u_l)dx. \tag{5.38}$$

Let $\alpha>0$ small enough. We define :

$$k_l^2=\|u_l\|_{L^2}^2+\int_{\mathbb{R}^N}(\bar{\kappa}c|\nabla q_l|^2+2\alpha\nabla q_l.u_l)dx\,. \tag{5.39}$$

In using the previous inequality and the fact that $a_1b_1\leq\frac{1}{2}a_1^2+\frac{1}{2}b_1^2$, we have in summing :

$$\frac{1}{2}\frac{d}{dt}k_l^2+\frac{1}{2}\int_{\mathbb{R}^N}(a|\nabla u_l|^2+\alpha b|\Delta q_l|^2)dx\lesssim(\|G_l\|_{L^2}+\|R_l\|_{L^2})$$
$$\times(\alpha\|\nabla q_l\|_{L^2}+\|u_l\|_{L^2})+\|\nabla F_l\|_{L^2}(\alpha\|u_l\|_{L^2}+\|c\nabla q_l\|_{L^2})+\frac{1}{2}\|\partial_t c\|_{L^\infty}\|\nabla q_l\|_{L^2}^2 \tag{5.40}$$
$$+\|\nabla c\|_{L^\infty}\|\nabla q_l\|_{L^2}\|\nabla u_l\|_{L^2}\,.$$

For small enough α, we have according (5.39) :

$$\frac{1}{2}k_l^2\leq\|u_l\|^2+\int_{\mathbb{R}^N}\bar{\kappa}c|\nabla q_l|^2dx\leq\frac{3}{2}k_l^2\,. \tag{5.41}$$

Hence according to (5.40) and (5.41) :

$$\frac{1}{2}\frac{d}{dt}k_l^2+K2^{2l}k_l^2\leq k_l\,(\|G_l\|_{L^2}+\|R_l\|_{L^2}+\|\nabla F_l\|_{L^2})\|\partial_t c\|_{L^\infty}\|\nabla q_l\|_{L^2}+2^{2l}k_l^2\|\nabla c\|_{L^2}\,.$$

By integrating with respect to the time, we obtain :

$$k_l(t)\leq e^{-K2^{2l}t}k_l(0)+C\int_0^t e^{-K2^{2l}(t-\tau)}(\|\partial_t c\|_{L^\infty}\|\nabla q_l(\tau)\|_{L^2}+\|\nabla F_l(\tau)\|_{L^2}+\|G_l(\tau)\|_{L^2}$$
$$+\|R_l(\tau)\|_{L^2}+2^l k_l(\tau)\|\nabla c(\tau)\|_{L^2})d\tau\,.$$

After convolution inequalities imply that :

$$\|k_l\|_{L^r([0,T])}\leq(2^{-\frac{2l}{r}}k_l(0)+(2^{-2l(1+1/r-1/r_1)}\|(\nabla F_l,G_l)\|_{L_T^{r_1}(L^2)}+2^{-\frac{2l}{r}}\|R_l\|_{L_T^1(L^2)}$$
$$+2^{-\frac{2l}{r}}\|\nabla q_l\|_{L_T^\infty(L^2)}\|\partial_t c\|_{L_T^1(L^\infty)}+\|\nabla c\|_{L_T^2(L^\infty)}\|k_l\|_{L^r([0,T])}\,. \tag{5.42}$$

Moreover we have :

$$C^{-1}\,k_l \leq \|\nabla q_l\|_{L^2} + \|u_l\|_{L^2} \leq C\,k_l.$$

Finally multiplying by $2^{(\frac{N}{2}-1+s+\frac{2}{r})l}$ and using (5.41), we end up with :

$$\|(\nabla q,u)\|_{L^r_T(B^{\frac{N}{2}-1+s+2/r})}(1-C\|\nabla c\|_{L^2(L^\infty)}) \leq \|(\nabla F,G)\|_{\widetilde{L}^{r_1}_T(B^{\frac{N}{2}-3+s+2/r_1})}$$

$$\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1+s}} + \|\nabla q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s})}\|\partial_t c\|_{L^1_T(L^\infty)} + \sum_{l\in\mathbb{Z}} 2^{l(\frac{N}{2}+s-1)}\|R_l\|_{L^1_T(L^2)}\,.$$

Finally, applying lemma 1 on the appendix to bound the remainder term completes the proof

$$\sum_{l\in\mathbb{Z}} 2^{l(\frac{N}{2}+s-1)}\|R_l\|_{L^1_T(L^2)} \leq C\|a\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\|u\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})} + C\|b\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\|u\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}$$

$$+ C\|c\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\|q\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\,.$$

□

5.2 Local existence Theorem for temperature independent coefficients

We recall the space we will work with :

$$F_T = [\widetilde{C}_T(B^{\frac{N}{2}}) \cap L^1_T(B^{\frac{N}{2}+2})] \times [\widetilde{C}_T(B^{\frac{N}{2}-1})^N \cap L^1_T(B^{\frac{N}{2}+1})^N] \times [\widetilde{C}_T(B^{\frac{N}{2}-2}) \cap L^1_T(B^{\frac{N}{2}})]$$

endowed with the following norm :

$$\|(q,u,\mathcal{T})\|_{F_T} = \|q\|_{L^1_T(B^{\frac{N}{2}+2})} + \|q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})} + \|u\|_{L^1_T(B^{\frac{N}{2}+1})} + \|u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})}$$

$$+ \|\mathcal{T}\|_{L^1_T(B^{\frac{N}{2}})} + \|\mathcal{T}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2})}.$$

We will now prove the local existence of a solution for general initial data with a linear specific intern energy and coefficients independent of the temperature. The functional space we shall work with is larger than previously, the reason why is that the low frequencies don't play an important role as far as one is interested in *local* results.

In what follows, $N \geq 3$ is assumed.

Proof of the theorem 2.4 :

Let :

$$q^n = q^0 + \bar{q}^n,\ \rho^n = \bar{\rho}(1+q^n),\ u^n = u^0 + \bar{u}^n,\ \mathcal{T}^n = \mathcal{T}^0 + \bar{\mathcal{T}}^n \ \text{ and } \ \theta^n = \bar{\theta} + \mathcal{T}^n$$

where $(q^0,u^0,\mathcal{T}^0)$ stands for the solution of :

$$\begin{cases}\partial_t q^0-\Delta q^0=0,\\ \partial_t u^0-\Delta u^0=0,\\ \partial_t\mathcal{T}^0-\Delta\mathcal{T}^0=0,\end{cases}$$

supplemented with initial data :

$$q^0(0)=q_0\ ,\ u^0(0)=u_0\ ,\ \mathcal{T}^0(0)=\mathcal{T}_0.$$

Let $(\bar{q}_n,\bar{u}_n,\bar{\mathcal{T}}_n)$ be the solution of the following system :

$$(N_1)\quad\begin{cases}\partial_t\bar{q}_{n+1}+\operatorname{div}(\bar{u}_{n+1})=F_n,\\ \partial_t\bar{u}_{n+1}-\operatorname{div}\Big(\dfrac{\mu(\rho^n)}{\rho^n}\nabla\bar{u}^{n+1}\Big)-\nabla\Big(\dfrac{\zeta(\rho^n)}{\rho^n}\operatorname{div}(\bar{u}^{n+1})\Big)-\nabla(K(\rho^n)\Delta\bar{q}^{n+1})=G_n,\\ \partial_t\bar{\mathcal{T}}_{n+1}-\operatorname{div}\Big(\dfrac{\chi(\rho^n)}{1+q^n}\bar{\mathcal{T}}_{n+1}\Big)=H_n,\\ (\bar{q}_{n+1},\bar{u}_{n+1},\bar{\mathcal{T}}_{n+1})_{t=0}=(0,0,0),\end{cases}$$

where :

$$\begin{aligned}F_n=&-\operatorname{div}(q^nu^n)-\Delta q^0-\operatorname{div}(u^0),\\ G_n=&-(u^n)^*.\nabla u^n+\nabla(\frac{K'_{\rho^n}}{2}|\nabla\rho^n|^2)-\nabla(\frac{\mu(\rho^n)}{\rho^n})\operatorname{div}u^n+\nabla(\frac{\zeta(\rho^n)}{\rho^n})\operatorname{div}u^n\\ &+\frac{\lambda'(\rho^n)\nabla\rho^n\operatorname{div}u^n}{1+q^n}+\frac{(du^n+\nabla u^n)\,\mu'(\rho^n)\nabla\rho^n}{1+q^n}+\Big[\frac{P_1(\rho^n)}{\rho^n\psi'(\widetilde{T}^n)}\Big]\nabla\theta^n\\ &+\frac{[P_0'(\rho^n)+\widetilde{T}^nP_1'(\rho^n)]\nabla q^n}{1+q^n}-\Delta u^0+\operatorname{div}\Big(\frac{\mu(\rho^n)}{1+q^n}\nabla u^0\Big)\\ &+\nabla\Big(\frac{\mu(\rho^n)+\lambda(\rho^n)}{1+q^n}\operatorname{div}(u^0)\Big)+\nabla(K(\rho^n)\Delta q^0),\\ H_n=&\nabla(\frac{1}{1+q^n}).\nabla\theta^n\chi(\rho^n)-\frac{\widetilde{T}^nP_1(\rho^n)}{\rho^n}\operatorname{div}u^n-(u^n)^*.\nabla\theta^n+\frac{D_n:\nabla u^n}{\rho^n}\\ &-\Delta\theta_0+\operatorname{div}(\frac{\chi(\rho^n)}{1+q^n}\nabla\theta^0).\end{aligned}$$

1) First Step , Uniform Bound

Let ε be a small parameter and choose T small enough so that in using the estimate of the heat equation stated in proposition 3.5 we have :

$$(\mathcal{H}_\varepsilon)\quad\begin{aligned}&\|\mathcal{T}^0\|_{L^1_T(B^{\frac{N}{2}})}+\|u^0\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^0\|_{L^1_T(B^{\frac{N}{2}+2})}\leq\varepsilon,\\ &\|\mathcal{T}^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-2})}+\|u^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1})}+\|q^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})}\leq A_0.\end{aligned}$$

We are going to show by induction that :

$$(\mathcal{P}_n) \qquad\qquad \|(\bar{q}^n,\bar{u}^n,\bar{\mathcal{T}}^n)\|_{F_T}\leq\varepsilon.$$

As $(\bar{q}_0,\bar{u}_0,\bar{\mathcal{T}}_0)=(0,0,0)$ the result is true for $n=0$. We suppose now $(\mathcal{P}_n)$ true and we are going to show $(\mathcal{P}_{n+1})$.
To begin with we are going to show that $1+q^n$ is positive. Using the fact that $B^{\frac{N}{2}}\hookrightarrow L^\infty$ and that we take ε small enough, we have for $t\in[0,T]$:

$$\begin{aligned}\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}&\lesssim\|\text{div}\bar{u}^n\|_{L^1_T(B^{\frac{N}{2}})}+\|\text{div}(q^{n-1}u^{n-1})\|_{L^1_T(B^{\frac{N}{2}})}+\|\text{div}u^0\|_{L^1_T(B^{\frac{N}{2}})},\\&\lesssim 2\varepsilon+\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})},\end{aligned}$$

and :

$$\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}\leq\|q^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L^2_T(B^{\frac{N}{2}})}.$$

Hence :

$$\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\leq C_1(2\varepsilon+(A_0+\varepsilon)\varepsilon).$$

Finally we thus have :

$$\begin{aligned}\|1+q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}-\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\leq 1+q^n\leq\|1+q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)}\\+\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)},\end{aligned}$$

whence if ε is small enough :

$$\frac{c}{2\bar{\rho}}\leq 1+q^n\leq 1+\frac{\|\rho_0\|_{L^\infty}}{\bar{\rho}}.$$

In order to bound $(\bar{q}^n,\bar{u}^n,\bar{\mathcal{T}}^n)$ in F_T, we shall use proposition 5.12. For that we must check that the different hypotheses of this proposition adapted to our system (N_1) are satisfied, so we study the following terms :

$$a^n=\frac{\mu(\rho^n)}{1+q^n}\;,\;b^n=\frac{\zeta(\rho^n)}{1+q^n}\;,\;c^n=K(\rho_n)\;,\;d^n=\frac{\chi(\rho^n)}{1+q^n}.$$

In using $(\mathcal{P}_n)$ and by continuity of μ and the fact that μ is positive on $\big[\bar{\rho}\big(1+\min(q_0)\big)-\alpha,\bar{\rho}\big(1+\max(q_0)\big)+\alpha\big]$, we have :

$$0<c_1\leq a^n=\frac{\mu(\rho^n)}{1+q^n}\leq M_1\;.$$

We proceed similarly for the others terms.
Next, notice that :

$$\|\nabla a^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\leq\|\frac{\mu(\rho^n)}{1+q^n}-\mu(\bar{\rho})\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\leq C\|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}.$$

$$\|\nabla b^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\leq\|\frac{\zeta(\rho^n)}{1+q^n}-\zeta(\bar{\rho})\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\leq C\|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}$$

$$\|\nabla c^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\leq C\|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\;.$$

To end on our hypotheses we have to control $\partial_t c^n$ in norm $\|\cdot\|_{L^1_T(L^\infty)}$. As $B^{\frac{N}{2}} \hookrightarrow L^\infty$, it actually suffices to bound $\|\partial_t c^n\|_{L^1_T(B^{\frac{N}{2}})}$. We have :

$$\partial_t c^n = K^{'}(\rho^n)\partial_t q^n = K^{'}(\rho^n)(\mathrm{div}(q^{n-1}u^{n-1}) - \mathrm{div}(u^n)) \,.$$

And we have in using the propositions 3.6 and 3.7 :

$$\begin{aligned}\|K^{'}(\rho^n)(\mathrm{div}(q^{n-1}u^{n-1}) - \mathrm{div}(u^n))\|_{L^1_T(B^{\frac{N}{2}})} &\leq \|K^{'}(\rho^n)\mathrm{div}(q^{n-1}u^{n-1})\|_{L^1_T(B^{\frac{N}{2}})} \\ &\quad + \|K^{'}(\rho^n)\mathrm{div}(u^n)\|_{L^1_T(B^{\frac{N}{2}})}, \\ \lesssim (1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})(\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})} &+ \|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}) \\ \lesssim (1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})(\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})} &+ \|q^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})} \\ &+ \|q^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L^2_T(B^{\frac{N}{2}})}) \,.\end{aligned}$$

We now use proposition 5.11 to get the bound on $\bar{\mathcal{T}}^n$, so we obtain in taking $\tau = \frac{N}{2} - 2$:

$$\begin{aligned}\|\mathcal{T}^n\|_{L^1_T(B^{\frac{N}{2}})\cap L^\infty_T(B^{\frac{N}{2}-2})} \lesssim \big(\|H_n\|_{L^1_T(B^{\frac{N}{2}-2})} + \|\nabla(\frac{\chi(\rho^n)}{\rho^n})\|_{L^\infty_T(B^{\frac{N}{2}-1})} \\ \times \|\mathcal{T}^n\|_{L^1_T(B^{\frac{N}{2}})}\big).\end{aligned} \tag{5.43}$$

So we need to bound d^n in $L^\infty_T(B^{\frac{N}{2}})$:

$$\|\nabla d^n\|_{L^\infty_T(B^{\frac{N}{2}-1})} \leq C\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}.$$

Now we show by induction $(\mathcal{P}_{n+1})$. Finally, applying the estimates of propositions 5.12 and 5.11, we conclude that :

$$\begin{aligned}\|(\bar{q}^{n+1}, \bar{u}^{n+1}, \bar{\mathcal{T}}^{n+1})\|_{F_T}\big(1 - C(\|a^n\|_{L^2_T(B^{\frac{N}{2}+1})} + \|b^n\|_{L^2_T(B^{\frac{N}{2}+1})} + \|c^n\|_{L^2_T(B^{\frac{N}{2}+1})} \\ + \|d^n\|_{L^\infty_T(B^{\frac{N}{2}})} + \|\partial_t c^n\|_{L^1_T(B^{\frac{N}{2}})})\big) \leq \|(\nabla F_n, G_n)\|_{L^1_T(B^{\frac{N}{2}-1})} + \|H_n\|_{L^1_T(B^{\frac{N}{2}-2})} \,.\end{aligned} \tag{5.44}$$

Bounding the right-hand side may be done by applying propositions 3.6 and 3.7. For instance, we have :

$$\|F_n\|_{L^1_T(B^{N/2})} \leq \|\mathrm{div}(q^n u^n)\|_{L^1_T(B^{N/2})} + \|\mathrm{div} u^0\|_{L^1_T(B^{N/2})} + \|\Delta q^0\|_{L^1_T(B^{N/2})}.$$

Since :

$$\|u^n q^n\|_{L^1_T(B^{N/2+1})} \lesssim \|q^n\|_{L^\infty_T(B^{N/2})}\|u^n\|_{L^1_T(B^{N/2+1})} + \|q^n\|_{L^2_T(B^{N/2+1})}\|u^n\|_{L^2_T(L^\infty)},$$

we can conclude that :

$$\|F_n\|_{L^1_T(B^{N/2})} \leq C(A_0 + \varepsilon + \sqrt{\varepsilon})^2.$$

Next we want to control the different terms of G_n. According to propositions 3.6 and 5.12, we have :

$$\|(u^n)^*.\nabla u^n\|_{L^1_T(B^{\frac{N}{2}-1})}\lesssim\|u_n\|_{L^\infty_T(B^{\frac{N}{2}-1})}\|u_n\|_{L^1_T(B^{\frac{N}{2}+1})}$$

$$\begin{aligned}\|\nabla\bigg(\frac{K^{'}_{\rho^n}}{2}|\nabla\rho_n|^2\bigg)\|_{L^1_T(B^{\frac{N}{2}-1})}&\lesssim\|L(q^n)|\nabla\rho_n|^2\|_{L^1_T(B^{\frac{N}{2}})}+\||\nabla\rho_n|^2\|_{L^1_T(B^{\frac{N}{2}})}\\&\lesssim(1+\|q^n\|^2_{L^\infty_T(B^{\frac{N}{2}})})\|q^n\|^2_{L^2_T(B^{\frac{N}{2}+1})}.\end{aligned}$$

After we have :

$$\begin{aligned}\|\nabla(\frac{\mu(\rho^n)}{\rho^n})\mathrm{div}u_n\|_{L^1_T(B^{\frac{N}{2}-1})}&\lesssim\|\mathrm{div}u_n\|_{L^1_T(B^{\frac{N}{2}})}\|\nabla(\frac{\mu(\rho^n)}{\rho^n}-\frac{\mu(\bar{\rho})}{\bar{\rho}})\|_{L^\infty_T(B^{\frac{N}{2}-1})}\\&\leq C\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\;.\end{aligned}$$

$$\|\nabla(\frac{\zeta(\rho^n)}{\rho^n})\mathrm{div}u_n\|_{L^1_T(B^{\frac{N}{2}-1})}\leq C\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\;.$$

After we study the term coming from $\mathrm{div}(D)$:

$$\begin{aligned}\|\frac{\lambda^{'}(\rho^n)\nabla\rho^n\mathrm{div(u^n)}}{1+q^n}\|_{L^1_T(B^{\frac{N}{2}-1})}&\lesssim\|L(q^n)\nabla\rho^n\,\mathrm{div}(u^n)\|_{L^1_T(B^{\frac{N}{2}-1})}+\|\nabla\rho^n\,\mathrm{div}(u^n)\|_{L^1_T(B^{\frac{N}{2}-1})}\\&\lesssim(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\;.\end{aligned}$$

We proceed similarly for the following term :

$$\frac{(du^n+\nabla u^n)\mu^{'}(\rho^n)\nabla\rho^n}{1+q^n}.$$

Next we study the last terms :

$$\begin{aligned}\|\frac{[P^{'}_0(\rho^n)+\mathcal{T}^nP^{'}_1(\rho^n)]\nabla q^n}{1+q^n}\|_{L^1_T(B^{\frac{N}{2}-1})}&\lesssim\|K(q^n)\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})}+\|K(q^n)\mathcal{T}^n\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})}\\&\quad+\|\mathcal{T}^n\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})},\\&\lesssim T\|q^n\|^2_{L^\infty_T(B^{\frac{N}{2}})}+\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}-1})}\|q^n\|_{L^2_T(B^{\frac{N}{2}+1})}(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})\;.\end{aligned}$$

$$\|[\frac{P_1(\rho^n)}{\rho^n}]\nabla\theta^n\|_{L^1_T(B^{\frac{N}{2}-1})}\leq C(\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+1)\|\mathcal{T}^n\|_{L^1_T(B^{\frac{N}{2}})},$$

$$\|\mathrm{div}(\frac{\mu(\rho^n)}{1+q^n}\nabla u^0)\|_{L^1_T(B^{\frac{N}{2}-1})}\lesssim(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})\|u^0\|_{L^1_T(B^{\frac{N}{2}+1})}.$$

We proceed similarly with the other terms :

$$-\Delta u^0,\quad\nabla\bigg(\frac{\zeta(\rho^n)}{1+q^n}\mathrm{div}(u^0)\bigg),\quad\nabla(K(\rho^n)\Delta q^0).$$

Let us estimate now $\|H_n\|_{L^1_T(B^{\frac{N}{2}-2})}$. We obtain :

$$\|\nabla(\frac{1}{1+q^n}).\nabla\theta^n\chi(\rho^n)\|_{L^1_T(B^{\frac{N}{2}-2})}\lesssim\|K(q^n)\nabla(\frac{q^n}{1+q^n}).\nabla\theta^n\|_{L^1_T(B^{N/2-2})}$$

$$+C\|\nabla(\frac{q^n}{1+q^n}).\nabla\theta^n\|_{L^1_T(B^{\frac{N}{2}-2})},$$

$$\lesssim(1+\|q^n\|_{L^2_T(B^{\frac{N}{2}})})\|q^n\|_{L^2_T(B^{\frac{N}{2}+1})}\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}-1})}\,.$$

We have after these last two terms :

$$\|\frac{\mathcal{T}^nP_1(\rho^n)}{\rho^n}{\rm div}u^n\|_{L^1_T(B^{\frac{N}{2}-2})}\lesssim\|K(q^n){\rm div}u^n\|_{L^1_T(B^{\frac{N}{2}-2})}+\|K(q^n)K_1(\mathcal{T}^n){\rm div}u^n\|_{L^1_T(B^{\frac{N}{2}-2})}$$

$$+\|K_1(\mathcal{T}^n){\rm div}u^n\|_{L^1_T(B^{\frac{N}{2}-2})},$$

with K and K_1 regular in sense of the proposition 3.7 and :

$$\|K(q^n){\rm div}u^n\|_{L^1_T(B^{\frac{N}{2}-2})}\lesssim T\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^n\|_{L^\infty_T(B^{\frac{N}{2}-1})}\,,$$

$$\|K_1(\mathcal{T}^n){\rm div}u^n\|_{L^1_T(B^{N/2-2})}\lesssim\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}-2})}\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}\,,$$

so finally :

$$\|\frac{\mathcal{T}^nP_1(\rho^n)}{\rho^n}{\rm div}u^n\|_{L^1_T(B^{\frac{N}{2}-2})}\lesssim\big(T\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}-2})}\big)\|u^n\|_{L^\infty_T(B^{\frac{N}{2}-1})}$$

$$+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}-2})}\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}.$$

and, since $N\geq 3$:

$$\|u^n.\nabla\theta^n\|_{L^1_T(B^{\frac{N}{2}-2})}\leq\|u^n\|_{L^\infty_T(B^{\frac{N}{2}-1})}\|\theta^n\|_{L^1_T(B^{\frac{N}{2}})},$$

$$\|\nabla u^n:\nabla u^n\|_{L^1_T(B^{\frac{N}{2}-2})}\leq\|u^n\|^2_{L^2_T(B^{\frac{N}{2}})}.$$

We obtain in using (5.44) and the different previous inequalities :

$$\|(\bar{q}_{n+1},\bar{u}_{n+1},\bar{\mathcal{T}}_{n+1})\|_{F_T}\big(1-C2\sqrt{\varepsilon}(A_0+\sqrt{\varepsilon})\big)\leq C_1\big(\varepsilon(A_0+\sqrt{\varepsilon})^2+T(A_0+\sqrt{\varepsilon})\big).$$

In taking T and ε small enough we have $(\mathcal{P}_{n+1})$, so we have shown by induction that $(q^n,u^n,\mathcal{T}^n)$ is bounded in F_T.

Second Step : Convergence of the sequence

We will show that $(q^n,u^n,\mathcal{T}^n)$ is a Cauchy sequence in the Banach space F_T, hence converges to some $(q,u,\mathcal{T})\in F_T$.
Let :

$$\delta q^n=q^{n+1}-q^n,\;\delta u^n=u^{n+1}-u^n,\;\delta\mathcal{T}^n=\mathcal{T}^{n+1}-\mathcal{T}^n.$$

The system verified by $(\delta q^n, \delta u^n, \delta \mathcal{T}^n)$ reads :

$$\begin{cases} \partial_t \delta q^n + \mathrm{div} \delta u^n = F_n - F_{n-1}, \\ \partial_t \delta u^n - \mathrm{div}\Big(\dfrac{\mu(\rho^n)}{\rho^n} \nabla \delta u^n\Big) - \nabla\Big(\dfrac{\zeta(\rho^n)}{\rho^n} \mathrm{div}(\delta u^n)\Big) - \nabla(K(\rho^n)\Delta \delta q^n) = \\ \hfill G_n - G_{n-1} + G^{'}_n - G^{'}_{n-1}, \\ \partial_t \delta \mathcal{T}^n - \mathrm{div}\Big(\dfrac{\chi(\rho^n)}{1+q^n} \nabla \delta \mathcal{T}^n\Big) = H_n - H_{n-1} + H^{'}_n - H^{'}_{n-1}, \\ \delta q^n(0) = 0\ ,\ \delta u^n(0) = 0\ ,\ \delta \mathcal{T}^n(0) = 0, \end{cases}$$

where we define :

$$G^{'}_n = -\mathrm{div}\big((\frac{\mu(\rho^{n+1})}{\rho^{n+1}} - \frac{\mu(\rho^n)}{\rho^n})\nabla u^{n+1}\big) - \nabla\big((K(\rho^{n+1}) - K(\rho^n))\Delta q^{n+1}\big)$$
$$- \nabla\big((\frac{\zeta(\rho^{n+1})}{\rho^{n+1}} - \frac{\zeta(\rho^n)}{\rho^n})\mathrm{div}(u^{n+1})\big).$$

In the same way we have :

$$H^{'}_n = \mathrm{div}\big((\frac{\chi(\rho^{n+1})}{1+q^{n+1}} - \frac{\chi(\rho^n)}{1+q^n})\nabla \theta^{n+1}\big).$$

Applying propositions 5.11, 5.12, and using $(\mathcal{P}_n)$, we get :

$$\|(\delta q^n, \delta u^n, \delta \mathcal{T}^n)\|_{F_T} \leq C(\|F_n - F_{n-1}\|_{L^1_T(B^{N/2})} + \|G_n - G_{n-1} + G^{'}_n - G^{'}_{n-1}\|_{L^1_T(B^{N/2-1})}$$
$$+ \|H_n - H_{n-1} + H^{'}_n - H^{'}_{n-1}\|_{L^1_T(B^{N/2-2})}),$$

And by the same type of estimates as before, we get :

$$\|(\delta q^n, \delta u^n, \delta \mathcal{T}^n)\|_{F_T} \leq C\sqrt{\varepsilon}(1+A_0)^3 \|(\delta q^{n-1}, \delta u^{n-1}, \delta \mathcal{T}^{n-1})\|_{F_T}.$$

So in taking ε enough small we have that $(q^n, u^n, \mathcal{T}^n)$ is Cauchy sequence, so the limit $(q, u, \mathcal{T})$ is in F_T and we verify easily that this is a solution of the system.

Third step : Uniqueness

Suppose that $(q_1, u_1, \mathcal{T}_1)$ and $(q_2, u_2, \mathcal{T}_2)$ are solutions with the same initial conditions, and $(q_1, u_1, \mathcal{T}_1)$ corresponds to the previous solution.
Assume moreover that we have :

$$\|q_1(t)\|_{L^\infty} \leq \alpha,\ \ \forall t \in [0, T].$$

We set then :

$$\delta q = q_2 - q_1,\ \delta u = u_2 - u_1,\ \delta \mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1.$$

The triplet $(\delta q, \delta u, \delta\mathcal{T})$ satisfies the following system :

$$\begin{cases} \partial_t \delta q + \mathrm{div}\delta u = F_2 - F_1, \\ \partial_t \delta u - \mathrm{div}\big(\dfrac{\mu(\rho_2)}{\rho^2}\nabla\delta u\big) - \nabla\big(\dfrac{\zeta(\rho_2)}{\rho_2}\mathrm{div}(\delta u)\big) - \nabla\big(K(\rho_2)\Delta\delta q\big) = G_2 - G_1 + G^{'} \\ \partial_t \delta\mathcal{T} - \mathrm{div}(\dfrac{\chi(\rho_2)}{1+q_2}\nabla\delta\mathcal{T}) = H_2 - H_1 + H^{'}, \\ \delta q(0) = 0\ ,\ \delta u(0) = 0\ ,\ \delta\mathcal{T}(0) = 0 \end{cases}$$

with :

$$\begin{aligned} G^{'} =& -\mathrm{div}\Big(\big(\frac{\mu(\rho_2)}{\rho_2} - \frac{\mu(\rho_1)}{\rho_1}\big)\nabla u_2\Big) - \nabla\big((K(\rho_2) - K(\rho_1))\Delta q_2\big) \\ & - \nabla\Big(\big(\frac{\zeta(\rho_2)}{\rho_2} - \frac{\zeta(\rho_1)}{\rho_1}\big)\mathrm{div}(u_1)\Big), \\ H^{'} =& \mathrm{div}\Big(\big(\frac{\chi(\rho_2)}{1+q_2} - \frac{\chi(\rho_1)}{1+q_1}\big)\nabla\theta_2\Big)\ . \end{aligned}$$

Let $\bar{T}$ the largest time such that : $\|q_2\|_{L^\infty((0,\bar{T})\times\mathbb{R}^N} \le \alpha$. As $q_2 \in C([0,T]; B^{N/2})$, we have by continuity $0 < \bar{T} \le T$.
We are going to work on the interval $[0, T_1]$ with $0 < T_1 \le \bar{T}$ and we use the proposition 5.12, so we obtain in using the same type of estimates than in the part on the contraction :

$$\|(\delta q, \delta u, \delta\mathcal{T})\|_{\widetilde{F}_T^{\frac{N}{2}}} \le Z(T)\|(\delta q, \delta u, \delta\mathcal{T})\|_{\widetilde{F}_T^{\frac{N}{2}}}$$

with $Z(T) \to_{T\to 0} 0$.
We have then for T_1 small enough : $(\delta q, \delta u, \delta\mathcal{T}) = (0,0,0)$ on $[0, T_1]$ and by connectivity we finally conclude that :

$$q_1 = q_2,\ u_1 = u_2,\ \mathcal{T}_1 = \mathcal{T}_2 \ \text{ on } \ [0,T].$$

□

Proof of the theorem 2.5

In the special case $N = 2$, we need to take more regular initial data for the same reasons as in theorem 2.2. Indeed some terms like $\Psi(\theta)\mathrm{div}u$ or $u^*.\nabla\theta$ can't be controlled without more regularity.
The proof is similar to the previous proof of theorem 2.4 except that we have changed the functional space $F_T(2)$, in which the fixed point theorem is going to be applied. As we explain above we can use the paraproduct because we have more regularity, so we just see the term $u^*.\nabla\theta$. The other terms and the details are left to the reader.
We then have :

$$\|u^*.\nabla\theta\|_{L^1_T(\widetilde{B}^{-1,-1+\varepsilon'})} \lesssim \|\mathcal{T}\|_{L^1_T(\widetilde{B}^{0,1+\varepsilon'})}\|u\|_{L^\infty_T(B^0)}.$$

□

5.3 Local existence theorem in the general case

Now we suppose that all the coefficients depend on the temperature and on the density, and that conditions (C) and (D) are satisfied with strict inequalities.
One of the problem in the general case is the control of the L^∞ norm of the temperature θ in order to have control on the non linear terms where the physical coefficients appear. Indeed in the theorem of composition we need to control the norm L^∞.
So we must impose that θ_0 is in $B^{\frac{N}{2}}$ to hope a L^∞ control. And in consequence the others initial data have to be also more regular.

Proof of theorem 2.6 :

We proceed exactly like in theorem 2.4 except that we ask more regularity for the initial data. We define then :

$$q^n = q^0 + \bar{q}^n,\ u^n = u^0 + \bar{u}^n,\ \mathcal{T}^n = \mathcal{T}^0 + \bar{\mathcal{T}}^n \ \text{ and }\ \theta^n = \mathcal{T}^n + \bar{\mathcal{T}}$$

where $(q^0, u^0, \mathcal{T}^0)$ stands for the solution of :

$$\begin{cases} \partial_t q^0 - \Delta q^0 = 0, \\ \partial_t u^0 - \Delta u^0 = 0, \\ \partial_t \mathcal{T}^0 - \Delta \mathcal{T}^0 = 0, \end{cases}$$

supplemented with initial data :

$$q^0(0) = q_0\ ,\ u^0(0) = u_0\ ,\ \theta^0(0) = \theta_0.$$

Let $(\bar{q}_n, \bar{u}_n, \bar{\theta}_n)$ be the solution of the following system :

$$\begin{cases} \partial_t \bar{q}^{n+1} + \operatorname{div}(\bar{u}^{n+1}) = F_n, \\ \partial_t \bar{u}^{n+1} - \operatorname{div}\Big(\dfrac{\mu(\rho^n,\theta^n)}{\rho^n}\nabla \bar{u}^{n+1}\Big) - \nabla\Big(\dfrac{\zeta(\rho^n,\theta^n)}{\rho^n}\operatorname{div}(\bar{u}^{n+1})\Big) - \nabla(\kappa(\rho^n)\Delta \bar{q}^{n+1}) = G_n, \\ \partial_t \bar{\mathcal{T}}^{n+1} - \operatorname{div}\Big(\dfrac{\chi(\rho^n,\theta^n)}{1+q^n}\bar{\mathcal{T}}^{n+1}\Big) = H_n, \\ (\bar{q}^{n+1}, \bar{u}^{n+1}, \bar{\mathcal{T}}^{n+1})_{t=0} = (0,0,0). \end{cases}$$

where :

$$F_n = -\operatorname{div}(q^n u^n) - \Delta q^0 - \operatorname{div}(u^0),$$

$$\begin{aligned}
G_n =& -(u^n)^*.\nabla u^n + \nabla\big(\frac{K'_{\rho^n}}{2}|\nabla\rho^n|^2\big) - \nabla\big(\frac{\mu(\rho^n,\theta^n)}{\rho^n}\big)\operatorname{div}u^n + \nabla\big(\frac{\zeta(\rho^n,\theta^n)}{\rho^n}\big)\operatorname{div}u^n \\
&+ \frac{\lambda'_1(\rho^n,\theta^n)\nabla\rho^n\operatorname{div}u^n}{1+q^n} + \frac{(du^n+\nabla u^n)\,\mu'_1(\rho^n,\theta^n)\nabla\rho^n}{1+q^n} \\
&+ \frac{\lambda'_2(\rho^n,\theta^n)\nabla\theta^n\operatorname{div}u^n}{1+q^n} + \frac{(du^n+\nabla u^n)\,\mu'_2(\rho^n,\theta^n)\nabla\theta^n}{1+q^n} \\
&+ \frac{[P'_0(\rho^n)+\widetilde{T}^n P'_1(\rho^n)]\nabla q^n}{1+q^n} + \Big[\frac{P_1(\rho^n)}{\rho^n\psi'(T^n)}\Big]\nabla\theta^n \\
&- \Delta u^0 + \operatorname{div}\Big(\frac{\mu(\rho^n,\theta^n)}{1+q^n}\nabla u^0\Big) + \nabla\Big(\frac{\mu(\rho^n)+\lambda(\rho^n)}{1+q^n}\operatorname{div}(u^0)\Big) + \nabla(\kappa(\rho^n)\Delta q^0),
\end{aligned}$$

$$\begin{aligned}
H_n =& \nabla\big(\frac{1}{1+q^n}\big).\nabla\theta^n\chi(\rho^n,\theta^n) - \frac{\widetilde{T}^n P_1(\rho^n)}{\rho^n}\operatorname{div}u^n - (u^n)^*.\nabla\theta^n + \frac{D^n:\nabla u^n}{\rho^n} \\
&- \Delta\theta_0 + \operatorname{div}\big(\frac{\chi(\rho^n,\theta^n)}{1+q^n}\nabla\theta^0\big).
\end{aligned}$$

1) First Step , Uniform Bound

Let ε be a small positive parameter and choose T small enough so that in using the estimate of the proposition 3.5 we have :

$$(\mathcal{H}_\varepsilon) \qquad \begin{aligned}
&\|\mathcal{T}^0\|_{L^1_T(B^{\frac{N}{2}+2})} + \|u^0\|_{L^1_T(B^{\frac{N}{2}+2})} + \|q^0\|_{L^1_T(\widetilde{B}^{\frac{N}{2}+2,\frac{N}{2}+3})} \leq \varepsilon, \\
&\|\mathcal{T}^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})} + \|u^0\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}})} + \|q^0\|_{\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})} \leq A_0.
\end{aligned}$$

After we are going to show by induction that :

$$(\mathcal{P}_n) \qquad \|(\bar{q}_n,\bar{u}_n,\bar{\mathcal{T}}_n)\|_{F_T} \leq \varepsilon.$$

As $(\bar{q}_0,\bar{u}_0,\bar{\mathcal{T}}_0)=(0,0,0)$ the result is true for $(\mathcal{P}_0)$. We suppose now $(\mathcal{P}_n)$ true and we are going to show $(\mathcal{P}_{n+1})$.

To begin with we are going to show that $1+q^n$ is positive. In using the fact that $B^{\frac{N}{2}}\hookrightarrow L^\infty$ and that we can take ε enough small, we have for $t\in[0,T]$:

$$\begin{aligned}
\|q^n-q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq\;& C_1\big(\|\operatorname{div}(\bar{u}^n)\|_{L^1_T(B^{\frac{N}{2}})} + \|\operatorname{div}(q^{n-1}u^{n-1})\|_{L^1_T(B^{\frac{N}{2}})} \\
&+ \|\operatorname{div}(u^0)\|_{L^1_T(B^{\frac{N}{2}})}\big), \\
\leq\;& C_1\big(2\sqrt{T}\varepsilon + \|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}\big),
\end{aligned}$$

and by induction hypothesis $(\mathcal{P}_{n-1})$:

$$\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})} \leq \sqrt{T}\|q^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})} + \sqrt{T}\|q^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})},$$

thus :

$$\|q^n - q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq C_1\sqrt{T}(2\varepsilon + (A_0+\varepsilon)\varepsilon).$$

Finally we have :

$$\|1+q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)} - \|q^n - q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq 1+q^n \leq \|1+q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)} + \|q^n - q_0\|_{L^\infty((0,T)\times\mathbb{R}^N)},$$

$$\frac{c}{2\bar{\rho}} \leq 1+q^n \leq 1+\frac{\|\rho_0\|_{L^\infty}}{\bar{\rho}}\ .$$

So we have shown that :

$$(*) \qquad \|q^n\|_{L^\infty} \leq 2A_0$$

and that ρ^n is bounded away from 0.

To verify the uniform bound we use the propositions 5.11 and 5.12. For that we have to verify the different hypotheses of these propositions, so that we study the following terms :

$$a^n = \frac{\mu(\rho^n,\theta^n)}{1+q^n}\ ,\ b^n = \frac{\zeta(\rho^n,\theta^n)}{1+q^n}\ ,\ c^n = K(\rho_n)\ ,\ d^n = \frac{\chi(\rho^n,\theta^n)}{1+q^n}.$$

In using $(\mathcal{P}_n)$ and by continuity of μ and the fact that μ is positive on $[\bar{\rho}(1+\min(q^0)) - \alpha, \bar{\rho}(1+\max(q^0))+\alpha] \times [\bar{\theta}(1+\min(\mathcal{T}^0)) - \alpha, \bar{\rho}(1+\max(\mathcal{T}^0))+\alpha]$, we have :

$$0 < c_1 \leq a^n = \frac{\mu(\rho^n,\theta^n)}{1+q^n} \leq M_1\ .$$

We proceed similarly to verify the bounds of the other terms.
After we use the proposition 3.6 and the fact that q^n is bounded. We get :

$$\|\nabla a^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})} \lesssim \|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|\mathcal{T}^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\ ,$$

$$\|\nabla b^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})} \lesssim \|q^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|\mathcal{T}^n\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}\ .$$

$$\|\nabla c^n\|_{\widetilde{L}^2(B^{\frac{N}{2}})} \lesssim \|q^n\|_{\widetilde{L}^2_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}.$$

Next we want to estimate $\partial_t c^n$ in $L^1_T(B^{\frac{N}{2}})$. For that, we use the fact that :

$$\partial_t c^n = K^{'}(\rho^n)\partial_t q^n = K^{'}(\rho^n)(\mathrm{div}(q^{n-1}u^{n-1}) - \mathrm{div}(u^n))$$

And we have :

$$\|K'(\rho^n)(\mathrm{div}(q^{n-1}u^{n-1})-\mathrm{div}(u^n))\|_{L^1_T(B^{\frac{N}{2}})}\leq\|K'(\rho^n)\mathrm{div}(q^{n-1}u^{n-1})\|_{L^1_T(B^{\frac{N}{2}})}+\|K'(\rho^n)\mathrm{div}(u^n)\|_{L^1_T(B^{\frac{N}{2}})},$$

$$\lesssim(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})(\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^{n-1}u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}),$$

$$\lesssim(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})(\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^{n-1}\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^{n-1}\|_{L^1_T(B^{\frac{N}{2}+1})}+\|q^{n-1}\|_{L^2_T(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L^2_T(B^{\frac{N}{2}})})\,.$$

Now we want to show $(\mathcal{P}_{n+1})$ by induction and in this goal we will apply the estimates of proposition 5.11 and proposition 5.12. This is possible as we have verified above the validity of the hypotheses. We obtain :

$$\begin{aligned}\|(\bar{q}^{n+1},\bar{u}^{n+1},\bar{\mathcal{T}}^{n+1})\|_{F_T}\big(1-C(\|a^n\|_{L^2_T(B^{\frac{N}{2}+1})}+\|b^n\|_{L^2_T(B^{\frac{N}{2}+1})}+\|c^n\|_{L^2_T(B^{\frac{N}{2}+1})}\\+\|d^n\|_{L^2_T(B^{\frac{N}{2}})}+\|\partial_t c^n\|_{L^1_T(B^{\frac{N}{2}})})\big)\lesssim\|(\nabla F_n,G_n)\|_{L^1_T(B^{\frac{N}{2}})}+\|H_n\|_{L^1_T(B^{\frac{N}{2}})}\,.\end{aligned}\tag{5.45}$$

We want to control now the part on the right-hand side of (5.45), for this we do like previously in using proposition 3.6. We have :

$$\|F_n\|_{L^1_T(B^{\frac{N}{2}+1})}\leq\|\mathrm{div}(q^nu^n)\|_{L^1_T(B^{\frac{N}{2}+1})}+\|\mathrm{div}(u^0)\|_{L^1_T(B^{\frac{N}{2}+1})}+\|\Delta q^0\|_{L^1_T(B^{\frac{N}{2}+1})},$$

with :

$$\begin{aligned}\|u^nq^n\|_{L^1_T(B^{\frac{N}{2}+2})}&\leq\|q^n\|_{L^\infty_T(L^\infty)}\|u^n\|_{L^1_T(B^{\frac{N}{2}+2})}+\|q^n\|_{L^2_T(B^{\frac{N}{2}+2})}\|u^n\|_{L^2_T(B^{\frac{N}{2}})}\\&\leq\|q^n\|_{L^\infty_T(\widetilde{B}^{\frac{N}{2},\frac{N}{2}+1})}\|u^n\|_{L^1_T(B^{\frac{N}{2}+2})}+\sqrt{T}\|q^n\|_{\widetilde{L}^2_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2})}\|u^n\|_{L^\infty_T(B^{\frac{N}{2}})}.\end{aligned}$$

One ends up with :

$$\|F_n\|_{L^1_T(B^{\frac{N}{2}+1})}\leq C(A_0+\varepsilon+\sqrt{\varepsilon})^2.$$

Next we want to control the different terms of G_n. We have :

$$\|(u^n)^*.\nabla u^n\|_{L^1_T(B^{\frac{N}{2}})}\leq\sqrt{T}\|u^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^n\|_{L^2_T(B^{\frac{N}{2}+1})},$$

$$\|\nabla\left(\frac{K'_{\rho^n}}{2}|\nabla\rho_n|^2\right)\|_{L^1_T(B^{\frac{N}{2}})}\lesssim\sqrt{T}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})})\|q^n\|_{L^2_T(B^{\frac{N}{2}+2})}.$$

After we have :

$$\|\nabla(\frac{\mu(\rho^n,\theta^n)}{\rho^n})\mathrm{div}u_n\|_{L^1_T(B^{\frac{N}{2}})}\leq\|u^n\|_{L^2_T(B^{\frac{N}{2}+1})}(\sqrt{T}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}+\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}+1})})\,.$$

We treat similarly the term :

$$\nabla\left(\frac{\zeta(\rho^n,\theta^n)}{\rho^n}\right)\mathrm{div}u^n.$$

Next we study the term :

$$\|\frac{\lambda^{'}(\rho^n,\theta^n)\nabla\rho^n\mathrm{div(u^n)}}{1+q^n}\|_{L^1_T(B^{\frac{N}{2}})}\lesssim(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})\sqrt{T}\|u^n\|_{L^2_T(B^{\frac{N}{2}+1})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}\,.$$

We proceed similarly for the following term :

$$\frac{(du^n+\nabla u^n)\mu^{'}(\rho^n,\theta^n)\nabla\rho^n}{1+q^n}\,.$$

Next we have :

$$\begin{aligned}\|\frac{[P_0^{'}(\rho^n)+\widetilde{T}^nP_1^{'}(\rho^n)]\nabla q^n}{1+q^n}\|_{L^1_T(B^{\frac{N}{2}})}&\lesssim T\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}\\&\quad+T\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}\big(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\big)\,,\\ \|[\frac{P_1(\rho^n)}{\rho^n\Psi'(\widetilde{T}^n)}]\nabla\theta^n\|_{L^1_T(B^{\frac{N}{2}})}&\lesssim\big(\sqrt{T}(\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}})})\\&\quad+\sqrt{T}\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\big)\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}})}\,,\\ \|\mathrm{div}(\frac{\mu(\rho^n,\theta^n)}{1+q^n}\nabla u^0)\|_{L^1_T(B^{\frac{N}{2}})}&\lesssim\big(\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}})}\big)\|u^0\|_{L^1_T(B^{\frac{N}{2}+2})}\\&\quad+\|u^0\|_{L^2_T(B^{\frac{N}{2}+1})}\big(\sqrt{T}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}+\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}+1})}\big)\,.\end{aligned}$$

We proceed similarly with the other terms :

$$-\Delta u^0,\quad\nabla\bigg(\frac{\zeta(\rho^n,\theta^n)}{1+q^n}\mathrm{div}(u^0)\bigg),\quad\nabla(\kappa(\rho^n)\Delta q^0).$$

After we want to estimate the term $\|H^n\|_{L^1_T(B^{\frac{N}{2}})}$. So we have :

$$\begin{aligned}\|\nabla(\frac{1}{1+q^n}).\nabla\theta^n\chi(\rho^n,\theta^n))\|_{L^1_T(B^{\frac{N}{2}})}&\lesssim(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}})})\\&\quad\times\sqrt{T}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}+1})}\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}+1})}.\end{aligned}$$

We have after these last terms :

$$\begin{aligned}\|\frac{\widetilde{T}^nP_1(\rho^n)}{\rho^n}\mathrm{div}u^n\|_{L^1_T(B^{\frac{N}{2}})}&\lesssim\ \sqrt{T}\,\big(\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}+\|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}})}(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})\big)\\&\quad\times\|u^n\|_{L^2_T(B^{\frac{N}{2}+1})}.\end{aligned}$$

and :

$$\|u^n.\nabla\theta^n\|_{L^1_T(B^{\frac{N}{2}})}\leq\sqrt{T}\|u^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}+1})}$$

$$\|\nabla u^n:\nabla u^n\|_{L^1_T(B^{\frac{N}{2}})}\leq\|u^n\|^2_{L^2_T(B^{\frac{N}{2}})}$$

we obtain in using (5.45), the hypothesis of recurrence to the state n and the previous inequalities :

$$\|(\bar{q}_{n+1},\bar{u}_{n+1},\bar{\mathcal{T}}_{n+1})\|_{F_T}(1-C2\sqrt{\varepsilon}(A_0+\sqrt{\varepsilon})\leq C_1(\varepsilon(A_0+\sqrt{\varepsilon})^2+T(A_0+\sqrt{\varepsilon})).$$

In taking T and ε small enough we have $(\mathcal{P}_{n+1})$, so $(q^n, u^n, \mathcal{T}^n)$ is bounded in F_T. To conclude we proceed like in the proof of theorem 2.4 and we show in the same way that $(\bar{q}^n, \bar{u}^n, \bar{\mathcal{T}}^n)$ is a Cauchy sequence in F_T, hence converges to some $(q, u, \mathcal{T})$ in F_T. We verify after that (ρ, u, θ) is a solution of the system.

Uniqueness :

We compare the difference between two solutions with the same initial data and we use essentially the same type of estimates than in the part on contraction. The details are left to the reader.

6 Appendix

This part consists in one commutator lemma which enables us to conclude in proposition 5.12. Moreover we give the proof of proposition 3.7 on the composition of function in hybrid spaces adapted from Bahouri-Chemin in [2].

Lemma 1. *Let $0 \leq s \leq 1$. Suppose that $A \in \widetilde{L}^2_T(B^{\frac{N}{2}+1})$ and $B \in \widetilde{L}^2_T(B^{\frac{N}{2}-1+s})$. Then we have the following result :*

$$\|\partial_k[A, \Delta_l]B\|_{L^1_T(L^2)} \leq C c_l 2^{-l(\frac{N}{2}-1+s)} \|A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})} \|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}$$

with $\sum_{l\in\mathbb{Z}} c_l = 1$.

Proof :

We have the following decomposition :

$$uv = T_u v + T'_v u$$

where : $T_u v = \sum_{l\in\mathbb{Z}} S_{l-1} u \Delta_l v$ and : $T'_v u = \sum_{l\in\mathbb{Z}} S_{l+2} v \Delta_l u$.

We then have :

$$\partial_k[A, \Delta_l]B = \partial_k T'_{\Delta_l B} A - \partial_k \Delta_l T'_B A + [T_A, \Delta_l]\partial_k B + T_{\partial_k A} \Delta_l B - \Delta_l T_{\partial_k A} B. \tag{6.46}$$

From now on, we will denote by $(c_l)_{l\in\mathbb{Z}}$ a sequence such that :

$$\sum_{l\in\mathbb{Z}} c_l \leq 1.$$

Now we are going to treat each term of (6.46). According to the properties of quasi-orthogonality and the definition of T' we have :

$$\partial_k T'_{\Delta_l B} A = \sum_{m\geq l-2} \partial_k (S_{m+2} \Delta_l B \Delta_m A).$$

Next, in using Bernstein inequalities, we have :

$$\begin{aligned}\|\partial_k T'_{\Delta_l B} A\|_{L^1_T(L^2)} &\lesssim \sum_{m\geq l-2} 2^m \|\Delta_l B\|_{L^2_T(L^\infty)} \|\Delta_m A\|_{L^2_T(L^2)} \\ &\lesssim 2^{l\frac{N}{2}} \|\Delta_l B\|_{L^2_T(L^2)} \sum_{m\geq l-2} 2^{-m\frac{N}{2}} (2^{m(\frac{N}{2}+1)} \|\Delta_m A\|_{L^2_T(L^2)}) \\ &\lesssim 2^{-l(N/2-1+s)} (2^{l(\frac{N}{2}-1+s)} \|\Delta_l B\|_{L^2_T(L^2)}) \sum_{m\geq l-2} (2^{m(\frac{N}{2}+1)} \|\Delta_m A\|_{L^2_T(L^2)}) \\ &\lesssim c_l 2^{-l(N/2-1+s)} \|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})} \|A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}.\end{aligned}$$

Next, we will use the classic estimates on the paraproduct to bound the second term of the right-hand side of (6.46). We obtain then :

$$\|T'_B A\|_{L^1_T(B^{\frac{N}{2}+s})} \lesssim \|B\|_{L^2_T(B^{\frac{N}{2}-1+s})} \|A\|_{L^2_T(B^{\frac{N}{2}+1})}.$$

After in using the spectral localization we have :

$$\begin{aligned}\|\partial_k \Delta_l T'_B A\|_{L^1_T(L^2)} &\lesssim 2^l \|\Delta_l T'_B A\|_{L^1_T(L^2)} \\ &\lesssim c_l 2^{-l(\frac{N}{2}-1+s)} \|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})} \|A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}+1})}.\end{aligned}$$

According to the properties of orthogonality of Littlewood-Payley decomposition we have :

$$[T_A, \Delta_l]\partial_k B = \sum_{|m-l|\leq 4} [S_{m-1}A, \Delta_l]\Delta_m \partial_k B\ .$$

In applying Taylor formula, we obtain for $x \in \mathbb{R}^N$:

$$[S_{m-1}A, \Delta_l]\Delta_m \partial_k B(x) = 2^{-l} \int_{\mathbb{R}^N} \int_0^1 h(y)(y.S_{m-1}\nabla A(x - 2^{-l}\tau y))\Delta_m \partial_k B(x - 2^{-l}y) d\tau dy\ .$$

By an inequality of convolution we have :

$$\|[S_{m-1}A, \Delta_l]\Delta_m \partial_k B\|_{L^2} \lesssim 2^{-l} \|\nabla A\|_{L^\infty} \|\Delta_m \partial_k B\|_{L^2}\ .$$

So we get :

$$\|[T_A, \Delta_l]\partial_k B\|_{L^1_T(L^2)} \lesssim c_l 2^{-l(\frac{N}{2}-1+s)} \|\nabla A\|_{L^2_T(L^\infty)} \|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}\ .$$

Finally we have :

$$T_{\partial_k A}\Delta_l B = \sum_{|l-m|\leq 4} S_{m-1}\partial_k A \Delta_l \Delta_m B.$$

Hence :

$$\|T_{\partial_k A}\Delta_l B\|_{L^1_T(L^2)} \leq \|\partial_k A\|_{L^2_T(L^\infty)}\|\Delta_l B\|_{L^2_T(L^2)}\,.$$

And the classic estimates on the paraproduct give :

$$\|T_{\partial_k A}\Delta_l B\|_{L^1_T(B^{\frac{N}{2}-1})} \lesssim c_l 2^{-l(\frac{N}{2}-1+s)}\|\partial_k A\|_{\widetilde{L}^2_T(B^{\frac{N}{2}})}\|B\|_{\widetilde{L}^2_T(B^{\frac{N}{2}-1+s})}\,.$$

The proof is complete. □

Proof of proposition 3.7 :

To show (i) we use “first linéarisation” method introduced by Y.Meyer in [24], which amounts to write that :

$$F(u_1,u_2,\cdots,u_d)=\sum_{p\in\mathbb{Z}}(F(S_{p+1}u_1,\cdots,S_{p+1}u_d)-F(S_pu_1,\cdots,S_pu_d)).$$

According to Taylor formula, we have :

$$F(S_{p+1}u_1,\cdots,S_{p+1}u_d)-F(S_pu_1,\cdots,S_pu_d)=m_p^1u_1^p+\cdots+m_p^du_d^p$$

with $u_i^p=\Delta_p u_i$ and

$$m_p^i=\int_0^1\partial_i F(S_pu_1+su_1^p,\cdots,S_pu_i+su_i^p,\cdots,S_pu_d+su_d^p)ds.$$

Observe that :

$$\|m_p^i\|_{L^\infty}\leq\|\nabla F\|_{L^\infty}.$$

We have :

$$\Delta_pF(u_1,u_2,\cdots,u_d)=\Delta_p^1+\Delta_p^2$$

where we have decomposed the sum into two parts :

$$\Delta_p^{(1)}=\sum_{q\geq p}\Big(\Delta_p(u_1^qm_q^1)+\cdots+\Delta_p(u_d^qm_q^1)\Big),$$

$$\Delta_p^{(2)}=\sum_{q\leq p-1}\Big(\Delta_p(u_1^qm_q^1)+\cdots+\Delta_p(u_d^qm_q^1)\Big).$$

Now we bound $\|\Delta_p^{(1)}\|_{L^\rho_T(L^p)}$ in this way :

$$\begin{aligned}\|\Delta_p^{(1)}\|_{L^\rho_T(L^p)}&\leq\sum_{q\geq p}\big(\|u_1^q\|_{L^\rho_T(L^p)}\|m_q^1\|_{L^\infty_T(L^\infty)}+\cdots+\|u_d^q\|_{L^\rho_T(L^p)}\|m_q^1\|_{L^\infty_T(L^\infty)}\big)\\&\leq\sum_{q\geq p}\|\nabla F\|_{L^\infty}2^{-qs}c_q(\|u_1\|_{\widetilde{L}^\rho_T(B^s_p)}+\cdots+\|u_d\|_{\widetilde{L}^\rho_T(B^s_p)})\end{aligned}$$

with $(c_q) \in l^1(\mathbb{Z})$.

Therefore, since $s > 0$:

$$\sum_{p\in\mathbb{Z}} 2^{ps} \|\Delta_p^{(1)}\|_{L_T^\rho(L^p)} \leq C\|\nabla F\|_{L^\infty}(\|u_1\|_{\widetilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\widetilde{L}_T^\rho(B_p^s)}).$$

To bound $\|\Delta_p^{(2)}\|_{L_T^\rho(L^p)}$ we use the fact that the support of the Fourier transform of $\Delta_p^{(2)}$ is included in the shell $2^p\mathcal{C}$, so that according to Bernstein inequality :

$$\begin{aligned}\|\Delta_p^{(2)}\|_{L_T^\rho(L^p)} \leq& \sum_{q\leq p-1} (\|\Delta_p(u_1^q m_q^1)\|_{L_T^\rho(L^p)} + \cdots + \|\Delta_p(u_d^q m_q^1)\|_{L_T^\rho(L^p)}),\\ &\leq C2^{-p([s]+1)} \sum_{q\leq p-1} (\|\partial^{[s]+1}(u_1^q m_q^1)\|_{L_T^\rho(L^p)} + \cdots + \|\partial^{[s]+1}(u_d^q m_q^1)\|_{L_T^\rho(L^p)}).\end{aligned}$$

Moreover we have according to Faà-di-Bruno formula :

$$\partial^k m_q^i = \int_0^1 \sum_{l_1+\cdots+l_m=k, l_m\neq 0} A_{l_1\cdots l_m}^k F^{m+1}(S_q(u) + su^q) \prod_{n=1}^m \partial_{l_n}(S_q(u) + su^q))ds.$$

Hence we get for all $k \in \mathbb{N}$:

$$\|\partial^k m_q^i\|_{L_T^\infty(L^\infty)} \leq C_{u_i,k} 2^{qk}$$

with : $C_{u_i,k} = C(1 + \|u_i\|_{L_T^\infty(L^\infty)})$.

We have then :

$$\|\Delta_p^{(2)}\|_{L_T^\rho(L^p)} \leq C2^{-p([s]+1)} \sum_{q\leq p-1} c_q 2^{q(-s+[s]+1)} C_{u_1,\cdots,u_d}(\|u_1\|_{\widetilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\widetilde{L}_T^\rho(B_p^s)}).$$

Hence the result :

$$\sum_{p\in\mathbb{Z}} 2^{ps} \|\Delta_p^{(2)}\|_{L_T^\rho(L^p)} \leq C_{u_1,\cdots,u_d}(\|u_1\|_{\widetilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\widetilde{L}_T^\rho(B_p^s)}).$$

So the first part of the proof is complete.

For proving (ii) we proceed in the same way as before. We get :

$$F(u) = \sum_{q\in\mathbb{Z}} m_q u_q.$$

And we have for $p > 0$:

$$\Delta_p F(u) = \Delta_p^1 + \Delta_p^2$$

so :

$$\|\Delta_p^1\|_{L_T^\rho(L^2)} \leq \sum_{q\geq p} \|F'\|_{L^\infty} 2^{-qs_2} c_q \|u\|_{\widetilde{L}_T^\rho(\widetilde{B}^{s_1,s_2})}.$$

Hence in using convolution inequality :

$$\sum_{p>0} 2^{ps_2} \|\Delta_p^1\|_{L^\rho_T(L^2)} \leq C\|F^{'}\|_{L^\infty}\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.$$

After we get for all $s>0$:

$$\begin{aligned}\|\Delta_p^2\|_{L^\rho_T(L^2)} &\leq C2^{-p([s]+1)}\sum_{q\leq p-1}\|\partial^{[s]+1}(m_q u_q)\|_{L^\rho_T(L^2)},\\ &\leq C2^{-p([s]+1)}\sum_{q\leq p-1}2^{q([s]+1-s(q))}c_q\|u_q\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.\end{aligned}$$

with $s(q)=s_1$ or s_2.
So we obtain :

$$\begin{aligned}\sum_{p>0} 2^{ps_2}\|\Delta_p^2\|_{L^\rho_T(L^2)} \leq &\sum_{p>0}2^{-p([s]+1-s_2)}\sum_{q\leq 0}c_q 2^{q([s]+1-s_1)}\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}\\ &+\sum_{p>0}2^{-p([s]+1-s_2)}\sum_{0<q\leq p-1}c_q 2^{q([s]+1-s_2)}\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2})}.\end{aligned} \tag{6.47}$$

We have to choose s, so for the first term of (6.47) we just need that : $[s]+1-s_2>0$ and $[s]+1-s_1>0$ and for the second term of (6.47) we just have a inequality of convolution. So we can take $s=1+\max(s_1,s_2)$.

We do the same for $p<0$ and we have :

$$\begin{aligned}\sum_{p\leq 0}2^{ps_1}\|\Delta_p^1\|_{L^\rho_T(L^2)} \leq &\sum_{p\leq 0}2^{ps_1}\sum_{q\geq p}\|F^{'}\|_{L^\infty}2^{-qs_1}c_q\|u\|_{\widetilde{L}^\rho(\widetilde{B}^{s_1,s_2})}\\ &+\sum_{p\leq 0}2^{ps_1}\sum_{p\leq q\leq 0}\|F^{'}\|_{L^\infty}2^{-qs_2}c_q\|u\|_{\widetilde{L}^\rho(\widetilde{B}^{s_1,s_2})}.\end{aligned}$$

We conclude by a inequality of convolution.
And for the term Δ_p^2 we get :

$$\sum_{p\leq 0}2^{ps_1}\|\Delta_p^2\|_{\widetilde{L}^\rho(L^2)} \leq \sum_{p\leq 0}2^{-p([s]+1-s_1)}\sum_{q\leq p-1}c_q 2^{q([s]+1-s_1)}\|u\|_{\widetilde{L}^\rho(\widetilde{B}^{s_1,s_2})}.$$

For proving (iii) and (iv), one just has to use the following identity :

$$G(v)-G(u)=(v-u)\int_0^1 H(u+\tau(v-u))d\tau + G^{'}(0)(v-u)$$

where $H(w)=G^{'}(w)-G^{'}(0)$, and we conclude by using (i), (ii) and proposition 3.6. □

7 Annex : Notations of differential calculus

If $f : \mathbb{R}^n \to \mathbb{R}$, we denote :

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i = \partial_i f dx_i$$

with the summation convention on repeated indices and the simplified notation :

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$

The vector field associated to the differential df is noted ∇f,

$$\nabla f = \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$. Let denote f_i the i th component of f, and :

$$(df)_{i,j} = \partial_j f_i.$$

By analogy with the case of the scalar, we denote :

$$\nabla f = (df)^* \text{ , so } (\nabla f)_{i,j} = \partial_i f_j.$$

The curl of f is given by :

$$(\text{curl} f)_{i,j} = \partial_i f_j - \partial_j f_i.$$

The divergence of the vector field f is given by :

$$\text{div } f = tr\, df = \partial_i f_i.$$

If $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, with coefficients $a_{i,j}$,we set :

$$(\text{div } A)_j = \partial_i a_{i,j} \text{ , div } A = \text{div}(A \frac{\partial}{\partial x_j})\, dx_j.$$

In particular, for f scalar, we have :

$$\text{div}(fI) = df.$$

And finally we set :

$$A : B = a_{i,j} b_{i,j} \ .$$

Bibliographie

[1] D.M. Anderson, G.B McFadden and A.A. Wheller. Diffuse-interface methods in fluid mech. In Annal review of fluid mechanics, Vol. 30, pages 139-165. Annual Reviews, Palo Alto, CA, 1998.

[2] H. Bahouri and J.-Y. Chemin, Équations d'ondes quasilinéaires et estimation de Strichartz, Amer. J. Mathematics 121 (1999) 1337-1377.

[3] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Annales Scientifiques de l'école Normale Supérieure 14 (1981) 209-246.

[4] G. Bourdaud, Réalisations des espaces de Besov homogènes, Arkiv fur Mathematik 26 (1998) 41-54.

[5] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D Viscous shallow water equations and convergence to the quasi-geostrophic model. Comm. Math. Phys., 238(1-2) : 211-223, 2003.

[6] D. Bresch and B. Desjardins, Existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, Journal de Mathématiques Pures et Appliqués Volume 87, Issue 1, January 2007, Pages 57-90.

[7] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models : Korteweg,lubrication and shallow water systems. Comm. Partial Differential Equations, 28(3-4) : 843-868, 2003.

[8] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I. Interfacial free energy, J. Chem. Phys. 28 (1998) 258-267.

[9] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel, J.d'Analyse Math. 77 (1999) 27-50.

[10] J.-Y. Chemin, About Navier-Stokes system, Prépublication du Laboratoire d'Analyse Numérique de Paris 6 R96023 (1996).

[11] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, J.Differential Equations 121 (1992) 314-328.

[12] R. Danchin, Global Existence in Critical Spaces for Flows of Compressible Viscous and Heat-Conductive Gases, Arch.Rational Mech.Anal.160 (2001) 1-39

[13] Danchin.R, Local Theory in critical Spaces for Compressible Viscous and Heat-Conductive Gases,Communication in Partial Differential Equations 26 (78),1183-1233 (2001)

[14] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP,Analyse non linéaire 18,97-133 (2001)

[15] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working ,Arch. Rational Mech. Anal. 88(2) (1985) 95-133.

[16] E. Feireisl, Dynmamics of Viscous Compressible Fluids-Oxford Lecture Series in Mathematics and its Applications-26.

[17] M.E. Gurtin, D. Poligone and J. Vinals, Two-phases binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci.. 6(6) (1996) 815–831.

[18] H. Hattori and D.Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differential Equations 9(4) (1996) 323-342.

[19] H. Hattori and D. Li, Global Solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198(1) (1996) 84-97.

[20] D. Jamet, O. Lebaigue, N. Coutris and J.M. Delhaye, The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. J. Comput. Phys, 169(2) : 624–651, (2001).

[21] D.J. Korteweg. Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité. Arch. Néer. Sci. Exactes Sér. II, 6 :1-24, 1901.

[22] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Oxford University Press (1996)

[23] A.Mellet and A.Vasseur, On the barotropic compressible Navier-Stokes equation, Comm. Partial Differential Equations 32 (2007), no. 1-3, 431–452.

[24] Y.Meyer, Ondelettes et opérateurs, tome 3, Hermann, Paris, 1991

[25] J.S. Rowlinson, Translation of J.D van der Waals "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". J.Statist. Phys., 20(2) : 197-244, 1979.

[26] C. Truedelland W. Noll. The nonlinear field theories of mechanics. Springer-Verlag, Berlin, second edition, 1992.

Chapitre 3

Existence of solutions for compressible fluid models of Korteweg type

This chapter is devoted to proving existence of global weak solutions for a general isothermal model of capillary fluids derived by J.E Dunn and J.Serrin (1985) [5], which can be used as a phase transition model.
We distinguish two cases when the dimension $N=2$ and $N=1$, in the first case we need that $\frac{1}{\rho} \in L^{\infty}$, when $N=1$ we get a weak solution in finite time in the energy space.

1 Introduction

1.1 Derivation of Korteweg model

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by van der Waals and Korteweg [10] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [9, 16].
Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density.
The model derives from a Cahn-Hilliard like free energy (see the pioneering work by J.E.Dunn and J.Serrin in [5] and also in [1, 3, 6]), the conservation of mass reads :

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla P(\rho) = \kappa\rho\nabla\Delta\rho, \end{cases} \tag{1.1}$$

and :

$$\mu > 0 \ \text{ and } \ 2\mu + \lambda > 0$$

where ρ represents the density, $u\in\mathbb{R}^N$ the velocity, P is a general pressure, μ, λ are the coefficients of viscosity and κ the coefficient of capillarity. The term $\kappa\rho\nabla\Delta\rho$ corresponds to the capillarity term and allows to describe the variation of density in the interfaces between two phases.
One can now rewrite $\rho\nabla\Delta\rho$ in the following form to understand the difficulty of the non linear terms in distribution sense :

$$\begin{aligned}&\kappa\nabla\Delta\rho=\mathrm{div}K,\\ &\text{with }\; K_{i,j}=\frac{\kappa}{2}(\Delta\rho^2-|\nabla\rho|^2)\delta_{i,j}-\kappa\partial_i\rho\partial_j\rho.\end{aligned}\tag{1.2}$$

One can recall the classical energy inequality for the Korteweg system. Let $\bar{\rho}>0$ be a constant reference density, and let Π be defined by :

$$\Pi(s)=s\Big(\int_{\bar{\rho}}^{s}\frac{P(z)}{z^2}dz-\frac{P(\bar{\rho})}{\bar{\rho}}\Big)$$

so that $P(s)=s\Pi^{'}(s)-\Pi(s)$, $\Pi^{'}(\bar{\rho})=0$ and :

$$\partial_t\Pi(\rho)+\mathrm{div}(u\Pi(\rho))+P(\rho)\mathrm{div}(u)=0 \text{ in } \mathcal{D}^{'}((0,T)\times\mathbb{R}^N).$$

Notice that Π is convex as far as P is non decreasing (since $P^{'}(s)=s\Pi^{''}(s)$), which is the case for γ-type pressure laws or for Van der Waals law above the critical temperature. Multiplying the equation of momentum conservation in the system (1.1) by ρu and integrating by parts over $\mathbb{R}^N$, we obtain the following estimate :

$$\begin{aligned}\int_{\mathbb{R}^N}\big(\frac{1}{2}\rho|u|^2+(\Pi(\rho)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho|^2\big)(t)dx+2\int_0^t\int_{\mathbb{R}^N}\big(2\mu D(u):D(u)\\ +(\lambda+\mu)|\mathrm{div}u|^2\big)dx\le\int_{\mathbb{R}^N}\big(\frac{|m_0|^2}{2\rho}+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho_0|^2\big)dx.\end{aligned}\tag{1.3}$$

We will note in the sequel :

$$\mathcal{E}(t)=\int_{\mathbb{R}^N}\big(\frac{1}{2}\rho|u|^2+(\Pi(\rho)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho|^2\big)(t)dx,\tag{1.4}$$

It follows that assuming that the initial total energy is finite :

$$\epsilon_0=\int_{\mathbb{R}^N}\big(\frac{|m_0|^2}{2\rho}+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa}{2}|\nabla\rho_0|^2\big)dx<+\infty\,,$$

then we have the a priori bounds :

$$\begin{aligned}&\Pi(\rho)-\Pi(\bar{\rho}),\;\text{ and }\;\rho|u|^2\in L^1(0,\infty,L^1(\mathbb{R}^N)),\\ &\nabla\rho\in L^\infty(0,\infty,L^2(\mathbb{R}^N))^N,\;\text{ and }\;\nabla u\in L^2(0,\infty,\mathbb{R}^N)^{N^2}.\end{aligned}$$

In the sequel, we aim at solving the problem of global existence of weak solution for the system (1.1). Assuming that we dispose from a smooth approximates sequel $(\rho_n, u_n)_{n\in\mathbb{N}}$ of system (1.1), one can remark easily that it is difficult to pass to the limit in the quadratic term $\nabla\rho_n \otimes \nabla\rho_n$ which belongs to $L^\infty(L^1)$. According to the classical theorems on weak topology, $\nabla\rho_n \otimes \nabla\rho_n$ converges up to extraction to a measure ν, the difficulty is to prove that $\nu = \nabla\rho \otimes \nabla\rho$ where ρ is a limit of the sequence $(\rho_n)_{n\in\mathbb{N}}$ in appropriate space.
Another difficulty in compressible fluid mechanics is to deal with the vacuum and we will see that this problem does appear in the model of Korteweg, when estimating $\nabla\rho$. As a matter of fact, the existence of global solution in time for the model of Korteweg is still an open problem.
The first ones to have studied the problem, are R. Danchin and B. Desjardins in [4]. They showed that if we take initial data close to a stable equilibrium in the energy space and assume that we control the vacuum and the norm L^∞ of the density ρ, then we get some weak solution globally in time. Controlling the vacuum amounts to bounding $\frac{1}{\rho}$ in L^∞.
Recently D. Bresch, B. Desjardins and C-K. Lin in [2] got some global weak solutions for the isotherm Korteweg model with some specific viscosity coefficients. In effect, they assume that $\mu(\rho) = C\rho$ with $C > 0$ and $\lambda(\rho) = 0$. In choosing these specific coefficients they can get a gain of derivative on the density ρ and obtain an estimate for ρ in $L^2(H^2)$. It is easy now with this type of estimate on ρ to pass to the limit in the term of capillarity. However a new difficulty appears with the loss of information on the velocity u and it becomes difficult to pass to the limit in the term $\rho u \otimes u$. Hence the solutions of D. Bresch, B. Desjardins and C-K. Lin require some specific test functions which depend on the density ρ. Indeed the loss of information is on the vacuum.

Our result is in the same spirit as the one by Danchin and Desjardins : we want to improve the energy inequality which allows a gain of derivative on the density ρ. We show that we don't need to control ρ in L^∞ norm to get global weak solution. However the control of the vacuum seems necessary.
In section 2 we recall some definitions on the Orlicz space and some energy inequality on the system in these spaces. In section 3 we show a theorem on the global existence of weak solutions in dimension two under some conditions which amount to controlling the vacuum. In the last section we investigate the case of the dimension one, and we get a theorem of local existence of solution in the energy space and a result of global existence with small initial data.

2 Classical a priori estimates and Orlicz spaces

2.1 Classical a priori estimates

We first want to explain how it is possible to obtain natural a priori bounds which correspond to energy when the density is close to a constant state.

We first rewrite the mass equation in using renormalized solutions, and the momentum equation. We get the following formal identities :

$$\begin{cases} \frac{1}{\gamma-1}\frac{\partial}{\partial t}\big(\rho^\gamma-\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}(\rho-\bar{\rho})\big)+\text{div}\big[u\frac{\gamma}{\gamma-1}(\rho^\gamma-\bar{\rho}^{\gamma-1}\rho)\big]=u\cdot\nabla(\rho^\gamma) \\ \rho\frac{\partial}{\partial t}\frac{|u|^2}{2}+\rho u\cdot\nabla\frac{|u|^2}{2}-\mu\Delta u\cdot u-\xi\nabla\text{div}u\cdot u+au\cdot\nabla\rho^\gamma=\kappa\rho u\cdot\nabla\Delta\rho\ , \end{cases} \tag{2.5}$$

where we note : $\xi=\mu+\lambda$.
Therefore we find in summing the two equalities of (2.5) :

$$\begin{aligned} \frac{\partial}{\partial t}\big[\rho\frac{|u|^2}{2}+\frac{a}{\gamma-1}(\rho^\gamma+(\gamma-1)\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}\rho)\big]+\text{div}\big[u(\frac{a\gamma}{\gamma-1}(\rho^\gamma-\bar{\rho}^{\gamma-1}\rho) \\ +\rho\frac{|u|^2}{2})\big]-\mu\Delta u\cdot u-\xi\nabla\text{div}u\cdot u=\kappa\rho\nabla\Delta\rho\cdot u. \end{aligned} \tag{2.6}$$

Notation 2. *In the sequel we will note :*

$$j_\gamma(\rho)=\rho^\gamma+(\gamma-1)\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}\rho.$$

We may then integrate in space the equality (2.6) and we get :

$$\begin{aligned} &\int_{\mathbb{R}^N}\big(\rho\frac{|u|^2}{2}+\frac{a}{\gamma-1}j_\gamma(\rho)+\kappa|\nabla\rho|^2\big)(t,x)dx+\mu\int_0^t ds\int_{\mathbb{R}^N}|Du|^2dx \\ &+\xi\int_0^t ds\int_{\mathbb{R}^N}|\text{div}u|^2dx\le\int_{\mathbb{R}^N}\big(\rho_0\frac{|u_0|^2}{2}+\frac{a}{\gamma-1}j_\gamma(\rho_0)+\kappa|\nabla\rho_0|^2\big)(x)\,dx. \end{aligned} \tag{2.7}$$

Notation 3. *In the sequel we will note :*

$$\mathcal{E}^\gamma(t)=\int_{\mathbb{R}^N}\big(\rho\frac{|u|^2}{2}+\frac{a}{\gamma-1}(\rho^\gamma+(\gamma-1)\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}\rho)+\kappa|\nabla\rho|^2\big)(t,x)dx$$

$$\mathcal{E}_0^\gamma=\int_{\mathbb{R}^N}\big(\rho_0\frac{|u_0|^2}{2}+\frac{a}{\gamma-1}(\rho_0^\gamma+(\gamma-1)\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}\rho_0)dx$$

We now want to estimate this quantity $j_\gamma(\rho)$ and in this goal we recall some properties of Orlicz spaces.

2.2 Orlicz spaces

We begin by describing the Orlicz space in which we will work :

$$L^q_p(\mathbb{R}^N)=\{f\in L^1_{loc}(\mathbb{R}^N)/f1_{\{|f|\le\delta\}}\in L^p(\mathbb{R}^N),\ \ f1_{\{|f|\ge\delta\}}\in L^q(\mathbb{R}^N)\}$$

where δ is fixed, $\delta>0$.
First of all, it is not difficult to check that L^q_p does not depend on the choice of $\delta>0$ since $\frac{x^p}{x^q}$ is bounded from above and from below on any interval $[\delta_1,\delta_2]$ with $0<\delta_1\le\delta_2<+\infty$. In particular we deduce that we have :

$$f^\varepsilon\in L^{\frac{q}{\varepsilon}}_{\frac{p}{\varepsilon}}(\mathbb{R}^N)\ \ \text{if}\ \ f\in L^q_p(\mathbb{R}^N)\ \ \text{and}\ \ p,\,q\ge\varepsilon.$$

Obviously we get meas$\{|f| \geq \delta\} < +\infty$ if $f \in L^q_p(\mathbb{R}^N)$ and thus we have the embedding :

$$L^q_p(\mathbb{R}^N) \subset L^{q_1}_{p_1}(\mathbb{R}^N) \quad \text{if } 1 \leq q_1 \leq q < +\infty, \;\; 1 \leq p \leq p_1 < +\infty.$$

Next, we choose Ψ a convex function on $[0, +\infty)$ which is equal (or equivalent) to x^p for x small and to x^q for x large, then we can define the space $L^q_p(\mathbb{R}^N)$ as follows :

Definition 2.8. *We define then the Orlicz space $L^q_p(\mathbb{R}^N)$ as follows :*

$$L^q_p(\mathbb{R}^N) = \{f \in L^1_{loc}(\mathbb{R}^N)/\Psi(f) \in L^1(\mathbb{R}^N)\}.$$

We can check that $L^q_p(\mathbb{R}^N)$ is a linear vector space. Now we endow $L^q_p(\mathbb{R}^N)$ with a norm so that $L^q_p(\mathbb{R}^N)$ is a separable Banach space :

$$\|f\|_{L^q_p(\mathbb{R}^N)} = \inf\{t > 0/ \;\; \Psi(\frac{f}{t}) \leq 1\}.$$

We recall now some useful properties of Orlicz spaces.

Proposition 2.13. *The following properties hold :*

1. *Dual space : If $p > 1$ and $q > 1$ then $(L^q_p(\mathbb{R}^N))' = L^{q'}_{p'}(\mathbb{R}^N)$ where $q' = \frac{q}{q-1}$, $p' = \frac{p}{p-1}$.*
2. *$L^q_p = L^p + L^q$ if $1 \leq q \leq p < +\infty$.*
3. *Composition : Let F be a continuous function on $\mathbb{R}$ such that $F(0) = 0$, F is differentiable at 0 and $F(t)|t|^{-\theta} \to \alpha \neq 0$ at $t \to +\infty$. Then if $q \geq \theta$,*

$$F(f) \in L^{\frac{q}{\theta}}_{\frac{p}{\theta}}(\mathbb{R}^N) \;\; \text{if} \;\; f \in L^q_p(\mathbb{R}^N).$$

Now we can recall a theorem on the Orlicz space concerning the inequality of energy

Proposition 2.14. *The function $j_\gamma(\rho)$ is in $L^1(\mathbb{R}^N)$ if and only if $\rho - \bar{\rho} \in L^\gamma_2$.*

Proof :

On the set $\{|\rho - \bar{\rho}| \leq \delta\}$, ρ is bounded from above, since $\gamma > 1$ we thus deduce that $j_\gamma(\rho)$ is equivalent to $|\rho - \bar{\rho}|^2$ on the set $\{|\rho - \bar{\rho}| \leq \delta\}$.
Next on the set $\{|\rho - \bar{\rho}| \geq \delta\}$, we observe that for some $\nu \in (0, 1)$ and $C \in (1, +\infty)$, we have :

$$\nu|\rho - \bar{\rho}|^\gamma \leq j_\gamma(\rho) \leq C|\rho - \bar{\rho}|^\gamma.$$

□

Link with our energy estimate

We recall the definition of the fractional derivative operator Λ^s :

Definition 2.9. *We define the operator Λ^s as follows :*

$$\widehat{\Lambda^s f} = |\xi|^s \widehat{f}$$

We recall now some useful results, we start with a proposition coming from the theorem of interpolation by Riesz-Thorin.

Proposition 2.15. *The Fourier transform is continuous from L^p in L^q with $p \in [1,2]$, $q \in [2,+\infty]$ and :*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We recall the definition of homogeneous Sobolev space.

Definition 2.10. *Let $s \in \mathbb{R}$. f is in the homogeneous space $\dot{H}^s$ if :*

$$|\xi|^s \hat{f} \in L^2(\mathbb{R}^N).$$

Proposition 2.16. *Let $f \in \dot{H}^s$ with $s > 0$ and $f \in L^p + L^2$ with $1 \leq p < 2$. Then $f \in L^2$.*

Proof :

Indeed we have as $f \in \dot{H}^s$:

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{f}|^2 d\xi < +\infty$$

so $\widehat{f} 1_{\{|\widehat{f}| \geq 1\}} \in L^2(\mathbb{R}^N)$. And as $f = f_1 + f_2$ with $f_1 \in L^p(\mathbb{R}^N)$ and $f_2 \in L^2$. In using the Riesz-Thorin theorem, we know that $\widehat{f_1} \in L^q(\mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$.
As $q \geq 2$ we then have $\widehat{f} 1_{\{|\widehat{f}| \leq 1\}} \in L^2(\mathbb{R}^N)$. This concludes the proof. □

According to the above theorem and our energy estimate we get that for all $T \in \mathbb{R}$, $\rho - \bar{\rho} \in L^\infty(0,T; L^\gamma_2(\mathbb{R}^N))$.

Remark 3. *We have then in using previous properties on Orlicz spaces and (2.7) :*

- *if $\gamma \geq 2$ then $L^\gamma_2(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and so $\rho - \bar{\rho} \in L^\infty(H^1(\mathbb{R}^N))$.*
- *if $\gamma \leq 2$ then following the proposition 2.16 and the fact that $L^\gamma_2 = L^\gamma + L^2$ we get $\rho - \bar{\rho} \in L^\infty(H^1(\mathbb{R}^N))$.*

We can now explain what we mean by weak solution of problem (1.1) in dimension $N = 1, 2$.

Definition 2.11. *Let the couple (ρ_0, u_0) satisfy ;*

1. *$\rho_0 \in L^\gamma_2(\mathbb{R}^N)$, $\nabla \rho_0 \in L^2(\mathbb{R}^N)$ and $\frac{1}{\rho_0} \in L^\infty(\mathbb{R}^N)$.*
2. *$\rho_0 |u_0|^2 \in L^1(\mathbb{R}^N)$*
3. *$\rho_0 u_0 = 0$ whenever $x \in \{\rho_0 = 0\}$,*

We have the following definition :

- *A couple (ρ, u) is called a weak solution of problem (1.1) on $I \times \mathbb{R}^N$ with I an interval of $\mathbb{R}$ if :*

- $\rho \in L^\infty(L^\gamma_2(\mathbb{R}^N))$, $\nabla\rho \in L^\infty(L^2(\mathbb{R}^N))$, $\varphi\rho \in L^2(H^{1+\alpha}(\mathbb{R}^N))$ $\forall\alpha \in]0,1[$, *and* $\forall\varphi \in C_0^\infty(\mathbb{R}^N)$.
  - $\frac{1}{\rho} \in L^\infty((0,+\infty)\times\mathbb{R}^N)$.
  - $\nabla u \in L^2(L^2(\mathbb{R}^N))$, $\rho|u|^2 \in L^\infty(L^1)$.
  - *Mass equation holds in* $\mathcal{D}'(I\times\mathbb{R}^N)$.
  - *Momentum equation holds in* $\mathcal{D}'(I\times\mathbb{R}^N)^N$.
  - $\lim_{t\to 0^+}\int_{\mathbb{R}^N}\rho(t)\varphi = \int_{\mathbb{R}^N}\rho_0\varphi$, $\forall\varphi\in\mathcal{D}(\mathbb{R}^N)$,
  - $\lim_{t\to 0^+}\int_{\mathbb{R}^N}\rho u(t)\cdot\phi = \int_{\mathbb{R}^N}(\rho u)_0\cdot\phi$, $\forall\phi\in\mathcal{D}(\mathbb{R}^N)^N$.
- *The quantity* $\mathcal{E}_0^\gamma$ *is finite and inequality (2.7) holds a.e in* I.

3 Existence of global weak solutions for $N = 2$

3.1 Gain of derivability in the case $N = 2$

In the following theorem we are interested by getting a gain of derivative on the density ρ. This will enable us to treat in distribution sense the quadratic term $\nabla\rho\otimes\nabla\rho$.

Theorem 3.7. *Let* $N = 2$ *and* (ρ, u) *be a smooth approximate solution of the system* (1.1) *such that* $\frac{1}{\rho}\in L^\infty((0,T)\times\mathbb{R}^N)$. *Then there exists a constant* $\eta > 0$ *depending only on the constant intervening in the Sobolev embedding such that if :*

$$\|\nabla\rho_0\|_{L^2(\mathbb{R}^2)} + \|\sqrt{\rho_0}|u_0|\|_{L^2(\mathbb{R}^2)} + \|j_\gamma(\rho_0)\|_{L^1} \le \eta$$

then we get for all $\varphi\in C_0^\infty(\mathbb{R}^N)$ *:*

$$\|\varphi\rho^2\|_{L^2_T(H^{1+\frac{s}{2}})} \le M \qquad \text{with } 0\le s<2,$$

where M *depends only on the initial conditions data, on* T, *on* φ, *on* s *and on* $\|\frac{1}{\rho}\|_{L^\infty}$.

Remark 4. *In fact instead of supposing that* $\frac{1}{\rho}$, *we can just assume that* $\nabla\log\rho\in L^\infty(L^2)$. *This will imply that* ρ *will be a weight of Muckenhoupt.*

Remark 5. *In the sequel the notation of space follows those by Runst, Sickel in [14].*

Proof of theorem 3.7 :

Our goal is to get a gain of derivative on the density in using energy inequalities and in taking advantage of the term of capillarity. We need to localize the argument to control the low frequencies. Let $\varphi\in C_0^\infty(\mathbb{R}^N)$ and we have then :

$$\begin{aligned}\partial_t\text{div}(\varphi\rho u)+&\partial_{i,j}(\varphi\rho u_i u_j) - (\lambda+2\mu)\Delta(\varphi\text{div}u) + \Delta(\varphi P(\rho))\\ &= \frac{\kappa}{2}\Delta\left(\Delta(\varphi\rho^2) - \varphi|\nabla\rho|^2\right) - \kappa\partial^2_{i,j}(\varphi\partial_i(\rho)\partial_j(\rho)) + R_\varphi\end{aligned} \tag{3.8}$$

with :

$$R_\varphi=\frac{\partial}{\partial t}(\rho u\cdot\nabla\varphi)+(\partial_{i,j}\varphi)\rho u_iu_j+2\partial_i\varphi\,\partial_j(\rho u_iu_j)-(2\mu+\lambda)\Delta\varphi\text{div}u$$
$$-2(2\mu+\lambda)\nabla\varphi\cdot\nabla\text{div}u+\Delta\varphi a\rho^\gamma+2a\nabla\varphi\cdot\nabla(\rho^\gamma)-\frac{\kappa}{2}\Delta\varphi(\Delta\rho^2-|\nabla\rho|^2)$$
$$-\kappa\nabla\varphi\cdot\nabla(\Delta\rho^2-|\nabla\rho|^2)+\kappa(\partial^2_{i,j}\varphi)\partial_i\rho\partial_j\rho+2\kappa\partial_i\varphi\,\partial_j(\partial_i\rho\partial_j\rho)-\kappa\Delta(\nabla\varphi\cdot\nabla\rho^2)$$
$$-\frac{\kappa}{2}\Delta(\rho^2\Delta\varphi).$$

We can apply to the momentum equation the operator $\Lambda^{-1}(\Delta)^{-1}\text{div}$ in order to make appear a term in $\Lambda\rho^2$ coming from the capillarity. Then we obtain :

$$\begin{aligned}&\frac{\kappa}{2}\Lambda(\varphi\rho^2)+\frac{\kappa}{2}\Lambda^{-1}(\varphi|\nabla\rho|^2)+\kappa\Lambda^{-1}R_iR_j(\varphi\partial_i\rho\partial_j\rho)=-\Lambda^{-3}\frac{\partial}{\partial t}\text{div}(\varphi\rho\,u)\\&+\Lambda^{-1}R_iR_j(\varphi\rho\,u_iu_j)-(2\mu+\lambda)\Lambda^{-1}(\varphi\text{div}u)+\Lambda^{-1}(\varphi P(\rho))+\Lambda^{-1}(\Delta)^{-1}R_\varphi,\end{aligned}\tag{3.9}$$

where R_i denotes the classical Riesz operator. We multiply now the previous equality by $\Lambda^{1+s}(\varphi\rho^2)$ and we integrate on space and in time :

$$\begin{aligned}&\frac{\kappa}{2}\int_0^T\int_{\mathbb{R}^N}|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)|^2dxdt+\frac{\kappa}{2}\int_0^T\int_{\mathbb{R}^N}(|\varphi\nabla\rho|^2)\Lambda^s(\varphi\rho^2)dxdt\\&+\kappa\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}R_iR_j(\varphi\partial_i\rho\partial_j\rho)\Lambda^s(\varphi\rho^2)dxdt=\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\,u)\Lambda^{1+s}(\varphi\rho^2)(T)dx\\&-\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho_0\,u_0)\Lambda^{1+s}(\varphi\rho_0^2)dx-\int_0^T\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\,u)\Lambda^{1+s}\frac{\partial}{\partial t}(\varphi\rho^2)dxdt\\&+(2\mu+\lambda)\int_0^T\int_{\mathbb{R}^N}\varphi\text{div}u\,\Lambda^s(\varphi\rho^2)dxdt-\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}R_iR_j(\varphi\rho\,u_iu_j)\Lambda^s(\varphi\rho^2)dxdt\\&+\int_0^T\int_{\mathbb{R}^N}\varphi P(\rho)\Lambda^s(\varphi\rho^2)dxdt+\int_0^T\int_{\mathbb{R}^N}(\Delta)^{-1}R_\varphi\Lambda^s(\varphi\rho^2)dxdt.\end{aligned}\tag{3.10}$$

Now we want to control the term $\int_0^T\int_{\mathbb{R}^N}|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)|^2$. Before getting in the heart of the proof we want to rewrite the inequality (3.10) in particular the term :

$$\int_0^T\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\,u)\Lambda^{1+s}\frac{\partial}{\partial t}(\varphi\rho^2).$$

In this goal we recall the renormalized equation for $\varphi\rho^2$:

$$\frac{\partial}{\partial t}(\varphi\rho^2)+\text{div}(\varphi\rho^2u)=-\varphi\rho^2\text{div}u+r_\varphi,\tag{3.11}$$

with $r_\varphi=-\nabla\varphi\cdot\rho^2u$.

So in using the renormalized equation (3.11) we have :

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\,u)\Lambda^{1+s}\frac{\partial}{\partial t}(\varphi\rho^2)dxdt=-\int_0^T\int_{\mathbb{R}^N}\Lambda^{-2}\text{div}(\varphi\rho\,u)\Lambda^s(\varphi\rho^2\text{div}u)dxdt\\&-\int_0^T\int_{\mathbb{R}^N}\Lambda^{-2}\text{div}(\varphi\rho\,u)\Lambda^s\text{div}(\varphi\rho^2u)dxdt+\int_0^T\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\,u)r_\varphi.\end{aligned}\tag{3.12}$$

In combining (3.10) and (3.12) we get :

$$
\begin{aligned}
&\frac{\kappa}{2}\int_0^T\int_{\mathbb{R}^N}|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)|^2dxdt+\frac{\kappa}{2}\int_0^T\int_{\mathbb{R}^N}(|\varphi\nabla\rho|^2)\Lambda^s(\varphi\rho^2)dxdt\\
&+\kappa\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}R_iR_j(\varphi\partial_i\rho\partial_j\rho)\Lambda^s(\varphi\rho^2)dxdt=\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho\,u)\Lambda^{1+s}(\varphi\rho^2)(T)dx\\
&\quad-\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho_0\,u_0)\Lambda^{1+s}(\varphi\rho_0^2)dx-\int_0^T\int_{\mathbb{R}^N}\Lambda^{-2}\text{div}(\varphi\rho\,u)\Lambda^s(\varphi\rho^2\text{div}u)dxdt\\
&-\int_0^T\int_{\mathbb{R}^N}\Lambda^{-2}\text{div}(\varphi\rho\,u)\Lambda^s\text{div}(\varphi\rho^2u)dxdt+(2\mu+\lambda)\int_0^T\int_{\mathbb{R}^N}\varphi\text{div}u\,\Lambda^s(\varphi\rho^2)dxdt\\
&\qquad+\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}R_iR_j(\varphi\rho\,u_iu_j)\Lambda^s(\varphi\rho^2)dxdt+\int_0^T\int_{\mathbb{R}^N}\varphi P(\rho)\Lambda^s(\varphi\rho^2)dxdt\\
&\qquad+\int_0^T\int_{\mathbb{R}^N}(\Delta)^{-1}R_\varphi\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)dxdt+\int_0^T\int_{\mathbb{R}^N}\Lambda^{-3}\text{div}(\varphi\rho u)\Lambda^{1+s}r_\varphi dxdt.
\end{aligned}
\tag{3.13}
$$

In order to control $\int_0^T\int_{\mathbb{R}^N}|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)|^2$, it suffices to bound all the other terms of (3.13). Next we will have a control of $\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)$ and so a gain of $\frac{s}{2}$ derivative on the density. We start with the the most complicated term which requires a control of $\frac{1}{\rho}$ in L^∞.

1) $\int_0^T\int_{\mathbb{R}^N}(\varphi|\nabla\rho|^2)\Lambda^s(\varphi\rho^2)$:

By induction we have $\nabla(\varphi\rho^2)\in L^2_T(\dot{H}^{\frac{s}{2}})$. So by Sobolev embedding we get $\nabla(\varphi\rho^2)\in L^2(L^p)$ with $\frac{1}{p}=\frac{1}{2}-\frac{s}{4}$ (we remark that the case $s=2$ is critical for Sobolev embedding) . Now we have $\varphi\nabla\rho=\frac{\nabla(\varphi\rho^2)}{2\rho}-\frac{1}{2}\rho\nabla\varphi$ and we recall that by hypothesis $\frac{1}{\rho}\in L^\infty$, so we have $\varphi\nabla\rho\in L^2(L^p)$ because $\rho\nabla\varphi\in L^\infty(L^r)$ for all $1\leq r\leq+\infty$ as $\rho-\bar{\rho}\in L^\infty(H^1)$.
We now consider $\Lambda^s(\varphi\rho^2)$. We have by induction $\Lambda^s(\varphi\rho^2)\in L^2(\dot{H}^{1-\frac{s}{2}})$ and $\Lambda^s(\varphi\rho^2)\in L^2(L^2)$ because $\varphi\rho^2\in L^2(L^2)$ which enables us to control the low frequencies of $\Lambda^s(\varphi\rho^2)$. We have then $\Lambda^s(\varphi\rho^2)\in L^2(H^{1-\frac{s}{2}})$.
Finally by Hölder inequality we get $\varphi|\nabla\rho|^2\Lambda^s(\varphi\rho^2)\in L^1_T(L^1(\mathbb{R}^N))$ because $\frac{1}{2}+\frac{1}{p}+\frac{1}{q}=\frac{1}{2}+\frac{1}{2}-\frac{s}{4}+\frac{s}{4}=1$ and we have :

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^N}(\varphi|\nabla\rho|^2)\Lambda^s(\varphi\rho^2)dxdt&\lesssim\|\nabla\rho\|_{L^\infty_T(L^2)}\|\Lambda^s(\varphi\rho^2)\|_{L^2_T(L^q)}\|\varphi\nabla\rho\|\|_{L^2_T(L^p)},\\
&\lesssim\|\frac{1}{\rho}\|_{L^\infty_T(L^\infty)}\|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)\|^2_{L^2_T(L^2)}\|\nabla\rho\|_{L^\infty_T(L^2)}.
\end{aligned}
\tag{3.14}
$$

We proceed similarly for the term :

$$\int_0^T\int_{\mathbb{R}^N}\sum_{i,j}R_iR_j(\varphi\partial_i\rho\partial_j\rho)\Lambda^s(\varphi\rho^2)dxdt$$

because we have in following the same lines $\varphi\partial_i\rho\partial_j\rho\in L^2(L^q)$ with $\frac{1}{q}=1-\frac{s}{4}$ and we have the Riesz operator which is continuous from L^p in L^p for $1<p<+\infty$.

We next study the term $\int_{\mathbb{R}^N} \Lambda^{-3}\text{div}(\varphi\rho\, u)\Lambda^{1+s}(\varphi\rho^2)(t)dxdt$.

2) $\int_{\mathbb{R}^N} \Lambda^{-3}\text{div}(\varphi\rho\, u)\Lambda^{1+s}(\varphi\rho^2)dx$ **:**

We rewrite the term $\int_{\mathbb{R}^N} \Lambda^{-3}\text{div}(\rho\, u)\Lambda^{1+s}(\varphi\rho^2)$ on the form :

$$\begin{aligned}\int_{\mathbb{R}^N} \Lambda^{-3}\text{div}(\varphi\rho\, u)\Lambda^{1+s}(\varphi\rho^2)dx &= \int_{\mathbb{R}^N} \Lambda^{-1}\text{div}(\varphi\rho\, u)\Lambda^{-1+s}(\varphi\rho^2)dx\\ &= \sum_{1\leq i\leq N}\int_{\mathbb{R}^N} R_i(\varphi\rho u_i)\Lambda^{-1+s}(\varphi\rho^2)dx.\end{aligned}$$

As $\frac{1}{\rho} \in L^\infty_T(L^\infty)$ then we have $u \in L^\infty_T(L^2)$. We recall that $\rho-\bar{\rho} \in L^\infty(H^1)$ then $\varphi\rho \in L^\infty(L^p)$ for all $1 \leq p < +\infty$. We deduce that $\varphi\rho u$ belongs to $L^\infty(L^{2-\beta}\cap L^1)$ for $\beta > 0$. So we have $R_i(\varphi\rho u_i) \in L^\infty_T(L^r)$ for all $1 < r < 2$ by continuity of the operator R_i from L^p to L^p when $1 < p < +\infty$.

Case $1 \leq s < 2$ **:**

Next we have :

$$\nabla(\varphi\rho^2) = 2\varphi\rho\nabla\rho + \rho^2\nabla\varphi$$

then we get $\nabla(\varphi\rho^2) \in L^\infty(L^{2-\beta})$, in using the fact that $\rho-\bar{\rho} \in L^\infty(H^1)$ and Sobolev embedding with Hölder inequalities. We get then that $\varphi\rho^2$ belongs to $L^\infty(W^1_{2-\beta}) = L^\infty(H^1_{2-\beta})$. We have then $\Lambda^{s-1}(\varphi\rho^2) \in L^\infty(H^{2-s}_{2-\beta})$. By Sobolev embedding we get $\Lambda^{s-1}(\varphi\rho^2) \in L^\infty(L^p)$ with $\frac{1}{p} = \frac{1}{2-\beta} - \frac{2-s}{2} = \frac{1}{2-\beta} + \frac{s}{2} - 1$ with β small enough to avoid critical embedding.
Finally we get $R_i(\varphi\rho u)\Lambda^{-1+s}(\varphi\rho^2) \in L^1_T(L^1(\mathbb{R}^N)$. Indeed we have $\frac{1}{p}+\frac{1}{2-\beta} = \frac{2}{2-\beta}+\frac{s}{2}-1 < 1$ in choosing β small enough and $\frac{1}{p} + \frac{1}{1+\beta} = \frac{1}{1+\beta} - 1 + \frac{2}{2-\beta} + \frac{s}{2} > 1$ in choosing β small enough if necessary, we conclude by interpolation.
We have finally :

$$\Big|\int_{\mathbb{R}^N} \Lambda^{-3}\text{div}(\varphi\rho\, u)\Lambda^{1+s}(\varphi\rho^2)dx\Big| \leq M_0$$

with M_0 depending only on the initial data.

Case $0 < s < 1$ **:**

In this case we conclude by interpolation with the previous case. We now want to study the other terms coming from the renormalised equation (3.11).

3) $\int_0^T\int_{\mathbb{R}^N} \Lambda^{-2}\text{div}(\varphi\rho\, u)\Lambda^s(\varphi\rho^2\text{div}u)dxdt$ **and** $\int_0^T\int_{\mathbb{R}^N} \Lambda^{-2}\text{div}(\varphi\rho\, u)\Lambda^s(\text{div}(\varphi\rho^2\, u))dxdt$ **:**

We start with :

$$\int_0^T\int_{\mathbb{R}^N} \Lambda^{-1}\text{div}(\varphi\rho\, u)\Lambda^{s-1}(\text{div}(\varphi\rho^2\, u)) = \int_0^T\int_{\mathbb{R}^N} \text{div}(\varphi\rho\, u)\Lambda^{s-2}(\text{div}(\varphi\rho^2\, u)).$$

Case $1 \leq s < 2$ **:**

We have :

$$\mathrm{div}(\varphi\rho\, u) = u \cdot \nabla(\varphi\rho) + \varphi\rho\mathrm{div}u.$$

By Hölder inequalities and Sobolev embedding we get that $\mathrm{div}(\varphi\rho\, u)$ belongs to $L^2_T(L^{2-\beta})$ for all $\beta \in]0,1]$.
Next we rewrite $\mathrm{div}(\varphi\rho^2\, u)$ on the form :

$$\mathrm{div}(\varphi\rho^2\, u) = u \cdot \nabla(\varphi\rho^2) + \varphi\rho^2\mathrm{div}u.$$

As previously $\mathrm{div}(\varphi\rho^2\, u)$ is in $L^2_T(L^{2-\beta})$ for all $\beta \in]0,1]$. Now by Sobolev embedding we have $\Lambda^{s-2}\mathrm{div}(\varphi\rho^2\, u) \in L^2_T(L^p)$ with $\frac{1}{p} = \frac{1}{2-\beta} - \frac{2-s}{2}$ with β small enough to avoid critical Sobolev embedding.
We conclude that $\mathrm{div}(\varphi\rho\, u)\Lambda^{s-2}(\mathrm{div}(\varphi\rho^2\, u))$ is in $L^1_T(L^1)$ because $\frac{1}{2-\beta} + \frac{1}{p} = \frac{2}{2-\beta} - \frac{2-s}{2} = \frac{2}{2-\beta} - 1 + \frac{s}{2} < 1$ with β small enough if necessary and $1 + \frac{1}{p} > 1$, so we obtain the result by interpolation. Finally we have :

$$|\int_0^T \int_{\mathbb{R}^N} \Lambda^{-2}\mathrm{div}(\varphi\rho\, u)\Lambda^s(\mathrm{div}(\varphi\rho^2\, u))dxdt| \leq M_0$$

with M_0 depending only on the initial data.

Case $0 < s < 1$ **:**

We have the result by interpolation with the previous case.

Next we proceed similarly for :

$$\int_0^T \int_{\mathbb{R}^N} \Lambda^{-1}\mathrm{div}(\varphi\rho\, u)\Lambda^{s-1}(\varphi\rho^2\mathrm{div}u)dxdt.$$

4) Last terms

We now want to concentrate us on the following term :

$$\int_0^T \int_{\mathbb{R}^N} \sum_{i,j} R_iR_j(\varphi\rho\, u_iu_j)\Lambda^s(\varphi\rho^2)dxdt$$

We know that $u \in L^\infty(L^2)$ and $Du \in L^2(L^2)$ then $u \in L^2_T(H^1)$ and by Hölder inequalities and Sobolev embedding we can show that $\varphi\rho\, u_iu_j \in L^2_T(L^2)$ and so $R_iR_j(\varphi\rho\, u_iu_j) \in L^2_T(L^2)$.
We have seen that $\Lambda^s(\varphi\rho^2) \in L^2(H^{1-\frac{s}{2}})$ then we have as $1 - \frac{s}{2} > 0$:

$$\|\Lambda^s(\varphi\rho^2)\|_{L^2_T(L^2)} \leq M_0 + \|\varphi\rho^2\|^{\beta}_{L^2_T(\dot{H}^{1-\frac{s}{2}})}$$

with $0 < \beta < 1$.
We have then :

$$|\int_0^T \int_{\mathbb{R}^N} \sum_{i,j} R_iR_j(\varphi\rho\, u_iu_j)\Lambda^s(\varphi\rho^2)dxdt| \leq M_0 + \|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)\|^{\beta}_{L^2(L^2)}$$

with $0<\beta<1$ and M_0 depending only on the initial data.
We are interested by the term :

$$\int_0^t\int_{\mathbb{R}^N}\varphi\text{div}u\,\Lambda^s(\varphi\rho^2)dxdt$$

We have then $\text{div}u\in L^2(L^2)$ and we have shown that $\Lambda^s(\varphi\rho^2)\in L^\infty(L^2)$ so we conclude in the same way than the previous term.
We finally conclude with the term :

$$\int_0^T\int_{\mathbb{R}^N}\varphi P(\rho)\Lambda^s(\varphi\rho^2)dxdt.$$

Similarly we have $\Lambda^s(\varphi\rho^2)\in L^2_T(L^2)$ and $\varphi P(\rho)\in L^2_T(L^2)$ because $\varphi\rho$ is in $L^\infty(H^1)$, and we conclude by Sobolev embedding.
To finish we have to control the term with R_φ and r_φ that we leave to the reader. These terms are easy because they are more regular than the preceding terms. We finally get in using all the previous inequalities :

$$\|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)\|^2_{L^2(L^2)}\leq C_0\|\nabla\rho\|_{L^\infty(L^2)}\|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)\|^2_{L^2(L^2)}+C_1\|\Lambda^{1+\frac{s}{2}}(\varphi\rho^2)\|^{2\beta}_{L^2(L^2)}+M_0$$

with $0<\beta<1$ and C_0, C_1, M_0 depends only of the norm of initial data.
As we have by energy inequalities $\|\nabla\rho\|_{L^\infty(L^2)}\leq\varepsilon<1$, we can conclude that :

$$\|\varphi\rho^2\|_{L^2(\dot{H}^{1+\frac{s}{2}})}\leq M_0$$

with M_0 depending only on the initial data. □

Corollary 1. *Let the assumptions of theorem 3.7 be satisfied with the exception of hypothesis on the viscosity coefficients which is replaced by :*
- *it exists $c>0$, $s_0>0$ such that $\forall s\geq s_0$ $\mu(s)>c$.*
- *it exists $c_1>0$, $m>1$ such that $\forall s\geq 0$ $\mu(s)\leq\frac{s^m}{c_1}$.*
- *it exists $c_2>0$, $m^{'}>1$ such that $\forall s\geq 0$ $\lambda(s)\leq\frac{s^{m^{'}}}{c_2}$.*

We get similar results as in theorem 3.7 for all $\varphi\in C_0^\infty(\mathbb{R}^N)$:

$$\|\varphi\rho^2\|_{L^2_T(\dot{H}^{1+\frac{s}{2}})}\leq M\quad\text{with }0\leq s<2,$$

where M depends only on the initial conditions data, on T, on φ, on s and on $\|\frac{1}{\rho}\|_{L^\infty}$.

Proof :

Indeed the only term which changes are the viscosity term. We have then in applying the same operation as in the proof of theorem 3.7 the following terms to control :

$$\int_0^T\int_{\mathbb{R}^N}\varphi\mu(\rho)D(u)\Lambda^s(\varphi\rho^2)dxdt\quad\text{and}\quad\int_0^T\int_{\mathbb{R}^N}\varphi R_{i,j}(\lambda(\rho)\partial_i u_j)\Lambda^s(\varphi\rho^2)dxdt.$$

We conclude easily by the fact that $\sqrt{\mu(\rho)}\nabla u$ and $\sqrt{\lambda(\rho)}\nabla u$ belong to $L^2(L^2)$ and as $\rho\in L^\infty(H^1)$ we get that $\sqrt{\mu(\rho)}$ and $\sqrt{\lambda(\rho)}$ belongs to $L^\infty(L^p)$ for all $p>2$ by Sobolev embedding. We can then conclude by Hölder inequality. □

3.2 Existence of global weak solutions for $N = 2$ away from vacuum

We may now turn to our compactness result. First, we assume that a sequence $(\rho_n, u_n)_{n\in\mathbb{N}}$ of approximate weak solutions has been constructed by a mollifying process, which have suitable regularity to justify the formal estimates like the energy estimate and the previous theorem.
Moreover this sequence $(\rho_n, u_n)_{n\in\mathbb{N}}$ has initial data $((\rho_0)_n, (u_0)_n))$ close to the energy space. In using the above energy inequalities, we assume that $j_\gamma((\rho_0)_n)$, $|\nabla(\rho_0)_n|$ and $(\rho_0)_n|(u_0)_n|^2$ are bounded in $L^1(\mathbb{R}^N)$ so that $(\rho_0)_n$ is bounded in $L^\gamma_2(\mathbb{R}^N)$.

Then it follows from the energy inequality that :

1. $j_\gamma(\rho_n), |\nabla\rho_n|^2$, $\rho_n|u_n|^2$ are bounded in $L^\infty(0,T,L^1(\mathbb{R}^N))$,
2. Du_n is bounded in $L^2(\mathbb{R}^N\times(0,T))$,
3. u_n is bounded in $L^2(0,T,H^1(B_R))$ for all $R,T\in(0,+\infty)$.

Extracting subsequences if necessary, we may assume that ρ_n, u_n converge weakly respectively in $L^\gamma((0,T)\times B_R)$, $L^2(0,T;H^1(B_R))$ to ρ, u for all $R,T\in(0,+\infty)$.We also extract subsequences for which $\sqrt{\rho_n}u_n$, $\rho_n u_n$, $\rho_n u_n\otimes u_n$ converge weakly as previously.
Our goal now is to verify that the non-linear terms converge in the sense of the distribution. The unique difficult term to treat is $\nabla\rho_n\otimes\nabla\rho_n$.
Moreover we assume that $\frac{1}{\rho_n}$ is bounded in $L^\infty(L^\infty)$ and in using the previous theorem we get :

$$\forall\varphi\in C_0^\infty(\mathbb{R}^N)\quad \varphi\rho_n^2 \text{ is bounded in } L^2(H^{1+\frac{s}{2}}) \text{ for all } 0<s<2.$$

We can now show the following theorem :

Theorem 3.8. *Let $N=2$. We assume that there exists $\beta>0$ such that for all $n\in\mathbb{N}$:*

$$\rho_n(t,x)\geq\beta \quad \textit{for a.a } (t,x)\in(0,+\infty)\times\mathbb{R}^2$$

Then there exists $\eta>0$ such that if :

$$\|\nabla\rho_0^n\|_{L^2}+\|\sqrt{\rho_0^n}|u_0^n|\|_{L^2}+\|j_\gamma(\rho_0^n)\|_{L^1}\leq\eta$$

then, up to a subsequence (ρ_n,u_n) converges strongly to a weak solution (ρ,u) (see definition 2.11) of the system (1.1). Moreover we have $\nabla\rho_n\otimes\nabla\rho_n$ converges strongly in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^N)$.

Proof of the theorem 3.8 :

According to theorem 3.7 we have seen that for all $\varphi\in C_0^\infty(\mathbb{R}^N)$ $\varphi(\rho_n^2-\bar{\rho}^2)\in L^2(H^{1+\frac{s}{2}})$. We can now use some results of compactness to show that $\nabla\rho_n^2$ converge strongly in $L^2(L^2_{loc})$ to $\nabla\rho^2$. We recall the following theorem from Aubin-Lions (see Simon for general results [15]).

Lemma 2. *Let $X \hookrightarrow Y \hookrightarrow Z$ be Hilbert spaces such that the embedding from X in Y is compact. Let $(f_n)_{n\in\mathbb{N}}$ a sequence bounded in $L^q(0,T;Y)$, (with $1 < q < +\infty$) and $(\frac{df_n}{dt})_{n\in\mathbb{N}}$ bounded in $L^p(0,T;Z)$ (with $1 < p < +\infty$), then $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $L^q(0,T;Y)$.*

We need to localize because we have some result of compactness for the local Sobolev space $H^{\frac{s}{2}}_{loc}$ which is compactly embedded in L^2_{loc}. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^N)$ cut-off functions supported in the ball $B(0,p+1)$ of $\mathbb{R}^N$ and equal to 1 in a neighborhood of $B(0,p)$.
We have then in using mass equation :

$$\frac{d}{dt}\nabla(\rho_n^2) + \nabla\text{div}(\rho_n^2 u_n) = -\nabla(\rho_n^2\text{div}u_n)$$

We can then show that $\big(\frac{d}{dt}(\chi_p\nabla(\rho_n^2))\big)_{n\in\mathbb{N}}$ is uniformly bounded for all p in $L^q(H^\alpha)$ for $\alpha < 0$ in using energy inequalities and $\big(\chi_p\nabla(\rho_n^2)\big)_{n\in\mathbb{N}}$ is uniformly bounded for all p in $L^2(H^{\frac{s}{2}})$. Apply lemma 2 with the family $\big(\chi_p\nabla(\rho_n^2)\big)_{n\in\mathbb{N}}$ and $X = \chi_p H^{\frac{s}{2}}$, $Y = \chi_p L^2$, $Z = \chi_p H^\alpha$, then use Cantor's diagonal process. This finally provides that :

$$\forall p > 0 \quad \chi_p\nabla(\rho_n^2) \to_{n\to+\infty} \chi_p\nabla(\rho^2) \ \text{ in } \ L^2(L^2). \tag{3.15}$$

We now want to show some results of strong convergence for $\nabla\rho_n$.
We have then for all $\phi \in C_0^\infty$:

$$|\phi(\nabla\rho_n - \nabla\rho)| \le \frac{1}{\rho_n}|\phi(\nabla\rho_n^2 - \nabla\rho^2)| + |\nabla\rho^2||\phi(\frac{1}{\rho_n} - \frac{1}{\rho})| = A_n + B_n.$$

We have A_n converges to 0 in $L^2(L^2)$ because $\frac{1}{\rho_n}$ is uniformly bounded in $L^\infty(L^\infty)$ and in using (3.15) for all $\phi \in C_0^\infty$, $\phi\nabla\rho_n^2$ converges to $\nabla\rho_n^2$ in $L^2(L^2)$. In the same way we have $\frac{1}{\rho_n}$ converges a.e to $\frac{1}{\rho}$ (because we can show in using again the theorem of Aubin-Lions that $\rho_n - \bar\rho \to \rho - \bar\rho$ in $L^2(L^q)$ up to an extraction with $1 \le q < +\infty$ then we can extract again a subsequence so that ρ_n converges a.e to ρ). Moreover we have :

$$|\frac{1}{\rho_n} - \frac{1}{\rho}| \le \frac{2}{\beta}$$

then by the theorem of dominated convergence we have B_n tends to 0 in $L^2(L^2)$.
We can conclude that :

$$\forall\phi \in C_0^\infty, \quad \phi\nabla\rho_n \otimes \nabla\rho_n \to_n \phi\nabla\rho \otimes \nabla\rho \ \text{ in } \ L^1(L^1).$$

□

4 Existence of weak solution in the case $N = 1$

We are now interested by the case $N = 1$. To start with, we focus on the gain of derivative for $\nabla\rho^2$.

4.1 Gain of derivative

We can now write a theorem where we expose a gain of derivative on the density ρ in using the same type of inequalities as in the case $N = 2$.

Theorem 4.9. *Let (ρ, u) be a regular solution of the system (1.1) with initial data in the energy space. Then we have :*

$$\|\rho^2\|_{L^2_T(H^{1+\frac{s}{2}}(\mathbb{R}))} \leq M_0$$

with $0 \leq s < \frac{1}{2}$ and M_0 depending only of the initial data.

Remark 6. *We observe the two important facts :*

1. *We don't need any hypothesis on the size of the initial data.*
2. *We don't need to localize because we know that $\rho \in L^\infty_{t,x}$.*
3. *We don't need to assume that $\frac{1}{\rho} \in L^\infty$, this is important. In fact if we assume that ρ_0 admits some vacuum, we could show that $\nabla\rho_n(t,x) \to \nabla\rho$ a.e on the set $A = \{\rho > 0\}$ because we have by compactness up to an extraction $\nabla\rho_n^2 \to \nabla\rho^2$ a.e.*

Proof of theorem 4.9 :

We use the same estimates as in the previous proof except for the delicate term : $\int_0^T \int_{\mathbb{R}} |\partial_x\rho|^2 \Lambda^s\rho^2$. We have then $\partial_x\rho \in L^\infty(L^2)$ and $\rho - \bar{\rho} \in L^\infty(L^2)$ so $\rho - \bar{\rho} \in L^\infty(H^1)$ and we have then by Sobolev embedding $\rho \in L^\infty(L^\infty)$. By composition theorem on the Sobolev space we get $(\rho - \bar{\rho})^2 \in L^\infty(H^1)$ and $\rho^2 - \bar{\rho}^2 = (\rho - \bar{\rho})^2 + 2\bar{\rho}(\rho - \bar{\rho}) \in L^\infty(H^1)$. Finally we get for $0 < s \leq 1$, $\Lambda^s\rho^2 \in L^\infty(H^{1-s})$. Now for $0 \leq s < \frac{1}{2}$ by Sobolev embedding we obtain :

$$\Lambda^s\rho^2 \in L^\infty(L^\infty).$$

So we can control the term $\int_0^T \int_{\mathbb{R}} |\partial_x\rho|^2 |\Lambda^s\rho^2$ as follows :

$$\int_0^T \int_{\mathbb{R}} |\partial_x\rho|^2 |\Lambda^s\rho^2| \lesssim \|\partial_x\rho\|^4_{L^\infty_T(L^2)}.$$

We treat the other terms similarly as in the previous proof. □

4.2 Results of compactness

We can now prove our result of stability of solution in the case $N = 1$ in using the previous gain of derivative. Let $(\rho_n, u_n)_{n\in\mathbb{N}}$ a sequel of approximate weak solutions of system (1.1).

Theorem 4.10. *Let (ρ_0^n, u_0^n) initial data of the system (1.1) in the energy space what it means that :*

$$\int_{\mathbb{R}} \rho_0^n |u_0^n|^2 + j_\gamma(\rho_0^n) + |\partial_x\rho_0^n|^2 dx \leq M$$

with $M > 0$.
Moreover we assume that $\rho_0^n \geq c > 0$. Then there exists a time T such that up to a subsequence, (ρ_n, u_n) converges strongly to a weak solution (ρ, u) on $(0,T)\times\mathbb{R}$ in the sense of the distribution (see definition 2.11).
Moreover $\partial_x \rho_n$ converges strongly in $L^2(\mathbb{R}\times\mathbb{R})$ to $\partial_x\rho$.

Proof of the theorem 4.10 :

We want now to control the vacuum of $\frac{1}{\rho}$. We recall that $\rho-\bar{\rho}\in L^\infty(H^1)$ and $\frac{\partial}{\partial t}\rho \in L^\infty(L^p)$, then we have by Aubin-Lions theorem for all $\phi\in C_0^\infty$, $\phi(\rho-\bar{\rho})\in C([0,T],L^\infty)$.
Moreover we set :

$$f_n(t) = \int_{\mathbb{R}} |\partial_x\rho|^2 1_{\mathbb{R}\setminus B(0,n)}$$

then we have f_n is a decreasing sequence and converges to 0. Then in using the theorem of Dini we get that on $[0,T]$, f_n converges uniformly to 0. Let $\varepsilon > 0$, then it exists $n_0\in\mathbb{N}$ enough big such that for all $n\geq n_0$:

$$\int_{\mathbb{R}} |\partial_x(\rho-\bar{\rho})|^2 1_{\mathbb{R}\setminus B(0,n)} \leq \varepsilon$$

we proceed similarly to show that for a $n_0^{'}$ big enough we have for all $n^{'}\geq n_0^{'}$:

$$\int_{\mathbb{R}} |\rho-\bar{\rho}|^2 1_{\mathbb{R}\setminus B(0,n^{'})} \leq \varepsilon$$

then we have for $\phi\in C_0^\infty$ with a big enough support $(\rho-\bar{\rho})(1-\phi)\in L^\infty(H^1)$ with a norm inferior to ε then it exists c such that by continuity with $0<\varepsilon\leq c<\bar{\rho}$, $|\rho-\bar{\rho}|\leq c$.
Finally we have $\frac{1}{\rho}\in L^\infty(L^\infty)$ and we can conclude as in the previous theorem 4.9 show that :

$$\forall\phi\in C_0^\infty \quad \phi\partial_x\rho_n \to_n \phi\partial_x\rho \ \text{ in } \ L^2(L^2).$$

□

Theorem 4.11. *Let (ρ_0^n, u_0^n) initial data of the system (1.1) in the energy space.*
Then it exists $\varepsilon > 0$ such that if :

$$\|\partial_x\rho_0^n\|_{L^2} + \|\sqrt{\rho_0^n}|u_0^n|\|_{L^2} + \|j_\gamma(\rho_0^n)\|_{L^1} \leq \varepsilon$$

then up to a subsequence (ρ_n, u_n) converges strongly to a weak solution (ρ, u) on $\mathbb{R}\times\mathbb{R}$ (see the definition 2.11).
Moreover $\partial_x\rho_n$ converges strongly to $\partial_x\rho$ in $L^2(\mathbb{R}\times\mathbb{R})$.

Proof of theorem 4.11 :

We can show easily that $\|\rho_n-\bar{\rho}\|_{L^\infty(H^1)}\leq C\varepsilon$ and so we have $\frac{1}{\rho_n}$ is uniformly bounded in $L^\infty(L^\infty)$. We can then conclude as in the proof of theorem 4.9.

□

Bibliographie

[1] D.M. Anderson, G.B McFadden and A.A. Wheller. Diffuse-interface methods in fluid mech. In Annal review of fluid mechanics, Vol. 30, pages 139-165. Annual Reviews, Palo Alto, CA, 1998.

[2] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models : Korteweg,lubrication and shallow water systems. Comm. Partial Differential Equations, 28(3-4) : 843-868, 2003.

[3] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I. Interfacial free energy, J. Chem. Phys. 28 (1998) 258-267.

[4] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP,Analyse non linéaire 18,97-133 (2001)

[5] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working ,Arch. Rational Mech. Anal. 88(2) (1985) 95-133.

[6] M.E. Gurtin, D. Poligone and J. Vinals, Two-phases binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci.. 6(6) (1996) 815–831.

[7] H. Hattori and D.Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differential Equations 9(4) (1996) 323-342.

[8] H. Hattori and D. Li, Global Solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198(1) (1996) 84-97.

[9] D. Jamet, O. Lebaigue, N. Coutris and J.M. Delhaye, The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. J. Comput. Phys, 169(2) : 624–651, (2001).

[10] D.J. Korteweg. Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité. Arch. Néer. Sci. Exactes Sér. II, 6 :1-24, 1901.

[11] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Oxford University Press (1996)

[12] A.Mellet and A.Vasseur, On the barotropic compressible Navier-Stokes equation, Comm. Partial Differential Equations 32 (2007), no. 1-3, 431–452.

[13] J.S. Rowlinson, Translation of J.D van der Waals "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". J.Statist. Phys., 20(2) : 197-244, 1979.

[14] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, volume 3 of De Gruyter series in nonlinear analysis and applications. Berlin 1996.

[15] J. Simon. Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl., 146, 65-96, 1987.

[16] C. Truedelland W. Noll. The nonlinear field theories of mechanics. Springer-Verlag, Berlin, second edition, 1992.

Chapitre 4

Study of compactness for compressible fluid models with a capillary tensor for discontinuous interfaces

This chapter is devoted to the global stability theory of solutions for a general isothermal model of capillary fluids derived by C. Rohde in [18], which can be used as a phase transition model.
This chapter is structured in the following way : first of all inspired by the result by P.-L. Lions in [16] on the Navier-Stokes compressible system we will show the global stability of weak solutions for our system with isentropic pressure and next with general pressure. Next we will consider perturbations close to a stable equilibrium as in the case of strong solutions.

1 Introduction

1.1 Presentation of the model

The correct mathematical description of liquid-vapor phase interfaces and their dynamical behavior in compressible fluid flow has a long history. We are concerned with compressible fluids endowed with internal capillarity. One of the first model which takes into consideration the variation of density on the interface between two phases, originates from the XIXth century work by Van der Waals and Korteweg [14]. It was actually derived in his modern form in the 1980s using the second gradient theory, see for instance [13, 19]. Korteweg suggests a modification of the Navier-Stokes system to account additionally for phase transition phenomena in introducing a term of capillarity. He assumed that the thickness of the interfaces was not null as in the *sharp interface approach*. This is called the *diffuse interface approach*.

Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. In terms of the free energy, this principle takes the form of a generalized Gibbs relation, see [19].

In the present chapter, we follow a new approach introduced by Coquel, Rohde and their collaborators in [4]. They remark that the local diffuse interface approach requires more regular solutions than in the original sharp interface approach. Indeed the interfaces are assumed of non zero thickness, so that the density vary continuously between the two interfaces, whereas in the sharp interface models, the interfaces represent zone of discontinuity for the density. Coquel, Rohde and their collaborators present an alternative model with a capillarity term which avoids spatial derivatives. The model reads :

$$(NSK)\qquad \begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho\, u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div} u + \nabla(P(\rho)) = \kappa\rho\nabla D[\rho] \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0) \end{cases}$$

with :

$$\mu > 0 \ \text{ and } \ \lambda + 2\mu > 0$$

where ρ denotes the density of the fluid and $u \in \mathbb{R}^N$ the velocity, μ and λ represent the viscosity coefficients, κ is a coefficient of capillarity, P is a general pressure function. We are particulary interested by Van der Waals type pressure :

$$P : (0,b) \to (0,+\infty)$$
$$P(\rho) = \frac{RT_*\rho}{b-\rho} - a\rho^2$$

where a, b, R, T_* are positive constants, R being the specific gas constant. For fixed values a, b we choose the constant reference temperature T_* so small as P to be monotone decreasing in some non-empty interval.

Further we impose the conditions :

$$u(t,x) \to 0, \ \ \rho(t,x) \to 0 \ \text{ as } \ |x| \to +\infty, \tag{1.1}$$

In the last section, we consider also more general situations : monotone pressure, other conditions at infinity than (1.1), namely :

$$u(t,x) \to 0, \ \ \rho(t,x) \to \bar{\rho} \ \text{ as } \ |x| \to +\infty, \tag{1.2}$$

where $\bar{\rho}$ is a given nonnegative constant.

The term $\kappa\rho\nabla D[\rho]$ corresponds to the capillarity which is supposed to model capillarity effects close to phase transitions in [14]. The classical Korteweg's capillarity term is $D[\rho] = \Delta\rho$.

Based on Korteweg's original ideas Coquel, Rohde and their collaborators in [4] and Rohde in [18] choose a nonlocal capillarity term D which penalizes rapid variations in the density field close from the interfaces. They introduce the following capillarity term :

$$D[\rho] = \phi * \rho - \rho$$

where ϕ is chosen so that :

$$\phi \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi(x)dx = 1, \quad \phi \text{ even, and } \phi \geq 0.$$

This choice of capillarity term allows to get solution with jumps, i.e with sharp interfaces. Before tackling the global stability theory for the system (NSK), let us derive formally the uniform bounds available on (ρ, u).

1.2 Energy spaces

Let Π (free energy) be defined by :

$$\Pi(s) = s\Big(\int_0^s \frac{P(z)}{z^2}dz\Big), \tag{1.3}$$

so that $P(s) = s\Pi'(s) - \Pi(s)$, $\Pi'(\bar{\rho}) = 0$ and if we renormalize the mass equation :

$$\partial_t \Pi(\rho) + \operatorname{div}(u\Pi(\rho)) + P(\rho)\operatorname{div}(u) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N).$$

Notice that Π is convex whenever P is nondecreasing. Multiplying the equation of momentum conservation by u and integrating by parts over $\mathbb{R}^N$, we obtain the following energy estimate :

$$\begin{aligned}\int_{\mathbb{R}^N} (\frac{1}{2}\rho|u|^2 + \Pi(\rho) + E_{global}[\rho(.,t)])(x)dx(t) + \int_0^t \int_{\mathbb{R}^N} (\mu D(u) : D(u)\\ + (\lambda+\mu)|\operatorname{div}u|^2)dx \leq \int_{\mathbb{R}^N} \big(\frac{|m_0|^2}{2\rho} + \Pi(\rho_0) + E_{global}[\rho_0]\big)dx,\end{aligned} \tag{1.4}$$

where we have :

$$E_{global}[\rho(.,t)](x) = \frac{\kappa}{4}\int_{\mathbb{R}^N} \phi(x-y)(\rho(y,t)-\rho(x,t))^2 dy.$$

The only non-standard term is the energy term E_{global} which comes from the product of u with the capillarity term $\kappa\rho\nabla(\phi*\rho-\rho)$. Indeed we have :

$$\begin{aligned}\kappa\int_{\mathbb{R}^N} & u(t,x)\rho(t,x)\cdot\nabla([\phi*\rho(t,\cdot)](x)-\rho(t,x))dx\\ &= -\kappa\int_{\mathbb{R}^N} \operatorname{div}(u(t,x)\rho(t,x))([\phi*\rho(t,\cdot)](x)-\rho(t,x))dx,\\ &= \kappa\int_{\mathbb{R}^N} \frac{\partial}{\partial t}\rho(t,x)([\phi*\rho(t,\cdot))](x)-\rho(t,x))dx,\\ &= -\frac{d}{dt}\int_{\mathbb{R}^N} E_{global}[\rho(t,\cdot)](x)dx\ .\end{aligned}$$

To derive the last equality we use the relation :

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} E_{global}[\rho(t,\cdot)](x)dx &= \frac{\kappa}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(x-y)(\rho(t,y)-\rho(t,x))\frac{\partial}{\partial t}\rho(t,y)dydx \\ &\quad+\frac{\kappa}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(y-x)(\rho(t,x)-\rho(t,y))\frac{\partial}{\partial t}\rho(t,x)dydx, \\ &=\kappa\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(x-y)(\rho(t,y)-\rho(t,x))\frac{\partial}{\partial t}\rho(t,y)dydx, \\ &=-\kappa\int_{\mathbb{R}^N}([\phi*\rho(t,\cdot)](x)-\rho(t,x))\frac{\partial}{\partial t}\rho(t,x)dx\ .\end{aligned}$$

where we just use integration by parts.
In the sequel we will note :

$$\mathcal{E}(\rho,\rho u)(t)=\int_{\mathbb{R}^N}\frac{1}{2}\rho|u|^2(t,x)+\Pi(\rho)(t,x)+E_{global}[\rho(.,t)])(t,x)dx, \qquad (1.5)$$

We are interested to use the above inequality energy to determine the functional space we must work with.
So if we expand $E_{global}[\rho(.,t)](x)$ we get :

$$E_{global}[\rho(t,\cdot)](x)=\frac{\kappa}{4}\big(\rho^2+\phi*\rho^2-2\rho\,(\phi*\rho)\big).$$

Because by the mass equation we obtain that ρ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$ if we suppose that $\rho_0\in L^1$ and we have supposed that $\phi\in L^\infty(\mathbb{R}^N)$, we obtain that $\rho\,(\phi*\rho)$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$. So we get that $\rho^2+\phi*\rho^2\in L^\infty(0,T;L^1(\mathbb{R}^N))$ and as $\phi\geq 0$ and $\rho\geq 0$ we get a control of ρ in $L^\infty(0,T;L^2(\mathbb{R}^N))$ (a property which turns out to be important to taking advantage of the theory of renormalized solutions, indeed ρ in $L^\infty(0,T;L^2(\mathbb{R}^N))$ implies that $\rho\in L^2_{loc}(\mathbb{R}^+\times\mathbb{R}^N)$ and we will can use the theorem of Diperna-Lions on renormalized solutions, see [15])
In view of (1.4), we can specify initial conditions on $\rho_{/t=0}=\rho_0$ and $\rho u_{/t=0}=m_0$ where we assume that :

- $\rho_0\geq 0$ a.e in $\mathbb{R}^N$, $\rho_0\in L^1(\mathbb{R}^N)\cap L^s(\mathbb{R}^N)$ with $s=\max(2,\gamma)$,
- $m_0=0$ a.e on $\rho_0=0$,
- $\dfrac{|m_0|^2}{\rho_0}$ (defined to be 0 on $\rho_0=0$) is in $L^1(\mathbb{R}^N)$.

(1.6)

We deduce the following a priori bounds which give us the energy space in which we will work :

- $\rho\in L^\infty(0,T;L^1(\mathbb{R}^N)\cap L^s(\mathbb{R}^N))$,
- $\rho|u|^2\in L^\infty(0,T;L^1(\mathbb{R}^N))$,
- $\nabla u\in L^2((0,T)\times\mathbb{R}^N)^N$.

We will use this uniform bound in our result of compactness. Let us emphasize at this point that the above a priori bounds do not provide any control on $\nabla\rho$ in contrast with the case of $D[\rho]=\Delta\rho$ (see in [6]).

1.3 Notion of weak solutions

We now explain what we mean by renormalized weak solutions, weak solutions, and bounded energy weak solution of problem (NSK).
Multiplying mass equation by $b^{'}(\rho)$, we obtained the so-called renormalized equation (see [15]) :

$$\frac{\partial}{\partial t}b(\rho)+\operatorname{div}(b(\rho)u)+(\rho b^{'}(\rho)-b(\rho))\operatorname{div}u=0. \tag{1.7}$$

with :

$$b\in C^{0}([0,+\infty))\cap C^{1}((0,+\infty)),\;\; |b^{'}(t)|\leq ct^{-\lambda_0},\;\; t\in(0,1],\;\; \lambda_0<1 \tag{1.8}$$

with growth conditions at infinity :

$$|b^{'}(t)|\leq ct^{\lambda_1},\;\; t\geq 1,\;\; \text{where}\;\; c>0,\; -1<\lambda_1<\frac{s}{2}-1. \tag{1.9}$$

Definition 1.12. *A couple (ρ,u) is called a renormalized weak solution of problem (NSK) if we have :*

- *Equation of mass holds in $\mathcal{D}^{'}(\mathbb{R}^N)$.*
- *Equation (1.7) holds in $\mathcal{D}^{'}(\mathbb{R}^N)$ for any function b verifying (1.8) and (1.9).*

Definition 1.13. *Let the couple (ρ_0,u_0) satisfy ;*

- $\rho_0\in L^1(\mathbb{R}^N)$, $\Pi(\rho_0)\in L^1(\mathbb{R}^N)$, $E_{global}[\rho_0]\in L^1(\mathbb{R}^N)$, $\rho_0\geq 0$ *a.e in* $\mathbb{R}^N$.
- $\rho_0 u_0\in (L^1(\mathbb{R}^N))^N$ *such that* $\rho_0|u_0|^2 1_{\rho_0>0}\in L^1(\mathbb{R}^N)$
- $\rho_0 u_0=0$ *whenever* $x\in\{\rho_0=0\}$,

where the quantity Π is defined in (1.3). We have the following definitions :

1. *A couple (ρ,u) is called a weak solution of problem (NSK) on $\mathbb{R}$ if :*
   (a) $\rho\in L^r(L^r(\mathbb{R}^N))$ *for* $s\leq r\leq +\infty$,
   (b) $P(\rho)\in L^\infty(L^1(\mathbb{R}^N))$, $\rho\geq 0$ *a.e in* $\mathbb{R}\times\mathbb{R}^N$,
   (c) $\nabla u\in L^2(L^2(\mathbb{R}^N))$, $\rho|u|^2\in L^\infty(L^1(\mathbb{R}^N))$.
   (d) *Mass equation holds in* $\mathcal{D}^{'}(\mathbb{R}\times\mathbb{R}^N)$.
   (e) *Momentum equation holds in* $\mathcal{D}^{'}(\mathbb{R}\times\mathbb{R}^N)^N$.
   (f) $\lim_{t\to 0^+}\int_{\mathbb{R}^N}\rho(t)\varphi=\int_{\mathbb{R}^N}\rho_0\varphi$, $\forall\varphi\in\mathcal{D}(\mathbb{R}^N)$,
   (g) $\lim_{t\to 0^+}\int_{\mathbb{R}^N}\rho u(t)\cdot\phi=\int_{\mathbb{R}^N}\rho_0 u_0\cdot\phi$, $\forall\phi\in\mathcal{D}(\mathbb{R}^N)^N$.
2. *A couple (ρ,u) is called a bounded energy weak solution of problem (NSK) if in addition to (1d), (1e), (1f), (1g) we have :*
   - *The quantity $\mathcal{E}_0$ is finite and inequality (1.4) with $\mathcal{E}$ defined by (1.5) and with $\mathcal{E}_0$ in place of $\mathcal{E}(\rho(0),\rho u(0))$ holds a.e in $\mathbb{R}$.*

1.4 Mathematical results

We wish to prove global stability results for (NSK) with $D[\rho] = \phi * \rho - \rho$ in functional spaces very close to energy spaces. In the non capillary case and $P(\rho) = a\rho^{\gamma}$, P-L. Lions in [16] proved the global existence of weak solutions (ρ, u) to (NSK) with $\kappa = 0$ (which becomes the compressible isotherm system of Navier-Stokes) for $\gamma > \frac{N}{2}$ if $N \geq 4$, $\gamma \geq \frac{3N}{N+2}$ if $N = 2, 3$ and initial data (ρ_0, m_0) such that :

$$\rho_0, \;\; \rho_0^{\gamma}, \;\; \frac{|m_0|^2}{\rho_0} \in L^1(\mathbb{R}^N).$$

where we agree that $m_0 = 0$ on $\{x \in \mathbb{R}^N / \;\rho_0(x) = 0\}$. More precisely, he obtains the existence of global weak solutions (ρ, u) to (NSK) with $\kappa = 0$ such that for all $t \in (0, +\infty)$:

- $\rho \in L^{\infty}(0, T; L^{\gamma}(\mathbb{R}^N))$ and $\rho \in C([0, T], L^p(\mathbb{R}^N))$ if $1 \leq p < \gamma$,
- $\rho \in L^q((0, T) \times \mathbb{R}^N)$ for $q = \gamma - 1 + \frac{2\gamma}{N} > \gamma$.
- $\rho|u|^2 \in L^{\infty}(0, T; L^1(\mathbb{R}^N))$ and $Du \in L^2((0, T) \times \mathbb{R}^N)$.

Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties of the density ρ in $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$ spaces required to pass to the limit in the pressure term $P(\rho) = a\rho^{\gamma}$.

Let us mention that Feireisl in [9] generalized the result to $\gamma > \frac{N}{2}$ in establishing that we can obtain renormalized solution without imposing that $\rho \in L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$ (what needed Lions in dimension $N = 2, 3$, that's why $\gamma - 1 + \frac{2\gamma}{N} \geq 2$), for this he introduces the concept of oscillation defect measure evaluating the loss of compactness. We refer to the book of Novotný and Straškraba for more details (see [17]).

Let us mention here that the existence of strong solution with $D[\rho] = \Delta\rho$ is known since the work by Hattori an Li in [10], [11] in the whole space $\mathbb{R}^N$. In [6], Danchin and Desjardins study the well-posedness of the problem for the isothermal case with constant coefficients in critical Besov spaces. We recall too the results by Rohde in [18] who obtains the existence and uniqueness in finite time for two-dimensional initial data in $H^4(\mathbb{R}^2) \times H^4(\mathbb{R}^2)$.

In the present chapter, we aim at showing the global stability of weak solutions in the energy spaces for the system (NSK). This work is composed of four parts, the first one concerns estimates on the density to get a gain of integrability on the density needed to pass to the weak limit in the term of pressure and of capillarity. The second part is the passage to the weak limit in the non-linear terms of the density and the velocity according to Lions' methods. The idea is to use renormalized solution to test the weak limit on convex test functions. In this part we will concentrate on the case of pressure laws of type $P(\rho) = a\rho^{\gamma}$. We get the following theorem where $(\rho_n, u_n)_{n\in\mathbb{N}}$ is a sequence of regular bounded energy weak solu- tions of (NSK) where, in addition, the sequence ρ_n is bounded in $L^r((0, T) \times \mathbb{R}^N)) \cap L^1((0, T) \times R^N)$ with a certain $r > \max(\gamma, 2)$.

Theorem 1.12. *Let $N \geq 2$. Let $\gamma > N/2$ if $N \geq 4$ and $\gamma \geq 1$ else.*
Let the couple (ρ_0^n, u_0^n) satisfy :

- *ρ_n^0 is uniformly bounded in $L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ with $s = \max(\gamma, 2)$ and $\rho_0^n \geq 0$ a.e in $\mathbb{R}^N$,*

- $\rho_n^0|u_0^n|^2$ *is uniformly bounded in* $L^1(\mathbb{R}^N)$,
- *and such that* $\rho_0^n u_0^n = 0$ *whenever* $x \in \{\rho_0 = 0\}$.

In addition we suppose that ρ_n^0 *converges in* $L^1(\mathbb{R}^N)$ *to* ρ_0.
Then up to a subsequence (ρ_n, u_n) *converges strongly to a weak solution* (ρ, u) *of the system* (NSK) *satisfying the initial condition* (ρ_0, u_0) *as in (1.6). Moreover we have the following convergence :*

- $\rho_n \to_n \rho$ *in* $C([0,T], L^p(\mathbb{R}^N)) \cap L^r((0,T)\times K)$ *for all* $1 \leq p < s$, $1 \leq r < q$, *with* $q = s + \frac{N\gamma}{2} - 1$ *if* $N \geq 3$, *where* $K = \mathbb{R}^N$ *except when* $N \geq 4$ *and* $\gamma < \frac{N}{2}(1+\frac{1}{N})$ *where* K *is an arbitrary compact.*
- $\rho_n \to_n \rho$ *in* $C([0,T], L^p(\mathbb{R}^2)) \cap L^r((0,T)\times K)$ *for all* $1 \leq p < s$, $1 \leq r < q$, *with* K *an arbitrary compact in* $\mathbb{R}^2$ *if* $N = 2$.

In addition we have :

- $\rho_n u_n \to \rho u$ *in* $L^p(0,T; L^r(\mathbb{R}^N))$ *for all* $1 \leq p < +\infty$ *and* $1 \leq r < \frac{2s}{s+1}$,
- $\rho_n (u_i)_n (u_j)_n \to \rho_n u_i u_j$ *in* $L^p(0,T; L^1(\Omega))$ *for all* $1 \leq p < +\infty$, $1 \leq i,j \leq N$ *if* $N \geq 3$, *where* $K = \mathbb{R}^N$ *except when* $N \geq 4$ *and* $\gamma < \frac{N}{2}(1+\frac{1}{N})$ *where* K *is an arbitrary compact.*
- $\rho_n (u_i)_n (u_j)_n \to \rho_n u_i u_j$ *in* $L^p(0,T; L^1(\Omega))$ *for all* $1 \leq p < +\infty$, $1 \leq i,j \leq N$ *with* Ω *an arbitrary bounded open set in* $\mathbb{R}^2$ *if* $N = 2$.

In the third part we will focus on general pressure laws, and particulary van der Waal's pressure. In the fourth part we concentrate on the case of initial data close to a constant $\bar{\rho}$, and we will work in Orlicz space, this case is the most adapted for the strong solution because it enables us to control the vacuum so that one can use the property of ellipticity of the momentum equation.

2 Existence of weak solution for a isentropic pressure law

2.1 A priori estimates on the density

In this part we are interested by getting a gain of integrability on the density and we consider the case where $P(\rho) = a\rho^\gamma$. This will enable us to pass to the weak limit in the pressure and the Korteweg terms. It is expressed by the following theorem :

Theorem 2.13. *Let* $N \geq 2$ *and* $\gamma \geq 1$, *with in addition* $\gamma > \frac{N}{2}$ *if* $N \geq 4$. *Let* (ρ, u) *be a regular bounded energy weak solution of the system* (NSK) *with* $\rho \geq 0$ *and* $\rho \in L^\infty(L^1 \cap L^{s+\varepsilon})$ *where we define* ε *below. Then we have if* $\gamma \geq \frac{N}{2}(1+\frac{1}{N})$ *for* $N \geq 4$ *:*

$$\int_{(0,T)\times\mathbb{R}^N} \big(\rho^{\gamma+\varepsilon} + \rho^{2+\varepsilon}\big) dxdt \leq M \;\; \textit{for any} \;\; 0 < \varepsilon \leq \frac{2}{N}\gamma - 1 \quad \textit{if} \;\; N \geq 4,$$

$$\textit{and} \quad 0 < \varepsilon \leq \frac{4}{N} - 1 \quad \textit{if} \;\; N = 2,3.$$

with M *depending only on the initial conditions and on the time* T.

If $\gamma < \frac{N}{2}(1+\frac{1}{N})$ *for* $N \geq 4$, *we have :*

$$\int_{(0,T)\times K} \big(\rho^{\gamma+\varepsilon} + \rho^{2+\varepsilon}\big) dxdt \leq M' \ \ \textit{for any} \ \ 0 < \varepsilon \leq \frac{2}{N}\gamma - 1.$$

for any arbitrary compact K *with* M' *depending only on the initial conditions, on* K *and on the time* T.

Proof :

We will begin with the case where $N \geq 3$ and we treat after the specific case $N = 2$.

Case $N \geq 3$ **:**

We apply to the momentum equation the operator $(-\Delta)^{-1}\text{div}$ in order to concentrate us on the pressure and we get :

$$\begin{aligned} a\rho^{\gamma} = \frac{\partial}{\partial t}(-\Delta)^{-1}\text{div}(\rho u) + (-\Delta)^{-1}\partial^2_{i,j}(\rho u_i u_j) &+ (2\mu+\lambda)\text{div} u \\ &- \kappa(-\Delta)^{-1}\text{div}\big(\rho\nabla(\phi * \rho - \rho)\big), \end{aligned} \tag{2.10}$$

and in multiplying by ρ^{ε} with $0 < \varepsilon \leq \min(\frac{1}{N}, \frac{2}{N}\gamma - 1)$ to estimate $\rho^{\gamma+\varepsilon}$, we get :

$$\begin{aligned} a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon} =& -\kappa\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho(\nabla\phi * \rho)) + \rho^{\varepsilon}(-\Delta)^{-1}\partial^2_{ij}(\rho u_i u_j) \\ &+ \frac{\partial}{\partial t}\big(\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)\big) - [\frac{\partial}{\partial t}\rho^{\varepsilon}](-\Delta)^{-1}\text{div}(\rho u) + (\mu+\zeta)\text{div} u \ , \end{aligned} \tag{2.11}$$

where we note $\xi = \lambda + \mu$. We now rewrite the previous equality as follows :

$$\begin{aligned} a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon} =& -\kappa\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho(\nabla\phi * \rho)) + \rho^{\varepsilon}(-\Delta)^{-1}\partial^2_{ij}(\rho(u_i)(u_j)) \\ &+ \frac{\partial}{\partial t}\big(\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)\big) + \text{div}[u\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)] + (\mu+\zeta)\text{div} u \\ &- (\rho)^{\varepsilon} u \cdot \nabla(-\Delta)^{-1}\text{div}(\rho u) + (1-\varepsilon)(\text{div} u)\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u) \ . \end{aligned} \tag{2.12}$$

Next we integrate (2.12) in time on $[0,T]$ and in space so we obtain :

$$\begin{aligned} \int_{(0,T)\times\mathbb{R}^N} (a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon}) dx\,dt = \int_{(0,T)\times\mathbb{R}^N} \Big(&\frac{\partial}{\partial t}[\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)] + (\mu+\zeta)(\text{div} u)\rho^{\varepsilon} \\ &+ (1-\varepsilon)(\text{div} u)\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u) + \rho^{\varepsilon}[R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j)] \\ &+ \text{div}[u\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)] - \kappa\rho^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho\nabla(\phi * \rho))\Big) dx\,dt \ , \end{aligned} \tag{2.13}$$

where R_i is the classical Riesz transform.
Now we want to control the term $\int_0^T \int_{\mathbb{R}^N} \big(\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon}\big) dxdt$. As ρ is positive, it will enable us to control $\|\rho\|_{L^{\gamma+\varepsilon}_{t,x}}$ and $\|\rho\|_{L^{2+\varepsilon}_{t,x}}$. This may be achieved by bounding each term on the right side of (2.13).

We start by treating the term $\int_{(0,T)\times\mathbb{R}^N}\frac{\partial}{\partial t}[\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)]$. So we need to control $\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)$ in $L^\infty(0,T;L^1(\mathbb{R}^N))$ and $\rho_0^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_0 u_0)$ because :

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^N}\frac{\partial}{\partial t}[\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)](t,x)dt\,dx=&\int_{\mathbb{R}^N}\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)](x)dx(t)\\&-\int_{\mathbb{R}^N}\rho_0^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_0 u_0)](x)dx,\end{aligned}$$

We recall that ρ, ρ^2, ρ^γ and $\rho|u|^2$ are bounded in $L^\infty(L^1)$ while Du is bounded in $L^2((0,T)\times\mathbb{R}^N)$ and u is bounded in $L^2(0,T;L^{\frac{2N}{N-2}}(\mathbb{R}^N))$ by Sobolev embedding. In particular by Hölder inequalities we get that ρu is bounded in $L^\infty(0,T,(L^{\frac{2\gamma}{\gamma+1}}\cap L^{\frac{4}{3}})(\mathbb{R}^N))$. Thus we get in using Hölder inequalities and Sobolev embedding :
$\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\in L^\infty(0,T,L^1\cap L^\alpha)$ with :

$$\frac{1}{\alpha}=\frac{\varepsilon}{s}+\min(\frac{\gamma+1}{2\gamma},\frac{3}{4})-\frac{1}{N}<1.$$

The fact that $\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho u)\in L^\infty(0,T,L^1)$ is obtained by interpolation because $\rho\in L^\infty(L^1)$ and in using less integrability in Sobolev embedding.
Next we have the same type of estimates for $\|\rho_0^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_0 u_0)\|_{L^1(\mathbb{R}^N)}$.
Finally (2.13) is rewritten on the following form in using Green formula :

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}\big(\rho^{\gamma+\varepsilon}+\frac{\kappa}{2}\rho^{2+\varepsilon}\big)dxdt\leq C\big(1+\int_0^T\int_{\mathbb{R}^N}\big[\,|\mathrm{div}u|\rho^\varepsilon(1+|(-\Delta)^{-1}\mathrm{div}(\rho u)|\,]\\+\rho^\varepsilon|R_iR_j(\rho u_iu_j)-u_iR_iR_j(\rho u_j)|+\kappa\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho\nabla(\phi*\rho))\big]dt\,dx\big).\end{aligned}$$

Now we will treat each term of the right hand side. We treat all the terms with the same type of estimates than P.-L. Lions in [16] , excepted the capillarity term.

We start with the term $|\mathrm{div}u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|$ where we have :

$$|\mathrm{div}u|\in L^2(L^2),\;\;\rho^\varepsilon\in L^\infty(L^{\frac{s}{\varepsilon}}),\;\;\rho\,u\in L^2(0,T,L^r(\mathbb{R}^N))$$

with $\frac{1}{r}=\frac{1}{s}+\frac{N-2}{2N}$ and by Sobolev embedding $|(-\Delta)^{-1}\mathrm{div}(\rho u)|\in L^2(L^{s'})$ with $\frac{1}{s'}=\frac{1}{r}-\frac{1}{N}$ (this is possible only if $r<N$). We are in a critical case for the Sobolev embedding (i.e $r\geq N$) only when $N=3$ and $\gamma\geq 6$, that's why for $N=3$ and $\gamma\geq 6$.
So by Hölder inequalities we get $|\mathrm{div}u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|\in L^1(L^{s_1})$ with : $\frac{1}{s_1}=\frac{1}{s'}+\frac{\varepsilon}{s}+\frac{1}{2}=1-\frac{2}{N}+\frac{1+\varepsilon}{s}\leq 1$ as we have $s>\frac{N}{2}$.
Moreover by interpolation $|\mathrm{div}u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|$ belongs to $L^1(0,T;L^1(\mathbb{R}^N))$.

We now treat the case $N = 3$ and $\gamma \geq 6$ where we choose case $\varepsilon = \frac{2}{N}\gamma - 1$. We have :

$$\begin{aligned}
\||\mathrm{div} u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|\|_{L^1} &\leq \|Du\|_{L^2(L^2)}\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|_{L^{\frac{2(\gamma+\varepsilon)}{\gamma-\varepsilon}}(L^{\frac{6(\gamma+\varepsilon)}{5\gamma-\varepsilon}})}, \\
&\leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|_{L^{\frac{10\gamma-6}{\gamma+3}}(L^{\frac{3(10\gamma-6)}{13\gamma+3}})} \leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|^{\frac{\gamma+3}{5\gamma-3}}_{L^2(L^{\frac{6\gamma}{\gamma+6}})}\|\rho u\|^{\frac{2(2\gamma-3}{5\gamma-3}}_{L^\infty(L^2)}, \\
&\leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}}\|\rho u\|^{\frac{5\gamma}{5\gamma-3}}_{L^2(L^{\frac{6\gamma}{\gamma+6}})}\|\rho u\|^{\frac{2(2\gamma-3)}{5\gamma-3)}}_{L^\infty(L^{\frac{2\gamma}{\gamma+1}})}, \\
&\leq C\|\rho\|^\varepsilon_{L^{\gamma+\varepsilon}},
\end{aligned}$$

since we have $\frac{1}{2} + \frac{\varepsilon}{\gamma+\varepsilon} + \frac{\gamma-\varepsilon}{2(\gamma+\varepsilon)} = 1$, $\frac{1}{2} + \frac{\varepsilon}{\gamma+\varepsilon} + \frac{5\gamma-\varepsilon}{6(\gamma+\varepsilon)} - \frac{1}{3} = 1$, and $\frac{6(\gamma+\varepsilon)}{5\gamma-\varepsilon} = 3\frac{10\gamma-6}{13\gamma+3} < 3$.

We now want to treat the term : $\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho\nabla(\phi * \rho))|$, so we have : $\rho\nabla(\phi * \rho) = \rho(\nabla\phi * \rho) \in L^\infty(L^1 \cap L^{\frac{s}{2}})$ by Hölder inequalities and the fact that we have $\rho \in L^\infty(L^1)$ and $\nabla\phi \in L^1$.
After we get that $\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho\nabla(\phi*\rho)) \in L^\infty(L^{r_1})$ with : $\frac{1}{r_1} = \frac{\varepsilon}{s} + \frac{2}{s} - \frac{1}{N} = \frac{2+\varepsilon}{s} - \frac{1}{N} < 1$.
We conclude that $\rho^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho\nabla(\phi*\rho))$ is $L^\infty(L^1)$ in using interpolation when $N = 2, 3$.
Indeed we have $\rho\nabla(\phi*\rho) \in L^\infty(L^1)$ and in choosing $\varepsilon = \frac{2}{N}s - 1$ we have : $1 - \frac{1}{N} + \frac{2}{N}s - 1 \geq 1$ when $s \geq \frac{N}{2}(1 + \frac{1}{N})$. This is the case when $N = 2, 3$, and this is the case when $N \geq 4$ and $\gamma \geq \frac{N}{2}(1 + \frac{1}{N})$.
In the other case we need to work in arbitrary compact.
We have after the term $(\mathrm{div}(u))\rho^\varepsilon$. We recall that ρ^ε is in $L^\infty(L^{\frac{1}{\varepsilon}} \cap L^{\frac{s}{\varepsilon}})$. If $\varepsilon \geq \frac{1}{2}$ (i.e $s \geq \frac{3}{4}N$), the bound is obvious because $\frac{1}{2} + \varepsilon \geq 1$ and $\frac{1}{2} + \frac{\varepsilon}{s} < 1$, we can then conclude by interpolation. On the other hand, this rather simple term presents a technical difficulty when $\varepsilon \leq \frac{1}{2}$ since we do not know in that case if $\mathrm{div} u\, \rho^\varepsilon \in L^1(\mathbb{R}^N \times (0, T))$. One way to get round the difficulty is to multiply (2.10) by $\rho^\varepsilon 1_{\{\rho\geq 1\}}$. Then we obtain an estimate on $\rho^{s+\varepsilon}1_{(\rho\geq 1)}$ in $L^1((0, T) \times \mathbb{R}^N)$ as $\rho^\varepsilon 1_{\{\rho\geq 1\}}|\mathrm{div} u| \leq \rho|\mathrm{div} u| \in L^1((0, T) \times \mathbb{R}^N)$ (where $\varepsilon \leq \frac{1}{2}$) and we can conclude since $0 \leq \rho^{s+\varepsilon}1_{\{\rho<1\}} \leq \rho$ on $(0, T) \times \mathbb{R}^N$ and $\rho \in L^\infty(L^1)$.

We end with the following term $\rho^\varepsilon(R_iR_j(\rho\, u_iu_j) - u_iR_iR_j(\rho\, u_j))$. In the same way than in the previous inequalities we have $\rho^\varepsilon R_iR_j(\rho\, u_iu_j)$ is bounded in $L^1(0, T; L^1(\mathbb{R}^N))$. Indeed we have by Hölder inequalities and the fact that R_i is continuous from L^p in L^p with $1 < p < +\infty$: $\frac{1}{s} + 2\frac{N-2}{2N} + \frac{\varepsilon}{s} = 1 - \frac{2}{N} + \frac{1+\varepsilon}{s} \leq 1$ (because $s > \frac{N}{2}$). And we conclude by interpolation. We treat the term $\rho^\varepsilon u_iR_iR_j(\rho\, u_j)$ similarly.

We have to treat now the case $N = 2$ where we have to modify the estimates when we are in critical cases for Sobolev embedding.

Case $N = 2$:

In the case $N = 2$ most of the proof given above stay exact except for the slightly more delicate terms $\rho^e\mathrm{div} u|(-\Delta)^{-1}\mathrm{div}(\rho u)|$ and $\rho^\varepsilon(R_iR_j(\rho\, u_iu_j) - u_iR_iR_j(\rho\, u_j))$.

We start with the term $|\rho^e \mathrm{div} u(-\Delta)^{-1}\mathrm{div}(\rho u)|$. In our previous estimate it was possible to use Sobolev embedding on the term $(-\Delta)^{-1}\mathrm{div}(\rho u)|$ only if $r \geq N$ (see above the notation), so in the case where $N=2$ we are in a critical case for the Sobolev embedding when $\gamma \geq 2$. This may be overcome by using that, by virtue of Sobolev embedding, we have :

$$\||\mathrm{div} u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|\|_{L^1} \leq C\|\rho\|^{\varepsilon}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}\|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}$$

Indeed by Hölder inequality, we have :

$$\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{\gamma+\varepsilon+1}{2(\gamma+\varepsilon)}-\frac{1}{2}=\frac{1}{2}+\frac{2\varepsilon+1}{2\varepsilon+2\gamma}\leq 1=\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{1}{2(\gamma+\varepsilon})\leq 1,$$

thus :

$$\frac{1}{2}+\frac{\varepsilon}{\gamma+\varepsilon}+\frac{1}{2(\gamma+\varepsilon)}\leq 1.$$

Moreover we have as $\rho u=\sqrt{\rho}\sqrt{\rho}u$, hence :

$$\|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}\leq C\|\rho\|^{\frac{1}{2}}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}$$

and thus :

$$\||\mathrm{div} u|\rho^\varepsilon|(-\Delta)^{-1}\mathrm{div}(\rho u)|\|_{L^1(L^1)}\leq C\|\rho\|^{\varepsilon+\frac{1}{2}}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}.$$

Next we are interested by the term $\rho^\varepsilon(R_iR_j(\rho\, u_iu_j)-u_iR_iR_j(\rho\, u_j))$. We use the fact that u is bounded in $L^2(0,T;\dot{H}^1)$ and thus in $L^2(0,T;BMO)$. Then by the Coifman-Rochberg-Weiss commutator theorem in [3], we have for almost all $t\in[0,T]$:

$$\|R_iR_j(\rho\, u_iu_j)-u_iR_iR_j(\rho\, u_j)\|_{L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}\leq C\|u\|_{L^2(BMO)}\|\rho\, u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}.$$

So we have :

$$\|\rho^\varepsilon(R_iR_j(\rho\, u_iu_j)-u_iR_iR_j(\rho\, u_j))\|_{L^1}\leq C\|\rho\|^{\varepsilon+\frac{1}{2}}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}.$$

In view of the previous inequalities we get finally :

$$\|\rho\|^{\gamma+\varepsilon}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}\leq C(1+\|\rho\|^{\frac{1}{2}+\varepsilon}_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})})$$

and the $L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})$ bound on ρ is proven since $\frac{1}{2}+\varepsilon<\gamma+\varepsilon$. □

2.2 Compactness results for compressible Navier-Stokes equations of Korteweg type in the case of isentropic pressure

In the sequel we are not going to treat in details the case with $N\geq 4$ and $\gamma<\frac{N}{2}(1+\frac{1}{N})$, we just remark that the proof is the same as in the case $N=2$, it suffices to localize because we can only apply the theorem 2.13 on the gain of integrability on any compact K.
So let follow the theorem 2.13 and assume that $\gamma>\frac{N}{2}$ if $N\geq 4$ and $\gamma\geq 1$ such that if

(ρ,u) is a regular solution then $\rho\in L^q((0,T)\times\mathbb{R}^N)$ with $q=\gamma+1-\frac{2\gamma}{N}$. We can observe that in this case $q>s=\max(\gamma,2)$. We will see that it will be very useful in the sequel to justify the passage to the weak limit in some terms to get a gain of integrability on the density. Indeed the key point to proving the existence of weak solutions is the passage to the limit in the term of pressure and in the term of capillarity $\rho\nabla(\phi*\rho-\rho)$.
First, we assume that a sequence $(\rho_n,u_n)_{n\in\mathbb{N}}$ of approximate weak solutions has been constructed by a mollifying process, which have suitable regularity to justify the formal estimates like the energy estimate (1.4) and the theorem 2.13. $(\rho_n,u_n)_{n\in\mathbb{N}}$ has the initial data of the theorem 1.12 with uniform bounds.
In addition :

- ρ_n is bounded uniformly in $L^\infty(0,T;L^1\cap L^s(\mathbb{R}^N))\cap C([0,T];L^p(\mathbb{R}^N))$ for $1\leq p<\max(2,\gamma)$,
- $\rho_n\geq 0$ a.e. and ρ_n is bounded uniformly in $L^q(0,T,\mathbb{R}^N)$ for some $q>s$,
- ∇u_n is bounded in $L^2(0,T;L^2(\mathbb{R}^N))$, $\rho_n|u_n|^2$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$,
- u_n is bounded in $L^2(0,T;L^{\frac{2N}{N-2}}(\mathbb{R}^N))$ for $N\geq 3$.

Passing to the weak limit in the previous bound in extracting subsequence if necessary, one can assume that :

- $\rho_n\rightarrow\rho$ weakly in $L^s((0,T)\times\mathbb{R}^N)$,
- $u_n\rightarrow u$ weakly in $L^2(0,T,\dot{H}^1(\mathbb{R}^N))$,
- $\rho_n^\gamma\rightarrow\overline{\rho^\gamma}$ weakly in $L^r((0,T)\times\mathbb{R}^N)$ for $r=\frac{q}{\gamma}>1$,
- $\rho_n^2\rightarrow\overline{\rho^2}$ weakly in $L^{r_1}((0,T)\times\mathbb{R}^N)$ for $r_1=\frac{q}{2}>1$.

Notation 4. *We will always write in the sequel $\overline{B(\rho)}$ to mean the weak limit of the sequence $B(\rho_n)$ bounded in appropriate space that we will precise.*

We recall that the main difficulty will be to pass to the limit in the pressure term and the capillary term. The idea of the proof will be to test the convergence of the sequence $(\rho_n)_{n\in\mathbb{N}}$ on convex functions B in order to use their properties of lower semi-continuity with respect to the weak topology in $L^1(\mathbb{R}^N)$. In this goal we will use the theory of renormalized solutions introduced by Diperna and Lions in [7]. So we will obtain strong convergence of ρ_n in appropriate spaces.

2.3 Idea of the proof

We here give a sketchy proof of the theorem 1.12. First, we can rewrite mass conservation of the regular solution $(\rho_n,u_n)_{n\in\mathbb{N}}$ on the form :

$$\frac{\partial}{\partial t}(B(\rho_n))+\mathrm{div}(u_nB(\rho_n))=(B(\rho_n)-\rho_nB'(\rho_n))\mathrm{div}u_n.$$

Supposing that $B(\rho_n)$ is bounded in appropriate space we can pass to the weak limit where we have in the energy space $\rho_n\rightharpoonup\rho$ and $u_n\rightharpoonup u$, so we get :

$$\frac{\partial}{\partial t}(\overline{B(\rho)})+\mathrm{div}(u\overline{B(\rho)})=\overline{b(\rho)\mathrm{div}u}\quad\text{with }\ b(\rho)=B(\rho)-\rho B'(\rho).\tag{2.14}$$

Arguing like P-L. Lions in [16] p 13 we will get :

$$\frac{\partial}{dt}\rho + \mathrm{div}(\rho u) = 0. \tag{2.15}$$

After we will just have to verify the passage to the limit for the product ρu. Next we will use the theorem on the renormalized solutions of Diperna-Lions in [15] on (2.15) in recalling that $\rho \in L^\infty(L^2)$. So we get :

$$\frac{d}{dt}(B(\rho)) + \mathrm{div}(uB(\rho)) = b(\rho)\mathrm{div}(u). \tag{2.16}$$

Next we subtract (2.14) to (2.16), so we obtain :

$$\frac{d}{dt}(\overline{B(\rho)} - B(\rho)) + \mathrm{div}(u(\overline{B(\rho)} - B(\rho))) = \overline{b(\rho)\mathrm{div}u} - b(\rho)\mathrm{div}u. \tag{2.17}$$

Consequently, in order to estimate the difference $\overline{B(\rho)} - B(\rho)$ which tests the convergence of ρ_n, we need to estimate the difference $\overline{b(\rho)\mathrm{div}(u)} - b(\rho)\mathrm{div}(u)$. We choose then B a concave function and we have :

$$\overline{B(\rho)} - B(\rho) \leq 0.$$

The goal will be now to prove the reverse inequality in order to justify that $B(\rho_n)$ tends to $B(\rho)$ a.e.

So now we aim at estimating the difference $\overline{b(\rho)\mathrm{div}u} - b(\rho)\mathrm{div}u$. This may be achieved by introducing the effective viscous pressure $\mathrm{P}_{eff} = P - (2\mu + \lambda)\mathrm{div}u$ after D. Hoff in [**?**], which satisfies some important properties of weak convergence.

In fact owing to the capillarity term we adapt Hoff's concept to our equation in setting :

$$\widetilde{P}_{eff} = P + \frac{\kappa}{2}\rho^2 - (2\mu + \lambda)\mathrm{div}u.$$

Proof of theorem 1.12

We begin with the case $N \geq 3$, and next we will complete the proof by the case $N = 2$ in specifying the changes to bring.

Before getting into the heart of the proof, we first recall that we obtain easily the convergence in distribution sense of $\rho_n u_n$ to ρu and $\rho_n (u_n)_i (u_n)_j$ to $\rho u_i u_j$. We refer to the classical result by Lions (see [16]) or the book of Novotný and Straškraba [17].

Case $N \geq 3$

As explained in section 2.3 our goal is to compare $\overline{B(\rho)}$ and $B(\rho)$ for certain concave functions B . From the mass equation we have obtained :

$$\partial_t(\overline{B(\rho)} - B(\rho)) + \mathrm{div}(u(\overline{B(\rho)} - B(\rho))) = \overline{b(\rho)\mathrm{div}(u)} - b(\rho)\mathrm{div}(u). \tag{2.18}$$

So before comparing $\overline{B(\rho)}$ and $B(\rho)$, we have to investigate the expression $\overline{b(\rho)\mathrm{div}(u)} -$

$b(\rho)\text{div}(u)$. By virtue of theorem 2.13 which gives a gain of integrability we can take the function $B(x) = x^{\varepsilon}$, as we control for ε small enough $\rho^{s+\varepsilon}$. Our goal now is to exhibit the effective pressure $\widetilde{P}_{eff}$, and to multiply it by ρ^{ε} to extract $\overline{\text{div}u\, b(\rho)}$. We will see in the sequel how to compare it with $b(\rho)\text{div}(u)$.

Control of the term $\overline{\text{div}u\, b(\rho)}$

Taking the div of the momentum equation satisfied by the regular solution yields :

$$\frac{\partial}{\partial t}\text{div}(\rho_n u_n) + \partial^2_{ij}(\rho_n u^i_n u^j_n) - \zeta\Delta\text{div}u_n + \Delta(a\rho_n^{\gamma}) = \kappa\text{div}(\rho_n(\nabla\phi * \rho_n)) - \frac{\kappa}{2}\Delta(\rho_n^2), \tag{2.19}$$

with $\zeta = \lambda + 2\mu$. Applying the operator $(-\Delta)^{-1}$ to (2.19), we obtain :

$$\begin{aligned} \frac{\partial}{\partial t}(-\Delta)^{-1}\text{div}(\rho_n u_n) + (-\Delta)^{-1}\partial^2_{ij}(\rho_n u^i_n u^j_n) + [\zeta\text{div}u_n - a\rho_n^{\gamma} - \frac{\kappa}{2}\rho_n^2] \\ = \kappa(-\Delta)^{-1}\text{div}(\rho_n(\nabla\phi * \rho_n)) \; . \end{aligned} \tag{2.20}$$

After we multiply (2.20) by ρ_n^{ε} with ε that we choose small enough in $(0,1)$:

$$\begin{aligned} [\zeta\text{div}u_n - a\rho_n^{\gamma} - \frac{\kappa}{2}\rho_n^2]\rho_n^{\varepsilon} &= \kappa\rho_n^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n(\nabla\phi * \rho_n)) \\ &- \rho_n^{\varepsilon}\frac{\partial}{\partial t}(-\Delta)^{-1}\text{div}(\rho_n u_n) - \rho_n^{\varepsilon}(-\Delta)^{-1}\partial_{ij}(\rho_n u^i_n u^j_n). \end{aligned} \tag{2.21}$$

So if we rewrite (2.21), we have :

$$\begin{aligned} [\zeta\text{div}u_n - a\rho_n^{\gamma} - \frac{\kappa}{2}\rho_n^2]\rho_n^{\varepsilon} &= \kappa\rho_n^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n(\nabla\phi * \rho_n)) - \rho_n^{\varepsilon}(-\Delta)^{-1}\partial^2_{ij}(\rho_n u^i n u^j_n) \\ &- \frac{\partial}{\partial t}\big((\rho_n)^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n)\big) + [\frac{\partial}{\partial t}(\rho_n)^{\varepsilon}](-\Delta)^{-1}\text{div}(\rho_n u_n), \end{aligned}$$

Next we have :

$$\begin{aligned} [\zeta\text{div}u_n - a\rho_n^{\gamma} - \frac{\kappa}{2}\rho_n^2]\rho_n^{\varepsilon} &= \kappa\rho_n^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n(\nabla\phi * \rho_n)) - \rho_n^{\varepsilon}(-\Delta)^{-1}\partial^2_{ij}(\rho_n u^i_n u^j_n) \\ &- \frac{\partial}{\partial t}[(\rho_n)^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n)] - \text{div}[u_n(\rho_n)^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n)] \\ &+ (\rho_n)^{\varepsilon}u_n\cdot\nabla(-\Delta)^{-1}\text{div}(\rho_n u_n) + (1-\varepsilon)(\text{div}u_n)(\rho_n)^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n), \end{aligned} \tag{2.22}$$

or finally :

$$\begin{aligned} [\zeta\text{div}u_n - a\rho_n^{\gamma} - \frac{\kappa}{2}\rho_n^2]\rho_n^{\varepsilon} &= \kappa\rho_n^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n(\nabla\phi * \rho_n)) \\ &- \frac{\partial}{\partial t}[\rho_n^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n)] - \text{div}[u_n(\rho_n)^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n)] \\ &+ (\rho_n)^{\varepsilon}[u_n.\nabla(-\Delta)^{-1}\text{div}(\rho_n u_n) - (-\Delta)^{-1}\partial^2_{ij}(\rho u^i_n u^j_n)] \\ &+ (1-\varepsilon)(\text{div}u_n)(\rho_n)^{\varepsilon}(-\Delta)^{-1}\text{div}(\rho_n u_n) \; . \end{aligned} \tag{2.23}$$

Now like in Lions [16] we want to pass to the limit in the distribution sense in (2.23) in order to estimate $\overline{\text{div}u(\rho)^{\varepsilon}}$.

Passage to the weak limit in (2.23)

We shall use the following lemma by P-L Lions in [16] :

Lemma 3. *Let Ω be an open set of $\mathbb{R}^N$. Let (g_n, h_n) converge weakly to (g, h) in $L^{p_1}(0, T, L^{p_2}(\Omega))\times L^{q_1}(0, T, L^{q_2}(\Omega))$ where $1 \le p_1, p_2, q_1, q_2 \le +\infty$ satisfy,*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1\,.$$

Assume in addition that :

$$\frac{\partial g^n}{\partial t} \text{ is bounded in } L^1(0, T, W^{-m,1}(\Omega)) \text{ for some } m \ge 0 \text{ independent of } n. \tag{2.24}$$

and that :

$$\|h^n - h^n(\cdot, \cdot + \xi)\|_{L^{q_1}(0,T,L^{q_2}(\Omega))} \to 0 \quad \text{as } |\xi| \to 0, \text{ uniformly in } n. \tag{2.25}$$

Then, $g^n h^n$ converges to gh (in the sense of distribution on $\Omega \times (0, T)$).

So we use the above lemma to pass to the weak limit in the following four non-linear terms of (2.23) :

$$T_n^1 = u_n\rho_n^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n), \qquad T_n^2 = \rho_n^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n),$$
$$T_n^3 = \rho_n^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n(\nabla\phi * \rho_n)), \qquad T_n^4 = (\mathrm{div}u_n)(\rho_n)^\varepsilon(-\Delta)^{-1}\mathrm{div}(\rho_n u_n)\,.$$

So we choose the different g_n^i and h_n^i as follows :

for T_n^1	$g_n^1 = u_n(\rho_n)^\varepsilon$	$g^1 = u\overline{\rho^\varepsilon}$	$h_n^1 = (-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$
for T_n^2	$g_n^2 = \rho_n^\varepsilon$	$g^2 = \overline{\rho^\varepsilon}$	$h_n^2 = (-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$
for T_n^3	$g_n^3 = \rho_n^\varepsilon$	$g^3 = \overline{\rho^\varepsilon}$	$h_n^3 = (-\Delta)^{-1}\mathrm{div}(\rho_n(\nabla\phi * \rho_n))$
for T_n^4	$g_n^4 = (\mathrm{div}u_n)(\rho_n)^\varepsilon$	$g^4 = \overline{\mathrm{div}u\,\rho^\varepsilon}$	$h_n^4 = (-\Delta)^{-1}\mathrm{div}(\rho_n u_n)$.

To show that $u_n(\rho_n)^\varepsilon$ converges in distribution sense to $u\overline{\rho^\varepsilon}$ we apply lemma 3 with $h_n = u_n$ and $g_n = \rho_n^\varepsilon$. We now want to examine each term and apply the above lemma to pass to the limit in the weak sense.
We start with the first term T_n^1. We have that $\rho_n^\varepsilon u_n \in L^\infty(L^q) \cap L^2(L^r)$ with $\frac{1}{q} = \frac{\varepsilon}{2s} + \frac{1}{2}$ and $\frac{1}{r} = \frac{(N-2)}{2N} + \frac{\varepsilon}{s} = \frac{1}{2} - \frac{1}{N} + \frac{\varepsilon}{s}$. In addition the hypothesis (2.24) is immediately verified (use the momentum equation).
We now want to verify the hypothesis (2.25), so we have h_n^1 belongs to $L^\infty(W_{loc}^{1,q'}(\mathbb{R}^N)) \cap L^2(W_{loc}^{1,r'}(\mathbb{R}^N))$ with $\frac{1}{q'} = \frac{1}{2} + \frac{1}{2s}$ and $\frac{1}{r'} = \frac{(N-2)}{2N} + \frac{1}{s} = \frac{1}{2} + \frac{1}{s} - \frac{1}{N}$. This result enables us to verify the hypothesis (2.25) by Sobolev embedding.
So we can choose (with the notation of the above lemma) $q_1 = 2$ and $q_2 \in (r', \frac{Nr'}{N-r'})$, $p_1 = 2$, and $p_2 = 1 - \frac{1}{q_2}$ which is possible by interpolation. Indeed we have : $\frac{1}{r'} + \frac{1}{r} = 1 - \frac{2}{N} + \frac{1+\varepsilon}{s} \le 1$.
We proceed in the same way for T_n^2 and T_n^4.
We can similarly examine T_n^3, because $\rho_n^\varepsilon \in L^\infty(L^{\frac{1}{\varepsilon}} \cap L^{\frac{s}{\varepsilon}})$ and $\rho_n(\nabla\phi * \rho_n) \in L^\infty(L^1 \cap L^{\frac{s}{2}})$,

we can choose $p_2=\frac{1}{\varepsilon}$, we have then $(\Delta)^{-1}\text{div}\rho_n(\nabla\phi*\rho_n)\in L^\infty(0,T;W^{1,\frac{s}{2}})$ so that we can choose $q_1=2$, $q_2\in(1,\frac{N\frac{s}{2}}{N-\frac{s}{2}})$. We can conclude by interpolation.

Finally we have to study the last non linear following term that we treat similarly as P-L.Lions in [16] :

$$A_n=(\rho_n)^\varepsilon[u_n.\nabla(-\Delta)^{-1}\text{div}(\rho_n u_n)-(-\Delta)^{-1}\partial^2_{ij}(\rho_n(u_i)_n(u_j)_n)].$$

We can express this term A_n as follows :

$$A_n=(\rho_n)^\varepsilon[u_n^j,R_{ij}](\rho_n u_n^i).$$

where $R_{ij}=(-\Delta)^{-1}\partial^2_{ij}$ with R_i the classical Riesz transform.
Next, we use a result by Coifman, Lions, Meyer, Semmes on this type of commutator (see [5]) to take advantage of the regularity of $[u_n^j,R_{ij}](\rho_n u_n^i)$.

Theorem 2.14. *The following map is continuous for any $N\geq 2$:*

$$\begin{aligned}W^{1,r_1}(\mathbb{R}^N)^N\times L^{r_2}(\mathbb{R}^N)&\rightarrow W^{1,r_3}(\mathbb{R}^N)^N\\(a,b)&\rightarrow[a_j,R_iR_j]b_i\end{aligned}\tag{2.26}$$

with : $\frac{1}{r_3}=\frac{1}{r_1}+\frac{1}{r_2}$.

To pass to the weak limit in A_n we will use the previous lemma. We start with the case with $s>3$. This quantity belongs to the space $L^1(W^{1,q})$ provided that $Du_n\in L^2(L^2)$ and $\rho u^j\in L^2(L^r)$ where $\frac{1}{r}=\frac{N-2}{2N}+\frac{1}{s}=\frac{1}{2}-\frac{1}{N}+\frac{1}{s}$ in which case $\frac{1}{q}=\frac{1}{r}+\frac{1}{2}=1-\frac{1}{N}+\frac{1}{s}\leq 1$. After we can use the above lemma applied to $h_n=[R_{ij},u_n^j](\rho_n u_n^i)$ and $g_n=\rho_n^\varepsilon$. We can show easily in using again lemma 3 that h_n converges in distribution sense to $[R_{ij},u_j](\rho_n u_i)$. So we can take : $q_1=1$, $p_1=+\infty$ and $q_2\in(q,\frac{qN}{N-q})$, $p_2=1-\frac{1}{q_2}$, this one because we can use interpolation and we can localize as we want limit in distribution sense.
In the case where $s\leq 3$, a simple interpolation argument can be used to accommodate the general case. It suffices to fix $L^{r_2}(\mathbb{R}^N)$ in the application (2.26) and use a result of Riesz-Thorin.
Finally passing to the limit in (2.21), we get :

$$\begin{aligned}[\zeta\overline{\text{div}u\,\rho^\varepsilon}-\overline{(a\rho^{\gamma+\varepsilon})}-\frac{\kappa}{2}\overline{\rho^{2+\varepsilon}}]=\overline{\rho^\varepsilon}(-\Delta)^{-1}\text{div}(\overline{\rho(\nabla\phi*\rho)})-\frac{\partial}{\partial t}[\overline{\rho^\varepsilon}(-\Delta)^{-1}\text{div}(\rho u)]\\-\text{div}[\overline{\rho^\varepsilon}u(-\Delta)^{-1}\text{div}(\rho u)]+\overline{\rho^\varepsilon}[u.\nabla(-\Delta)^{-1}\text{div}(\rho u)-(-\Delta)^{-1}\partial_{ij}(\rho u_i u_j)]\\+(1-\varepsilon)\overline{\text{div}u\,\rho^\varepsilon}(-\Delta)^{-1}\text{div}(\rho u).\end{aligned}\tag{2.27}$$

Inequality between the terms $\overline{\rho^\varepsilon}\,\text{div}u$ and $\overline{\text{div}u\,\rho^\varepsilon}$

Now we are interested in estimating the term $\overline{\rho^\varepsilon}\text{div}u$ in order to describe the quantity $\overline{\rho^\varepsilon}\,\text{div}u-\overline{\text{div}u\,\rho^\varepsilon}$ before considering the quantity $\rho^\varepsilon\,\text{div}u-\overline{\text{div}u\,\rho^\varepsilon}$. We pass to the weak limit directly in (2.20) and we get in using again lemma 3 :

$$\begin{aligned}\frac{\partial}{\partial t}(-\Delta)^{-1}\mathrm{div}(\rho u)+(-\Delta)^{-1}\partial^2_{ij}(\rho u_i u_j)+&[(\mu+\xi)\mathrm{div}u-\overline{a\rho^\gamma}]=\\&-(-\Delta)^{-1}\mathrm{div}(\overline{\rho(\nabla\phi*\rho)})+\frac{\kappa}{2}\overline{\rho^2}.\end{aligned}\tag{2.28}$$

Now we just multiply (2.28) with $\overline{\rho^\varepsilon}$ and we can see that each term has a distribution sense. So we get in proceeding in the same way as before :

$$\begin{aligned}&[\zeta\mathrm{div}u\,\overline{\rho^\varepsilon}-\overline{(a\rho^\gamma)}\,\overline{\rho^\varepsilon})-\frac{\kappa}{2}\overline{\rho^2}\,\overline{\rho^\varepsilon}]=\overline{\rho^\varepsilon}(-\Delta)^{-1}\mathrm{div}(\overline{\rho(\nabla\phi*\rho)})\\&-\overline{\rho^\varepsilon}\frac{\partial}{\partial t}[\rho(-\Delta)^{-1}\mathrm{div}(\rho u)]+\overline{\rho^\varepsilon}[u.\nabla(-\Delta)^{-1}\mathrm{div}(\rho u)-(-\Delta)^{-1}\partial_{ij}(\rho u_i u_j)]\\&\qquad-\mathrm{div}[\rho^\varepsilon u(-\Delta)^{-1}\mathrm{div}(\rho u)]+(1-\varepsilon)\overline{\mathrm{div}u(\rho)^\varepsilon}\,(-\Delta)^{-1}\mathrm{div}(\rho u).\end{aligned}\tag{2.29}$$

Subtracting (2.29) from (2.27), we get :

$$\zeta\,\overline{\mathrm{div}u(\rho)^\varepsilon}-a\overline{\rho^{\gamma+\varepsilon}}-\frac{\kappa}{2}\overline{\rho^{2+\varepsilon}}=\zeta\mathrm{div}u\,\overline{\rho^\varepsilon}-a\overline{\rho^\gamma}\,\overline{\rho^\varepsilon}-\frac{\kappa}{2}\overline{\rho^2}\,\overline{\rho^\varepsilon}\quad\text{a.e .}$$

Next we observe that by convexity :

$$(\overline{\rho^{\gamma+\varepsilon}})^{\frac{\varepsilon}{\gamma+\varepsilon}}\geq(\overline{\rho^\varepsilon}),\quad(\overline{\rho^{\gamma+\varepsilon}})^{\frac{\gamma}{\gamma+\varepsilon}}\geq(\overline{\rho^\gamma})\quad\text{a.e .}$$

So we get :

$$\overline{\mathrm{div}u\,(\rho)^\varepsilon}\geq\mathrm{div}u\,\overline{\rho^\varepsilon}.\tag{2.30}$$

Comparison between ρ and $\overline{\rho^\varepsilon}^{\frac{1}{\varepsilon}}$

As (ρ_n,u_n) is a smooth approximate solution, applying equality (2.16) to $B(x)=x^\varepsilon$ yields :

$$\frac{\partial}{\partial t}(\rho_n)^\varepsilon+\mathrm{div}(u_n(\rho_n)^\varepsilon)=(1-\varepsilon)\mathrm{div}u_n(\rho_n)^\varepsilon.\tag{2.31}$$

Passing to the weak limit in (2.31), we get :

$$\frac{\partial}{\partial t}\overline{\rho^\varepsilon}+\mathrm{div}(u\,\overline{\rho^\varepsilon})=(1-\varepsilon)\overline{\mathrm{div}u\rho^\varepsilon}.\tag{2.32}$$

Combining with (2.30) we thus get :

$$\frac{\partial}{\partial t}\overline{(\rho)^\varepsilon}+\mathrm{div}(u\overline{(\rho)^\varepsilon})\geq(1-\varepsilon)\mathrm{div}u\overline{(\rho)^\varepsilon}.\tag{2.33}$$

Now we wish to conclude about the pointwise convergence of ρ_n in proving that $(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}=\rho$ and to finish we will use the following theorem (see [9] p 34) applied to $B(x)=x^{\frac{1}{\varepsilon}}$ which is convex.

Theorem 2.15. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions bounded in $L^1(\mathbb{R}^N)$ such that :*

$$v_n\rightharpoonup v\quad\textit{weakly in }\;L^1(\mathbb{R}^N).$$

Let $\varphi : \mathbb{R} \longrightarrow [-\infty, +\infty)$ be a upper semi-continuous strictly concave function such that $\varphi(v_n) \in L^1(\mathbb{R}^N)$ for any n, and :

$$\varphi(v_n) \rightharpoonup \overline{\varphi(v)} \quad \text{weakly in } L^1(\mathbb{R}^N).$$

Then :

$$\varphi(v) \geq \overline{\varphi(v)}.$$

and if $\varphi(v) = \overline{\varphi(v)}$ then :

$$v_n(y) \rightarrow v(y) \ \ a.e.$$

extracting a subsequence as the case may be.

Now we want to use a type of Diperna-Lions theorem on inequality (2.33). Our goal is to renormalize this inequality with the function $B(x) = x^{\frac{1}{\varepsilon}}$ so that one can compare ρ and $\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}$. Although (2.33) doesn't correspond exactly to the mass equation, we can use the same technics to renormalize the solution provided that $\rho \in L^\infty(L^2)$ which is the case. [1] We recall Diperna-Lions theorem on renormalized solution for the mass equation.

Theorem 2.16. *Suppose that $\rho \in L^\infty(L^2)$, $\beta \in C[0,\infty) \cap C^1(0,\infty);\mathbb{R})$ and the function $b(z) = z\beta'(z) - \beta(z)$ is bounded on $[0,\infty)$ with moreover $\beta(0) = b(0) = 0$.*
We have then :

$$\frac{\partial \beta(\rho)}{\partial t} + \mathrm{div}(\beta(\rho)\, u) = (\beta(\rho) - \rho\beta'(\rho))\mathrm{div} u$$

in distribution sense.

We now want to adapt this theorem for our equation (2.33) with $\beta(x) = x^{\frac{1}{\varepsilon}}$, so we may regularize by ω_α (with $\omega_\alpha = \frac{1}{\alpha^N}\omega(\frac{\cdot}{\alpha})$ where $\omega \in C_0^\infty(\mathbb{R}^N)$, supp $\omega \in B_1$ and $\int \omega dx = 1$) and find for all $\beta \in C_0^\infty([0, +\infty))$:

$$\frac{\partial}{\partial t}(\overline{\rho^{\varepsilon}} * \omega_\alpha) + \mathrm{div}[u\,\overline{\rho^{\varepsilon}} * \omega_\alpha] \geq (1-\varepsilon)\mathrm{div} u\,\overline{\rho^{\varepsilon}} * \omega_\alpha + R_\alpha$$

where we have :

$$R_\alpha = \mathrm{div}[u\,\overline{\rho^{\varepsilon}} * \omega_\alpha] - \mathrm{div}(u\,\overline{\rho^{\varepsilon}}) * \omega_\alpha + (1-\varepsilon)[\mathrm{div} u\,\overline{\rho^{\varepsilon}}] * \omega_\alpha - (1-\varepsilon)\mathrm{div} u\,\overline{\rho^{\varepsilon}} * \omega_\alpha$$

We get :

$$\begin{aligned}\frac{\partial}{\partial t}(\beta(\overline{\rho^{\varepsilon}} * \omega_\alpha)) + \mathrm{div}[u\,\beta(\overline{\rho^{\varepsilon}} * \omega_\alpha)] &\geq (1-\varepsilon)\mathrm{div} u\,\overline{\rho^{\varepsilon}} * \omega_\alpha\,\beta'(\overline{\rho^{\varepsilon}} * \omega_\alpha) \\ &\quad + (\mathrm{div} u)[\beta(\overline{\rho^{\varepsilon}} * \omega_\alpha) - \overline{\rho^{\varepsilon}} * \omega_\alpha \beta'(\overline{\rho^{\varepsilon}} * \omega_\alpha)] + R_\alpha \beta'(\overline{\rho^{\varepsilon}} * \omega_\alpha) \\ &= -\varepsilon(\mathrm{div} u)\overline{(\rho)^{\varepsilon}}\beta'(\overline{\rho^{\varepsilon}}) + (\mathrm{div} u)\,\beta(\overline{\rho^{\varepsilon}}) + R_\alpha \beta'(\overline{\rho^{\varepsilon}} * \omega_\alpha)\ .\end{aligned}$$

After we pass to the limit when $\alpha \rightarrow 0$ and we see that R_α tends to 0 in using lemma on regularization in [15] p 43. This looks like a rather harmless manipulation but it's at this

[1] In our case it is very important that $\rho \in L^\infty(L^2)$, indeed it avoids to have supplementary conditions on the index γ like for the compressible Navier-Stokes system in [16].

point that we require to control ρ in $L^2(0,T;\mathbb{R}^N)$. And in our case we don't need to impose $\gamma>\frac{N}{2}$ for $N=2,3$. Hence :

$$\frac{\partial}{\partial t}(\beta(\overline{(\rho)^\varepsilon}))+\text{div}[u\,\beta(\overline{(\rho)^\varepsilon})]\geq -\varepsilon(\text{div}u)\overline{\rho^\varepsilon}\beta'(\overline{\rho^\varepsilon})+(\text{div}u)\beta(\overline{\rho^\varepsilon}).$$

We then choose $\beta=(\Psi_M)^{\frac{1}{\varepsilon}}$ where $\Psi_M=M\Psi(\frac{\cdot}{M})$, $M\geq 1$, $\Psi\in C_0^\infty([0,+\infty))$, $\Psi(x)=x$ on $[0,1]$, supp$\Psi\subset[0,2]$, and we obtain :

$$\begin{aligned}\frac{\partial}{\partial t}(\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}})&+\text{div}[u\,\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}]\\
&\geq(\text{div}u)\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}-1}\Psi_M'(\overline{\rho^\varepsilon})\overline{\rho^\varepsilon}+(\text{div}u)\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}\\
&\geq \text{div}u\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}-1}[\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}-\Psi_M'(\overline{\rho^\varepsilon})\overline{\rho^\varepsilon}]1_{(\overline{\rho^\varepsilon}>M)}\\
&\geq -C_0|\text{div}u|M^{\frac{1}{\varepsilon}}1_{(\overline{\rho^\varepsilon}>M)}.\end{aligned}$$

where $C_0=\sup\{|\Psi(x)|^{\frac{1}{\varepsilon}-1}|\Psi(x)-x\Psi'(x)|,\ x\in[0,+\infty)\}$.
Now we claim that :

$$\frac{\partial}{\partial t}(\overline{(\rho)^\varepsilon})^{\frac{1}{\varepsilon}})+\text{div}(u\,\overline{(\rho)^\varepsilon})^{\frac{1}{\varepsilon}})\geq 0. \tag{2.34}$$

For proving that, we notice that by convexity $\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}\leq\rho$, so we get :

$$\||\text{div}u|M^{\frac{1}{\varepsilon}}1_{\overline{(\rho)^\varepsilon}>M}\|_{L^1_T(L^1(\mathbb{R}^N))}\leq\|\text{div}u\|_{L^2_T(L^2(\mathbb{R}^N))}\|\rho\,1_{\rho>M^{\frac{1}{\varepsilon}}}\|_{L^2_T(L^2(\mathbb{R}^N))}\rightarrow 0\ \text{ as }\ M\rightarrow+\infty.$$

We have concluded by dominated convergence.
At this stage we subtract the mass equation to (2.34) and we get in setting $r=\rho-\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}$:

$$\frac{\partial}{\partial t}(r)+\text{div}(ur)\leq 0. \tag{2.35}$$

We now want to integrate and to use the fact that $r\geq 0$ to get that $r=0$ a.a. To justify the integration we test our inequality against a cut-off function $\varphi_R=\varphi(\frac{\cdot}{R})$ where $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi=1$ on $B(0,1)$, Supp$\varphi\subset B(0,2)$ and $R>1$. We get :

$$\int_{[0,T]\times\mathbb{R}^N}\frac{\partial}{\partial t}[r(t,x)]\varphi_R(x)-u(t,x)r(t,x)\frac{1}{R}\nabla\varphi(\frac{x}{R})dt\,dx\leq 0. \tag{2.36}$$

Next we notice that :

$$\begin{aligned}|\int_{[0,T]\times\mathbb{R}^N}u(t,x)r(t,x)\frac{1}{R}\nabla\varphi(\frac{x}{R})dt\,dx|\leq\|u\|_{L^1(0,T;L^{\frac{2N}{N-2}}(\mathbb{R}^N))}&\|r\|_{L^1(0,T;L^{\frac{2N}{N+2}}(\mathbb{R}^N))}\\
&\times\frac{1}{R}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^N)}.\end{aligned}$$

It implies that :

$$\int_{[0,T]\times\mathbb{R}^N}u(t,x)r(t,x)\frac{1}{R}\nabla\varphi(\frac{x}{R})dt\,dx\rightarrow 0\quad\text{as }\ R\rightarrow+\infty.$$

We have then :

$$\int_{[0,T]\times\mathbb{R}^N}\frac{\partial}{\partial t}r(t,x)\varphi_R(x)dt\,dx=\int_{\mathbb{R}^N}r(T,x)\varphi_R(T,x)dx-\int_{\mathbb{R}^N}r(0,x)\varphi_R(0,x)dx.$$

In order to conclude, it suffices to verify that $r(0,\cdot)=0$. Indeed we will obtain that :

$$\lim_{R\to+\infty}\int_{\mathbb{R}^N}r(T,x)\varphi_R(T,x)dx\to\int_{\mathbb{R}^N}r(T,x)dx\leq 0\quad\text{and}\quad r\geq 0.$$

then $r=0$.
We know that ρ_n is uniformly bounded in $L^\infty(L^1\cap L^s(\mathbb{R}^N))$, then ρ_n^ε is relatively compact in $C([0,T];L^p-w)$ with $1<p<s$ (where L^p-w denote the space L^p endowed with weak topology). As moreover $(\rho_0^\varepsilon)_n$ converges to ρ_0^ε, we deduce that $r(0)=0$ a.a.
Now as $r=0$ we conclude in using the theorem 2.15 that ρ_n converges a.a to ρ and that ρ_n converges to ρ in $L^p([0,T]\times B_R)$ for all $p\in[1,q)$ and in $L^{p_1}(0,T,L^{p_2}(B_R))$ for all $p_1\in[1,+\infty)$, $p_2\in[1,s)$ and for all $R\in(0,+\infty)$.

Conclusion

We wish now conclude and get the convergence of our theorem in the total space.
We aim at proving here the convergence of ρ_n in $C([0,T],L^p(\mathbb{R}^N))\cap L^{q'}(\mathbb{R}^N\times(0,T))$ for all $1\leq p<s$, $1\leq q'<q$. We have just to show the convergence of ρ_n to ρ in $C([0,T],L^1(\mathbb{R}^N))$. To this end, we introduce $d_n=\sqrt{\rho_n}$ which clearly converges to $\sqrt{\rho}$ in $L^{2p_1}(0,T,L^{2p_2}(B_R))\cap L^{2p}(B_R\times(0,T))$ to $d=\sqrt{\rho}$ for all $R\in(0,+\infty)$.
We next remark that $\rho\in C([0,T],L^1(\mathbb{R}^N))$ and thus $d\in C([0,T],L^2(\mathbb{R}^N)$. Indeed, using once more the regularization lemma in [15] we obtain the existence of a bounded $\rho_\alpha\in C([0,T],L^1(\mathbb{R}^N))$ smooth in x for all t satisfying :

$$\frac{\partial\rho_\alpha}{\partial t}+\operatorname{div}(u\rho_\alpha)=r_\alpha\quad\text{in}\quad L^1((0,T)\times\mathbb{R}^N)\quad\text{as}\quad\alpha\to 0_+,$$

with $r_\alpha=\operatorname{div}(u\rho_\alpha)-\operatorname{div}(\rho u)*\omega_\alpha$ (where ω is defined as in the previous part).
$\rho_\alpha\to\rho$ in $L^1(\mathbb{R}^N\times(0,T))$, $\rho_\alpha/_{t=0}\to\rho/_{t=0}$ in $L^1(\mathbb{R}^N)$ as $\alpha\to 0_+$.
From these facts, it is straightforward to deduce that :

$$\frac{\partial}{\partial t}|\rho_\alpha-\rho_\eta|+\operatorname{div}(u|\rho_\alpha-\rho_\eta|)=|r_\alpha-r_\eta|$$

and thus :

$$\sup_{[0,T]}\int_{\mathbb{R}^N}|\rho_\alpha-\rho_\eta|dx=\int_0^T\int_{\mathbb{R}^N}|r_\alpha-r_\eta|dx.$$

Since $\rho\in C([0,T],L^p(B_R)-w)$ (for all $R\in(0,+\infty)$, $1<p<s$), we may then deduce that ρ_α converges to ρ in $C([0,T],L^1(\mathbb{R}^N))$.
Next, we observe that we can justify as we did above that d_n and d satisfy :

$$\frac{\partial d_n}{\partial t}+\operatorname{div}(u_nd_n)=\frac{1}{2}d_n\operatorname{div}(u_n),$$

$$\frac{\partial d}{\partial t} + \mathrm{div}(ud) = \frac{1}{2} d\,\mathrm{div}(u).$$

Therefore once more, d_n converges to d in $C([0,T], L^2(\mathbb{R}^N) - w)$.
Thus in order to conclude, we just have to show that whenever $t_n \in [0,T]$, $t_n \to t$, then $d_n(t_n) \to d(t)$ in $L^2(\mathbb{R}^N)$ or equivalently that :

$$\int_{\mathbb{R}^N} d_n(t_n)^2 dx = \int_{\mathbb{R}^N} \rho_n(t_n) dx \to_n \int_{\mathbb{R}^N} d(t)^2 dx = \int_{\mathbb{R}^N} \rho(t) dx.$$

This is the case since we deduce from the mass equation, integrating this equation over $\mathbb{R}^N$ and justifying the integration exactly like previously that :

$$\int_{\mathbb{R}^n} \rho_n(t_n) dx = \int_{\mathbb{R}^n} (\rho_0)_n dx \to_n \int_{\mathbb{R}^n} \rho_0 dx = \int_{\mathbb{R}^n} \rho(t) dx.$$

We then conclude by uniform continuity that $\|\rho_n(t_n) - \rho_n(t)\|_{L^1}$ tends to 0.

Case $N = 2$

First of all, the main difficulty is the fact that we no longer have global L^p bounds on u_n. That's why most of the proof is in fact local and we know that u_n is bounded in $L^2(0,T; L^p(B_R))$ for all $p \in [1, +\infty)$, $R \in (0, +\infty)$.
As we need to localize the argument, we get the following limit :

$$(\zeta \mathrm{div} u_n - a\rho_n^\gamma - \frac{1}{2}\rho_n^2)\rho_n^\varepsilon \rightharpoonup_n (\zeta \mathrm{div} u - a\overline{\rho^\gamma} - \frac{1}{2}\overline{\rho^2})\,\overline{\rho^\varepsilon} \;\text{ in }\; \mathcal{D}'(\mathbb{R}^N \times [0,T]).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $0 \le \varphi \le 1$, $\mathrm{supp}\varphi \subset K$ for an arbitrary compact set $K \in \mathbb{R}^N$. We apply the operator $(\Delta)^{-1}\mathrm{div}$ to the momentum equation that we have localized and we pass directly to the weak limit :

$$\begin{aligned} &\frac{\partial}{\partial t}(-\Delta)^{-1}\mathrm{div}(\varphi\rho u) + R_{ij}(\varphi\rho u_i u_j) + [\zeta \mathrm{div} u\,\varphi - \overline{a\rho^\gamma}\varphi] \\ &\qquad\qquad = \kappa(-\Delta)^{-1}\mathrm{div}(\varphi\overline{\rho(\nabla\phi * \rho)} - \frac{\kappa}{2}\varphi\overline{\rho^2})) + (-\Delta)^{-1}\overline{R}. \end{aligned} \tag{2.37}$$

with :

$$\begin{aligned} \overline{R} = &\, \partial_i\varphi\partial_j(\rho u_i u_j) + (\partial_{ij}\varphi)\rho u_i u_j - \zeta\Delta\varphi \mathrm{div} u - \zeta\nabla\varphi\cdot\nabla\mathrm{div} u + \mu\Delta u\cdot\nabla\varphi \\ &+ \Delta\varphi a\overline{\rho^\gamma} + a\nabla\varphi\cdot\nabla\overline{\rho^\gamma} + \frac{\kappa}{2}\Delta\varphi\overline{\rho^2} + \frac{\kappa}{2}\nabla\varphi\cdot\nabla\overline{\rho^2}. \end{aligned}$$

Now we multiply (2.37) with $\overline{\rho^\varepsilon}$ and we verify that each term has a sense.
So we get in proceeding in the same way as before, we can verify that $(\rho_n)^\varepsilon(-\Delta)^{-1}R_n$ converges in distribution sense to $\overline{\rho^\varepsilon}(-\Delta)^{-1}\overline{R}$ for small enough ε. We get as in the previous case for $N \ge 3$:

$$\varphi[\zeta\,\overline{\mathrm{div} u(\rho)^\varepsilon} - a\overline{\rho^{\gamma+\varepsilon}} - \frac{\kappa}{2}\overline{\rho^{2+\varepsilon}}] = \varphi[\zeta(\mathrm{div} u)\varphi\,\overline{\rho^\varepsilon} - a\overline{\varphi\rho^\gamma}\,\overline{\rho^\varepsilon} - \frac{\kappa}{2}\overline{\varphi\rho^2}\overline{\rho^\varepsilon}] \quad \text{a.e .}$$

We then deduce the following inequalities as in the previous proof :

$$\frac{d}{dt}(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}} + \mathrm{div}(u(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}) \ge 0 \;\text{ in }\; \mathcal{D}'((0,T)\times\mathbb{R}^N).$$

We see that the only point left to check is the justification of the integration over $\mathbb{R}^2$ of terms like $\mathrm{div}(\overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}}u)$ or $\mathrm{div}(\rho u)$ and more precisely that the integral vanishes. This is in fact straightforward provided we use the bounds on $\rho \in L^\infty(L^1(\mathbb{R}^N))$ and $\rho|u|^2 \in L^\infty(L^1(\mathbb{R}^N))$ and so $\rho u \in L^\infty(L^1(\mathbb{R}^N))$. Then, letting $\varphi \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B(0,1)$ and $\varphi = 0$ on ${}^cB(0,2)$. We set $\varphi_R(\cdot) = \varphi(\frac{\cdot}{R})$ for $R \geq 1$, we have similarly as in the previous case :

$$|\int_0^T dt \int_{\mathbb{R}^2} ru \cdot \nabla\varphi_R(x)dx| \leq \|\nabla\varphi\|_{L^\infty(\mathbb{R}^2)}\frac{1}{R}\|\rho u\|_{L^1(0,T)\times\mathbb{R}^2)} \to 0 \quad \text{as} \quad R \to +\infty$$

We can then conclude as in the previous proof in using the fact that that $0 \leq \overline{\rho^{\varepsilon}}^{\frac{1}{\varepsilon}} \leq \rho$. □

Proof of the convergence assertion on $\rho_n u_n$

We now want to show the convergence of $\rho_n u_n$ to have informations on strong convergence of u_n modulo the vacuum. We recall in this part some classical inequalities to get the convergence of $\rho_n u_n$, for more details see Lions in [16]. We use once more a mollifier $k_\alpha = \frac{1}{\alpha^N}k(\frac{\cdot}{\alpha})$ where $k \in C_0^\infty(\mathbb{R}^N)$ and we let $g_\alpha = g * k_\alpha$ for an arbitrary function g. We first observe that we have for all $\frac{N}{2} < p < s$:

$$|\big((\rho_n u_n)_\alpha - \rho_n u_n\big)(x)| = |\int_{\mathbb{R}^N}[\rho_n(t,y) - \rho_n(t,x)]u_n(t,y)k_\alpha(x-y)dy$$
$$+ \rho_n(t,x)\big((u_n)_\alpha - u_n\big)(t,x)|$$

We have in using Hölder inequalities with the measure $k_\alpha(x-y)dy$:

$$|\big((\rho_n u_n)_\alpha - \rho_n u_n\big)(x)| \leq \big[\int_{\mathbb{R}^N} |\rho_n(t,y) - \rho_n(t,x)|^p k_\alpha(x-y)dy\big]^{\frac{1}{p}} \big(|u_n|^{\frac{p}{p-1}}\big)_\alpha^{\frac{p-1}{p}}$$
$$+ \rho_n|(u_n)_\alpha - u_n|(t,x).$$

Hence for all $t \geq 0$

$$\int_{\mathbb{R}^N} |\big((\rho_n u_n)_\alpha - \rho_n u_n\big)(x)|dx \leq \big[\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} |\rho_n(t,y) - \rho_n(t,x)|^p k_\alpha(x-y)dy\big]^{\frac{1}{p}}$$
$$\times \|\big(|u_n|^{\frac{p}{p-1}}\big)_\alpha\|_{L^1}^{\frac{p-1}{p}} + \|\rho_n\|_{L^p}\|(u_n)_\alpha - u_n\|_{L^{\frac{p-1}{p}}},$$
$$\leq \big[\sup_{|z|\leq\alpha} \|\rho_n(\cdot+z) - \rho_n\|_{L^p}\big]\|u_n\|_{L^{\frac{p-1}{p}}} + \|\rho_n\|_{L^p}\|(u_n)_\alpha - u_n\|_{L^{\frac{p-1}{p}}}.$$

Next if we choose $p > \frac{2N}{N+2}$, so that $\frac{p}{p-1} < \frac{2N}{N-2}$ then $\|(u_n)_\alpha - u_n\|_{L^2(0,T;L^{\frac{p-1}{p}})}$ converges to 0 as α goes to 0_+ uniformly in n. In addition, the convergence on ρ_n assure that $\sup_{|z|\leq\alpha}\|\rho_n(\cdot+z) - \rho_n\|_{L^p}$ converge to 0 as α goes to 0_+ uniformly in n. Therefore in conclusion, $(\rho_n u_n)_\alpha - \rho_n u_n$ converge to 0 in $L^2(0,T;L^1)$ as α goes to 0_+ uniformly in n.
Next $(\rho_n u_n)_\alpha$ is smooth in x, uniformly in n and in $t \in [0,T]$. Therefore, remarking that $\frac{\partial}{\partial t}(\rho_n u_n)_\alpha$ is bounded in a $L^2(0,T;H^m)$ for any $m \geq 0$, we deduce that $(\rho_n u_n)_\alpha$ converge to $(\rho u)_\alpha$ as n goes to $+\infty$ in $L^1((0,T)\times\mathbb{R}^N)$ for each α. Then using the bound on $\rho_n u_n$ in $L^\infty(L^{\frac{2s}{s+1}})$, we deduce that $\rho_n u_n$ converges to ρu in $L^1((0,T)\times\mathbb{R}^N)$ and we can conclude by interpolation.
The last convergence result is a consequence of the strong convergence of ρ_n and $\rho_n u_n$. □

3 Existence of weak solution with general pressure

In the sequel we focus on the cases $N=2,3$. We now want to extend our previous result to more general and physical pressure laws. In particular we are now interested by two cases, the first one concerns monotonous pressure law (close in a certain sense that we will precise to ρ^γ pressure) , the second one is the case of a slightly modified Van der Waals pressure.

The technics of proof will be very similar to the previous proof, only technical points change.

3.1 Monotonous pressure

In this section, we shall investigate an extension of the preceding results to the case of a general monotonous pressure P, i.e P is assumed to be a C^1 non-decreasing function on $[0,+\infty)$ vanishing at 0.

We want here to mention in the general situation our new energy inequality, we recall the inequality (1.4) :

$$\int_{\mathbb{R}^N}(\frac{1}{2}\rho|u|^2+\Pi(\rho)+E_{global}[\rho(.,t)])(x)dx(t)+\int_0^t\int_{\mathbb{R}^N}(\mu D(u):D(u)\\
+(\lambda+\mu)|\text{div}u|^2)dx\leq\int_{\mathbb{R}^N}\big(\frac{|m_0|^2}{2\rho}+\Pi(\rho_0)+E_{global}[\rho_0]\big)dx.$$

where we define Π by : $\frac{\partial}{\partial t}(\frac{\Pi(t)}{t})=\frac{P(t)}{t^2}$ for all $t>0$.

There are two cases worth considering : first of all if $P(t)$ is such that $\int_0^1\frac{P(s)}{s^2}ds<+\infty$ then we can choose $\Pi(\rho)=\rho\int_0^\rho\frac{P(s)}{s^2}ds$.

In the other case, i.e $\lim_{t\to 0}\frac{P(t)}{t}=c>0$, we can choose $\Pi(\rho)=\rho\int_\rho^1\frac{p(s)}{s^2}ds$ and Π behaves like $\rho\log\rho$ as ρ goes to 0.

We now consider a sequence of solutions (ρ_n,u_n) and we make the same assumptions on this sequence as in the previous section except that we need to modify the assumptions on ρ_n. We assume always that ρ_n is bounded in $C([0,T],L^1(\mathbb{R}^N))$, $P(\rho_n)$ is bounded in $L^\infty(0,T,L^1(\mathbb{R}^N))$,

$$(\rho_n)_{n\geq 1}\ \text{ is bounded in }\ L^\infty(0,T,L^2),$$

and we also assume that we have :

$$(\rho_n^\varepsilon P(\rho_n))_{n\geq 1}\ \text{ is bounded in }\ L^1(K\times(0,T))$$

for some $\varepsilon>0$, where K is an arbitrary compact set included in $\mathbb{R}^N$.

Theorem 3.17. *Let the assumptions of theorem 1.12 be satisfied with in addition P a monotone pressure.*
Then there exists a renormalized finite energy weak solution to problem (NSK) in the sense of definitions 1.12 and 1.13. Moreover $P(\rho_n)$ converges to $P(\rho)$ in $L^1(K\times(0,T))$ for any compact set K.

Proof :

The proof of theorem 3.17 is based on the same compactness argument as in the theorem 1.12. In particular, there is essentially one observation which allows us to adapt the proof of theorem 1.12. Namely we still obtain the following identity for the effective viscous flux :

$$\beta(\rho_n)(\zeta \mathrm{div} u_n - P(\rho_n) - \frac{\kappa}{2}\rho_n^2) \rightharpoonup_n \overline{\beta(\rho)}(\zeta \mathrm{div} u - \overline{P(\rho)} - \frac{\kappa}{2}\overline{\rho}^2),$$

with $\beta(\rho) = \rho^\varepsilon$ for ε small enough. We have then :

$$\zeta\overline{\rho^\varepsilon \mathrm{div} u} - \overline{P(\rho)\rho^\varepsilon} - \frac{\kappa}{2}\overline{\rho^{2+\varepsilon}}) = \zeta \overline{\mathrm{div} u}\,\overline{\rho^\varepsilon} - \overline{P(\rho)}\,\overline{\rho^\varepsilon} - \frac{\kappa}{2}\overline{\rho^2}\,\overline{\rho^\varepsilon}), \tag{3.38}$$

Now we can recall a lemma coming from P.-L. Lions in [16] :

Lemma 4. *Let p_1, $p_2 \in C([0,\infty))$ be non-decreasing functions. We assume that $p_1(\rho_n)$, $p_2(\rho_n)$ and $p_1(\rho_n)p_2(\rho_n)$ are relatively weakly compact in $L^1(K \times (0,T))$ for any compact set $K \subset \mathbb{R}^N$. Then, we have :*

$$\overline{p_1(\rho)p_2(\rho)} \geq \overline{p_1(\rho)}\,\overline{p_2(\rho)} \quad a.e.$$

We get finally as in the proof of theorem 1.12 in using lemma 4 :

$$\overline{\mathrm{div} u\, \rho^\varepsilon} \geq \mathrm{div} u \overline{\rho^\varepsilon},$$

All remaining argumentation of the proof of theorem 1.12 can be performed to conclude. □

3.2 Pressure of Van der Waals type

In this section we are interested by pressure of Van der Waals type which consequently are not necessarily non-decreasing. That's why in the following proof we will proceed slightly differently.

So we consider the pressure law :

$$P(\rho) = \frac{RT_*\rho}{b-\rho} - a\rho^2 \ \text{ for } \ \rho \leq b - \theta \quad \text{for some small } \ \theta > 0$$

and we extend the function P to be strictly increasing on $\rho \geq b - \theta$.

We have then :

1. P' is bounded from below, that is :

$$P'(\rho) \geq -\bar{\rho} \quad \text{for all } \ \rho > 0.$$

2. P is a strictly increasing function for ρ large enough.

Under the above conditions, it is easy to see that the pressure can be written as :

$$P(\rho) = P_1(\rho) - P_2(\rho).$$

with P_1 a non-decreasing function of ρ, and

$$P_2 \in C^2[0,+\infty), \quad P_2 \geq 0, \quad P_2 = 0 \;\text{ for }\; \rho \geq \overline{\rho}.$$

Remark 1. *The a priori energy estimate give us the bound of ρ in $L^\infty(L^2)$. We have thus :*

$$|\{(t,x) \in (0,T) \times \mathbb{R}^N / |\rho(t,x)| > b\}| \leq \frac{T\|\rho\|^2_{L^\infty(L^2(\mathbb{R}^N))}}{b^2}.$$

Hence the set where P is different from the Van der Waals law is of finite measure.

We obtain the following theorem.

Theorem 3.18. *If in addition to the above assumptions, we assume that ρ_n^0 converges in $L^1(\mathbb{R}^N)$ to ρ_0 then (ρ,u) is a weak solution of the system (NSK) satisfying the initial condition and we have :*

$$\rho_n \to \rho \;\text{ in }\; C([0,T], L^p(\mathbb{R}^N) \cap L^r((0,T)\times\mathbb{R}^N) \;\text{ for all }\; 1 \leq p < 2,\, 1 \leq r < 1 + \frac{4}{N},$$

Proof :

Most of the proof of theorem 3.18 is similar as theorem 1.12. We will use a approximated sequel T_k (introduced by Feireisl in [9]) of ρ by some concave bounded function.

Definition 3.14. *We define the function $T \in C^\infty(\mathbb{R}^N)$ as follows :*

$$\begin{aligned} &T(z) = z \quad \text{for } z \in [0,1],\\ &T(z) \;\text{ concave on }\; [0,+\infty),\\ &T(z) = 2 \quad \text{for } z \geq 3,\\ &T(z) = -T(-z) \quad \text{for } z \in (-\infty,0], \end{aligned}$$

And T_k is the cut-off function :

$$T_k(z) = kT(\frac{z}{k}).$$

In following the proof of theorem 1.12, we get :

$$\begin{aligned} \frac{\partial}{\partial t}(\overline{L_k(\rho)} - L_k(\rho)) + \mathrm{div}((\overline{T_k(\rho)} - T_k(\rho))u) + \overline{T_k(\rho)\mathrm{div}u} - \overline{T_k(\rho)}\mathrm{div}u&\\ = (T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}u \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^N)&, \end{aligned}$$

where $L_k(\rho) = \rho\log(\rho)$ for $0 \leq \rho \leq k$, and $0 \leq L_k(\rho) \leq \rho\log(\rho)$ otherwise.
So we get in integrating in time on $[t_1,t_2]$:

$$\begin{aligned} \int_{\mathbb{R}^N}\big(\overline{L_k(\rho)} - L_k(\rho)\big)(t_2)dx - \int_{\mathbb{R}^N}\big(\overline{L_k(\rho)} - L_k(\rho)\big)(t_1)dx&\\ + \int_{t_1}^{t_2}\int_{\mathbb{R}^N}\overline{T_k(\rho)\mathrm{div}u} - \overline{T_k(\rho)}\mathrm{div}u\,dxdt = \int_{t_1}^{t_2}\int_{\mathbb{R}^N}(T_k(\rho) - \overline{T_k(\rho)})\mathrm{div}u\,dxdt&. \end{aligned}$$

We can show that :

$$\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2((0,T)\times\mathbb{R}^N)}\to_{k\to+\infty}0$$

For proving the previous inequality, we see that $\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^1((0,T)\times\mathbb{R}^N)}\to 0$ for $k\to+\infty$. We then conclude by interpolation with $T_k(\rho)-\overline{T_k(\rho)}\in L^q(0,T)\times\mathbb{R}^N)$ with $q>2$. By Hölder inequality we obtain that :

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(T_k(\rho)-\overline{T_k(\rho)})\mathrm{div}u\,dxdt\to 0\quad\text{for } k\to+\infty.$$

We have then :

$$\begin{aligned}\lim_{k\to+\infty}\int_{\mathbb{R}^N}\big(\overline{L_k(\rho)}-L_k(\rho)\big)(t_2)dx-\int_{\mathbb{R}^N}\big(\overline{L_k(\rho)}-L_k(\rho)\big)(t_1)dx\\ =-\lim_{k\to+\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\overline{T_k(\rho)\mathrm{div}u}-\overline{T_k(\rho)}\mathrm{div}u\,dxdt.\end{aligned}\tag{3.39}$$

We set :

$$dft[\rho_n\to\rho](t)=\lim_{k\to+\infty}\int_{\mathbb{R}^N}\big(\overline{L_k(\rho)}-L_k(\rho)\big)(t)dx$$
$$A(k,\rho)=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\overline{T_k(\rho)\mathrm{div}u}-\overline{T_k(\rho)}\mathrm{div}u\,dxdt.$$

We can show as in the previous proof of theorem 1.12 that :

$$\begin{aligned}&\int_{t_1}^{t_2}\int_K\overline{T_k(\rho)\mathrm{div}u}-\overline{T_k(\rho)}\mathrm{div}u\,dxdt\\ &\qquad=\lim_{n\to+\infty}\int_{t_1}^{t_2}\int_K\big((P(\rho_n)+\frac{\kappa}{2}\rho_n^2)T_k(\rho_n)-\overline{(P(\rho)+\frac{\kappa}{2}\rho^2)}\overline{T_k(\rho)}\big)dxdt.\end{aligned}$$

for any compact $K\subset\mathbb{R}^N$.
Using the lemma 4 we deduce that :

$$\lim_{n\to+\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(P_1(\rho_n)+\frac{\kappa}{2}\rho_n^2)T_k(\rho_n)-(\overline{P_1(\rho)}+\frac{\kappa}{2}\overline{\rho^2})\overline{T_k(\rho)}dxdt)\leq 0.$$

We have then :

$$\begin{aligned}dft[\rho_n\to\rho](t_2)-dft[\rho_n\to\rho](t_1)\leq\lim_{k\to+\infty}\Big(\lim_{n\to+\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}P_2(\rho_n)T_k(\rho_n)\\ -\overline{P_2(\rho)}\,\overline{T_k(\rho)}dxdt\Big).\end{aligned}$$

As the sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T,L^2(\mathbb{R}^N))$, and P_2 is a bounded function, we have :

$$\begin{aligned}\lim_{k\to+\infty}\Big(\lim_{n\to+\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}P_2(\rho_n)T_k(\rho_n)-\overline{P_2(\rho)T_k(\rho)}dxdt\Big)\\ =\lim_{n\to+\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}P_2(\rho_n)\rho_n-\overline{P_2(\rho)}\,\overline{\rho}dxdt.\end{aligned}$$

Since the function P_2 is twice continuously differentiable and compactly supported in $[0,+\infty)$, there exists $\Lambda>0$ big enough such that both $\rho\to\Lambda\rho\log\rho-\rho P_2(\rho)$ and

$\rho \to \Lambda\rho\log\rho + \rho P_2(\rho)$ are convex functions of ρ, indeed the second derivative are positive.

As a consequence of weak lower semi-continuity of convex functionals, we obtain :

$$\lim_{n\to+\infty}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} P_2(\rho_n)\rho_n - \overline{P_2(\rho)}\,\overline{\rho}\,dx\,dt$$
$$\leq \Lambda\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(\overline{\rho\,\log\rho} - \rho\,\log\rho)dxdt + \int_{t_1}^{t_2}\int_{\mathbb{R}^N}(P_2(\rho) - \overline{P_2(\rho)})\rho\,dx\,dt.$$

Futhermore we have :

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(P_2(\rho)-\overline{P_2(\rho)})\rho dxdt &\leq \int_{t_1}^{t_2}\int_{\rho\leq\rho_r}(P_2(\rho)-\overline{P_2(\rho)})\rho\\ &\leq \Lambda\int_{t_1}^{t_2}\int_{\rho\leq\rho_r}(\overline{\rho\,\log\rho}-\rho\,\log\rho)dxdt\\ &\leq \Lambda\rho_r\int_{t_1}^{t_2}\int_{\mathbb{R}^N}(\overline{\rho\,\log\rho}-\rho\,\log\rho)dxdt\end{aligned}$$

The previous relation gives :

$$dft[\rho_n\to\rho](t_2) \leq dft[\rho_n\to\rho](t) + \omega\int_{t_1}^{t_2} dft[\rho_n\to\rho](t).$$

Applying Grönwall's lemma we infer :

$$dft[\rho_n\to\rho](t_2) \leq dft[\rho_n\to\rho](t_1)\exp(\omega(t_2-t_1)).$$

We conclude that $dft[\rho_n\to\rho](t) = 0\ \forall t$, because ρ_n^0 converges strongly in L^1 to ρ_0.

4 Weak solutions with data close to a stable equilibrium

We consider in this section one situation which is rather different from the three cases considered in the preceding sections. This situation is relevant for practical applications and realistic flow and they involve conditions at infinity different from those studied.

We wish to investigate the system (NSK) with hypothesis close from these of the theorem for strong solutions. We want then to study the system with a density close from a stable equilibrium in the goal to can choose initial data avoiding the vacuum. We look now for a solution (ρ,u) defined on $\mathbb{R}\times\mathbb{R}^N$ of the system (NSK) (where $P(\rho)=a\rho^\gamma$) with $\rho\geq 0$ on $\mathbb{R}\times\mathbb{R}^N$.

In addition we require (ρ,u) to satisfy the following limit conditions :

$$(\rho,u)(x,t)\to(\bar\rho,0)\ \text{ as }\ |x|\to+\infty,\ \text{ for all }\ t>0$$

where $\bar\rho>0$.

Such an analysis requires the use of the Orlicz spaces. We define the Orlicz space $L^q_p(\mathbb{R}^N)$ as follows :

$$L^q_p(\mathbb{R}^N) = \{f\in L^1_{loc}(\mathbb{R}^N)/f1_{\{|f|\leq\delta\}}\in L^p(\mathbb{R}^N),\ \ f1_{\{|f|\geq\delta\}}\in L^q(\mathbb{R}^N)\}.$$

Definition 4.15. *We define Ψ as a convex function on $[0,+\infty)$ which is equal (or equivalent) to x^p for x small and to x^q for x large.*

$$L^q_p(\mathbb{R}^N)=\{f\in L^1_{loc}(\mathbb{R}^N)/\Psi(f)\in L^1(\mathbb{R}^N)\}.$$

We can check that $L^q_p(\mathbb{R}^N)$ is a linear space. Now we endow $L^q_p(\mathbb{R}^N)$ with a norm so that $L^q_p(\mathbb{R}^N)$ is a separable Banach space, by setting :

$$\|f\|_{L^q_p(\mathbb{R}^N)}=\inf\big(t>0/\ \ \Psi(\frac{f}{t})\leq 1\big).$$

We recall now some useful properties of the Orlicz space :

Corollary 1. *We have :*

1. *Embedding :*

$$L^q_p(\mathbb{R}^N)\subset L^{q_1}_{p_1}(\mathbb{R}^N)\ \ if\ \ 1\leq q_1\leq q<+\infty,\ \ 1\leq p\leq p_1<+\infty.$$

2. *Topology : $f_n\rightarrow_n 0$ in $L^q_p(\mathbb{R}^N)$ if and only if $\Psi(f_n)\rightarrow_n 0$ and that :*

$$\Psi(\frac{f}{\|f\|_{L^q_p(\mathbb{R}^N)}})=1\quad if\ \ f\neq 0$$

We recall now some useful properties of the Orlicz space :

Proposition 4.17. *We then have :*

- *Dual space : If $p>1$ then $(L^q_p(\mathbb{R}^N))^{'}=L^{q'}_{p'}(\mathbb{R}^N)$ where $q^{'}=\frac{q}{q-1}$, $p^{'}=\frac{p}{p-1}$.*
- *Let F be a continuous function on $\mathbb{R}$ such that $F(0)=0$, F is differentiable at 0 and $F(t)|t|^{-\theta}\rightarrow\alpha\neq 0$ at $t\rightarrow+\infty$. Then if $q\geq\theta$,*

$$F(f)\in L^{\frac{q}{\theta}}_p(\mathbb{R}^N)\ \ if\ \ f\in L^q_p(\mathbb{R}^N).$$

Our goal is now to get energy estimate. We have to face a new difficulty. Indeed ρ, $\rho|u|^2$, ρ^γ need not belong to L^1.
We first want to explain how it is possible to obtain natural a priori bounds which correspond to energy-like identities.
Next we write the following formal identities :

$$\begin{aligned}&\frac{1}{\gamma-1}\frac{d}{dt}(\rho^\gamma-\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}(\rho-\bar{\rho})+\text{div}[u\frac{\gamma}{\gamma-1}(\rho^\gamma-\bar{\rho}^{\gamma-1}\rho)]=u\cdot\nabla(\rho^\gamma)\\&\rho\frac{d}{dt}\frac{|u|^2}{2}+\rho u\cdot\nabla\frac{|u|^2}{2}-\mu\Delta u\cdot u-\xi\nabla\text{div}u\cdot u+au\cdot\nabla\rho^\gamma=\kappa\rho u\nabla(\phi*\rho-\rho).\end{aligned}\tag{4.40}$$

We may then integrate in space the equality (4.40) and we get :

$$\begin{aligned}&\Big(\int_{\mathbb{R}^N}\rho\frac{|u|^2}{2}+\frac{a}{\gamma-1}(\rho^\gamma+(\gamma-1)\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}\rho)+E_{global}[\rho-\bar{\rho}]dx\Big)(t)\\&+\int_0^t ds\int_{\mathbb{R}^N}2\mu|Du|^2+2\xi|\text{div}u|^2dx\leq\int_{\mathbb{R}^N}\rho_0\frac{|u_0|^2}{2}+\frac{a}{\gamma-1}(\rho_0^\gamma+(\gamma-1)\bar{\rho}^\gamma-\gamma\bar{\rho}^{\gamma-1}\rho_0)\\&\qquad+E_{global}[\rho_0-\bar{\rho}]dx.\end{aligned}\tag{4.41}$$

Notation 5. *In the sequel we will note :*

$$j_\gamma(\rho) = \rho^\gamma + (\gamma - 1)\bar{\rho}^\gamma - \gamma\bar{\rho}^{\gamma-1}\rho.$$

Now we can recall a theorem (see [16]) on the Orlicz space concerning this quantity :

Theorem 4.19. $j_\gamma(\rho) \in L^1(\mathbb{R}^N)$ *if and only if* $\rho - \bar{\rho} \in L_2^\gamma$.

In following this theorem and our energy estimate we get that $\rho - \bar{\rho} \in L^\infty(0,T; L_2^\gamma(\mathbb{R}^N))$ for all $T \in \mathbb{R}$.
Moreover we have :

$$E_{global}[\rho(t,\cdot) - \bar{\rho}](x) = \frac{\kappa}{4}(\rho - \bar{\rho})^2 + \phi * (\rho - \bar{\rho})^2 - 2(\rho - \bar{\rho})\,(\phi * (\rho - \bar{\rho})).$$

Then in using the fact that $\rho - \bar{\rho} \in L^\infty(0,T; L_2^\gamma(\mathbb{R}^N)) \hookrightarrow L^\infty(L^2(\mathbb{R}^N) + L^\gamma(\mathbb{R}^N))$ and interpolation on $\nabla\phi$, we get that $\rho - \bar{\rho} \in L^\infty(0,T; L^2(\mathbb{R}^N))$.
We may now turn to our compactness result. First of all, we consider sequences of approximate smooth solutions (ρ_n, u_n) of the system corresponding to some initial conditions (ρ_n^0, u_n^0).
In using the above energy inequalities, we assume that $j_\gamma(\rho_n^0)$, $E_{global}[\rho_n^0 - \bar{\rho}]$ and $\rho_n^0|u_n^0|^2$ are bounded in $L^\infty(L^1(\mathbb{R}^N))$ and that $\rho_n^0 - \bar{\rho}$ converges weakly in $L_2^\gamma(\mathbb{R}^N)$ to some $\rho_0 - \bar{\rho}$.

We now assume that :

$$j_\gamma(\rho_n),\ E_{global}[\rho_n - \bar{\rho}],\ \rho_n|u_n|^2 \ \text{ are bounded in } \ L^\infty(0,T,L^1(\mathbb{R}^N)),$$

Moreover we have for all $, T \in (0,+\infty)$ and for all compact sets $K \subset \mathbb{R}^N$:

$$\rho_n - \bar{\rho} \in L^\infty(L^2(\mathbb{R}^N) \quad \text{and} \quad \rho_n \ \text{ is bounded in } \ L^q((0,T) \times K),$$

for some $q > s$.

$$Du_n \ \text{ is bounded in } \ L^2(\mathbb{R}^N \times (0,T))$$
$$u_n \ \text{ is bounded in } \ L^2(0,T,H^1(B_R)) \ \text{ for all } \ R,T \in (0,+\infty)\ .$$

Extracting subsequences if necessary, we may assume that ρ_n, u_n converge weakly respectively in $L^2((0,T) \times B_R)$, $L^2(0,T; H^1(B_R))$ to ρ, u for all $R,T \in (0,+\infty)$. We also extract subsequences for which $\rho_n u_n$, $\rho_n u_n \otimes u_n$ converge weakly as previously.

Remark 2. *We notice that the conditions at infinity are implicitly contained in the fact that* $(\rho_n - \bar{\rho})^2$ *and* $\rho_n|u_n|^2 \in L^1(\mathbb{R}^N)$.

We then have the following theorem.

Theorem 4.20. *Let* $\gamma \geq 1$. *We assume that* ρ_n^0 *converges in* $L^1(B_R)$ *(for all* $R \in (0,+\infty)$*) to* ρ_0. *Then* $(\rho_n, u_n)_{n\in\mathbb{N}}$ *converges in distribution sense to* (ρ, u) *a solution of* (NSK). *Moreover we have for all* $R,T \in (0,+\infty)$ *:*

$$\rho_n \to_n \rho \ \text{ in } \ C([0,T], L^p(B_R \times (0,T)) \cap L^{s_1}(B_R \times (0,T)) \ \text{ for all } \ 1 \leq p < s, 1 \leq s_1 < q.$$

with $q = s + \frac{4}{N} - 1$.

Proof :

As in the theorem 1.12, we want test the strong convergence of ρ_n on concave function B. Since the proof is purely local, we have again for small enough $\varepsilon > 0$:

$$(\rho_n)^\varepsilon\big((\mu+\xi)\mathrm{div}u_n - a(\rho_n)^\gamma - \frac{\kappa}{2}\rho_n^2\big) \rightharpoonup_n \overline{(\rho)^\varepsilon\big((\mu+\xi)\mathrm{div}u - a\overline{\rho^\gamma} - \frac{\kappa}{2}\overline{\rho^2}\big)}$$
$$\text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^N\big),$$

so we obtain :

$$\frac{d}{dt}(\overline{\rho^\varepsilon}) + \mathrm{div}(u\overline{\rho^\varepsilon}) \geq (1-\varepsilon)(\mathrm{div}u)\overline{\rho^\varepsilon} \quad \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^N\big). \tag{4.42}$$

Next since $(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}} \in L^2(B_R\times(0,T))$ for all $R,T\in(0,+\infty)$, as in the theorem 1.12 in using a result of type Diperna-Lions on renormalized solutions, we get :

$$\frac{d}{dt}(\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}) + \mathrm{div}(u\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}) \geq 0 \quad \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^N\big). \tag{4.43}$$

while we have $\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}} \leq \rho$ a.e in $\mathbb{R}^N\times(0,+\infty)$ and $\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}_{/t=0} = \rho_{/t=0}$ in $\mathbb{R}^N$.
Now in subtracting the second equality of (4.43) from the mass equation and setting $f = \rho - \overline{\rho^\varepsilon}^{\frac{1}{\varepsilon}}$, we have :

$$\frac{d}{dt}(f) + \mathrm{div}(uf) \leq 0,\ f\geq 0 \text{ a.e, } f_{/t=0} = 0 \text{ in } \mathbb{R}^N. \tag{4.44}$$

Next we want again to show from (4.44) that $f=0$, in integrating (4.44) and in using the fact that $f\leq 0$ to conclude that $f=0$. The difference with the proof of theorem 1.12 is to justify the integration by parts as we work in different energy space.
We need a cut-off function. We introduce $\varphi\in C_0^\infty(\mathbb{R}^N)$, $0\leq\varphi\leq 1$, supp$\varphi\subset B_2$, $\varphi = 1$ on B_1 and we set $\varphi_R = \varphi(\frac{x}{R})$ for $R\geq 1$. Multiplying (4.44) by $\varphi_R(x)$, we obtain :

$$\frac{d}{dt}\int_{\mathbb{R}^N} f\varphi_R(x)dx = \int_{\mathbb{R}^N}\frac{1}{R}fu\cdot\nabla\varphi(\frac{x}{R}). \tag{4.45}$$

Next, if $T>0$ is fixed, we see that supp $\nabla\varphi(\frac{\cdot}{R})\subset\{R\leq|x|\leq 2R\}$, therefore, for R large enough we have :

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} f\varphi_R(x)dx &= \int_{\mathbb{R}^N}\frac{1}{R}fu.\nabla\varphi(\frac{x}{R}),\\ &\leq \frac{C}{R}\int_{\mathbb{R}^N} f|u|1_{(R\leq|x|\leq 2R)}dx, \quad \text{for } t\in(0,T),\end{aligned} \tag{4.46}$$

To conclude that $f=0$, we only have to prove that :

$$\frac{1}{R}\int_{\mathbb{R}^N} f|u|1_{(R\leq|x|\leq 2R)}dx \to 0 \text{ as } R\to+\infty. \tag{4.47}$$

We now use the fact that $f\in L^\infty(0,T,L^2(\mathbb{R}^N))$ and $f|u|^2\in L^\infty(0,T,L^1(\mathbb{R}^N)$ for all $T\in(0,+\infty)$ to control (4.47). The second fact is obvious since $0\leq\rho$ and $\rho|u|^2\in L^\infty(0,T;L^1(\mathbb{R}^N))$.

In order to prove the first claim, we only have to show that $\overline{(\rho)^{\varepsilon}}^{\frac{1}{\varepsilon}} - \bar{\rho} \in L^\infty(0,T,L^2(\mathbb{R}^N))$. We rewrite $(\rho_n)^\varepsilon - (\bar{\rho})^\varepsilon = (\bar{\rho} + (\rho_n - \bar{\rho}))^\varepsilon - (\bar{\rho})^\varepsilon$ is bounded in $L^\infty(0,T,L^2(\mathbb{R}^N))$ in using proposition 4.17 with $F(x) = (\bar{\rho}+x)^\varepsilon - (\bar{\rho})^\varepsilon$. So we have $\sqrt{f} \in L^\infty(L^4(\mathbb{R}^N))$ and we get :

$$\frac{1}{R}\int_0^T dt \int_{\mathbb{R}^N} f|u|1_{(R\leq|x|\leq 2R)}dx \leq \frac{C_0}{R}\text{meas}(C(0,R,2R))^{\frac{1}{4}}.$$

We recall that :

$$\text{meas}(C(0,R,2R)) \sim_{R\to+\infty} C(n)R^N$$

Then we get :

$$\frac{d}{dt}\int_{\mathbb{R}^N} f\varphi_R(x)dx \to_{R\to+\infty} 0$$

and we conclude as in the proof of theorem 1.12.

At this stage, it only remains to show that, for instance, ρ_n converges to ρ in $C([0,T],L^1(B_R))$ for all $R,T \in (0,+\infty)$. In order to do so, we just have to localize the corresponding argument in the proof of theorem 1.12.

Therefore we choose for $R,T \in (0,+\infty)$ fixed, $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi = 1$ on B_R, $0 \leq \varphi$ on $\mathbb{R}^N$. Then, we observe that we have :

$$\begin{aligned}
&\frac{\partial}{\partial t}(\varphi^2\rho_n) + \text{div}(u_n(\varphi^2\rho_n)) = \rho_n u_n\cdot\nabla\varphi^2,\ \frac{\partial}{\partial t}(\varphi^2\rho) + \text{div}(u(\varphi^2\rho)) = \rho u\cdot\nabla\varphi^2\\
&\frac{\partial}{\partial t}(\varphi\sqrt{\rho_n}) + \text{div}(u_n(\varphi\rho_n)) = \frac{1}{2}(\text{div}u_n)\varphi\sqrt{\rho_n} + \sqrt{\rho_n}u_n\cdot\nabla\varphi,\\
&\frac{\partial}{\partial t}(\varphi\sqrt{\rho}) + \text{div}(u(\varphi\rho)) = \frac{1}{2}(\text{div}u)\varphi\sqrt{\rho} + \sqrt{\rho}u\cdot\nabla\varphi.
\end{aligned}$$

From these equations, we deduce as in the proof of the previous theorem, that $\varphi^2\rho \in C([0,+\infty),L^1(\mathbb{R}^N))$, $\varphi\sqrt{\rho} \in C([0,+\infty),L^2(\mathbb{R}^N))$ and that $\varphi\sqrt{\rho_n}$ converges weakly in $L^2(\mathbb{R}^N)$, uniformly in $t \in [0,T]$. Therefore, in order to conclude, we just have to show that we have :

$$\int_{\mathbb{R}^N}\varphi^2\rho_n(t_n)dx \to \int_{\mathbb{R}^N}\varphi^2\rho(\bar{t})dx$$

whenever $t_n \in [0,T]$, $t_n \to_n \bar{t}$, and this is straightforward since we have, in view of the above equation :

$$\begin{aligned}
\int_{\mathbb{R}^N}\varphi^2\rho_n(t_n)dx &= \int_{\mathbb{R}^N}\varphi^2(\rho_0)_n dx + \int_0^{t_n}ds\int_{\mathbb{R}^N}\rho_n u_n\cdot\nabla\varphi^2\,dx\\
&\longrightarrow_n \int_{\mathbb{R}^N}\varphi^2\rho_0 dx + \int_0^T ds\int_{\mathbb{R}^N}\rho u\cdot\nabla\varphi^2\,dx = \int_{\mathbb{R}^N}\varphi\rho(\bar{t})dx.
\end{aligned}$$

□

Bibliographie

[1] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models : Korteweg, lubrication and shallow water systems. Comm. Partial Differential Equations, 28(3-4) : 843-868, 2003.

[2] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I. Interfacial free energy, J. Chem. Phys. 28 (1998) 258-267.

[3] R. Coiffman, R. Rochberg and G. Weiss, Ann. Math., 103 (1976), pp. 611-635.

[4] F. Coquel, D. Diehl, C. Merkle and C. Rohde, Sharp and diffuse interface methods for phase transition problems in liquid-vapour flows. Numerical Methods for Hyperbolic and Kinetic Problems, 239-270, IRMA Lect. Math. Theor. Phys,7,Eur. Math. Soc, Zürich, 2005.

[5] R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes, Compensated-compactness and Hardy spaces, J. Math. Pures Appl., 72 (1993), p 247-286.

[6] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP,Analyse non linéaire 18, 97-133 (2001)

[7] R.J. Diperna, P.-L. Lions - Global solutions of Boltzmann's equation and the entropy inequality, Arch. Rational Mech. Anal. 114 (1991), 47-55.

[8] J.E. Dunn and J. Serrin, On the thermomechanics of interstitial working ,Arch. Rational Mech. Anal. 88(2) (1985) 95-133.

[9] E. Feireisl, Dynmamics of Viscous Compressible Fluids-Oxford Lecture Series in Mathematics and its Applications-26 (2004).

[10] H. Hattori and D.Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differential Equations 9(4) (1996) 323-342.

[11] H. Hattori and D. Li, Global Solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198(1) (1996) 84-97.

[12] David Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Rational Mech. Anal., 132(1) : 1-14, 1995.

[13] D. Jamet, O. Lebaigue, N. Coutris and J.M. Delhaye, The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. J. Comput. Phys, 169(2) : 624–651, (2001).

[14] D.J. Korteweg. Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité. Arch. Néer. Sci. Exactes Sér. II, 6 :1-24, 1901.

[15] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 1, Inompressible models, Oxford lecture series in mathematics and its application.3 (1996)

[16] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Oxford lecture series in mathematics and its application. 10 (1998).

[17] A. Novotný and I. Straškraba. Introduction to the mathematical theory of compressible flow, Oxford lecture series in mathematics and its application. 27 (2004)

[18] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. ZAMM Vol. 85, No. 12 (2005).

[19] C. Truedell and W. Noll. The nonlinear field theories of mechanics. Springer-Verlag, Berlin, second edition, 1992.

Chapitre 5

Cauchy problem for viscous Navier-Stokes equations with a term of capillarity

Abstract

In this chapter, we consider the compressible Navier-Stokes equation with density dependent viscosity coefficients and a term of capillarity introduced by Coquel et al in [13]. This model includes at the same time the barotropic Navier-Stokes equations with variable viscosity coefficients, shallow-water system and the model of Rohde in [36].
We first study the well-posedness of the model in spaces with critical regularity for the scaling of the associated equations. In a functional setting as close as possible to the physical energy spaces, we prove global existence of solutions close to a stable equilibrium, and local in time existence for solutions with general initial data. Uniqueness is also obtained.

1 Introduction

This chapter is devoted to the Cauchy problem for the compressible Navier-Stokes equation with viscosity coefficients depending on the density and with a capillary term coming from the works of Coquel, Rohde and theirs collaborators in [13], [36]. Let ρ and u denote the density and the velocity of a compressible viscous fluid. As usual, ρ is a non-negative function and u is a vector valued function defined on $\mathbb{R}^N$. Then, the Navier-Stokes equation for compressible fluids endowed with internal capillarity introduced in [36] reads :

$$(SW)\quad \begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0, \\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)-\operatorname{div}(2\mu(\rho)Du)-\nabla(\lambda(\rho)\operatorname{div}u)+\nabla P(\rho)=\kappa\rho\nabla D[\rho], \end{cases}$$

supplemented by the initial condition :

$$\rho_{/t=0}=\rho_0,\ \ \rho u_{/t=0}=\rho_0 u_0$$

and :

$$D[\rho] = \phi * \rho - \rho$$

where ϕ is chosen so that :

$$\phi \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi(x)dx = 1, \;\; \phi \text{ even, and } \phi \geq 0$$

and where $P(\rho)$ denotes the pressure, μ and λ are the two Lamé viscosity coefficients (they depend regularly on the density ρ) satisfying :

$$\mu > 0 \qquad 2\mu + N\lambda \geq 0.$$

(μ is sometimes called the shear viscosity of the fluid, while λ is usually referred to as the second viscosity coefficient).
several physical models arise as a particular case of system (SW) :

- when $\kappa = 0$ (SW) represents compressible Navier-Stokes model with variable viscosity coefficients.
- when $\kappa = 0$ and $\mu(\rho) = \rho$, $\lambda(\rho) = 0$, $P(\rho) = \rho^2$, $N = 2$ then (SW) describes the system of shallow-water.
- when $\kappa = 0$ and μ, λ are constant, (SW) reduce to the Rohde model of chapter four.

One of the major difficulty of compressible fluid mechanics is to deal with the vacuum. The problem of existence of global solution in time for Navier-Stokes equations was addressed in one dimension for smooth enough data by Kazhikov and Shelukin in [31], and for discontinuous ones, but still with densities away from zero, by Serre in [38] and Hoff in [24]. Those results have been generalized to higher dimension by Matsumura and Nishida in [33] for smooth data close to equilibrium and by Hoff in the case of discontinuous data in [26, 27].
Concerning large initial data, Lions showed in [32] the global existence of weak solutions for $\gamma \geq \frac{3}{2}$ for $N = 2$ and $\gamma \geq \frac{9}{5}$ for $N = 3$. Let us mention that Feireisl in [20] generalized the result to $\gamma > \frac{N}{2}$ in establishing that we can obtain renormalized solution without imposing that $\rho \in L^2_{loc}$, for this he introduces the concept of oscillation defect measure evaluating the loss of compactness.
Other results provide the full range $\gamma > 1$ under symmetries assumptions on the initial datum, see Jiang and Zhang [29]. All those results do not require to be far from the vacuum. However they rely strongly on the assumption that the viscosity coefficients are bounded below by a positive constant. This non physical assumption allows to get some estimates on the gradient of the velocity field.
The main difficulty when dealing with vanishing viscosity coefficients on vacuum is that the velocity cannot even be defined when the density vanishes and so we cannot use some properties of parabolicity of the momentum equation, see [11], [12].
The first result handling this difficulty is due to Bresch, Desjardins and Lin in [7]. They show the existence of global weak solution for Korteweg system in choosing specific type of viscosity where μ and λ are linked.

The result was later improved by Bresch and Desjardins in [4] to include the case of vanishing capillarity ($\kappa = 0$), but with an additional quadratic friction term $r\rho u|u|$ (see also [6]). However, those estimates are not enough to treat the case without capillarity and friction effects $\kappa = 0$ and $r = 0$ (which corresponds to equation (1) with $h(\rho) = \rho$ and $g(\rho) = 0$).
The main difficulty, to prove the stability of (SW) , is to pass to the limit in the term $\rho u \otimes u$ (which requires the strong convergence of $\sqrt{\rho}u$). Note that this is easy when the viscosity coefficients are bounded below by a positive constant. On the other hand, the new bounds on the gradient of the density make the control of the pressure term far simpler than in the case of constant viscosity coefficients.
In [6] Bresch and Desjardins show a result of global existence of weak solution for the non isothermal Navier-Stokes equation by imposing some condition between the viscosity coefficient and a bound by below on the viscosity coefficient. A. Mellet and A. Vasseur in using the same entropy inequality get a very interesting result of stability. They get more general estimate, which hold for any viscosity coefficients $\mu(\rho)$, $\lambda(\rho)$ satisfying the relation :

$$\mu(\rho) = \rho\lambda^{'}(\rho) - \lambda(\rho). \tag{1.1}$$

Mellet and Vasseur show in [34] the L^1 stability of weak solutions of Navier-Stokes compressible isotherm under some conditions on the viscosity coefficients (including (1.1)) but without any additional regularizing terms. The interest of this result is to consider conditions where the viscosity coefficients vanish on the vacuum set. It includes the case $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ (when $N = 2$ and $\gamma = 2$, where we recover the Saint-Venant model for Shallow water). The key to the proof is a new energy inequality on the velocity and a gain of integrability, which allows to pass to the limit.
The existence and uniqueness of local classical solutions for (SW) with smooth initial data such that the density ρ_0 is bounded and bounded away from zero (i.e., $0 < \underline{\rho} \leq \rho_0 \leq M$) has been stated by Nash in [35]. Let us emphasize that no stability condition was required there.
On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [33]. Many recent works have been devoted to the qualitative behavior of solutions for large time (see for example [24, 31]). Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [39], [40], [25], [30]. The most important result on the system of Navier-Stokes compressible isothermal comes from R. Danchin in [15] and [18] who show the existence of global solution and uniqueness with initial data close from a equilibrium, and he has the same result in finite time. The interest is that he can work in *critical* Besov space (*critical* in the sense of the scaling of the equation.)
We generalize the result of R. Danchin in considering general viscosity coefficient and in connecting this result with those of A. Mellet and A. Vasseur. This result improves too the

case of strong solution for the shallow-water system, where W.Wang and C-J Xu in [41] have got global existence in time for small initial data with $h_0,\ u_0 \in H^{2+s}$ with $s > 0$.

1.1 Notations and main results

We will mainly consider the global well-posedness problem for initial data close enough to stable equilibria. Here we want to investigate the well-posedness of the system (SW) problem in critical spaces, that is, in spaces which are invariant by the scaling of the equations. Let us explain precisely the scaling of the system. We can easily check that, if (ρ, u) solves (SW), so does $(\rho_\lambda, u_\lambda)$, where :

$$\rho_\lambda(t,x) = \rho(\lambda^2 t, \lambda x) \;\text{ and }\; u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

provided the pressure law P has been changed into $\lambda^2 P$.

Definition 1.16. *We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation :*

$$(\rho, u) \longrightarrow (\rho_\lambda, u_\lambda)$$

(up to a constant independent of λ).

This suggests us to choose initial data (ρ_0, u_0) in spaces whose norm is invariant for all $\lambda > 0$ by $(\rho_0, u_0) \longrightarrow (\rho_0(\lambda\cdot), \lambda u_0(\lambda\cdot))$.

A natural candidate is the homogeneous Sobolev space $\dot{H}^{N/2} \times (\dot{H}^{N/2-1})^N$, but since $\dot{H}^{N/2}$ is not included in L^∞, we cannot expect to get L^∞ control on the density when $\rho_0 \in \dot{H}^{N/2}$. This is the reason why as in the chapter two, instead of the classical homogeneous Sobolev space, we will consider homogeneous Besov spaces $B^{N/2}_{2,1} \times (B^{N/2-1}_{2,1})^N$ with the same derivative index. This allows to control the density from below and from above, without requiring more regularity on derivatives of ρ. In the sequel, we will need to control the vacuum, this motivates the following definition :

Definition 1.17. *Let $\bar{\rho} > 0$, $\bar{\theta} > 0$. We will note in the sequel :*

$$q = \frac{\rho - \bar{\rho}}{\bar{\rho}}$$

Let us first state a result of global existence and uniqueness of (SW) for initial data close to a equilibrium.

Theorem 1.21. *Let $N \geq 2$. Let $\bar{\rho} > 0$ be such that : $P'(\bar{\rho}) > 0$, $\mu(\bar{\rho}) > 0$ and $2\mu(\bar{\rho}) + \lambda(\bar{\rho}) > 0$. There exist two positive constants ε_0 and M such that if $q_0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}$, $u_0 \in B^{\frac{N}{2}-1}$ and :*

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}-1}} \leq \varepsilon_0$$

then (SW) *has a unique global solution* (q,u) *in* $E^{\frac{N}{2}}$ *which satisfies :*

$$\|(q,u)\|_{E^{\frac{N}{2}}}\leq M\big(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_0\|_{B^{\frac{N}{2}-1}}\big),$$

for some M *independent of the initial data where :*

$$E^{\frac{N}{2}}=[C_b(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})\cap L^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})]\times[C_b(\mathbb{R}_+,B^{\frac{N}{2}-1})^N \cap L^1(\mathbb{R}^+,B^{\frac{N}{2}+1})^N].$$

In the following theorems, we want to show some result of existence and uniqueness in finite time for large initial velocity and initial density close to some constant.
The following result shows the existence and uniqueness in finite time with initial data in critical Besov space for the scaling of the equations. However as we said, we need for some technical reasons of an hypothesis of smallness on q_0. We note that we work on Besov space B^s_p with general index p on the integrability.

Theorem 1.22. *Let* $p\in[1,+\infty[$*. Let* $q_0\in B_p^{\frac{N}{p}}$ *and* $u_0\in B_p^{\frac{N}{p}-1}$*. Assume also that :* $\|q_0\|_{B_p^{\frac{N}{p}}}\leq\varepsilon$ *for a suitably small positive constant* $\varepsilon>0$*.*
Under the assumptions of the theorem 1.21 for the physical coefficients, there exists a time $T>0$ *such that the following results hold :*

1. *Existence : If* $p\in[1,2N[$ *then system* (SW) *has a solution* (q,u) *in* $F_p^{\frac{N}{p}}$ *with :*

$$F_p^{\frac{N}{p}}=\widetilde{C}_T(B_p^{\frac{N}{p}})\times\big(L^1_T(B_p^{\frac{N}{p}+1})\cap\widetilde{C}_T(B_p^{\frac{N}{p}-1}\big).$$

2. *Uniqueness : If in addition* $1\leq p\leq N$ *then uniqueness holds in* $F_p^{\frac{N}{p}}$*.*

Moreover we have a control on the time T *which may be bounded from below by :*

$$\min\Big(\eta,\max\big(t>0,\sum_{q\in\mathbb{Z}}2^{q(\frac{N}{p}-1)}\|\Delta_q u_0\|_{L^p}(\frac{1-e^{-c\widetilde{\nu}t2^{2q}}}{c\widetilde{\nu}})\leq\frac{\varepsilon\widetilde{\nu}^2}{\widetilde{\nu}+U_0})\Big).$$

where $\widetilde{\nu}=\min(\mu(\bar{\rho}),\lambda(\bar{\rho})+2\mu(\bar{\rho}))$ *and* $U_0=\|u_0\|_{B^{\frac{N}{p}}}$*.*

In the next theorem, we consider the case when the initial variational density q_0 belongs to $\bar{\rho}+\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}+\varepsilon}$ with $\varepsilon>0$ and satisfies $0<\underline{\rho}<\rho$. Here we do not suppose a smallness condition on $\|q_0\|_{B^{\frac{N}{p}}}$ but we impose more regularity.

Theorem 1.23. *Let* $\varepsilon\in]0,1[$ *and* $p\in[1,\frac{N}{1-\varepsilon}[$ *and we assume that the physical coefficients verify the same hypothesis as in theorem 1.21. Let* $\rho_0\in\bar{\rho}+\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}+\varepsilon}$ *for a constant* $\bar{\rho}>0$*,* $u_0\in B^{\frac{N}{p}+\varepsilon-1}$*. Assume that there is a constant* $\underline{\rho}$ *such that :*

$$0<\underline{\rho}\leq\rho_0.$$

There exists a time $T>0$ *such that system* (SW) *has a unique solution* (q,u) *in* $F_{p+\varepsilon}^{\frac{N}{p}}$*.*
Moreover we have a control on the time T *which may be bounded from below by :*

$$\min\Big(\eta,\max\big(t>0,\sum_{q\in\mathbb{Z}}2^{q(\frac{N}{p}-1)}\|\Delta_q u_0\|_{L^p}(\frac{1-e^{-c\widetilde{\nu}t2^{2q}}}{c\widetilde{\nu}})\leq\frac{\varepsilon\widetilde{\nu}^2}{\widetilde{\nu}+U_0})\Big).$$

where : $\widetilde{\nu}=\min(\mu(\bar{\rho}),\lambda(\bar{\rho})+2\mu(\bar{\rho}))$ *and* $U_0=\|u_0\|_{B^{\frac{N}{p}}}$*.*

The present chapter is structured as follows.
In the Section 2, we recall some basic facts about Littlewood-Paley decomposition and Besov spaces.
In the Section 3 we prove the theorem 1.21 and in the Section 4 we show the theorem 1.22.
We conclude in the Section 5 by the proof of the theorem 1.23.

2 Littlewood-Paley theory and Besov spaces

2.1 Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables.
We can use for instance any $\varphi \in C^{\infty}(\mathbb{R}^N)$, supported in $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that :

$$\sum_{l\in\mathbb{Z}} \varphi(2^{-l}\xi) = 1 \ \text{ if } \ \xi \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by :

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN}\int_{\mathbb{R}^N} h(2^l y)u(x-y)dy \ \text{ and } \ S_l u = \sum_{k\leq l-1}\Delta_k u\,.$$

Formally, one can write that :

$$u = \sum_{k\in\mathbb{Z}}\Delta_k u\,.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^N)$ for two reasons :

1. The right hand-side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^N)$.
2. Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^N)$ (consider the case of the polynomials).

However, this equality holds true modulo polynomials hence homogeneous Besov spaces will be defined modulo the polynomials, according to [15].

2.2 Homogeneous Besov spaces and first properties

Definition 2.18. *For $s \in \mathbb{R},\ p \in [1,+\infty],\ q \in [1,+\infty],$ and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set :*

$$\|u\|_{B^s_{p,q}} = \Big(\sum_{l\in\mathbb{Z}}(2^{ls}\|\Delta_l u\|_{L^p})^q\Big)^{\frac{1}{q}}.$$

A difficulty due to the choice of homogeneous spaces arises at this point. Indeed, $\|.\|_{B^s_{p,q}}$ cannot be a norm on $\{u \in \mathcal{S}'(\mathbb{R}^N), \|u\|_{B^s_{p,q}} < +\infty\}$ because $\|u\|_{B^s_{p,q}} = 0$ means that u is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces, see [15].

Definition 2.19. *Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $q \in [1, +\infty]$.*
Denote $m = [s - \frac{N}{p}]$ if $s - \frac{N}{p} \notin \mathbb{Z}$ or $q > 1$ and $m = s - \frac{N}{p} - 1$ otherwise.

- *If $m < 0$, then we define $B^s_{p,q}$ as :*

$$B^s_{p,q} = \Big\{ u \in \mathcal{S}'(\mathbb{R}^N) / \ \|u\|_{B^s_{p,q}} < \infty \ \text{ and } \ u = \sum_{l \in \mathbb{Z}} \Delta_l u \ \text{ in } \mathcal{S}'(\mathbb{R}^N) \Big\}.$$

- *If $m \geq 0$, we denote by $\mathcal{P}_m[\mathbb{R}^N]$ the set of polynomials of degree less than or equal to m and we set :*

$$B^s_{p,q} = \Big\{ u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \ / \ \|u\|_{B^s_{p,q}} < \infty \ \text{ and } \ u = \sum_{l \in \mathbb{Z}} \Delta_l u \ \text{ in } \mathcal{S}'(\mathbb{R}^N)\mathcal{P}_m[\mathbb{R}^N] \Big\}.$$

The definition of $B^s_{p,r}$ does not depend on the choice of the Littlewood-Paley decomposition.

Remark 7. *In the sequel, we will use only Besov space $B^s_{p,q}$ with $q = 1$ and we will denote them by B^s_p or even by B^s if there is no ambiguity on the index p.*
Let us now state some basic properties for those Besov spaces.

Proposition 2.18. *The following properties holds :*

1. *Density : If $p < +\infty$ and $|s| \leq N/p$, then C_0^∞ is dense in B^s_p.*
2. *Derivatives : there exists a constant universal C such that :*
$$C^{-1}\|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}.$$
3. *Sobolev embeddings : If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*
4. *Algebraic properties : For $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. Moreover, for any $p \in [1, +\infty]$ then $B^{\frac{N}{p}}_{p,1} \hookrightarrow B^{\frac{N}{p}}_{p,\infty} \cap L^\infty$, and $B^{\frac{N}{p}}_{p,1}$ is an algebra if p is finite.*
5. *Real interpolation : $(B^{s_1}_{p,r}, B^{s_2}_{p,r})_{\theta,r'} = B^{\theta s_1 + (1-\theta)s_2}_{p,r'}$.*

2.3 Hybrid Besov spaces and Chemin-Lerner spaces

Hybrid Besov spaces are functional spaces where regularity assumptions are different in low frequency and high frequency, see [15]. We are going to give the definition of this this new spaces and give some of their main properties.

Definition 2.20. *Let $s, t \in \mathbb{R}$. We set :*

$$\|u\|_{\widetilde{B}^{s,t}_{p,r}} = \big(\sum_{q \leq 0} (2^{qs}\|\Delta_q u\|_{L^p})^r \big)^{\frac{1}{r}} + \big(\sum_{q > 0} (2^{qt}\|\Delta_q u\|_{L^p})^r \big)^{\frac{1}{r}}.$$

Denote $m = [s - \frac{N}{p}]$ if $s - \frac{N}{p} \notin \mathbb{Z}$ or $r > 1$ and $m = s - \frac{N}{p} - 1$ otherwise, we then define :

- $\widetilde{B}^{s,t}_p = \{u \in \mathcal{S}'(\mathbb{R}^N) \ / \ \|u\|_{\widetilde{B}^{s,t}_p} < +\infty\}$, *if $m < 0$*
- $\widetilde{B}^{s,t}_p = \{u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}_m[\mathbb{R}^N] \ / \ \|u\|_{\widetilde{B}^{s,t}_p} < +\infty\}$ *if $m \geq 0$.*

Let now give some properties of these hybrid spaces and some results on how they behave with respect to the product. The following results come directly from the paradifferential calculus.

Proposition 2.19. *We give here some results of inclusion :*

1. *We have* $\widetilde{B}^{s,s}_{p,r} = B^s_{p,r}$.
2. *If* $s \leq t$ *then* $\widetilde{B}^{s,t}_{p,r} = B^s_{p,r} \cap B^t_{p,r}$ *or if* $s > t$ *then* $\widetilde{B}^{s,t}_{p,r} = B^s_{p,r} + B^t_{p,r}$.
3. *If* $s_1 \leq s_2$ *and* $t_1 \geq t_2$ *then* $\widetilde{B}^{s_1,t_1}_{p,r} \hookrightarrow \widetilde{B}^{s_2,t_2}_{p,r}$.

Proposition 2.20. *For all* $s,\, t > 0$, $1 \leq r, p \leq +\infty$, *the following inequality holds true :*

$$\|uv\|_{\widetilde{B}^{s,t}_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{\widetilde{B}^{s,t}_{p,r}} + \|v\|_{L^\infty}\|u\|_{\widetilde{B}^{s,t}_{p,r}})\,. \tag{2.2}$$

For all $s_1,\, s_2,\, t_1,\, t_2 \leq \frac{N}{p}$ *such that* $\min(s_1+s_2,\, t_1+t_2) > 0$ *we have :*

$$\|uv\|_{\widetilde{B}^{s_1+t_1-\frac{N}{p},s_2+t_2-\frac{N}{p}}_{p,r}} \leq C\|u\|_{\widetilde{B}^{s_1,t_1}_{p,r}}\|v\|_{\widetilde{B}^{s_2,t_2}_{p,\infty}}\,. \tag{2.3}$$

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty} \quad \text{if } |s| < \frac{N}{p}. \tag{2.4}$$

For a proof of this proposition see [15]. The limit case $s_1+s_2 = t_1+t_2 = 0$ in (2.3) is of interest. When $p \geq 2$, the following estimate holds true whenever s is in the range $(-\frac{N}{p}, \frac{N}{p}]$ (see e.g. [37]) :

$$\|uv\|_{B^{-\frac{N}{p}}_{p,\infty}} \leq C\|u\|_{B^s_{p,1}}\|v\|_{B^{-s}_{p,\infty}}. \tag{2.5}$$

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r = p$). We are now going to define the spaces of Chemin-Lerner in which we will work (see [8]), which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.21. *Let* $\rho \in [1,+\infty]$, $T \in [1,+\infty]$ *and* $s_1, s_2 \in \mathbb{R}$. *We then denote :*

$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}_{p,r})} = \big(\sum_{l\leq 0} 2^{lrs_1}(\|\Delta_l u(t)\|^r_{L^\rho_T(L^p)}\big)^{\frac{1}{r}} + \big(\sum_{l>0} 2^{lrs_2}(\int_0^T \|\Delta_l u(t)\|^\rho_{L^p}dt)^{\frac{r}{\rho}})\big)^{\frac{1}{r}}\,.$$

We note that thanks to Minkowsky inequality we have :

$$\|u\|_{L^\rho_T(\widetilde{B}^{s_1,s_2}_{p,r})} \leq \|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}_{p,r})} \quad \text{if } \rho \leq r,$$
$$\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}_{p,r})} \leq \|u\|_{L^\rho_T(\widetilde{B}^{s_1,s_2}_{p,r})} \quad \text{if } \rho \geq r.$$

We then define the space :

$$\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}_p) = \{u \in L^\rho_T(\widetilde{B}^{s_1,s_2}_p)/\|u\|_{\widetilde{L}^\rho_T(\widetilde{B}^{s_1,s_2}_p)} < \infty\}\,.$$

We denote moreover by $\widetilde{C}_T(\widetilde{B}^{s_1,s_2}_p)$ the set of those functions of $\widetilde{L}^\infty_T(\widetilde{B}^{s_1,s_2}_p)$ which are continuous from $[0,T]$ to $\widetilde{B}^{s_1,s_2}_p$. In the sequel we are going to give some properties of this spaces concerning the interpolation and their relationship with the heat equation.

Proposition 2.21. *Let s, t, s_1, $s_2 \in \mathbb{R}$, r, ρ, ρ_1, $\rho_2 \in [1,+\infty]$. We have :*

1. *Interpolation :*

$$\|u\|_{\tilde{L}^\rho_T(\tilde{B}^{s,t}_{p,r})} \leq \|u\|^\theta_{\tilde{L}^{\rho_1}_T(\tilde{B}^{s_1,t_1}_{p,r})} \|u\|^{1-\theta}_{\tilde{L}^{\rho_2}_T(\tilde{B}^{s_2,t_2}_{p,r})}$$

with $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}$, $s = \theta s_1 + (1-\theta)s_2$ and $t = \theta t_1 + (1-\theta)t_2$.

2. *Embedding :*

$$\tilde{L}^\rho_T(\tilde{B}^{s,t}_p) \hookrightarrow L^\rho_T(C_0) \ \ and \ \ \tilde{C}_T(B^{\frac{N}{p}}_p) \hookrightarrow C([0,T]\times\mathbb{R}^d)$$

Here we recall a result of interpolation which explains the link of the space $B^s_{p,1}$ with the homogeneous spaces , see [14].

Proposition 2.22. *There exists a constant C such that for all $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \leq p \leq +\infty$, we have*

$$\|u\|_{B^s_{p,1}} \leq C\frac{1+\varepsilon}{\varepsilon}\|u\|_{B^s_{p,\infty}}\left(1 + \log\frac{\|u\|_{B^{s-\varepsilon}_{p,\infty}} + \|u\|_{B^{s+\varepsilon}_{p,\infty}}}{\|u\|_{B^s_{p,\infty}}}\right).$$

To finish with we adapt the results of the paradifferential calculus on the product of Besov function to the spaces of Chemin-Lerner. So we have the following properties :

Proposition 2.23. *Let $p, r \in [1,+\infty]$. We have the two following properties :*

- *Let $s > 0$, $t > 0$, $1/\rho_2 + 1/\rho_3 = 1/\rho_1 + 1/\rho_4 = 1/\rho \leq 1$, $u \in \tilde{L}^{\rho_3}_T(\tilde{B}^{s,t}_{p,r}) \cap \tilde{L}^{\rho_1}_T(L^\infty)$ and $v \in \tilde{L}^{\rho_4}_T(\tilde{B}^{s,t}_{p,r}) \cap \tilde{L}^{\rho_2}_T(L^\infty)$. Then $uv \in \tilde{L}^\rho_T(\tilde{B}^{s,t}_{p,r})$ and we have :*

$$\|uv\|_{\tilde{L}^\rho_T(\tilde{B}^{s,t}_{p,r})} \lesssim \|u\|_{\tilde{L}^{\rho_1}_T(L^\infty)}\|v\|_{\tilde{L}^{\rho_4}_T(\tilde{B}^{s,t}_{p,r})} + \|v\|_{\tilde{L}^{\rho_2}_T(L^\infty)}\|u\|_{\tilde{L}^{\rho_3}_T(\tilde{B}^{s,t}_{p,r})}$$

- *If $s_1, s_2, t_1, t_2 \leq \frac{N}{p}$, $s_1 + s_2 > 0$, $t_1 + t_2 > 0$ $1/\rho_1 + 1/\rho_2 = 1/\rho \leq 1$, $u \in \tilde{L}^{\rho_1}_T(B^{s_1,t_1}_{p,r})$ and $v \in \tilde{L}^{\rho_2}_T(B^{s_2,t_2}_{p,r})$ then $uv \in \tilde{L}^\rho_T(B^{s_1+s_2-d/2}_2)$ and*

$$\|uv\|_{\tilde{L}^\rho_T(B^{s_1+s_2-\frac{N}{p},t_1+t_2-\frac{N}{p}}_{p,r})} \lesssim \|u\|_{\tilde{L}^{\rho_1}_T(B^{s_1,t_1}_{p,r})}\|v\|_{\tilde{L}^{\rho_2}_T(B^{s_2,t_2}_{p,r})}.$$

The analogous of the endpoint estimate (2.5) reads (for $p \geq 2$) :

$$\|uv\|_{\tilde{L}^\rho_T(B^{-\frac{N}{p}}_{p,\infty})} \lesssim \|u\|_{\tilde{L}^{\rho_1}_T(B^s_{p,1})}\|u\|_{\tilde{L}^{\rho_2}_T(B^{-s}_{p,\infty})}, \tag{2.6}$$

whenever s is in the range $(-\frac{N}{p}, NN]$ and $\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho} \leq 1$ (see the proof in [19]). For a proof of this proposition see [15]. Finally we need an estimate on the composition of functions in the spaces $\tilde{L}^\rho_T(\tilde{B}^s_p)$.

Proposition 2.24. *Let $s > 0$, $r \in [1,+\infty]$ and $F \in W^{s+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0) = 0$. There exists a function C depending only on s, p, N and F, and such that :*

$$\|F(u)\|_{\tilde{L}^\rho_T(\tilde{B}^{s_1,s_2}_{p,r})} \leq C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\tilde{L}^\rho_T(\tilde{B}^{s_1,s_2}_{p,r})}.$$

If $v,\ u\ \in\ \tilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p})\cap L^{\infty}_{T}(L^{\infty})$ *and* $G\in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ *then* $G(u)-G(v)$ *belongs to* $\tilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p})$ *and it exists a constant* C *depending only of* s,p,N *and* G *such that :*

$$\|G(u)-G(v)\|_{\tilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p,r})}\leq\ C(\|u\|_{L^{\infty}_{T}(L^{\infty})},\|v\|_{L^{\infty}_{T}(L^{\infty})})(\|v-u\|_{\tilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p,r})}$$

$$(1+\|u\|_{L^{\infty}_{T}(L^{\infty})}+\|v\|_{L^{\infty}_{T}(L^{\infty})})+\|v-u\|_{L^{\infty}_{T}(L^{\infty})}(\|u\|_{\tilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p,r})}+\|v\|_{\tilde{L}^{\rho}_{T}(B^{s_1,s_2}_{p,r})}).$$

The proof is a adaptation of a theorem by J.Y. Chemin and H. Bahouri in [1].
We end this section by recalling some estimates in Besov spaces for transport and heat equations. For more details, the reader is referred to [8] and [17].

Proposition 2.25. *Let* $(p,r)\in[1,+\infty]^2$ *and* $s\in(-\min(\frac{N}{p},\frac{N}{p'}),\frac{N}{p}+1)$. *Let* u *be a vector field such that* ∇u *belongs to* $L^1(0,T;B^{\frac{N}{p}}_{p,r}\cap L^{\infty})$. *Suppose that* $q_0\in B^s_{p,r}$, $F\in L^1(0,T,B^s_{p,r})$ *and that* $q\in L^{\infty}_{T}(B^s_{p,r})\cap C([0,T];\mathcal{S}')$ *solves the following transport equation :*

$$\begin{cases}\partial_t q+u\cdot\nabla q=F,\\ q_{t=0}=q_0.\end{cases}$$

Let $U(t)=\int_0^t\|\nabla u(\tau)\|_{B^{\frac{N}{p}}_{p,r}\cap L^{\infty}}d\tau$. *There exits a constant* C *depending only on* s, p *and* N, *and such that for all* $t\in[0,T]$, *the following inequality holds :*

$$\|q\|_{\widetilde{L}^{\infty}_{t}(B^s_{p,r})}\leq\exp^{CU(t)}\Big(\|q_0\|_{B^s_{p,r}}+\int_0^t\exp^{-CU(\tau)}\|F(\tau)\|_{B^s_{p,r}}\,d\tau\Big)$$

If $r<+\infty$ *then* q *belongs to* $C([0,T];B^s_{p,r})$.

Actually, in [17], the proposition below is proved for non-homogeneous Besov spaces. The adaptation to homogeneous spaces is straightforward. Let us now some estimates for the heat equation :

Proposition 2.26. *Let* $s\in\mathbb{R}$, $(p,r)\in[1,+\infty]^2$ *and* $1\leq\rho_2\leq\rho_1\leq+\infty$. *Assume that* $u_0\in B^s_{p,r}$ *and* $f\in\widetilde{L}^{\rho_2}_{T}(\widetilde{B}^{s-2+2/\rho_2}_{p,r})$. *Let* u *be a solution of :*

$$\begin{cases}\partial_t u-\mu\Delta u=f\\ u_{t=0}=u_0\,.\end{cases}$$

Then there exists $C>0$ *depending only on* N,μ,ρ_1 *and* ρ_2 *such that :*

$$\|u\|_{\widetilde{L}^{\rho_1}_{T}(\widetilde{B}^{s+2/\rho_1}_{p,r})}\leq C\big(\|u_0\|_{B^s_{p,r}}+\mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}^{\rho_2}_{T}(\widetilde{B}^{s-2+2/\rho_2}_{p})}\big)\,.$$

If in addition r *is finite then* u *belongs to* $C([0,T],B^s_{p,r})$.

The proof of local well-posedness for initial density bounded away from zero requires estimates in B^s_p spaces for the following linear system :

$$\begin{cases}\partial_t u-\bar{\mu}\mathrm{div}(a\nabla u)-(\bar{\lambda}+\bar{\mu})\nabla(a\mathrm{div}u)=G,\\ u_{t=0}=u_0,\end{cases}$$

where $u(t,x)\in\mathbb{R}^N$, $\bar{\nu}=2\bar{\mu}+\bar{\lambda}>0$ and the diffusion coefficient is assumed to satisfy :

$$0<\underline{a}\leq a(t,x)\leq\overline{a}. \tag{2.7}$$

We can prove that the solution of the previous system satisfy estimates analogous to those of proposition 2.26, see [15].

Proposition 2.27. *Let $1<p<+\infty$, $1\leq\alpha_1\leq r\leq+\infty$ and s be such that $\max(1,\frac{N}{p})\leq s\leq\frac{N}{p}+1$. Set $\alpha_2'=\frac{2}{s-\frac{N}{p}}$ and let α_2 be such that $\frac{1}{\alpha_1}=\frac{1}{\alpha_2}+\frac{1}{\alpha_2'}$. Let assumption (2.7) be fulfilled and u be a solution of (2.3). We suppose that the regularity index τ satisfies :*

$$1-\frac{2}{\alpha_1}-\frac{N}{p}<\tau\leq1-\frac{2}{\alpha_1}+s.$$

Then the following estimate holds for all $\alpha\in[r,+\infty]$:

$$\|u\|_{\widetilde{L}^{\alpha}_T(B_p^{\tau+\frac{2}{\alpha}})}\lesssim\|u_0\|_{B_p^{\tau}}+\|G\|_{\widetilde{L}^{r}_T(B_p^{\tau+\frac{2}{r}-2})}+\|\nabla a\|_{\widetilde{L}^{\alpha_2'}_T(B_p^{s-1})}\|\nabla u\|_{\widetilde{L}^{\alpha_2'}_T(B_p^{\tau-1+\frac{2}{\alpha_2}})}.$$

3 Proof of theorem 1.21

3.1 Sketch of the Proof

In this section, we give the sketch of the proof of theorem 1.21 on the global existence result with small initial data.
We will suppose that ρ is close to a constant state $\bar{\rho}$, so that ρ will be strictly superior to a positive constant, we will use the parabolicity of the momentum equation to get a gain of derivatives on the velocity u. The density ρ has a behavior similar to the solution of a transport equation. Let us rewrite the (SW) system in a non conservative form in using the definition 1.17.

$$(SW1)\quad\begin{cases}\partial_t q+u.\nabla q+\mathrm{div}u=F\\ \partial_t u+u.\nabla u-\dfrac{\mu(\bar{\rho})}{\bar{\rho}}\Delta u-\dfrac{\mu(\bar{\rho})+\lambda(\bar{\rho})}{\bar{\rho}}\nabla\mathrm{div}u+(\kappa\bar{\rho}+P'(\bar{\rho}))\nabla q\\ \hfill -\kappa\bar{\rho}\phi*\nabla q=G\end{cases}$$

where we have :

$$F=-q\,\mathrm{div}u,$$

$$G=\mathcal{A}(\rho,u)+K(\rho)\nabla q\,,$$

where we set :

$$\mathcal{A}(\rho,u)=\Big[\frac{\mathrm{div}\big(\mu(\rho)D(u)\big)}{\rho}-\frac{\mu(\bar{\rho})}{\bar{\rho}}\Delta u\Big]+\Big[\frac{\nabla\big((\mu(\rho)+\lambda(\rho))\mathrm{div}u\big)}{\rho}-\frac{\mu(\bar{\rho})+\lambda(\bar{\rho})}{\bar{\rho}}\nabla\mathrm{div}u\Big],$$

$$K(\rho)=\frac{\bar{\rho}\,P'(\rho)}{\rho}-P'(\bar{\rho}).$$

For $s\in\mathbb{R}$, we denote $\Lambda^s h=\mathcal{F}^{-1}(|\xi|^s\widehat{h})$. We set now :

$$d=\Lambda^{-1}\mathrm{div}u\ \text{ and }\ \Omega=\Lambda^{-1}\mathrm{curl}u$$

where d represents the compressible part of the velocity and Ω the incompressible part. We rewrite now the system $(SW1)$ in using these previous notations on a linear form :

$$(SW2)\qquad \begin{cases} \partial_t q + \Lambda d = F_1, \\ \partial_t d - \bar{\nu}\Delta d - \bar{\delta}\Lambda q + \bar{\kappa}\Lambda(\phi * q) = G_1 \\ \partial_t \Omega - \bar{\mu}\Delta\Omega = H_1 \\ u = -\Lambda^{-1}\nabla d - \Lambda \mathrm{div}\Omega \end{cases}$$

where we have :

$$\bar{\mu} = \frac{\mu(\bar{\rho})}{\bar{\rho}}, \quad \bar{\lambda} = \frac{\lambda(\bar{\rho})}{\bar{\rho}}, \quad \bar{\nu} = 2\bar{\mu} + \bar{\lambda}, \quad \bar{\delta} = \kappa\bar{\rho} + P^{'}(\bar{\rho}), \quad \text{and} \quad \bar{\kappa} = \kappa\bar{\rho}.$$

We have in our case :

$$\begin{aligned} F_1 &= -q\,\mathrm{div}u - u\cdot\nabla q, \\ G_1 &= -\Lambda^{-1}\mathrm{div}(G), \\ H_1 &= -\Lambda^{-1}\mathrm{curl}(G). \end{aligned}$$

The first idea will be to study the linear system associated to $(SW2)$. We concentrate on the first two equations because the third equation is just a heat equation with a non linear term. The system we want to study reads :

$$\begin{cases} \partial_t q + \Lambda d = F^{'}, \\ \partial_t d - \bar{\nu}\Delta d - \bar{\delta}\Lambda q + \bar{\kappa}\Lambda(\phi * q) = G^{'}. \end{cases}$$

This system has been studied by D. Hoff and K. Zumbrum in [28] in the case $\bar{\kappa} = 0$. There, they investigate the decay estimates, and exhibit the parabolic smoothing effect on d and on the low frequencies of q, and a damping effect on the high frequencies of q.
The problem is that if we focus on this linear system, it appears impossible to control the term of convection $u\cdot\nabla q$ which is one derivative less regular than q. Hence we shall include the convection term in the linear system. We thus have to study :

$$(SW2)^{'}\qquad \begin{cases} \partial_t q + v\cdot\nabla q + \Lambda d = F, \\ \partial_t d + v\cdot\nabla d - \bar{\nu}\Delta d - \bar{\delta}\Lambda q + \bar{\kappa}\Lambda(\phi * q) = G, \end{cases}$$

where v is a function and we will precise its regularity in the next proposition. System $(SW2)^{'}$ has been studied in the case where $\phi = 0$ by R. Danchin in [15], we then adapt the proof in taking into consideration the term coming from the capillarity.
In the sequel we will assume $\bar{\nu} > 0$ and $\bar{\delta} - \bar{\kappa}\|\widehat{\phi}\|_{L^\infty} \geq c > 0$. We get then the following proposition.

Proposition 3.28. *Let (q,d) a solution of the system $(SW2)^{'}$ on $[0,T[\,$, $1-\frac{N}{2} < s \leq 1+\frac{N}{2}$ and $V(t) = \int_0^t \|v(\tau)\|_{B^{\frac{N}{2}+1}} d\tau$. We have then the following estimate :*

$$\begin{aligned} &\|(q,d)\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} + \int_0^t \|(q,d)(\tau)\|_{\widetilde{B}^{s+1,s}\times B^{s+1}} d\tau \\ &\qquad\qquad \leq Ce^{CV(t)}\big(\|(q_0,d_0)\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} + \int_0^t e^{-CV(\tau)}\|(F,G)(\tau)\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} d\tau\big), \end{aligned}$$

where C depends only on $\bar{\nu}$, $\bar{\delta}$, $\bar{\kappa}$, ϕ, N and s.

Proof of Proposition 3.28 :

Let (q, u) be a solution of $(SW2)^{'}$ and we set :

$$\widetilde{q} = e^{-KV(t)}q,\ \widetilde{u} = e^{-KV(t)}u,\ \widetilde{F} = e^{-KV(t)}F \text{ and } \widetilde{G} = e^{-KV(t)}G. \tag{3.8}$$

We are going to separate the case of the low and high frequencies, which have a different behavior concerning the control of the derivative index for the Besov spaces.
In this goal we will consider the two different expressions in low and high frequencies where $l_0 \in \mathbb{Z}$, A, B and K_1 will be fixed later in the proof :

$$\begin{aligned} f_l^2 &= \bar{\delta}\|\widetilde{q}_l\|_{L^2}^2 - \bar{\kappa}(\widetilde{q}_l, \phi * \widetilde{q}_l) + \|\widetilde{d}_l\|_{L^2}^2 - 2K_1(\Lambda\widetilde{q}_l, \widetilde{d}_l) \quad \text{for } l \leq l_0,\\ f_l^2 &= \|\Lambda\widetilde{q}_l\|_{L^2}^2 + A\|\widetilde{d}_l\|_{L^2}^2 - \frac{2}{\bar{\nu}}(\Lambda\widetilde{q}_l, \widetilde{d}_l) \quad \text{for } l > l_0. \end{aligned} \tag{3.9}$$

In the first two steps, we show that K_1 and A may be chosen such that :

$$2^{l(s-1)} f_l^2 \approx 2^{ls} \max(1, 2^{-l})\|\widetilde{q}_l\|_{L^2}^2 + 2^{l(s-1)}\|\widetilde{d}_l\|_{L^2}^2, \tag{3.10}$$

and we will show the following inequality :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}f_l^2 + \alpha \min(2^{2l}, 1) f_l^2 \leq C 2^{-l(s-1)}\alpha_l f_l \big(&\|(\widetilde{F}, \widetilde{G})\|_{\widetilde{B}^{s-1,s} \times B^{s-1}} \\ &+ V^{'}\|(\widetilde{q}, \widetilde{d})\|_{\widetilde{B}^{s-1,s} \times B^{s-1}}\big) - KV^{'} f_l^2. \end{aligned} \tag{3.11}$$

where $\sum_{l \in \mathbb{Z}} \alpha_l \leq 1$ and α is a positive constant.
This inequality enables us to get a decay for q and d which will be used to show a smoothing parabolic effect on d.

Case of low frequencies

Applying operator Δ_l to the system $(SW2)^{'}$, we obtain then in setting :

$$\widetilde{q}_l = \Delta_l \widetilde{q},\ \widetilde{d}_l = \Delta_l \widetilde{d}.$$

the following system :

$$\begin{cases} \dfrac{d}{dt}\widetilde{q}_l + \Delta_l(v \cdot \nabla\widetilde{q}) + \Lambda\widetilde{d}_l = \widetilde{F}_l - KV^{'}(t)\widetilde{q}_l,\\ \dfrac{d}{dt}d_l + \Delta_l(v \cdot \nabla\widetilde{d}_l) - \bar{\nu}\Delta\widetilde{d}_l - \bar{\delta}\Lambda\widetilde{q}_l + \bar{\kappa}\Lambda(\phi * \widetilde{q}_l) = \widetilde{G}_l - KV^{'}(t)\widetilde{d}_l. \end{cases} \tag{3.12}$$

We set :

$$f_l^2 = \bar{\delta}\|\widetilde{q}_l\|_{L^2}^2 + \|\widetilde{d}_l\|_{L^2}^2 - 2K_1(\Lambda\widetilde{q}_l, \widetilde{d}_l) \tag{3.13}$$

for some $K_1 \geq 0$ to be fixed hereafter and $(\cdot, \cdot)$ noting the L^2 inner product.
To begin with, we consider the case where $F = G = 0$, $v = 0$ and $K = 0$. Taking the L^2

scalar product of the first equation of (3.12) with $\tilde{q}_l$ and of the second equation with $\tilde{d}_l$, we get the following two identities :

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|q_l\|_{L^2}^2 + (\Lambda d_l, q_l) = 0, \\ \frac{1}{2}\frac{d}{dt}\|d_l\|_{L^2}^2 + \bar{\nu}\|\Lambda d_l\|_{L^2}^2 - \bar{\delta}(\Lambda q_l, d_l) + \bar{\kappa}(\Lambda(\phi * q_l), d_l) = 0. \end{cases} \tag{3.14}$$

In the same way we have :

$$\frac{1}{2}\frac{d}{dt}(q_l, q_l * \phi) + (\Lambda d_l, \phi * q_l) = 0, \tag{3.15}$$

because we have by the theorem of Plancherel :

$$(\frac{d}{dt}q_l, q_l * \phi) = (\frac{d}{dt}\widehat{q_l}, \widehat{q_l}\widehat{\phi}) = \frac{1}{2}\frac{d}{dt}(\widehat{q_l}, \widehat{q_l}\widehat{\phi}) = \frac{1}{2}\frac{d}{dt}(q_l, q_l * \phi).$$

We want now get an equality involving $\bar{\nu}(\Lambda d_l, q_l)$. To achieve it, we apply $\bar{\nu}\Lambda$ to the first equation of (3.12) and take the L^2-scalar product with d_l, then take the scalar product of the second equation with Λq_l and sum both equalities, which yields :

$$\frac{d}{dt}(\Lambda q_l, d_l) + \|\Lambda d_l\|_{L^2}^2 - \bar{\delta}\|\Lambda q_l\|_{L^2}^2 + \bar{\kappa}\|\phi * \Lambda q_l\|_{L^2}^2 + \bar{\nu}(\Lambda^2 d_l, \Lambda q_l) = 0. \tag{3.16}$$

By linear combination of (3.14) and (3.16), we get :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + (\bar{\nu} - K_1)\|\Lambda d_l\|_{L^2}^2 + K_1(\bar{\delta}\|\Lambda q_l\|_{L^2}^2 - \bar{\kappa}\|\phi * \Lambda q_l\|_{L^2}^2) - \bar{\nu}K_1(\Lambda^2 d_l, \Lambda q_l) = 0. \tag{3.17}$$

And as we have assumed that : $\delta - \bar{\kappa}\|\widehat{\phi}\|_{L^\infty} \geq c > 0$ we get :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + (\bar{\nu} - K_1)\|\Lambda d_l\|_{L^2}^2 + K_1 c\|q_l\|_{L^2}^2 - \bar{\nu}K_1(\Lambda^2 d_l, \Lambda q_l) \leq 0. \tag{3.18}$$

Using spectral localization for d_l and convex inequalities, we find for every $a > 0$:

$$|(\Lambda^2 d_l, \Lambda q_l)| \leq \frac{a2^{2l_0}}{2}\|\Lambda d_l\|_{L^2}^2 + \frac{1}{2a}\|\Lambda q_l\|_{L^2}^2.$$

In using the previous inequality and (3.17), we get :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + (\bar{\nu} - K_1 - \frac{a2^{2l_0}}{2})\|\Lambda d_l\|_{L^2}^2 + (K_1 c - \frac{1}{2a})\|\Lambda q_l\|_{L^2}^2 \leq 0. \tag{3.19}$$

From (3.13) and (3.19) we get in choosing $a = \bar{\nu}$ and $K_1 < \min(\frac{1}{2^{2l_0}}, \frac{\bar{\nu}}{2+2^{2l_0}\bar{\nu}^2})$, then :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + \alpha 2^{2l} f_l^2 \leq 0, \tag{3.20}$$

for a constant α depending only on $\bar{\nu}$ and K_1.

In the general case where F, G, K and v are not zero, we have :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + (\alpha 2^{2l} + KV')f_l^2 \leq (\widetilde{F}_l, \widetilde{q}_l) + (\widetilde{G}_l, \widetilde{d}_l) - K(\Lambda\widetilde{F}_l, \widetilde{d}_l) - K(\Lambda\widetilde{G}_l, \widetilde{q}_l) - (\Delta_l(v\cdot\nabla\widetilde{q}), \widetilde{q}_l)$$
$$- (\Delta_l(v\cdot\nabla\widetilde{d}), \widetilde{d}_l) + K\big((\Lambda\Delta_l(v\cdot\nabla\widetilde{q}), \widetilde{d}_l\big) + \big((\Lambda\Delta_l(v\cdot\nabla\widetilde{d}), \widetilde{q}_l\big).$$

Now we can use a lemma of harmonic analysis in [15] to estimate the last terms, and get the existence of a sequence $(\alpha_l)_{l\in\mathbb{Z}}$ such that $\sum_{l\in\mathbb{Z}}\alpha_l \leq 1$ and :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + (\alpha 2^{2l} + KV')f_l^2 \lesssim \alpha_l f_l 2^{-l(s-1)}\big(\|(\widetilde{F}, \widetilde{G})\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} + V'\|(\widetilde{q}, \widetilde{d})\|_{\widetilde{B}^{s-1,s}\times B^{s-1}}\big). \tag{3.21}$$

Case of high frequencies

We consider now the case where $l \geq l_0 + 1$ and we recall that :

$$f_l^2 = \|\Lambda\widetilde{q}_l\|_{L^2}^2 + A\|\widetilde{d}_l\|_{L^2}^2 - \frac{2}{\bar{\nu}}(\widetilde{q}_l, \widetilde{d}_l).$$

For the sake of simplicity, we suppose here that $F = G = 0$, $v = 0$ and $K = 0$. We now want a control $\|\Lambda q_l\|_{L^2}^2$ on e apply the operator Λ to the first equation of (3.12), multiply by Λq_l and integrate over $\mathbb{R}^N$, so we obtain :

$$\frac{1}{2}\frac{d}{dt}\|\Lambda q_l\|_{L^2}^2 + (\Lambda^2 d_l, \Lambda q_l) = 0. \tag{3.22}$$

Moreover we have :

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|d_l\|_{L^2}^2 + \bar{\nu}\|\Lambda d_l\|_{L^2}^2 - \bar{\delta}(\Lambda q_l, d_l) + \bar{\kappa}(\Lambda(\phi * q_l), d_l) = 0. \\ &\frac{d}{dt}(\Lambda q_l, d_l) + \|\Lambda d_l\|_{L^2}^2 - \bar{\delta}\|\Lambda q_l\|_{L^2}^2 + \bar{\kappa}\|\phi * \Lambda q_l\|_{L^2}^2 + \bar{\nu}(\Lambda^2 d_l, \Lambda q_l) = 0. \end{aligned} \tag{3.23}$$

By linear combination of (3.22)-(3.23) we have :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + \frac{1}{\bar{\nu}}\|\Lambda q_l\|_{L^2}^2 + \big(A\bar{\nu} - \frac{1}{\bar{\nu}}\big)\|\Lambda d_l\|_{L^2}^2 - A\bar{\delta}(\Lambda q_l, d_l) + A\bar{\kappa}(\Lambda(\phi * q_l), d_l) = 0. \tag{3.24}$$

Moreover we have :

$$| - A\bar{\delta}(\Lambda q_l, d_l) + A\bar{\kappa}(\Lambda(\phi * q_l), d_l)| \leq A(\bar{\delta} + \bar{\kappa}\|\widehat{\phi}\|_{L^\infty})|(\Lambda q_l, d_l)|$$

We have now in using Young inequalities for all $a > 0$:

$$|(d_l, \Lambda q_l)| \leq \frac{a}{2}\|\Lambda q_l\|_{L^2}^2 + \frac{1}{2a}\|d_l\|_{L^2}^2,$$

So we get :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + 2^{2l_0}\big(A\bar{\nu} - \frac{1}{\bar{\nu}} - \frac{1}{2a}\big)\|d_l\|_{L^2}^2 + (\frac{1}{\bar{\nu}} - \frac{a}{2})\|\Lambda q_l\|_{L^2}^2. \leq 0 \tag{3.25}$$

So in choosing :

$$a = \frac{1}{\bar{\nu}A} \quad \text{and} \quad A > \max(\frac{2}{\bar{\nu}}, 1)$$

there exists a constant α such that for $l \geq l_0 + 1$ we have :

$$\frac{1}{2}\frac{d}{dt}f_l^2 + \alpha f_l^2 \leq 0. \tag{3.26}$$

In the general case where F, G, H, K and v are not necessarily zero, we use a lemma of harmonic analysis in [15] to control the convection terms. We finally get :

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}f_l^2 + (\alpha + KV^{'})f_l^2 \lesssim \alpha_l f_l 2^{-l(s-1)}\big(&\|(\widetilde{F}, \widetilde{G})\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} \\ &+ V^{'}\|(\widetilde{q}, \widetilde{d})\|_{\widetilde{B}^{s-1,s}\times B^{s-1}}\big). \end{aligned} \tag{3.27}$$

This finish the proof of (3.9) and (3.11).

The damping effect

We are now going to show that inequality (3.11) entails a decay for q and d. In fact we get a parabolic decay for d, while q has a behavior similar to a transport equation.
Using $h_l^2 = f_l^2 + \delta^2$, integrating over $[0,t]$ and then having δ tend to 0, we infer :

$$\begin{aligned} f_l(t) + \alpha \min(2^{2l},1) \int_0^t f_l(\tau) d\tau & \\ \le f_l(0) + C2^{-l(s-1)} \int_0^t \alpha_l(\tau) \|(\widetilde{F}(\tau), \widetilde{G}(\tau))\|_{\tilde{B}^{s-1,s}\times B^s} d\tau & \qquad (3.28) \\ + \int_0^t V'(\tau)\big(C2^{-l(s-1)}\alpha_l(\tau)\|(\widetilde{q},\widetilde{d})\|_{\tilde{B}^{s-1,s}\times B^s} - K f_l(\tau)\big) d\tau. & \end{aligned}$$

Thanks to (3.10), we have in taking K large enough :

$$\sum_{l\in\mathbb{Z}} \big(C2^{-l(s-1)}\alpha_l(\tau)\|(\widetilde{q},\widetilde{d})\|_{\tilde{B}^{s-1,s}\times B^s} - K f_l(\tau)\big) \le 0,$$

In multiplying (3.28) by $2^{l(s-1)}$ and in using the last inequality, we conclude after summation on $\mathbb{Z}$, that :

$$\begin{aligned} \|\widetilde{q}(t)\|_{\tilde{B}^{s-1,s}} + \|\widetilde{d}\|_{\tilde{B}^{s-1}} + \alpha \int_0^t \|\widetilde{q}(\tau)\|_{\tilde{B}^{s-1,s}} d\tau + \sum_{l\in\mathbb{Z}} \int_0^t \alpha 2^{l(s-1)} \min(2^{2l},1) \|\widetilde{d}_l(\tau)\|_{L^2} d\tau & \\ \lesssim \|(\widetilde{q}_0, \widetilde{d}_0)\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} + \int_0^t \|(\widetilde{F},\widetilde{G})\|_{\widetilde{B}^{s-1,s}\times B^{s-1}} d\tau. & \end{aligned} \qquad (3.29)$$

The smoothing effect

Once stated the damping effect for q, it is easy to get the smoothing effect on d by considering the last two equations where the term Λq is considered as a source term .
Thanks to (3.29), it suffices to prove it for high frequencies only. We therefore suppose in this subsection that $l \ge l_0$ for a l_0 big enough.
We set $g_l = \|\widetilde{d}_l\|_{L^2}$ and in using the previous inequalities, we have :

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{d}_l\|_{L^2}^2 + \bar{\nu}\|\Lambda\widetilde{d}_l\|_{L^2}^2 - \bar{\delta}(\Lambda\widetilde{q}_l, \widetilde{d}_l) + \bar{\kappa}(\Lambda(\phi * \widetilde{q}_l), \widetilde{d}_l) = \widetilde{G}_l \cdot \widetilde{d}_l - KV'(t)\|\widetilde{d}_l\|_{L^2}^2.$$

We get finally with $\alpha > 0$:

$$\frac{1}{2}\frac{d}{dt} g_l^2 + \alpha 2^{2l} g_l^2 \le g_l\big(\|\Lambda\widetilde{q}_l\|_{L^2} + \|\widetilde{G}_l\|_{L^2}\big) + g_l V'(t)(C\alpha_l 2^{-l(s-1)}\|\widetilde{d}\|_{B^{s-1}} - K g_l\big).$$

We therefore get in using standard computations :

$$\begin{aligned} \sum_{l\ge l_0} 2^{l(s-1)}\|\widetilde{d}_l(t)\|_{L^2} + \alpha \int_0^t \sum_{l\ge l_0} 2^{l(s+1)}\|\widetilde{d}_l(\tau)\|_{L^2} d\tau \le \|d_0\|_{B^{s-1}} + \int_0^t \|\widetilde{G}(\tau)\|_{B^{s-1}} d\tau & \\ + \int_0^t \sum_{l\ge l_0} 2^{ls}\|\widetilde{q}_l(\tau)\|_{L^2} + CV(t) \sup_{\tau\in[0,t]} (\|\widetilde{d}(\tau)\|_{B^{s-1}}). & \end{aligned}$$

Using the above inequality and (3.29), we have :

$$\int_0^t \sum_{l\geq l_0} 2^{l(s+1)} \|\widetilde{d}_l(\tau)\|_{L^2} d\tau \lesssim (1+V(t))\big(\|q_0\|_{\widetilde{B}^{s-1,s}} + \|d_0\|_{B^{s-1}}\big) + \int_0^t (\|\widetilde{F}(\tau)\|_{\widetilde{B}^{s-1,s}} + \|\widetilde{G}(\tau)\|_{B^{s-1}}) d\tau. \tag{3.30}$$

Combining that last inequality (3.30) with (3.29), we achieve the proof of proposition 3.28. □

3.2 Proof of theorem 1.21

This section is devoted to the proof of the theorem 1.21. The principle of the proof is a very classical one. We want to construct a sequence $(q^n, u^n)_{n\in\mathbb{N}}$ of approximate solutions of the system (SW), and we will use the proposition 3.28 to get some uniform bounds on $(q^n, u^n)_{n\in\mathbb{N}}$. We will conclude by stating some properties of compactness, which will guarantee that up to an extraction, $(q^n, u^n)_{n\in\mathbb{N}}$ converges to a solution (q, u) of the system (SW).

First step : Building the sequence $(q^n, u^n)_{n\in\mathbb{N}}$

We start with the construction of the sequence $(q^n, u^n)_{n\in\mathbb{N}}$, in this goal we use the Friedrichs operators $(J_n)_{n\in\mathbb{N}}$ defined by :

$$J_n g = \mathcal{F}^{-1}(1_{B(\frac{1}{n},n)}\widehat{g}),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Let us consider the approximate system :

$$\begin{cases} \partial_t q^n + J_n(J_n u^n\cdot\nabla J_n q^n) + \Lambda J_n d^n = F^n \\ \partial_t d^n + J_n(J_n u^n\cdot\nabla J_n d^n) - \bar{\nu}\Delta J_n d^n - \bar{\delta}\Lambda J_n q^n - \bar{\kappa}\phi * \Lambda J_n q^n = G^n \\ \partial_t \Omega^n - \bar{\nu}\Delta J_n\Omega^n = H^n \\ u^n = -\Lambda^{-1}\nabla d^n - \Lambda^{-1}\mathrm{div}\Omega^n \\ (q^n, d^n, \Omega^n)_{/t=0} = (J_n q_0, J_n d_0, J_n\Omega_0) \end{cases} \tag{3.31}$$

with :

$$F^n = -J_n\big((J_n q^n)\mathrm{div} J_n u^n\big),$$

$$G^n = J_n\Lambda^{-1}\mathrm{div}\big[\mathcal{A}(\varphi\big(\bar{\rho}(1+J_n q^n)\big), J_n u^n) + K(\varphi\big(\bar{\rho}(1+J_n q^n)\big)\nabla q^n\big],$$

$$H^n = J_n\Lambda^{-1}\mathrm{curl}\big[\mathcal{A}(\varphi\big(\bar{\rho}(1+J_n q^n)\big), J_n u^n) + K(\varphi\big(\bar{\rho}(1+J_n q^n)\big)\nabla q^n\big].$$

where φ is a smooth function verifying $\varphi(s) = s$ for $\frac{1}{n}\leq s\leq n$ and $\varphi\geq\frac{1}{4}$.
We want to show that (3.31) is only an ordinary differential equation in $L^2\times L^2\times L^2$. We can observe easily that all the source term in (3.31) turn out to be continuous in

$L^2 \times L^2 \times L^2$. As a example, we consider the term $J_n\mathcal{A}(\varphi(\bar{\rho}(1+J_nq^n)), J_nu^n)$. We have then by Plancherel theorem :

$$\|J_n\big(\frac{\text{div}\big(\mu(\varphi(\bar{\rho}(1+J_nq^n))DJ_nu^n\big)}{\varphi(\bar{\rho}(1+J_nq^n))}\big)\|_{L^2} \leq n\|\mu(\varphi(\bar{\rho}(1+J_nq^n))DJ_nu^n\|_{L^2}$$
$$\times \|\frac{1}{\varphi(\bar{\rho}(1+J_nq^n))}\|_{L^\infty},$$
$$\leq 4M_n n^2\|u^n\|_{L^2}.$$

where $M_n = \|\mu(\varphi(\bar{\rho}(1+J_nq^n)\|_{L^\infty}$.
According to the Cauchy-Lipschitz theorem, a unique maximal solution exists in $C([0,T_n);L^2)$ with $T_n > 0$. Moreover, since $J_n = J_n^2$ we show that $(J_nq^n, J_nd^n, J_n\Omega^n)$ is also a solution and then by uniqueness we get that $(J_nq^n, J_nu^n) = (q^n, u^n)$. This implies that (q^n, d^n, Ω^n) is solution of the following system :

$$\begin{cases} \partial_t q^n + J_n(u^n.\nabla q^n) + \Lambda d^n = F_1^n \\ \partial_t d^n + J_n(u^n.\nabla d^n) - \bar{\nu}\Delta d^n - \bar{\delta}\Lambda q^n - \bar{\kappa}\phi * \Lambda q^n = G_1^n \\ \partial_t \Omega^n - \bar{\nu}\Delta\Omega^n = H_1^n \\ u^n = -\Lambda^{-1}\nabla d^n - \Lambda^{-1}\text{div}\Omega^n \\ (q^n, d^n, \Omega^n)_{/t=0} = (J_nq_0, J_nd_0, J_n\Omega_0) \end{cases} \tag{3.32}$$

and :

$$F_1^n = -J_n\big(q^n\text{div}u^n\big),$$
$$G_1^n = J_n\Lambda^{-1}\text{div}\big[\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + K(\varphi(\bar{\rho}(1+q^n))\big],$$
$$H_1^n = J_n\Lambda^{-1}\text{curl}\big[\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + K(\varphi(\bar{\rho}(1+q^n))\big].$$

And the system (3.32) is again an ordinary differential equation in L_n^2 with :

$$L_n^2 = \{g \in L^2(\mathbb{R}^N)/\text{supp}\widehat{g} \subset B(\frac{1}{n}, n)\}.$$

Due to the Cauchy-Lipschitz theorem again, a unique maximal solution exists in $C^1([0,T_n^{'});L_n^2)$ with $T_n^{'} \geq T_n > 0$.

Second step : Uniform estimates

In this part, we want to get uniform estimates independent of T on $\|(q^n,u^n)\|_{E_T^{\frac{N}{2}}}$ for all $T < T_n^{'}$. This will show that $T_n^{'} = +\infty$ by Cauchy-Lipchitz because the norms $\|\cdot\|_{E^{\frac{N}{2}}}$ and L^2 are equivalent on L_n^2. $E_T^{\frac{N}{2}}$ Let us set :

$$E(0) = \|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}}},$$
$$E(q,u,t) = \|q\|_{L_t^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})} + \|q\|_{L_t^\infty(B^{\frac{N}{2}-1})} + \|q\|_{L_t^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})} + \|q\|_{L_t^\infty(B^{\frac{N}{2}+1})},$$

and :

$$\bar{T}_n = \sup\{t \in [0,T_n^{'}), E(q^n,u^n,t) \leq 3CE(0)\}$$

C corresponds to the constant in the proposition 3.28 and as $C > 1$ we have $3C > 1$ so by continuity we have $\bar{T}_n > 0$.
We are going to prove that $\bar{T}_n = T_n'$ for all $n \in \mathbb{N}$ and we will conclude that $\forall n \in \mathbb{N}$ $T_n' = +\infty$. To achieve it, one can use the proposition 3.28 to the system (3.32) to obtain uniform bounds, so we get in setting $V_n(t) = \|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}$:

$$\|(q^n,u^n)\|_{E_T^{\frac{N}{2}}} \leq C\, e^{CV_n(t)}\big(\,\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}}} + \int_0^T e^{-CV_n(\tau)}(\|F_1^n(\tau)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}} + \|G_1^n(\tau)\|_{B^{\frac{N}{2}-1}} + \|H_1^n(\tau)\|_{B^{\frac{N}{2}-1}})d\tau.\big)$$

Therefore, it is only a matter of proving appropriate estimates for F_1^n, G_1^n and H_1^n in using properties of continuity on the paraproduct.

We estimate now $\|F_1^n\|_{L^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}$ in using proposition 2.23 and 2.24 :

$$\|F_1^n\|_{L^1_T(B^{\frac{N}{2}-1,\frac{N}{2}})} \leq C\|q^n\|_{L^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}})}\|\mathrm{div}u^n\|_{L^1_T(B^{\frac{N}{2}})},$$

We now want to estimate G_1^n :

$$\|\mathcal{A}(\varphi(\bar{\rho}(1+q^n)),u^n)\|_{L^1_T(B^{\frac{N}{2}-1})} \leq C\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}(1+\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}),$$

We can verify that K fulfills the hypothesis of the proposition 2.24, so we get :

$$\|K(\varphi(\bar{\rho}(1+q^n))\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})} \leq C\|q^n\|^2_{L^2_T(B^{\frac{N}{2}})}\|q^n\|_{L^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})},$$

Moreover we recall that according to proposition 2.23 :

$$\|q^n\|^2_{L^2_T(B^{\frac{N}{2}})} \leq \|q^n\|_{L^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})}\|q^n\|_{L^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}})}.$$

We proceed similarly to estimate $\|H_1^n\|_{L^1_T(B^{\frac{N}{2}-1})}$ and finally we have :

$$\|F_1^n\|_{L^1_T(B^{\frac{N}{2}-1})} + \|G_1^n\|_{L^1_T(B^{\frac{N}{2}-1})} + \|H_1^n\|_{L^1(B_T^{\frac{N}{2}-1})} \leq 2C(E^2(q^n,u^n,T) + E^3(q^n,u^n,T)),$$

whence :

$$\|(q^n,u^n)\|_{E_T^{\frac{N}{2}}} \leq Ce^{C^2 3E(0)}E(0)(1+18CE(0)(1+3E(0))),$$

We want now to get :

$$e^{3C^2E(0)}(1+18CE(0)(1+3E(0))) \leq 2$$

for this it suffices choose $E(0)$ small enough, let $E(0) < \varepsilon$ such that :

$$1+18CE(0)(1+3E(0)) \leq \frac{3}{2} \quad \text{and} \quad e^{3C^2E(0)} \leq \frac{4}{3}.$$

So we get $\bar{T}_n = T_n^{'}$, indeed we have shown that $\forall T$ such that $T < \bar{T}_n$:

$$E(q^n, u^n, T) \leq 2CE(0).$$

Then we have $\bar{T}_n = T_n^{'}$, because if $\bar{T}_n < T_n^{'}$ we have seen that $E(q^n, u^n, \bar{T}_n) \leq 2CE(0)$ and so by continuity for $\bar{T}_n + \varepsilon$ with ε small enough we obtain again $E(q^n, u^n, \bar{T}_n + \varepsilon) \leq 3CE(0)$ and stands in contradiction with the definition of $\bar{T}_n$.
So if $\bar{T}_n = T_n^{'} < +\infty$ we have seen that :

$$E(q^n, u^n, T_n^{'}) \leq 3CE(0).$$

As $\|q_n\|_{L^{\infty}_{T_n^{'}}(\widetilde{B}^{\frac{N}{2}})} < +\infty$ and $\|u_n\|_{L^{\infty}_{T_n^{'}}(\widetilde{B}^{\frac{N}{2}-1})} < +\infty$, it implies that $\|q_n\|_{L^{\infty}_{T_n^{'}}(L^2_n)} < +\infty$ and $\|u_n\|_{L^{\infty}_{T_n^{'}}(L^2_n)} < +\infty$, so by Cauchy-Lipschitz theorem, one may continue the solution beyond $T_n^{'}$ which contradicts the definition of $T_n^{'}$.
Finally the approximate solution $(q^n, u^n)_{n \in \mathbb{N}}$ is global in time.

Second step : existence of a solution

In this part, we shall show that, up to an extraction, the sequence $(q^n, u^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}^{'}(\mathbb{R}^+ \times \mathbb{R}^N)$ to a solution (q, u) of (SW) which has the desired regularity properties. The proof lies on compactness arguments. To start with, we show that the time first derivative of (q^n, u^n) is uniformly bounded in appropriate spaces. This enables us to apply Ascoli's theorem and get the existence of a limit (q, u) for a subsequence. Now, the uniform bounds of the previous part provide us with additional regularity and convergence properties so that we may pass to the limit in the system.
It is convenient to split (q^n, u^n) into the solution of a linear system with initial data (q_n, u_n) and forcing term, and the discrepancy to that solution.
More precisely, we denote by (q^n_L, u^n_L) the solution to :

$$\begin{aligned} &\partial_t q^n_L + \mathrm{div} u^n_L = 0 \\ &\partial_t u^n_L - \mathcal{A} u^n_L + \nabla q^n_L = 0 \\ &(q^n_L, v^n_L)_{/t=0} = (J_n q_0, J_n u_0) \end{aligned} \tag{3.33}$$

where : $\mathcal{A} = \bar{\mu}\Delta + (\bar{\lambda} + \bar{\mu})\nabla \mathrm{div}$ and we set $(\bar{q}^n, \bar{u}^n) = (q^n - q^n_L, u^n - u^n_L)$.
Obviously, the definition of $(q^n_L, v^n_L)_{/t=0}$ entails :

$$(q^n_L)_{/t=0} \to q_0 \text{ in } \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}},\ (u^n_L)_{/t=0} \to u_0 \text{ in } \widetilde{B}^{\frac{N}{2}-1}.$$

The proposition 2.26 insures that (q^n_L, u^n_L) converges to the solution (q_L, u_L) of the linear system associated to (3.33) in $E^{\frac{N}{2}}$. We now have to prove the convergence of $(\bar{q}^n, \bar{u}^n)$. This is of course a trifle more difficult and requires compactness results. Let us first state the following lemma.

Lemma 5. *$(q^n, u^n))_{n\in\mathbb{N}}$ is uniformly bounded in $C^{\frac{1}{2}}(\mathbb{R}^+; B^{\frac{N}{2}-1}) \times (C^{\frac{1}{4}}(\mathbb{R}^+; B^{\frac{N}{2}-\frac{3}{2}}))^N$.*

Proof :

In all the proof, we will note u.b for uniformly bounded.
We first prove that $\frac{\partial}{\partial t}q^n$ is u.b in $L^2(\mathbb{R}^+, B^{\frac{N}{2}-1})$, which yields the desired result for q^n.
Let us observe that q^n verifies the following equation

$$\frac{\partial}{\partial t}q^n = \mathrm{div}u^n - J_n(u^n.\nabla q^n) - J_n(q^n \mathrm{div}u^n).$$

According to the first part, $(u_n)_{n\in\mathbb{N}}$ is u.b in $L^2(B^{\frac{N}{2}})$, so we can conclude that $\frac{\partial}{\partial t}q^n$ is u.b in $L^2(B^{\frac{N}{2}-1})$. Indeed we have :

$$\|J_n(q^n \mathrm{div}u^n)\|_{L^2(B^{\frac{N}{2}-1})} \leq \|u^n\|_{L^2(B^{\frac{N}{2}})}\|q^n\|_{L^\infty(B^{\frac{N}{2}})},$$
$$\|J_n(u^n.\nabla q^n)\|_{L^2(B^{\frac{N}{2}-1})} \leq \|u^n\|_{L^2(B^{\frac{N}{2}})}\|q^n\|_{L^\infty(B^{\frac{N}{2}})}.$$

And we recall that we use the fact that $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}} \hookrightarrow B^{\frac{N}{2}}$.
Let us prove now that $\frac{\partial}{\partial t}d^n$ is u.b in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}}) + L^4(B^{\frac{N}{2}-\frac{3}{2}})$ and that $\partial_t\Omega^n$ is u.b in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$ (which gives the required result for u^n in using the relation $u^n = -\Lambda^{-1}\nabla d^n - \Lambda^{-1}\mathrm{div}\Omega^n$).
Let us recall that :

$$\begin{aligned}\frac{\partial}{\partial t}d^n = J_n(u^n \cdot \nabla d^n) + J_n\Lambda^{-1}\mathrm{div}\big[\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + J_n(K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n)\big]\\ + \bar{\nu}\Delta d^n + \bar{\delta}\Lambda q^n - \bar{\kappa}\phi * \Lambda q^n,\end{aligned}$$

$$\frac{\partial}{\partial t}\Omega^n = J_n\Lambda^{-1}\mathrm{curl}\big[\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + J_n(K(\varphi(\bar{\rho}(1+q^n))\nabla q^n))\big] + \bar{\mu}\Delta\Omega^n.$$

Results of step one and an interpolation argument yield uniform bounds for u^n in $L^\infty(B^{\frac{N}{2}-1}) \cap L^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}})$, we infer in proceeding as for $\frac{\partial}{\partial t}q^n$ that :

$$\begin{aligned}A_n = J_n(u^n \cdot \nabla d^n) + J_n\Lambda^{-1}\mathrm{div}\big[\mathcal{A}(\varphi(\bar{\rho}(1+q^n)), u^n) + J_n(K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n)\big] + \bar{\nu}\Delta d^n\\ \text{is u.b in } L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}}).\end{aligned}$$

Using the bounds for q^n in $L^2(B^{\frac{N}{2}}) \cap L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}})$, we get q^n u.b in $L^4(B^{\frac{N}{2}-\frac{1}{2}})$ in using proposition 2.23. We thus have $J_n(K(\varphi(\bar{\rho}(1+q^n))\nabla q^n$ u.b in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$.
Using the bounds for u^n in $L^\infty(B^{\frac{N}{2}-1}) \cap L^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}})$ we finally get A_n is u.b in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$.
To conclude $\phi * \Lambda q^n$ is u.b in $L^4(B^{\frac{N}{2}-\frac{3}{2}})$, so $\frac{\partial}{\partial t}d^n$ is u.b in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}}) + L^4(B^{\frac{N}{2}-\frac{3}{2}})$.

The case of $\frac{\partial}{\partial t}\Omega^n$ goes along the same lines. As the terms corresponding to Λq^n and $\phi * \Lambda \bar{q}^n$ do not appear, we simply get $\partial_t\Omega^n$ u.b in $L^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{3}{2}})$.

□

We can now turn to the proof of the existence of a solution and using Ascoli theorem to get strong convergence. We proceed similarly to the theorem of Aubin-Lions.

Theorem 3.24. *Let X a compact metric space and Y a complete metric space. Let A be an equicontinuous part of $C(X,Y)$. Then we have the two equivalent proposition :*

1. *A is relatively compact in $C(X,Y)$*
2. *$A(x)=\{f(x);\ \ f\in A\}$ is relatively compact in Y*

We need to localize because we have some result of compactness for the local Sobolev space. Let $(\chi_p)_{p\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^N)$ cut-off functions supported in the ball $B(0,p+1)$ of $\mathbb{R}^N$ and equal to 1 in a neighborhood of $B(0,p)$.
For any $p\in\mathbb{N}$, lemma 5 tells us that $((\chi_p q^n,\chi_p u^n))_{n\in\mathbb{N}}$ is uniformly equicontinuous in $C(\mathbb{R}^+;B^{\frac{N}{2}-1}\times(B^{\frac{N}{2}-\frac{3}{2}})^N)$. In using Ascoli's theorem we just need to show that $((\chi_p q^n(t,\cdot),\chi_p u^n)(t,\cdot))_{n\in\mathbb{N}}$ is relatively compact in $B^{\frac{N}{2}-1}\times(B^{\frac{N}{2}-\frac{3}{2}})^N\ \forall t\in[0,p]$.
Let us observe now that the application $u\rightarrow\chi_p u$ is compact from $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}=B^{\frac{N}{2}}\cap B^{\frac{N}{2}-1}$ into $\dot{H}^{\frac{N}{2}-1}$, and from $B^{\frac{N}{2}-1}\cap B^{\frac{N}{2}-\frac{3}{2}}$ into $\dot{H}^{\frac{N}{2}-\frac{3}{2}}$.
After we apply Ascoli's theorem to the family $((\chi_p q^n,\chi_p u^n))_{n\in\mathbb{N}}$ on the time interval $[0,p]$. We then use Cantor's diagonal process.This finally provides us with a distribution (q,u) belonging to $C(\mathbb{R}^+;\dot{H}^{\frac{N}{2}-1}\times(\dot{H}^{\frac{N}{2}-\frac{3}{2}})^N)$ and a subsequence (which we still denote by $(q^n,u^n)_{n\in\mathbb{N}}$ such that, for all $p\in\mathbb{N}$, we have :

$$(\chi_p q^n,\chi_p u^n)\rightarrow_{n\mapsto+\infty}(\chi_p q,\chi_p u)\text{ in }C([0,p];\,\dot{H}^{\frac{N}{2}-1}\times(\dot{H}^{\frac{N}{2}-\frac{3}{2}})^N)\tag{3.34}$$

This obviously entails that (q^n,u^n) tends to (q,u) in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^N)$.

Coming back to the uniform estimates of step one, we moreover get that (q,u) belongs to :

$$L^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}\times(B^{\frac{N}{2}+1})^N)\cap L^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}\times(B^{\frac{N}{2}+1})^N)$$

and to $C^{\frac{1}{2}}(\mathbb{R}^+;B^{\frac{N}{2}-1})\times(C^{\frac{1}{4}}(\mathbb{R}^+;B^{\frac{N}{2}-\frac{3}{2}})^N)$. Obviously, we have the bounds provided of the firts step.
Let us now prove that (q,u) solves the system (SW), we first recall that (q^n,u^n) solves the following system :

$$\begin{cases}\partial_t q^n+J_n(u^n\cdot\nabla q^n)+\text{div}u^n=-J_n(q^n\text{div}u^n)\\ \partial_t u^n-\bar{\nu}\Delta u^n+\bar{\delta}\nabla q^n-\bar{\kappa}\phi*\nabla q^n+J_n(u^n\cdot\nabla u^n)+J_n(K(\varphi(\bar{\rho}(1+q^n))\nabla q^n)\\ \qquad\qquad\qquad\qquad+J_n(\mathcal{A}(\varphi(\bar{\rho}(1+q^n)),u^n))=0\end{cases}$$

The only problem is to pass to the limit in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^N)$ in the non linear terms. This can be done by using the convergence results coming from the uniform estimates (3.34).
As it is just a matter of doing tedious verifications, we show as a example the case of the term $J_n(K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n)$ and $J_n(\mathcal{A}(\varphi(\bar{\rho}(1+q^n)),u^n))$.
We decompose :

$$J_n(K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n)-K(\rho^n)\nabla q^n=J_n(K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n)-K(\varphi(\bar{\rho}(1+q)))\nabla q.$$

(Note that for n big enough, we have $K(\varphi(\bar{\rho}(1+q^n)))=K(\rho^n)$ as we control $\|\rho^n\|_{L^\infty}$ and $\|\frac{1}{\rho^n}\|_{L^\infty}$). Next we have :

$$J_n(K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n)-K(\varphi(\bar{\rho}(1+q)))\nabla q=J_nA_n+(J_n-I)K(\varphi(\bar{\rho}(1+q)))\nabla q,$$

$$\text{where}\quad A_n=K(\varphi(\bar{\rho}(1+q^n)))\nabla q^n-K(\varphi(\bar{\rho}(1+q)))\nabla q.$$

We have then $(J_n-I)K(\varphi(\bar{\rho}(1+q)))\nabla q$ tends to zero as $n\rightarrow+\infty$ due to the property of J_n and the fact that $K(\varphi(\bar{\rho}(1+q)))\nabla q$ belongs to $L^\infty(B^{\frac{N}{2}-1})\hookrightarrow L^\infty(L^q)$ for some $q\geq 2$. Choose $\psi\in C_0^\infty([0,T)\times\mathbb{R}^N)$ and $\varphi^{'}\in C_0^\infty([0,T)\times\mathbb{R}^N)$ such that $\varphi^{'}=1$ on $\operatorname{supp}\psi$, we have :

$$|<(J_n-I)K(\varphi(\bar{\rho}(1+q)))\nabla q,\psi>|\leq\|\varphi^{'}K(\varphi(\bar{\rho}(1+q)))\nabla q\|_{L^\infty(L^2)}\|(J_n-I)\psi\|_{L^2},$$

because $L^q_{loc}\hookrightarrow L^2_{loc}$ and we conclude by the fact that $\|(J_n-I)\psi\|_{L^2}\rightarrow 0$ as n tends to $+\infty$.

Next :

$$<J_nA_n,\psi>=I_n^1+I_n^2,$$

with :

$$\begin{aligned}I_n^1&=<(K(\varphi(\bar{\rho}(1+q^n)))-K(\varphi(\bar{\rho}(1+q))))\nabla q^n,J_n\psi>,\\ I_n^2&=<K(\varphi(\bar{\rho}(1+q)))\nabla(q^n-q),J_n\psi>.\end{aligned}$$

We have then :

$$I_n^1\leq\|\varphi^{'}q^n\|_{L^\infty(B^{\frac{N}{2}})}\|\varphi^{'}(q^n-q)\|_{L^\infty(\dot{H}^{\frac{N}{2}-1})}\|\psi\|_{L^\infty},$$

Indeed we just use the fact that $\varphi^{'}B^{\frac{N}{2}-1}$ and $\varphi^{'}\dot{H}^{\frac{N}{2}-1}$ are embedded in L^2. Next we conclude as we have seen that $q^n\rightarrow_{n\rightarrow+\infty}q$ in $C_{loc}(H_{loc}^{\frac{N}{2}-1})$. So we obtain :

$$I_n^1\rightarrow_{n\rightarrow+\infty}0\quad\text{in}\quad\mathcal{D}^{'}((0,T^*)\times\mathbb{R}^N).$$

We proceed similarly for I_n^2, indeed we have :

$$I_n^2=<\varphi^{'}(q^n-q),\varphi^{'}\operatorname{div}(K(\varphi(\bar{\rho}(1+q)))J_n\psi)>$$

and we have $K(\varphi(\bar{\rho}(1+q)))J_n\psi\in L^\infty(B^{\frac{N}{2}})$ so :

$$I_n^2\leq\|\varphi^{'}(q^n-q)\|_{L^\infty(\dot{H}^{\frac{N}{2}-1})}\|K(\varphi(\bar{\rho}(1+q)))J_n\psi\|_{L^\infty(B^{\frac{N}{2}})}.$$

We conclude then that :

$$I_n^2\rightarrow_{n\rightarrow+\infty}0\quad\text{in}\quad\mathcal{D}^{'}((0,T^*)\times\mathbb{R}^N).$$

We concentrate us now on the term $J_n(\mathcal{A}(\varphi(\bar{\rho}(1+q^n)),u^n))$. Let $\varphi^{'}\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^N)$ and $p\in\mathbb{N}$ be such that $\operatorname{supp}\varphi^{'}\subset[0,p]\times B(0,p)$. We use the decomposition for n big enough :

$$\begin{aligned}\varphi^{'}J_n\mathcal{A}(\varphi(\bar{\rho}(1+q^n)),u^n)-\varphi^{'}\mathcal{A}(\rho,u)&=\varphi^{'}\chi_p\mathcal{A}(\varphi(\bar{\rho}(1+q^n)),\chi_p(u^n-u))\\&\quad+\varphi^{'}\mathcal{A}(\chi_p\varphi(\bar{\rho}(1+q^n))-\chi_p\bar{\rho}(1+q)),u).\end{aligned}$$

According to the uniform estimates and (3.34), $\chi_p(u^n-u)$ tends to 0 in $L^1([0,p];\dot{H}^{\frac{N}{2}+1})$ by interpolation so that the first term tends to 0 in $L^1(\dot{H}^{\frac{N}{2}-1})$ and we conclude for the second term in remarking that $\frac{\varphi}{\rho_n}$ tends to $\frac{\varphi}{\rho}$ as ρ_n in $L^\infty(L^\infty\cap\dot{H}^{\frac{N}{2}})$.

The other nonlinear terms can be treated in the same way.

3.3 Proof of the uniqueness in the critical case

Theorem 3.25. *Let $N \geq 2$, and (q_1, u_1) and (q_2, u_2) be solutions of (SW) with the same data (q_0, u_0) on the time interval $[0, T^*)$. Assume that for $i = 1, 2$:*

$$(q_i, u_i) \in C([0, T^*), B^1_{N,1}) \quad \text{and} \quad u_i \in \big(C([0, T^*), B^0_{N,1}) \cap L^1_{loc}([0, T^*), B^2_{N,1})\big)^N.$$

There exists a constant $\alpha > 0$ depending only on N and physical constants such that if :

$$\|q_1\|_{\widetilde{L}^\infty_{T^*}(B^1_{N,1})} \leq \alpha, \tag{3.35}$$

then $(q_1, u_1) = (q_2, u_2)$ on $[0, T^*)$.

Let (q_1, u_1), (q_2, u_2) belong to $E^{\frac{N}{2}}$ with the same initial data, we set $(\delta q, \delta u) = (q_2 - q_1, u_2 - u_1)$. We can then write the system (SW) as follows :

$$\begin{cases} \dfrac{\partial}{\partial t}\delta q + u_2 \cdot \nabla \delta q = H_1, \\ \dfrac{\partial}{\partial t}\delta u - \bar{\nu}\Delta \delta u = H_2 \end{cases} \tag{3.36}$$

with :

$$H_1 = -\mathrm{div}\delta u - \delta u \cdot \nabla q_1 - \delta q \mathrm{div} u_2 - q_1 \mathrm{div} u,$$

$$H_2 = -\bar{\delta}\nabla \delta q - \bar{\kappa}\phi * \nabla \delta q - u_2 \cdot \nabla \delta u - \delta u \cdot \nabla u_1 + \mathcal{A}(q_1, \delta u) + \mathcal{A}(\delta q, u_2).$$

Due to the term $\delta u \cdot \nabla q^1$ in the right-hand side of the first equation, we loose one derivative when estimating δq : one only gets bounds in $L^\infty(B^0_{N,1})$.
Now, the right hand-side of the second equation contains a term of type $\mathcal{A}(\delta q, u_2)$ so that the loss of one derivative for δq entails a loss of one derivative for δu. Therefore, getting bounds in :

$$C(\mathbb{R}^+; B^{-1}_{N,1}) \cap L^1(\mathbb{R}^+; B^1_{N,1})$$

for δu is the best that one can hope. If enough regularity were available, we would not have to worry about this loss of derivative. But in the present case, the above heuristic fails because we have reached some limit cases for the product laws. Indeed, a term such as $\delta u \cdot \nabla u_1$ cannot be estimated properly : the product does not map $B^0_{N,1} \times B^0_{N,1}$ into $B^{-1}_{N,1}$ but in the somewhat *larger space* $B^{-1}_{N,\infty}$. At this point, we could try instead to get bounds for δu in :

$$C([0, T^*); B^{-1}_{N,\infty}) \cap L^1_{loc}([0, T^*); B^1_{N,\infty}),$$

but we then have to face the lack of control on δu in $L^1(0, T; L^\infty)$ (because in contrast with $B^1_{N,1}$, the space $B^1_{N,\infty}$ is not imbedded in L^∞) so that we run into troubles when estimating $\delta u \cdot \nabla q_1$. The key to that difficulty relies on the following logarithmic interpolation inequality (see the proposition 2.22) :

$$\|u\|_{L^1_T(B^1_{N,1})} \lesssim \|u\|_{\widetilde{L}^1_T(B^1_{N,\infty})} \log\left(e + \frac{\|u\|_{\widetilde{L}^1_T(B^0_{N,\infty})} + \|u\|_{\widetilde{L}^1_T(B^2_{N,\infty})}}{\|u\|_{\widetilde{L}^1_T(B^1_{N,\infty})}}\right),$$

and a well-known generalization of Grönwall the Osgood's lemma (see [14]) that we recall.

Lemma 6. *Let F be a measurable positive function and γ a positive locally integrable function, each defined on the domain $[t_0, t_1]$. Let $\mu : [0,+\infty) \to [0,+\infty)$ be a continuous nondecreasing function, with $\mu(0)=0$. Let $a \geq 0$, and assume that for all $t \in [t_0, t_1]$,*

$$F(t) \leq a + \int_{t_0}^{t} \gamma(s)\mu(F(s))ds.$$

If $a > 0$, then :

$$-\mathcal{M}(F(t)) + \mathcal{M}(a) \leq \int_{t_0}^{t} \gamma(s)ds, \quad \textit{where } \mathcal{M}(x) = \int_x^1 \frac{ds}{\mu(s)}.$$

If $a = 0$ and $\mathcal{M}(0) = +\infty$, then $F = 0$.

Proof of the theorem 3.25 :

First step : in which space do we work ?

Let us observe first that in view of Sobolev embedding, $q_i \in C(\mathbb{R}^+; L^\infty)$. Therefore, if α is small enough, by embedding and continuity we get :

$$|q_i(t,x)| \leq \frac{1}{2}$$

for $x \in \mathbb{R}^N$ and t in a small nontrivial time interval $[0,T]$.
That observation will enable us to apply proposition 2.24 to the non-linear terms involving q_i.
We shall further assume that $T \in (0,+\infty)$ has been chosen so small as to satisfy :

$$C\|\nabla u_2\|_{L^1_T(B^1_{N,1})} \leq \log 2, \tag{3.37}$$

for some appropriate constant C whose meaning will be clear from the computations below.
To begin with, we shall prove uniqueness on the time interval $[0,T]$ by estimating $(\delta q, \delta u)$ in the following functional space :

$$F_T = L^\infty([0,T]; B^0_{N,\infty}) \times (L^\infty([0,T]; B^{-1}_{N,\infty}) \cap \widetilde{L}^1_T(B^1_{N,\infty}))^N.$$

Indeed as explained below, in this space we can control the remainder because it is appropriate to the result of paraproduct.

Why $(\delta q, \delta u)$ is in F_T ?

Of course, we have to state that $(\delta q, \delta u) \in F_T$, a fact which is not entirely obvious. We want now to show that $(\delta q, \delta u)$ belongs to F_T.
According to our assumption on (q_i, u_i), the estimates of paraproduct yield $\partial_t q_i \in L^2_T(B^0_{N,1})$.
Therefore $\bar{q}_i = q_i - q_0$ belongs to $C^{\frac{1}{2}}([0,T], B^0_{N,1})$, which clearly entails by embedding $\delta q \in C([0,T], B^0_{N,\infty})$.
Let $\bar{u}_i = u_i - u_L$ with u_L solution to the following linear heat equation :

$$\begin{cases} \partial_t u_L - \mu\Delta u_L = -\bar{\delta}\nabla q_0 + \bar{\kappa}\nabla(\phi * q_0), \\ u_L(0) = u_0. \end{cases}$$

We obviously have $(\bar{u}_i)_0 = 0$ and :

$$\partial_t \bar{u}_i - \bar{\nu}\Delta \bar{u}_i = -\bar{u}_i \cdot \nabla \bar{u}_i - \bar{\delta}\nabla q_i + \bar{\kappa}\nabla(\phi * q_i) + \mathcal{A}(\rho_i, q_i) + K(\rho_i, u_i).$$

The product and composition laws in Besov spaces insure that the right-hand side belongs to $L^2_T(B^{-1}_{N,\infty})$ (because $B^0_{N,1} \times B^0_{N,\infty} \hookrightarrow B^{-1}_{N,\infty}$) thus to $\widetilde{L}^1_T(B^{-1}_{N,\infty})$) (for the last term, we use that $\bar{q}_i \in L^\infty_T(B^0_{N,1})$.
Now Proposition 2.26 implies that :

$$\bar{u}_i \in L^\infty_T(B^{-1}_{N,\infty}) \cap \widetilde{L}^1_T(B^1_{N,\infty})).$$

Second step : Estimates on $(\delta q, \delta u)$

Let us turn to estimate δq. Proposition 2.25 combined with (3.37) yields for $t \leq T$:

$$\|\delta q\|_{L^\infty_t(B^0_{N,\infty})} \lesssim \int_0^t \big(\|\delta u \cdot \nabla q\|_{B^0_{N,\infty}} + \|\delta q \;\text{div} u_2\|_{B^0_{N,\infty}} + \|\text{div}\delta u\|_{B^0_{N,\infty}}\big)\big) d\tau$$

Estimate of type $B^{\frac{N}{p}}_{p,\infty} \cap L^\infty \times B^s_{p,\infty} \hookrightarrow B^s_{p,\infty}$ with $s + \frac{N}{p} > 0$ enables us to get the following inequality :

$$\|\delta q\|_{L^\infty_t(B^0_{N,\infty})} \lesssim \int_0^t \big(\|\delta q\|_{B^0_{N,\infty}} \|\text{div} u_2\|_{B^1_{N,\infty} \cap L^\infty} + \|\delta u\|_{B^1_{N,\infty} \cap L^\infty}(1 + \|q_1\|_{B^1_{N,1}}\big) d\tau,$$

whence, according to Gronwall inequality, to the embedding $B^1_{N,1} \hookrightarrow B^1_{N,\infty} \cap L^\infty$ and to (3.35) we get :

$$\|\delta q\|_{L^\infty_t(B^0_{N,\infty})} \lesssim \|\delta u\|_{L^1_t(B^1_{N,1})}(1 + \|q_1\|_{L^\infty_t(B^1_{N,1})}).$$

Making use of (3.35) and proposition 2.21, we end up with :

$$\|\delta q\|_{L^1_t(B^0_{N,\infty})} \lesssim \|\delta u\|_{\widetilde{L}^1_t(B^1_{N,\infty})} \log\left(e + \frac{\|\delta u\|_{\widetilde{L}^1_t(B^0_{N,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(B^2_{N,\infty})}}{\|\delta u\|_{\widetilde{L}^1_t(B^1_{N,\infty})}}\right).$$

Remark that :

$$\|\delta u\|_{\widetilde{L}^1_t(B^0_{N,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(B^2_{N,\infty})} \leq V(t) = V_1(t) + V_2(t)$$

with :

$$V_i(t) = \int_0^t (\|u_i(\tau)\|_{B^0_{N,1}} + \|u_i(\tau)\|_{B^2_{N,1}}) d\tau < +\infty$$

since $\widetilde{L}^\infty_t(B^0_{N,1}) \hookrightarrow \widetilde{L}^1_t(B^0_{N,1})$ for finite t.
We finally get :

$$\|\delta q\|_{L^1_t(B^0_{N,\infty})} \lesssim \|\delta u\|_{\widetilde{L}^1_t(B^1_{N,\infty})} \log\left(e + \frac{V(t)}{\|\delta u\|_{\widetilde{L}^1_t(B^1_{N,\infty})}}\right), \tag{3.38}$$

with V non-decreasing bounded function of $t \in [0, +\infty)$.

Let us now turn to the proof of estimates for δu. According to proposition 2.26, we have :

$$\|\delta u\|_{L_t^\infty(B^{-1}_{N,\infty})} + \|\delta u\|_{L_t^1(B^1_{N,\infty})} \lesssim \|u_2 \cdot \nabla \delta u\|_{\widetilde{L}_t^1(B^{-1}_{N,\infty})} + \|\mathcal{A}(q_1, \delta u)\|_{\widetilde{L}_t^1(B^{-1}_{N,\infty})}$$
$$+ \|\delta u \cdot \nabla u_1\|_{\widetilde{L}_t^1(B^{-1}_{N,\infty})} + \|\mathcal{A}(\delta q, u_2)\|_{\widetilde{L}_t^1(B^{-1}_{N,\infty})} + \|K(\delta q)\nabla q_2\|_{\widetilde{L}_t^1(B^{-1}_{N,\infty})}$$
$$+ \|K(q_2)\nabla \delta q\|_{\widetilde{L}_t^1(B^{-1}_{N,\infty})}.$$

Let us assume that the α appearing in (3.35) is small enough so that the second term in the right-hand side may be absorbed by the left-hand side. $\|u_2\|_{\widetilde{L}_t^2(B^1_{N,1})}$ tends to 0 when t goes to 0, so if we choose T small enough, the first term may also be absorbed. Using interpolation of proposition 2.21, we obtain for all $t \in [0, T]$,

$$\|\delta u\|_{L_t^\infty(B^{-1}_{N,\infty})} + \|\delta u\|_{L_t^1(B^1_{N,\infty})} \lesssim \int_0^t \left[\|u_1\|_{B^2_{N,1}} \|\delta u\|_{B^{-1}_{N,\infty}} + (1 + \|u_2\|_{B^2_{N,1}})\|\delta q\|_{B^0_{N,\infty}}\right] d\tau$$

Let us now plug (3.38) in the above inequality. Denoting :

$$X(t) = \|\delta u\|_{L_t^\infty(B^{-1}_{N,\infty})} + \|\delta u\|_{L_t^1(B^1_{N,\infty})}$$

we get for $t \leq T$,

$$X(t) \lesssim \int_0^t \big(1 + V'(\tau)\big) X(\tau) \log\left(e + \frac{V(\tau)}{X(\tau)}\right) d\tau$$

As :

$$V' \in L^1(0, T) \quad \text{and} \quad \int_0^1 \frac{dr}{r \log(e + \frac{V(T)}{r}} = +\infty,$$

Osgood's lemma (see lemma 6) implies that $X = 0$ on $[0, T]$, whence also $\delta q = 0$. Standard arguments of connexity then yield uniqueness on the whole interval $[0, +\infty)$.

4 Proof of theorem 1.22

We will proceed similarly to the proof of the theorem 1.21. To begin with, let us observe that under the definition 1.17, system reads :

$$\begin{cases} \partial_t q + u \cdot \nabla u = H \\ \partial_t u - \bar{\nu}\Delta u = -K \\ (q, u)_{t=0} = (q_0, u_0) \end{cases} \tag{4.39}$$

with :

$$H = -(1 + q)\text{div} u,$$
$$K = G - u \cdot \nabla u + (P'(\bar{\rho}) + \kappa)\nabla\rho - \kappa\phi * \nabla\rho.$$

As previously we can build approximate smooth solutions (q^n, u^n) of (4.40) in studying the Korteweg system with a capillarity coefficient $\kappa_n = \frac{1}{n}$. It is convenient to split (q^n, u^n) into the solution of a linear system with initial data (q_0^n, u_0^n), and the discrepancy to that solution. More precisely we denote by (q_L^n, u_L^n) the solution of the linearized pressure-less system on the intervall $[0,T]$:

$$\begin{cases} \partial_t q_L^n + \mathrm{div} u_L^n = 0, \\ \partial_t u_L^n - \bar{\mu}\Delta u_L^n - (\bar{\lambda}+\bar{\mu})\nabla \mathrm{div} u_L^n = 0, \\ (q_L^n, u_L^n)_{/t=0} = (q_0^n, u_0^n), \end{cases}$$

with :

$$(q_0^n, u_0^n) = (\sum_{|l|\leq n} \Delta_l q_0, \sum_{|l|\leq n} \Delta_l u_0).$$

We set :

$$(q^n, u^n) = (q_L^n + \bar{q}^n, u_L^n + \bar{u}^n)$$

We can state now that $(q^n, \bar{u}^n)$ verifies the following linear system :

$$\begin{cases} \partial_t \bar{q}^n + \mathrm{div} \bar{u}^n = F^n, \\ \partial_t \bar{u}^n - \bar{\mu}\Delta \bar{u}^n - (\bar{\lambda}+\bar{\mu})\nabla \mathrm{div} \bar{u}^n - \dfrac{1}{n}\nabla\Delta \bar{q}^n = G^n, \\ (\bar{q}^n, \bar{u}^n)_{/t=0} = (0,0) \end{cases} \tag{4.40}$$

with :

$$\begin{aligned} F^n &= -q^n \mathrm{div} u^n, \\ G^n &= -u^n\cdot\nabla u^n + \mathcal{A}(\rho^n, u^n) - K(\rho^n)\nabla q^n + (P'(\bar{\rho}) + \kappa)\nabla\rho^n - \kappa\phi * \nabla\rho^n. \end{aligned}$$

We want show that such solution (q^n, u^n) exists, in this goal we recall some theorem by R. Danchin and B. Desjardins in [19].

Theorem 4.26. *Let $p \in [1,+\infty[$. Then there exists $\eta > 0$ such that if $q_0 \in B^{\frac{N}{p}}$, $u_0 \in (B^{\frac{N}{p}-1})^N$ and :*

$$\|q_0\|_{B^{\frac{N}{p}}} \leq \eta,$$

then there exists $T > 0$ such that system (4.40) has a unique solution (q,u) in $\widetilde{E}_T^{p,\kappa}$.

In fact, we can extend this result to the case where the viscosity coefficients are variable. And we can show that the uniform estimates are independent of the capillarity coefficient. We can obtain the following result on our solution $(q^n, u^n)_{n\in\mathbb{N}}$.

Theorem 4.27. *Let $p \in [1,+\infty[$. Then there exists $\eta > 0$ such that if $q_0^n \in B^{\frac{N}{p}}$, $u_0^n \in (B^{\frac{N}{p}-1})^N$ and :*

$$\|q_0^n\|_{B^{\frac{N}{p}}} \leq \eta,$$

then there exists $T > 0$ such that system (4.40) has a unique solution (q^n, u^n) in $\widetilde{E}_T^{p,\frac{1}{n}}$ and (q^n, u^n) are uniformly bounded in F_T^p.

Proof :

We recall a proposition (see [19]) on the following linearized pressure-less system :

$$\begin{aligned}&\partial_t q+\mathrm{div}u=F,\\&\partial_t u-\bar{\mu}\Delta u-(\bar{\lambda}+\bar{\mu})\nabla\mathrm{div}u-\bar{\kappa}\nabla\Delta q=G.\end{aligned}\tag{4.41}$$

Proposition 4.29. *Let $s\in\mathbb{R}$, $p\in[1,+\infty]$, $1\leq\rho_1\leq+\infty$ and $T\in]0,+\infty]$. If $(q_0,u_0)\in B^s_p\times(B^{s-1}_p)^N$ and $(F,G)\in\widetilde{L}^{\rho_1}_T(B_p^{s-2+\frac{2}{\rho_1}}\times(B_p^{s-3+\frac{2}{\rho_1}})^N)$, then the above linear system has a unique solution $(q,u)\in\widetilde{C}_T(B^s_p\times(B^{s-1}_p)^N)\cap\widetilde{L}^{\rho_1}_T(B_p^{s+\frac{2}{\rho_1}}\times(B_p^{s-1+\frac{2}{\rho_1}})^N)$.*
Moreover for all $\rho\in[\rho_1,+\infty]$, there exists a constant C depending only on $\bar{\mu}$, $\bar{\lambda}$, $\bar{\kappa}$, p, ρ_1 and N such that the following inequality holds :

$$\begin{aligned}\|q\|_{\widetilde{L}^{\rho}_T(B_p^{s+\frac{2}{\rho}})}+\|u\|_{\widetilde{L}^{\rho}_T(B_p^{s-1+\frac{2}{\rho}})}\leq C(&\|q_0\|_{B^s_p}+\|u_0\|_{B^{s-1}_p}\\&+\|F\|_{\widetilde{L}^{\rho_1}_T(B_p^{s-2+\frac{2}{\rho_1}})}+\|G\|_{\widetilde{L}^{\rho_1}_T(B_p^{s-3+\frac{2}{\rho_1}})}).\end{aligned}$$

Remark 8. *More precisely we have :*

$$\|q\|_{\widetilde{L}^{\infty}_T(B^s_p)}+\kappa\|q\|_{\widetilde{L}^{1}_T(B^{s+2}_p)}\leq C(\|q_0\|_{B^s_p}+\|u_0\|_{B^{s-1}_p}+\|F\|_{\widetilde{L}^{1}_T(B^s_p)}+\|G\|_{\widetilde{L}^{1}_T(B^{s-1}_p)}).$$

Uniform Estimates for $(q^n,u^n)_{n\in\mathbb{N}}$

The existence of solutions in the case of general viscosity coefficient follow the same line as the proof in [19]. It suffices to solve a problem of fixed point.
Denoting by $V(t)$ the semi-group generated by system (4.41), we have :

$$(q^n_L,u^n_L)(t)=V(t)(q^n_0,u^n_0).$$

Let us define :

$$\phi_{q^n_L,u^n_L}(\bar{q}^n,\bar{u}^n)=\int_0^t V(t-s)(F(q^n_L+\bar{q}^n,u^n_L+\bar{u}^n)(s),G(q^n_L+\bar{q}^n,u^n_L+\bar{u}^n)(s))\,ds.$$

where we have set :

$$\begin{aligned}&F(q,u)=-\mathrm{div}(qu),\\&G(q,u)=-u\cdot\nabla u++\mathcal{A}(\rho,u)-K(\rho)\nabla q+(P^{'}(\bar{\rho})+\kappa)\nabla\rho-\kappa\phi*\nabla\rho.\end{aligned}$$

In order to prove the existence part of the theorem 4.27, ther's just have to show that $\phi_{q^n_L,u^n_L}$ has a fixed point in F^p_T. Since F^p_T is a Banach space, we can prove that $\phi_{q^n_L,u^n_L}$ satisfies the hypothesis of Picard's theorem in a ball $(B(0,R)$ of F^p_T for sufficiently small R. Moreover R depends only of mathematical constants. We can find a time T independent of n such that for all initial data verifying $\|q^n_0\|_{B^{\frac{N}{p}}}\leq\frac{R}{2}$, we have existence of solution $(q^n,u^n$ at less on the interval $(0,T)$. The end of proof consists to verify that (q^n,u^n) is uniform in F^p_T, it suffices to use the proposition 4.29.
We now have builded approximated solution (q^n,u^n) of system (SW) and we can conclude in using technics of compactness.

2) Existence of a solution

The existence of a solution stems from compactness properties for the sequence $(q^n,u^n)_{n\in\mathbb{N}}$ an we want use some result of type Ascoli as in the proof of theorem 1.21.

Lemma 7. *The sequence* $(\partial_t\bar{q}^n,\partial_t\bar{u}^n)_{n\in\mathbb{N}}$ *is uniformly bounded in :*

$$L^2(0,T;\widetilde{B}^{\frac{N}{p},\frac{N}{p}-1})\times(L^\alpha(0,T;\widetilde{B}^{\frac{N}{p}-1,\frac{N}{p}-2}))^N,$$

for some $\alpha>1$.

Proof :

Throughout the proof, we will extensively use that $\widetilde{L}^\rho_T(B^s_p)\hookrightarrow L^\rho_T(B^s_p)$. The notation u.b will stand for uniformly bounded.
We have :

$$\begin{aligned}&\partial_t q^n=-u^n\cdot\nabla q^n-(1+q^n)\mathrm{div}u^n,\\&\partial_t\bar{u}^n=-u^n\cdot\nabla u^n-q^n\mathcal{A}(\rho^n,\bar{u}^n)-K(q^n)\nabla q^n+\frac{1}{n}\nabla\Delta\bar{q}^n.\end{aligned}\tag{4.42}$$

We start with show that $\partial_t\bar{q}^n$ is u.b in $L^2(0,T;\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}-1})$.
Since u^n is u.b in $L^2_T(B^{\frac{N}{p}})$ and ∇q^n is u.b in $L^\infty_T(B^{\frac{N}{p}-1})$, then $u^n\cdot\nabla q^n$ is u.b in $L^2_T(\widetilde{B}^{\frac{N}{p},\frac{N}{p}-1})$. Similar arguments enable us to conclude for the term $(1+q^n)\mathrm{div}u^n$ which is u.b in $L^2_T(\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}-1})$ because q^n is u.b in $L^\infty_T(B_p^{\frac{N}{p}})$ and $\mathrm{div}u^n$ is u.b in $L^2_T(B_p^{\frac{N}{p}-1})$.

Let us now study $\partial_t\bar{u}^{n+1}$. According to step one and to the definition of u^n_L, the term $\mathcal{A}\bar{u}^{n+1}$ is u.b in $L^2(B_p^{\frac{N}{p}-2})$. Since u^n is u.b in $L^\infty(B_p^{\frac{N}{p}-1})$ and ∇u^n is u.b in $L^2(B_p^{\frac{N}{p}-1})$, so $u^n\cdot\nabla u^n$ is u.b in $L^2(B_p^{\frac{N}{p}-2})$ thus in $L^2(\widetilde{B}_p^{\frac{N}{p}-1,\frac{N}{p}-2})$.
Moreover we have q^n is u.b in $L^\infty(B_p^{\frac{N}{p}})$ and q^n is u.b in L^∞, so by proposition 2.24 $\nabla K_0(q^n)$ is u.b in $L^\infty(B_p^{\frac{N}{p}-1})$ thus in $L^2(\widetilde{B}_p^{\frac{N}{p}-1,\frac{N}{p}-2})$. This concludes the lemma. □

Now, let us turn to the proof of the existence of a solution for the system (SW). We want now use some results of type Ascoli to conclude in use the properties of compactness of the lemma 7.
According lemma 7, $(q^n,u^n)_{n\in\mathbb{N}}$ is u.b in :

$$C^{\frac{1}{2}}([0,T];\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}-1})\times(C^{1-\frac{1}{\alpha}}([0,T];\widetilde{B}_p^{\frac{N}{p}-1,\frac{N}{p}-2}))^N,$$

thus is uniformly equicontinuous in $C(([0,T];\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}-1})\times(\widetilde{B}_p^{\frac{N}{p}-1,\frac{N}{p}-2})^N)$. On the other hand we have the following result of compactness, for any $\phi\in C_0^\infty(\mathbb{R}^N)$, $s\in\mathbb{R}$, $\delta>0$ the application $u\rightarrow\phi u$ is compact from B^s_p to $\widetilde{B}_p^{s,s-\delta}$. Applying Ascoli's theorem, we infer that up to an extraction $(q^n,u^n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}'([0,T]\times\mathbb{R}^N)$ to a limit $(\bar{q},\bar{u})$ which belongs to :

$$C^{\frac{1}{2}}([0,T];\widetilde{B}_p^{\frac{N}{p},\frac{N}{p}-1})\times(C^{1-\frac{1}{\alpha}}([0,T];\widetilde{B}_p^{\frac{N}{p}-1,\frac{N}{p}-2}))^N$$

Let $(q,u)=(\bar q,\bar u)+(q_0,u_L)$. Using again uniform estimates of step one and proceeding as, we gather that (q,u) solves (SW) and belongs to :

$$\bar\rho+\widetilde L^\infty_T(\widetilde B_p^{\frac Np,\frac Np-1})\times\big(\widetilde L^1_T(B_p^{\frac Np+1})\cap\widetilde L^\infty_T(B_p^{\frac Np-1})\big)^N.$$

Applying proposition, we get the continuity results :

$$\rho-\bar\rho\in C([0,T],\widetilde B_p^{\frac Np,\frac Np-1}),\quad u\in C([0,T],B_p^{\frac Np-1}).$$

5 Proof of theorem 1.23

In this section, we consider the case when the initial density belongs to $\bar\rho+\widetilde B^{\frac Np,\frac Np+\varepsilon}$ and satisfies $0<\bar\rho\le\rho_0$. We consider again the study of a approximate sequence verifying (4.40) and we proceed as previously.

Construction of approximate solutions

We consider the following system :

$$\begin{aligned}&\partial_t q^{n+1}+u^n\cdot\nabla q^{n+1}=q^n\mathrm{div}u^n,\\&\partial_t u^{n+1}+u^n\cdot\nabla u^{n+1}-\mathcal A(u^{n+1})+\nabla q^{n+1}=G^n,\end{aligned}\tag{5.43}$$

with :

$$G^n=\mathcal A(u^n)-\mathcal A(\rho^n,u^n)+K(\rho^n)\nabla q^n.$$

Uniform estimates for $(q^n,u^n)_{n\in\mathbb N}$

Let us show that the sequence $(q^n,u^n)_{n\in\mathbb N}$ is uniformly bounded in provided that T and η have been chosen small enough.
Let us remark first that according to proposition, there exists a universal condition K such that for all $n\in\mathbb N$, we have :

$$\|u_L^n\|_{\widetilde L^\infty_T(B^{\frac Np})}\le U_0,$$

with $U_0=K\|u_0\|_{B^{\frac N2}}$,

$$\|u_L^n\|_{\widetilde L^1_T(B^{\frac Np+1})}\le K\sum_{q\in\mathbb Z}2^{q(\frac Np-1)}\|\Delta_q u_0\|$$

Suppose that $\|q_0\|_{B^{\frac Np}}\le\eta$ for a small constant η and let $C_0=1+\|q_0\|_{B^{\frac Np}}$. According to , we can choose a positive time T such that the following property holds for all $n\in\mathbb N$:

$$\|u_L^n\|_{\widetilde L^\infty_T(B^{\frac Np})}\le U_0,\quad\text{and}\quad\|u_L^n\|_{\widetilde L^r_T(B^{\frac Np+1})}\le\eta^{\frac 2r}U_0^{1-\frac1r}$$

Let us now show by induction that the following estimates are satisfied :

$$\begin{aligned}&\|q^n\|_{\widetilde L^\infty_T(B^{\frac Np})}\le\sqrt\eta\\&\|\bar u^n\|_{\widetilde L^\infty_T(B^{\frac Np-1})}+\|\bar u^n\|_{\widetilde L^1_T(B^{\frac Np+1})}\le\eta\end{aligned}$$

From now, we suppose that $\eta \leq (2C_1)^{-2}$ where C_1 is the norm of the injection $B^{\frac{N}{p}} \hookrightarrow L^\infty$. This ensures us that the following inequality is satisfied :

$$\frac{1}{2} \leq 1 + q^n \leq \frac{3}{2}$$

According to proposition, we have :

$$\|q^{n+1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}})} \leq \exp^{C\|u^n\|_{L^1_T(B^{\frac{N}{p}+1})}} \big(\|q_0\|_{B^{\frac{N}{p}}} + \|F^n\|_{L^1_T(B^{\frac{N}{p}})}\big)$$

Moreover we have by Proposition :

$$\|u^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}})} \leq \exp^{C\|u^n\|_{L^1_T(B^{\frac{N}{p}+1})}} \big(\|q_0\|_{B^{\frac{N}{p}}} + \|F^n\|_{L^1_T(B^{\frac{N}{p}})}\big)$$

whence according to proposition :

$$\|F_n\|_{L^1_T(B^{\frac{N}{p}})} \lesssim T^{1-\frac{1}{r_1}}(1 + \|q^n\|_{L^\infty_T(B^{\frac{N}{p}})}) \|\mathrm{div} u^n\|_{L^{r_1}_T(B^{\frac{N}{p}})} \lesssim (1+U_0)^{\frac{1}{2}}\eta,$$

and we get finally :

$$\|q^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}})} \leq (1+U_0)^{\frac{1}{2}}\eta \exp^{C(1+U_0)^{\frac{1}{2}}\eta}$$

Obviously if η has been chosen small enough then q^{n+1} satisfies the estimate in $(\mathcal{P}_{n+1})$. Applying proposition to the second equation of yields :

$$\|\bar{u}^{n+1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}})} + \|\bar{u}^{n+1}\|_{L^1_T(B^{\frac{N}{p}+1})} \lesssim \|u^n \cdot \nabla u^n\|_{L^1_T(B^{\frac{N}{p}-1})} + \|K_0(q^n)\nabla q^n\|_{L^1_T(B^{\frac{N}{p}-1})}$$
$$+ \|\mathcal{A}(q^n, u^n)\|_{L^1_T(B^{\frac{N}{p}-1})}.$$

We now use proposition 2.24 as in the proof of theorem 1.22 to conclude.

Existence of a solution

We can now easily show that (q^n, u^n) is a Cauchy sequel in our space F_T of uniqueness and so $(q_n, u_n) \to (q, u)$ in F_T.
It rests to verify by compactness that (q, u) is a solution of the system (SW). □

Bibliographie

[1] H. Bahouri and J.-Y. Chemin, Équations d'ondes quasilinéaires et estimation de Strichartz, Amer. J. Mathematics 121 (1999) 1337-1377.

[2] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Annales Scientifiques de l'école Normale Supérieure 14 (1981) 209-246.

[3] G. Bourdaud, Réalisations des espaces de Besov homogènes, Arkiv fur Mathematik 26 (1998) 41-54.

[4] D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D Viscous shallow water equations and convergence to the quasi-geostrophic model. Comm. Math. Phys., 238(1-2) : 211-223, 2003.

[5] D. Bresch and B. Desjardins, Existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids,Journal de Mathématiques Pures et Appliqués Volume 87, Issue 1, January 2007, Pages 57-90.

[6] D. Bresch and B. Desjardins, Some diffusive capillary models of Koretweg type. C. R. Math. Acad. Sci. Paris, Section Mécanique, 332(11) :881-886, 2004.

[7] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models : Korteweg, lubrification and shallow water systems. Comm. Partial Differential Equations, 28(3-4) : 843-868, 2003.

[8] J.-Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel, J.d'Analyse Math. 77 (1999) 27-50.

[9] J.-Y. Chemin, About Navier-Stokes system, Prépublication du Laboratoire d'Analyse Numérique de Paris 6 R96023 (1996).

[10] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, J.Differential Equations 121 (1992) 314-328.

[11] H. J. Choe and H. Kim, Strong solution of the Navier-Stokes equations for isentropic compressible fluids, J. Differential Equations 190 (2003), 504-523.

[12] H. J. Choe and H. Kim, Strong solution of the Navier-Stokes equations for nonhomogeneous incompressible fluids, Math. Meth. Appl. Sci. 28 (2005), 1-28.

[13] F. Coquel, D. Diehl, C. Merkle and C. Rohde, Sharp and diffuse interface methods for phase transition problems in liquid-vapour flows. Numerical Methods for Hyperbolic and

Kinetic Problems, 239-270, IRMA Lect. Math. Theor. Phys,7,Eur. Math. Soc, Zürich, 2005.

[14] R. Danchin, Fourier analysis method for PDE's, Preprint Novembre 2005.

[15] R. Danchin, Global Existence in Critical Spaces for Flows of Compressible Viscous and Heat-Conductive Gases, Arch.Rational Mech.Anal.160 (2001) 1-39

[16] Danchin.R, Density-dependent incompressible viscous fluids in critical spaces. Proc. Roy. Soc. Edimburgh Sect. A, 133(6) : 1311-1334, 2003.

[17] Danchin.R, A few remarks on the Camassa-Holm equation, Differential and Integral Equations 14 (2001), 953-988.

[18] Danchin.R, Local Theory in critical Spaces for Compressible Viscous and Heat-Conductive Gases,Communication in Partial Differential Equations 26 (78),1183-1233 (2001)

[19] R. Danchin and B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Annales de l'IHP,Analyse non linéaire 18,97-133 (2001)

[20] E. Feireisl, Dynmamics of Viscous Compressible Fluids-Oxford Lecture Series in Mathematics and its Applications-26.

[21] E. Feireisl. Compressible Navier-Stokes equations with a non-monotone pressure law. J. Differential Equations, 184(1) : 97-108, 2002.

[22] E. Feireisl. On the motion of a viscous, compressible, and heat conducting equation. Indiana Univ. Math. J., 53(6) : 1705-1738, 2004.

[23] E. Feireisl, Antonín Novotný, and Hana Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech., 3(4) : 358-392, 2001.

[24] David Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. Trans. Amer. Math. Soc, 303(1) : 169-181, 1987.

[25] David Hoff. Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of the heat conducting fluids. Arch. Rational Mech. Anal., 139, (1997), P. 303-354.

[26] David Hoff. Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differential Equations, 120(1) : 215-254, 1995.

[27] David Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. Arch. Rational Mech. Anal., 132(1) : 1-14, 1995.

[28] David Hoff and Kevin Zumbrum. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow, Indiana University Mathematics Journal, 1995, 44, 603-676.

[29] Song Jiang and Ping Zhang. Axisymetrics solutions of the 3D Navier-Stokes equations for compressible isentropic fluids. J. Math. Pures Appl. (9), 82(8) : 949-973, 2003.

[30] A. V. Kazhikov. The equation of potential flows of a compressible viscous fluid for small Reynolds numbers : existence, uniqueness and stabilization of solutions. Sibirsk. Mat. Zh., 34 (1993), no. 3, p. 70-80.

[31] A. V. Kazhikov and V. V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one- dimensional equations of a viscous gas. Prikl. Mat. Meh., 41(2) : 282-291, 1977.

[32] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Oxford University Press (1998)

[33] Akitaka Matsumura and Takaaki Nishida. The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. Proc. Japan Acad. Ser. A Math. Sci, 55(9) : 337-342, 1979.

[34] A.Mellet and A.Vasseur, On the barotropic compressible Navier-Stokes equation, Comm. Partial Differential Equations 32 (2007), no. 1-3, 431–452.

[35] J. Nash, Le problème de Cauchy pour les équations différentielles d'un fluide général, Bulletin de la Société Mathématique de France, 1962, 90, 487-497.

[36] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. ZAMMZ. Angew. Math. Mech. 85(2005), no. 12, 839-857.

[37] T. Runst and W. Sickel : Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. de Gruyter Series in Nonlinear Analysis and Applications, 3. Walter de Gruyter and Co., Berlin (1996)

[38] Denis Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible., 303(13) : 639-642, 1986

[39] V.A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes systems. Zap. Nauchn. Sem. LOMI, 38, (1973), p.153-231 ; J. Soviet Math. 8, (1977), p. 467-529.

[40] V. Valli, W. Zajaczkowski. Navier-Stokes equations for compressible fluids : global existence and qualitative properties of the solutions in the general case. Commun. Math. Phys., 103 (1986) no 2., p. 259-296.

[41] Weike Wang and Chao-Jiang Xu. The Cauchy problem for viscous shallow water equations. Rev. Mat. Iberoamericana 21, no. 1 (2005), 1-24.

Chapitre 6

Approximate solutions for Navier Stokes equation with capillarity term

Abstract

This chapter is devoted proving global existence of solutions for an general isothermal model of capillary fluids derived by Rohde in [6], which can be used as a phase transition model.
This chapter is structured in the following way : first of all inspired by the result by P-L. Lions in [4] on the compressible Navier-Stokes system we will show the global existence of weak solutions for our system with isentropic pressure and next with general pressure. Next we state global existence for data close to a stable equilibrium as in the case of strong solutions.

1 Introduction

We want to construct approximate solutions for the system (SW) and we use a similar scheme than those in [2] and [5]. Here we employ a three level approximation scheme based on solving the following system of equations :

Continuity equation with vanishing viscosity

$$\partial_t \rho + \operatorname{div}(\rho u) = \varepsilon \Delta \rho \quad \text{on} \;\; (0,T) \times \Omega, \;\; \varepsilon > 0, \tag{1.1}$$

with the homogeneous Neumann boundary condition :

$$\nabla \rho \cdot n = 0 \quad \text{on} \;\; \partial\Omega, \tag{1.2}$$

and the initial condition :

$$\rho(0) = \rho_{0,\delta} \quad \text{on} \;\; \Omega. \tag{1.3}$$

Momentum equation with artificial pressure

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u + \nabla(P(\rho) + \delta\rho^\beta)$$
$$+\,\varepsilon\nabla u\cdot\nabla\rho = \kappa\rho\nabla(\rho*\phi - \rho) \quad \text{on } (0,T)\times\Omega,\ \delta>0,\ \beta>N. \tag{1.4}$$

with :

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

$$(\rho u)(0) = m_{0,\delta} \quad \text{on } \Omega. \tag{1.6}$$

We suppose that :

$$P(s) = s^\gamma,\ \gamma > \frac{N}{2}. \tag{1.7}$$

We also assume that :

$$\mu > 0,\ \lambda + \frac{2}{N}\mu \geq 0. \tag{1.8}$$

The extra term $\varepsilon\Delta\rho$ in (1.1) represents a vanishing viscosity with no specific physical meaning. From the mathematical viewpoint, however, it converts the hyperbolic equation (1.1) into a parabolic one. As a result, one can expect better regularity properties of the densities ρ constructed at this level of approximation.

The quantity $\delta\rho^\beta$ added to the momentum equation (1.4) can be considered as an artificial pressure, which was introduced to make the pressure estimates compatible with the vanishing viscosity regularization of (1.1). More precisely, the pressure estimates based on multiplication of (1.4) by the quantity $\nabla\Delta^{-1}\rho^w$ will still remain in force for the modified system (1.1), (1.4) only if $w = 1$. Accordingly, one must take $\beta = \beta(N)$ large enough to be able to exploit the ideas of the proof of proposition 2.13. We assume then that :

$$\delta > 0 \ \text{ and } \ \beta > N. \tag{1.9}$$

In the same spirit, the new quantity $\varepsilon\nabla u\cdot\nabla\rho$ was introduced in (1.4) in order to eliminate the extra term arising in the energy inequality to save the a priori estimates.

The initial data is modified as follows :

1. The density $\rho_{0,\delta} \in C^{2+\nu}(\overline{\Omega})$, $\nu > 0$, satisfies the homogeneous Neumann boundary condition :

$$\nabla\rho_{0,\delta}\cdot n_{/\partial\Omega} = 0. \tag{1.10}$$

   Furthermore, we suppose :

$$0 < \delta \leq \rho_{0,\delta}(x) \leq \delta^{-\frac{1}{2\beta}} \quad \text{for all } x\in\Omega, \tag{1.11}$$

2. The initial momentum $m_{0,\delta}$ are defined as

$$\begin{aligned} m_{0,\delta}(x) &= m_0 \quad \text{if } \rho_{0,\delta}(x) \geq 0, \\ &= 0 \quad \text{for } \rho_{0,\delta}(x) < \rho_0(x). \end{aligned} \tag{1.12}$$

In particular, the initial value of the modified total energy :

$$E(0) = E_\delta(0) = \int_\Omega \frac{1}{2}\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + P(\rho_{0,\delta}) + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta dx, \tag{1.13}$$

is bounded by a constant independent of $\delta > 0$.
The principal strategy of the proof of the theorem 2.28 will be to solve first the system (1.1)-(1.6) for positive values of the parameters ε and δ, then to let $\varepsilon \to 0$ to get rid of the artificial viscosity in (1.1) ; and finally, evoking the full strength of the pressure, we pass to the limit for $\delta \to 0$ to recover the original Rohde system.

Mathematical result on continuity equation with dissipation

In the first part of this section, we deal with the heat equation complemented with the Neumann boundary conditions. We recall the classical results about existence, uniqueness and regularity in the L^p setting.
The continuity equation with dissipation forms one part of the system (1.1)-(1.6), we then need of several properties of this equation such as existence, uniqueness and regularity of solutions, pointwise lower and upper bounds. For these results we refer to Novotny, Straškraba in [5].

Regularity for the parabolic Neumann problem

We recall that Ω is a bounded domain of $\mathbb{R}^N$ and that $I = (0, T)$, where $T > 0$. We consider the following parabolic initial and boundary value problem :

$$\begin{aligned} &\partial_t \rho - \varepsilon\Delta\rho = h \ \text{ in } I \times \Omega, \\ &\rho(0, x) = \rho_0(x), \ x \in \Omega, \\ &\partial_n \rho = 0 \ \text{ in } I \times \partial\Omega, \end{aligned} \tag{1.14}$$

where $\varepsilon > 0$, ρ_0 and h are given functions on Ω and $I \times \Omega$ respectively.
We have then the well-known statements about the regularity of parabolic systems, see Chapter III in Amman [1].

Proposition 1.30. *Let $0 < \theta \leq 1$, $1 < p, q < +\infty$ and Ω be a bounded domain. If :*

$$\Omega \in C^{2,\theta}, \quad \rho_0 \in \widetilde{W}^{2-\frac{2}{p},q}(\Omega), \ h \in L^p(I, L^q(\Omega)),$$

(where $\widetilde{W}^{2-\frac{2}{p},q}(\Omega)$ is the completion of the space $\{z \in C^\infty(\overline{\Omega}); \partial_n z_{/\partial\Omega} = 0\}$ in $W^{2-\frac{2}{p},q}(\Omega)$) then there exists a unique $\rho \in L^p(I, W^{2,q}(\Omega)) \cap C^0(\bar{I}, W^{2-\frac{2}{p},q}(\Omega))$, $\partial_t\rho \in L^p(I, L^q(\Omega))$ satisfying equation (1.14) and which verifies estimate :

$$\begin{aligned} \varepsilon^{1-\frac{1}{p}}\|\rho\|_{L^\infty(I,W^{2-\frac{2}{p},q}(\Omega))} + \|\partial_t\rho\|_{L^p(I,L^q(\Omega))} + \varepsilon\|\rho\|_{L^p(I,W^{2,q}(\Omega))} \\ \leq c(p, q, \Omega)[\varepsilon^{1-\frac{1}{p}}\|\rho_0\|_{2-\frac{2}{p},q} + \|h\|_{L^p(I,L^q(\Omega))}]. \end{aligned} \tag{1.15}$$

In the sequel, we shall need the following existence and uniqueness result for the Neumann problem (1.1)-(1.6) with the right-hand side in divergence form $h = \nabla b$. Suppose that :

$$\Omega \in C^{2,\theta}, \ \ \rho_0 \in L^q(\Omega), \ b \in L^p(I, L^q(\Omega)),$$

Then there exists a unique $\rho \in L^p(I, W^{1,q}(\Omega)) \cap C^0(\bar{I}, L^q(\Omega))$ which satisfies the second equation of (1.14) with :

$$\frac{d}{dt}\int_\Omega \rho\eta + \varepsilon \int_\Omega \nabla\rho \cdot \nabla\eta = -\int_\Omega b \cdot \nabla\eta, \ \eta \in C^\infty(\overline{\Omega}), \text{ in } \mathcal{D}'(I),$$

and :

$$\varepsilon^{1-\frac{1}{p}}\|\rho\|_{L^\infty(I,L^q(\Omega))} + \varepsilon\|\nabla\rho\|_{L^p(I,L^q(\Omega))} \leq c(p,q,\Omega)[\varepsilon^{1-\frac{1}{p}}\|\rho_0\|_{L^q} + \|b\|_{L^p(I,L^q(\Omega))}]. \tag{1.16}$$

Continuity equation with dispersion

Next we investigate the equation :

$$\partial_t\rho + \operatorname{div}(\rho u) - \varepsilon\Delta\rho = 0 \quad \text{in } (0,T) \times \Omega, \tag{1.17}$$

completed with initial conditions :

$$\rho(0) = \rho_0 \quad \text{in } \Omega, \tag{1.18}$$

and boundary conditions :

$$\partial_n\rho = 0 \quad \text{in } (0,T) \times \partial\Omega. \tag{1.19}$$

Ω is a bounded domain, $\varepsilon > 0$ is a given constant, ρ_0 is a given function. We recall the following proposition, see in [5].

Proposition 1.31. *Let $0 < \theta \leq 1$, Ω be a bounded domain of class $C^{2,\theta}$, $0 < \underline{\rho} \leq \bar{\rho} < \infty$, and :*

$$\rho_0 \in W^{1,\infty}(\Omega), \ \textit{such that} \ \underline{\rho} \leq \rho_0 \leq \bar{\rho}.$$

Then there exists a unique mapping :

$$S_{\rho_0} : L^\infty(I, (W_0^{1,\infty}(\Omega))^N) \to C^0(\bar{I}, W^{1,2}(\overline{\Omega})),$$

where $W_0^{1,\infty}(\Omega) = W^{1,\infty}(\Omega) \cap \{\xi_{/\partial\Omega} = 0\}$ such that :

1. *$S_{\rho_0}(u) \in R_T$ with :*

$$R_T = \{\rho \in L^2(I, W^{2,p}(\Omega)) \cap C^0(\bar{I}, W^{1,p}(\Omega)), \partial_t\rho \in L^2(I, L^p(\Omega))\}, \tag{1.20}$$

   where $1 < p < \infty$.

2. *The function $\rho = S_{\rho_0}(u)$ satisfies (1.17) a.e in $(0,T) \times \Omega$, ((1.18) a.e in Ω and (1.19) in the sense of the trace a.e in I).*

3. Moreover we control the vacuum :

$$\underline{\rho} e^{-\int_0^t \|u(\tau)\|_{1,\infty} d\tau} \leq [S_{\rho_0}(u)](t,x) \leq \bar{\rho} e^{\int_0^t \|u(\tau)\|_{1,\infty} d\tau}, \ t \in \bar{I}, \ \text{for a.a } x \in \Omega. \qquad (1.21)$$

4. If $\|u\|_{L^\infty(I,W^{1,\infty}(\Omega))} \leq K$, *where* $K > 0$ *then :*

$$\begin{aligned} &\|S_{\rho_0}(u)\|_{L^\infty(I,W^{1,2}(\Omega))} \leq c\|\rho_0\|_{1,2} e^{\frac{c}{2\varepsilon}(K+K^2)t}, \ t \in \bar{I}, \\ &\|\nabla^2 S_{\rho_0}(u)\|_{L^2((0,T)\times\Omega)} \leq \frac{C}{\varepsilon}\sqrt{t}\|\rho_0\|_{1,2} K e^{\frac{C}{2\varepsilon}(K+K^2)t}, \ t \in \bar{I}, \\ &\|\partial_t S_{\rho_0}(u)\|_{L^2((0,T)\times\Omega)} \leq C\sqrt{t}\|\rho_0\|_{1,2} K e^{\frac{C}{2\varepsilon}(K+K^2)t}, \ t \in \bar{I}, \end{aligned} \qquad (1.22)$$

5. For $t \in \bar{I}$, *we have :*

$$\|[S_{\rho_0}(u_1) - S_{\rho_0}(u_2)](t)\|_{0,2,\Omega} \leq c(K,\varepsilon,T)t\|\rho_0\|_{1,2}\|u_1 - u_2\|_{L^\infty(I_t,W^{1,\infty}(\Omega))}. \qquad (1.23)$$

The constant C *in estimates (1.22) depends at most on* Ω*, in particular, it is independent of* ε, K, T, ρ_0, u.

We now give a proposition for renormalized solution of (1.17) see [5].

Proposition 1.32. *Assume that* Ω *is a domain in* $\mathbb{R}^N$. *Let* $2 \leq \beta < \infty$ *and let* $1 \leq p < \infty$. *Suppose that a couple* (ρ, u) *satisfies :*

$$\rho \in L^\infty(I, L^\beta_{loc}(\Omega)), \ \Delta\rho \in L^p_{loc}(I \times \Omega), \ \rho \geq 0 \ \text{ a.e in } I \times \Omega, \ u \in L^2(I, (W^{1,2}_{loc}(\Omega))^N), \ (1.24)$$

and

$$\partial_t \rho + \operatorname{div}(\rho u) - \varepsilon\Delta\rho = 0 \quad \text{in } \mathcal{D}'(I \times \Omega).$$

Then for any convex function :

$$b \in C^1([0,\infty)) \cap C^2((0,\infty)),$$

satisfying growth condition of Diperna-Lions theorem on renormalized equation for mass equation (see [3]), we have :

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + (\rho b'(\rho) - b(\rho))\operatorname{div} u - \varepsilon\Delta b(\rho) \leq 0 \quad \text{in } \mathcal{D}'(I \times \Omega). \qquad (1.25)$$

2 Construction of approximate solutions

Consider a finite-dimensional (Hilbert) space :

$$X_n = \operatorname{span}\{\eta_j\}_{j=1}^n$$

where $\eta_j \in \mathcal{D}(\Omega)^N$ are linearly independent vector functions ranging in the N-dimensional Euclidean space $\mathbb{R}^N$.

The approximate velocities $u_n \in C([0,T]; X_n)$ are looked for to satisfy an integral identity :

$$\begin{aligned} &\int_\Omega \rho u_n(t) \cdot \eta dx - \int_\Omega m_{0,\delta} \cdot \eta dx = \int_0^t \int_\Omega [\rho u_n \otimes u_n - S_n] : \nabla\eta + [P(\rho) + \delta\rho^\beta]\operatorname{div}\eta \, dx \\ &+ \int_0^t \int_\Omega [\kappa\rho\nabla(\phi * \rho - \rho) - \varepsilon\nabla u_n.\nabla\rho] \cdot \eta \, dxds \end{aligned} \qquad (2.26)$$

with :

$$S_n = \mu(\nabla u_n + \nabla u_n^t) + \lambda \mathrm{div} u_n \mathbb{I}$$

for any test function $\eta \in X_n$, and all $t \in [0,T]$. The system can be interpreted as a projection of the infinite-dimensional dynamical system represented by the equation onto a finite system of differential equations on X_n.
The density $\rho = \rho[u_n]$ appearing in the integral on the right hand side of (2.26) is determined as the unique solution of the problem (1.1) with u replaced by u_n. We know (see [5]) that there exist countable sets :

$$(\lambda_n)_{n\mathbb{N}},\ \ 0 < \lambda_1 \le \lambda_2 \le \cdots,\ \text{and}\ \ (\eta_j)_{j\in\mathbb{N}} \in (W_0^{1,p}(\Omega)^N \cap W^{2,p}(\Omega)^N), 1 \le p < \infty,$$

such that :

$$-\mu\Delta\eta_j - (\mu+\lambda)\nabla \mathrm{div}\eta_j = \lambda_j \eta_j.$$

First, we want to find an approximate solution of system (1.1)-(1.6). We have the following proposition.

Proposition 2.33. *Let μ, λ satisfy (1.8), γ satisfy (1.7), δ, β satisfy (1.9) and ε, $\underline{\rho}$, $\bar{\rho}$ satisfy :*

$$\varepsilon > 0,\ \ 0 < \underline{\rho} < \bar{\rho} < +\infty. \tag{2.27}$$

Assume that Ω is a bounded domain of class $C^{2,\theta}$, $\theta \in (0,1]$ and :

$$0 < \underline{\rho} \le \rho_0 \le \bar{\rho},\ \rho_0 \in W^{1,\infty}(\Omega),\ u_0 \in X_n.$$

Then there exists a unique couple (ρ_n, u_n) with the following properties :

1.
$$\begin{aligned} &\rho_n \in C^0(\bar{I}, W^{1,p}(\Omega)) \cap L^2(I, W^{2,p}(\Omega)), \partial_t\rho_n \in L^2(I, L^p(\Omega)),\\ &\rho_n > 0\ \ in\ I\times\Omega,\ u_n \in C^0(\bar{I}, X_n), \partial_t u_n \in L^2(I, X_n),\\ &\nabla\rho_n \in L^2(I, E_p^0(\Omega)),\ \rho_n u_n \in C^0(\bar{I}, E_p^0(\Omega)). \end{aligned} \tag{2.28}$$

2.
$$\begin{aligned} &\int_\Omega \partial_t(\rho u_n(t))\cdot\eta dx + \int_\Omega [\partial_j(\rho_n u_n u_n^j) - \mu\Delta u_n - (\mu+\lambda)\nabla\mathrm{div} u_n + \nabla\rho_n^\gamma\\ &\quad + \delta\nabla\rho_n^\beta + \varepsilon\nabla\rho_n\cdot\nabla u_n]\cdot\eta = \int_\Omega \kappa\rho_n\nabla(\phi*\rho_n - \rho_n)\cdot\eta,\ t\in I,\ \eta\in X_n. \end{aligned} \tag{2.29}$$

3.
$$\partial_t\rho_n + \mathrm{div}(\rho_n u_n) - \varepsilon\Delta\rho_n = 0\quad a.e.\ in\ Q_T. \tag{2.30}$$

4.
$$\rho_n(0) = \rho_0,\quad u_n(0) = u_0. \tag{2.31}$$

5. If $\delta \in (0,1)$, the following estimates hold :

$$\begin{cases} \|\rho_n\|_{L^\infty(0,T,L^{\max(2,\gamma)})} \le L(\mathcal{E}_0,T), \\ \delta^{\frac{1}{\beta}}\|\rho_n\|_{L^\infty(0,T,L^\beta)} \le L(\mathcal{E}_0,T), \\ \sqrt{\varepsilon}\|\nabla\rho_n\|_{L^2(0,T\times\Omega)} \le L(\mathcal{E}_0,\delta,\Omega,T), \\ \|\rho_n\|_{L^{\beta+1}((0,T)\times\Omega} \le L(\mathcal{E}_0,\varepsilon,\delta,\Omega,T), \\ \|\rho_n^{\frac{\beta}{2}}\|_{L^2(0,T,W^{1,2}(\Omega))} \le L(\mathcal{E}_0,\varepsilon,\delta,\Omega,T), \\ \|u_n\|_{L^2(0,T,W^{1,2}(\Omega))} \le L(\mathcal{E}_0,T), \\ \|\sqrt{\rho_n}u_n\|_{L^\infty(0,T,L^2)} \le L(\mathcal{E}_0,T). \end{cases} \tag{2.32}$$

Here L is a positive constant which is, in particular independent of n and we have set :

$$\mathcal{E}_0 = \int_{\mathbb{R}^N} \big(\frac{|m_0|^2}{2\rho} + \Pi(\rho_0) + E_{global}[\rho_0]\big)dx,$$

where E_{global} is defined in chapter 4.

Next in passing to the limit for $n \to +\infty$ in the equation (2.29) and (2.30) we obtain a sequence $(\rho_{\delta,\varepsilon}, u_{\delta,\varepsilon})$ which verifies system (1.1)-(1.6). In the following proposition we give some estimates of $(\rho_{\delta,\varepsilon}, u_{\delta,\varepsilon})$ independent of ε

Proposition 2.34. *Let ε, $\underline{\rho}$ and $\bar{\rho}$ satisfies (2.27) and :*

$$\underline{\rho} \le \rho_0 \le \bar{\rho},\ \rho_0 \in W^{1,\infty}.$$

Then there exists a couple $(\rho_{\delta,\varepsilon}, u_{\delta,\varepsilon})$ which verifies (1.1)-(1.6) with the following estimate if $\delta \in (0,1)$:

$$\begin{aligned} &\int_\Omega \rho_n(\tau)\big(\frac{1}{2}|u_{\delta,\varepsilon}|^2 + E_{global}[\rho_{\delta,\varepsilon}(\cdot,t)]) + \Pi(\rho_{\delta,\varepsilon}) + \frac{\delta}{\beta-1}\rho_{\delta,\varepsilon}^{\beta-1}\big)dx \\ &+ \delta\int_0^\tau\int_\Omega S_{\delta,\varepsilon} : \nabla u_{\delta,\varepsilon}dx\,dt + \varepsilon\int_0^\tau\int_\Omega |\nabla\rho_{\delta,\varepsilon}|^2\big(\frac{P'(\rho_{\delta,\varepsilon})}{\rho_{\delta,\varepsilon}} + \delta\beta\rho_{\delta,\varepsilon}^{\beta-2} + 1\big)dx\,dt \\ &\le \int_\Omega \frac{1}{2}m_{0,\delta}\cdot u_{\delta,\varepsilon}(0) + \rho_{0,\delta}\Pi(\rho_{0,\delta}) + \int_\Omega E_{global}[\rho_{0,\delta}] + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta dx, \end{aligned} \tag{2.33}$$

$$\begin{cases} \|u_{\delta,\varepsilon}\|_{L^2(I,W^{1,2}(\Omega))} \le L(\mathcal{E}_0,T), \\ \|\sqrt{\rho_{\delta,\varepsilon}}u_{\delta,\varepsilon}\|_{L^\infty(I,L^2(\Omega))} \le L(\mathcal{E}_0,T), \\ \delta^{\frac{1}{\beta}}\|\rho_{\delta,\varepsilon}\|_{L^\infty(I,L^\beta(\Omega))} \le L(\mathcal{E}_0,T), \\ \|\rho_{\delta,\varepsilon}\|_{L^s((0,T)\times\Omega)} \le L(\mathcal{E}_0,\Omega,T), \quad \text{with } s = \gamma + \frac{2}{N}\gamma - 1, \\ \delta^{\frac{1}{\beta+\theta}}\|\rho_{\delta,\varepsilon}\|_{L^{s'}((0,T)\times\Omega)} \le L(\mathcal{E}_0,\Omega,T)\ \ mboxwith\ s' = \gamma + \frac{2}{N}\gamma - 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \theta = \frac{2}{N}\gamma - 1, \\ \sqrt{\varepsilon}\|\nabla\rho_{\delta,\varepsilon}\|_{L^2((0,T)\times\Omega)} \le L(\mathcal{E}_0,\Omega,T). \end{cases} \tag{2.34}$$

Here L is a positive constant independent of ε.

Next we show that we can pass to the limit in ε and obtain a sequence (ρ_δ, u_δ) of solution for the following system :

$$\begin{aligned}
&\partial_t\rho + \text{div}(\rho u) = 0,\\
&\partial_t(\rho u) + \text{div}(\rho u \otimes u) - \mu\Delta u - (\mu+\lambda)\text{div}u + \nabla\rho^\gamma + \delta\nabla\rho^\beta = \kappa\rho\nabla(\phi * \rho - \rho),
\end{aligned} \tag{2.35}$$

in $(0,T)\times\Omega$, with initial conditions (1.3), (1.6), with boundary conditions (1.2), (1.5) and with :

$$\delta > 0,\ \beta \geq \max(\gamma, 2, N). \tag{2.36}$$

A weak solution of the original problem (SW) will be obtained as a weak limit (ρ, u) as $\delta \to 0$ to the sequence (ρ_δ, u_δ). This limit process has for main ingredient the following proposition which states estimates independent of δ.

Proposition 2.35. *We assume that (ρ_0, u_0) satisfies the energy estimate and :*

$$\rho_0 \in L^\beta(\Omega),\ \rho_0 \geq 0\ \ a.e\ in\ \Omega.$$

Then there exists a couple (ρ_δ, u_δ) which verifies (2.35) on $\mathcal{D}'((0,T)\times\Omega)$ with the following properties :

1. *(ρ_δ, u_δ) are renormalized solutions.*
2. *For $\delta \in (0,1)$, $\theta = \frac{2}{N}\gamma - 1$, the following estimates hold :*

$$\begin{aligned}
&\int_\Omega \rho_\delta(\tau)(\frac{1}{2}|u_\delta|^2 + E_{global}[\rho_\delta(\cdot,t)]) + \Pi(\rho_\delta) + \frac{\delta}{\beta-1}\rho_\delta^{\beta-1})dx + \delta\int_0^\tau\int_\Omega S_\delta : \nabla u_\delta dx\,dt\\
&\leq \int_\Omega (\frac{1}{2}m_{0,\delta}\cdot u_n(0) + \rho_{0,\delta}\Pi(\rho_{0,\delta}))dx + \int_\Omega \big(E_{global}[\rho_{0,\delta}] + \frac{\delta}{\beta-1}\rho_{0,\delta}^\beta\big)dx,
\end{aligned} \tag{2.37}$$

$$\begin{cases}
\|u_\delta\|_{L^2(I,W^{1,2}(\Omega))} \leq L(\mathcal{E}_0, T),\\
\|\sqrt{\rho_\delta}u_\delta\|_{L^\infty(I,L^2(\Omega))} \leq L(\mathcal{E}_0, T),\\
\delta^{\frac{1}{\beta}}\|\rho_\delta\|_{L^\infty(I,L^\beta(\Omega))} \leq L(\mathcal{E}_0, T),\\
\|\rho_\delta\|_{L^s((0,T)\times\Omega)} \leq L(\mathcal{E}_0,\Omega,T), \quad with\ s = \gamma + \dfrac{2}{N}\gamma - 1,\\
\delta^{\frac{1}{\beta+\theta}}\|\rho_\delta\|_{L^{s'}((0,T)\times\Omega)} \leq L(\mathcal{E}_0,\Omega,T), \quad with\ s' = \gamma + \dfrac{2}{N}\gamma - 1.
\end{cases} \tag{2.38}$$

L is a positive constant, independent of δ.

Theorem 2.28. *Let $N \geq 2$. Let $\gamma > N/2$ if $N \geq 4$ and $\gamma \geq 1$ else.*
Let the couple (ρ_0, u_0) satisfy :

- *ρ^0 belongs to $L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ with $s = \max(\gamma, 2)$ and $\rho_0 \geq 0$ a.e in $\mathbb{R}^N$.*
- *$\frac{|\rho_0 u_0|^2}{\rho_0}$ belongs to $L^1(\mathbb{R}^N)$,*
- *and $\rho_0 u_0 = 0$ whenever $x \in \{\rho_0 = 0\}$.*

Under the above conditions there exists a solution (ρ, u) of the Rohde system satisfying the above initial conditions and such that $\rho \in L^{\max(2,\gamma)}_{loc}((0,T)\times\mathbb{R}^N)$. In addition (ρ, u) satisfies the following energy inequality for almost all $t \in [0,T]$:

$$\begin{aligned}&\int_{\mathbb{R}^N}(\frac{1}{2}\rho|u|^2(t)+\Pi(\rho)(t)+E_{global}[\rho(.,t)])(x)dx+\int_0^t\int_{\mathbb{R}^N}(\mu D(u):D(u)(t)\\&\quad+(\lambda+\mu)|\mathrm{div}u(t)|^2)dx\leq\int_{\mathbb{R}^N}(\frac{|m_0|^2}{2\rho}+\Pi(\rho_0)+E_{global}[\rho_0])(x)dx\end{aligned}\tag{2.39}$$

2.1 Proof of proposition 2.33

Local existence

Following the standard approach we solve (1.1)-(1.6) on a possibly short time interval via a fixed point argument, establish estimates of the solution that are independent of the length of this interval, and iterate this procedure to obtain, after a finite time number of steps, a solution defined on the whole time interval $[0,T]$.
Given

$$\rho\in C^0([0,T],L^1(\Omega)),\ \ \partial_t\rho\in L^1((0,T)\times\Omega),\ \ \text{ess}\inf_{(t,x)\in(0,T)\times\Omega}\rho(t,x)\geq\underline{\rho}>0,\tag{2.40}$$

consider a family of linear operator :

$$M(\rho):X_n\to X_n^*,\ \ <M(\rho)v,w>=\int_\Omega\rho v\cdot wdx,$$

Here X_n is considered as a (finite dimensional) Hilbert space with a scalar product induced by the standard L^2-norm. As always, the symbol X_n^* stands for the dual space of X_n.
Realizing that the $W^{k,p}(\Omega)$ norms, $k=0,1,\cdots,\ 1\leq p\leq\infty$ are equivalent on X_n as X_n is a finite dimensional space, we obtain :

$$\|\mathcal{M}[\rho]\|_{\mathcal{L}(X_n,X_n^*)}\leq c(n)\int_\Omega\rho(t,x)dx.$$

Moreover, it is easy to see that the operator M is invertible provided ρ is strictly positive on Ω, and

$$\|\mathcal{M}^{-1}(\rho)\|\leq\frac{1}{\underline{\rho}}.\tag{2.41}$$

Moreover, the identity :

$$\mathcal{M}^{-1}[\rho^1]-\mathcal{M}^{-1}[\rho^2]=\mathcal{M}^{-1}[\rho^2](\mathcal{M}[\rho^2]-\mathcal{M}[\rho^1])\mathcal{M}^{-1}[\rho^1],$$

can be used to obtain :

$$\|\mathcal{M}^{-1}[\rho^1]-\mathcal{M}^{-1}[\rho^2]\|_{\mathcal{L}(X_n^*;X_n)}\leq\frac{c(n)}{\underline{\rho}^2}\|\rho_1-\rho_2\|_{L^1(\Omega)},\tag{2.42}$$

for any ρ^1, ρ^2 such that :

$$\inf_\Omega\rho^1,\ \inf_\Omega\rho^2\geq\underline{\rho}.$$

Now the integral equation (2.26) can be rephrased as :

$$u_n(t) = \mathcal{M}^{-1}[S_{\rho_0}(u_n)]\Big(P_n m_0 + \int_0^t P[\mathcal{N}(S_{\rho_0}(u_n), u_n]ds\Big), \tag{2.43}$$

where P is the orthogonal projection of $L^2(\Omega)$ onto X_n and :

$$\begin{aligned}\mathcal{N}(\rho,u) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u - \nabla\rho^\gamma - \delta\nabla\rho^\beta - \mathrm{div}(\rho u\otimes u) - \varepsilon\nabla\rho\cdot\nabla u \\ - \kappa\rho\nabla(\phi*\rho-\rho).\end{aligned} \tag{2.44}$$

We denote :

$$\begin{aligned}A_{\rho_0,m_0} = \{t\in(0,T]\setminus \text{there exists a unique } (\rho,u)\in R_t\times C^0(\bar I_t, X_n) \\ \text{satisfying (2.43) and (1.1)} - (1.3)\},\end{aligned}$$

where R_t is defined in (1.20).
Consider a bounded ball B in the space $C([0,T],X_n)$:

$$B_{K,\tau_0} = \{v\in C([0,T],X_n); \|v(t)\|_{C^0(\bar I_{\tau_0},X_n)}\le K\},$$

where we will precise later τ_0 and K.
Finally, define a mapping :

$$\begin{aligned}\mathcal{T}_n : B \to C([0,T];X_n), \\ \mathcal{T}_n[u_n] = \mathcal{M}^{-1}_{S_{\rho_0}(u_n)}\Big(m^*_{0,n} + \int_0^t P\mathcal{N}(u_n(s), S_{\rho_0}(u_n)(s))ds\Big).\end{aligned}$$

We now want to use the fixed point theorem to establish the existence of u_n. We have to show that $\mathcal{T}_n$ maps the ball B_{K,τ_0} into itself and that $\mathcal{T}_n$ is contractant.
Firts we derive some auxiliary estimates which concerns couple $(S_{\rho_0}(u_n), u_n)$ where u_n belongs to $C^0(\bar I, X_n)$ and $\|u_n\|_{C^0(\bar I,X_n)}\le K$.
From (2.44), by using equivalence norms for a finite dimensional space, we get :

$$\|P\mathcal{N}(\rho,v)\|_{X_n} \le c(n)[\|v\|_{X_n} + \|\rho\|_{0,\infty}(\|v\|_{X_n} + \|v\|^2_{X_n} + \|\rho\|^\gamma_{0,\infty} + \|\rho\|^\beta_{0,\infty} + \|\rho\|^2_{0,\infty}). \tag{2.45}$$

From (2.45) and (1.21) we deduce that :

$$\|P\mathcal{N}(\rho,v)\|_{X_n} \le M(K,\bar\rho,T,n), \ \ t\in I. \tag{2.46}$$

where M is non decreasing in the second variable.
Using (2.44) along with Taylor formula we obtain :

$$\begin{aligned}&< \mathcal{N}(\rho_1,v_1) - \mathcal{N}(\rho_2,v_2),\varphi >= \\ &\int_\Omega [\mu\Delta(v_1-v_2)] + (\mu+\lambda)\nabla\mathrm{div}(v_2-v_1)]\cdot\varphi + \int_\Omega\int_{\rho_1}^{\rho_2}(\gamma s^{\gamma-1} + \delta\beta s^{\beta-1})ds\,\mathrm{div}\varphi \\ &+ \int_\Omega(\rho_1-\rho_2)u_1^iu_1^j\partial_j\varphi^i + \int_\Omega \rho_2(u_1^i-u_2^i)u_1^j\partial_j\varphi^i + \int_\Omega\rho_2(u_1^j-u_2^j)u_2^i\partial_j\varphi^i \\ &+ \varepsilon\int_\Omega(\rho_1-\rho_2)(\Delta v_1\cdot\varphi + \partial_j u_1^i\partial_j\varphi^i) + \varepsilon\int_\Omega\rho_2[\Delta(v_2-v_1)\cdot\varphi + \partial_j(u_1^i-u_2^i)\partial_j\varphi^i] \\ &+ \kappa\int_\Omega(\rho_1-\rho_2)\nabla(\phi*\rho_1-\rho_1)\cdot\varphi + \rho_2\nabla(\phi*(\rho_1-\rho_2) - (\rho_1-\rho_2))\cdot\varphi.\end{aligned} \tag{2.47}$$

By virtue of (2.47), (1.21), due to elementary properties of the projection $P = P_n$ and using the equivalence of $W^{k,p}$ norms on X_n, we get :

$$\|P[\mathcal{N}(\rho_1, v_1) - \mathcal{N}(\rho_2, v_2)](t)\|_{X_n} \leq M(K, \bar{\rho}, T, n)\big(\|v_1 - v_2\|_{X_n} + \|(\rho_1 - \rho_2)(t)\|_{0,1}\big), \quad t \in I, \tag{2.48}$$

where M is again nondecreasing in the second variable. Due to (2.42) and (1.21), (1.23), we have :

$$\|\mathcal{M}^{-1}_{[S_{\rho_0}(v_2)](t)} - \mathcal{M}^{-1}_{[S_{\rho_0}(v_1)](t)}\|_{\mathcal{L}(X_n, X_n)} \leq \frac{M(K, T, n)}{\underline{\rho}^2}\|\rho_0\|_{1,2}t\|v_1 - v_2\|_{C^0(\bar{I}_t, X_n)}, \quad v_1, v_2 \in C^0(\bar{I}, X_n), \ \ t \in I, \ I_t = (0, t). \tag{2.49}$$

Since :

$$\mathcal{M}^{-1}_{[S_{\rho_0}(v)](t_1)} \int_0^{t_1} P[\mathcal{N}(S_{\rho_0}(v), v)] - \mathcal{M}^{-1}_{[S_{\rho_0}(v)](t_2)} \int_0^{t_2} P[\mathcal{N}(S_{\rho_0}(v), v)]$$
$$= \mathcal{M}^{-1}_{[S_{\rho_0}(v)](t_1)} \int_{t_2}^{t_1} P[\mathcal{N}(S_{\rho_0}(v), v)] + (\mathcal{M}^{-1}_{[S_{\rho_0}(v)](t_1)} - \mathcal{M}^{-1}_{[S_{\rho_0}(v)](t_2)}) \int_0^{t_2} P[\mathcal{N}(S_{\rho_0}(v), v)],$$

thanks to (2.41), (2.46) on one hand, and due to (2.42), (2.46), (1.20) on the other hand, we find that $\mathcal{M}^{-1}_{[S_{\rho_0}(v)](\cdot)} \int_0^{t_1} P[\mathcal{N}(S_{\rho_0}(v), v)] \in C^0(\bar{I}, X_n)$. Even more simply, we observe that :

$$\|\mathcal{M}^{-1}_{[S_{\rho_0}(v)]} \int_0^t P[\mathcal{N}(S_{\rho_0}(v), v)]\|_{C^0(\bar{I}, X_n)} \leq M_1(K, \underline{\rho}, \bar{\rho}, T, n)t, \ t \in I, \tag{2.50}$$

where M_1 is nonincreasing in the second and nondecreasing in the third variable.
Similarly, $\mathcal{M}^{-1}_{[S_{\rho_0}(v)]}(Pm_0) \in C^0(\bar{I}, X_n)$ and :

$$\|\mathcal{M}^{-1}_{[S_{\rho_0}(v)]}(Pm_0)\|_{C^0(\bar{I}, X_n)} \leq \underline{\rho}e^{Kt}\|Pm_0\|_{X_n}, \ t \in I. \tag{2.51}$$

We now can choose K and T_0 such that :

$$4 \max\left(\frac{Pm_0\|_{X_n}}{\underline{\rho}}, \|u_0\|_{X_n}\right) < K, \tag{2.52}$$

and we take :

$$T_0 = T_0(K, \underline{\rho}, \bar{\rho}, T, n) = \min\left(\frac{\ln 2}{K}, \frac{K}{2M_1}, T\right), \tag{2.53}$$

so that T_0 is nondecreasing in the second and nonincreasing in the third variable. With this choice we have $\underline{\rho}^{-1}e^{KT_0}\|Pm_0\|_{X_n} < \frac{K}{2}$ and $M_1T_0 < \frac{K}{2}$. Therefore by virtue of (2.43), (2.50), (2.51), the mapping $\mathcal{T}_{\rho_0, m_0}$ maps :

$$B_{K,\tau_0} = \{u \in C^0(\bar{I}_{\tau_0}, X_n); \ \ \|u\|_{C^0(\bar{I}_{\tau_0}, W^{1,\infty})} \leq K\} \tag{2.54}$$

into itself, for any $0 < \tau_0 \leq T_0$.
We now prove that $\mathcal{T}_{\rho_0, m_0}$ is a contraction. Due to the formula :

$$\mathcal{M}^{-1}_{\rho_1}(v_1) - \mathcal{M}^{-1}_{\rho_2}(v_2) = (\mathcal{M}^{-1}_{\rho_1} - \mathcal{M}^{-1}_{\rho_2})(v_1) + \mathcal{M}^{-1}_{\rho_2}(v_1 - v_2),$$

and due to (2.43), we get the identity :

$$\mathcal{T}_{\rho_0,m_0}(v_1)-\mathcal{T}_{\rho_0,m_0}(v_1)=(\mathcal{M}^{-1}_{S_{\rho_0}(v_1)}-\mathcal{M}^{-1}_{S_{\rho_0}(v_2)})\Big(Pm_0+\int_0^t P[\mathcal{N}(S_{\rho_0}(v_1),v_1)]\Big)$$
$$+\mathcal{M}^{-1}_{S_{\rho_0}(v_2)})\int_0^t\big(P[\mathcal{N}(S_{\rho_0}(v_1),v_1)]-P[\mathcal{N}(S_{\rho_0}(v_2),v_2)]\big). \tag{2.55}$$

We apply (2.46), (2.49), (2.52) to bound the first term and (2.41), (2.48), (1.21) and (1.23) to bound the second term. We finally get :

$$\|\mathcal{T}_{\rho_0,m_0}(v_1)-\mathcal{T}_{\rho_0,m_0}(v_1)\|_{X_n}(t)\leq M_2(K,\underline{\rho},\bar{\rho},n)(1+\|\rho_0\|_{1,2})t\|v_1-v_2\|_{C^0(\bar{I}_t,X_n)},$$
$$t\in I_{\tau_0},v_1,v_2\in B_{K,\tau_0}, \tag{2.56}$$

where M_2 is nonincreasing in the second and nondecreasing in the third variable. If we take :

$$\widetilde{T}=\frac{\min(\tau_0,\tau_1)}{1+\|\rho_0\|_{1,2}},\ \text{ where }\ \tau_1<\frac{1}{M_2},$$

then $\mathcal{T}_{\rho_0,m_0}$ maps $B_{K,\widetilde{T}}$ into itself and is a contraction. It therefore possesses in $B_{K,\widetilde{T}}$ a unique fixed point u, which satisfies (2.43).
The couple ($\rho=S_{\rho_0}u,u$) fulfils (2.26) and (1.1)-(1.3), or equivalently (2.43) and (1.1)-(1.3). Existence is thus proved. We observe that $\widetilde{T}$ has the form :

$$\widetilde{T}=\frac{M_3(K,\underline{\rho},\bar{\rho},T,n)}{1+\|\rho_0\|_{1,2}}, \tag{2.57}$$

where M_3 is nondecreasing in $\underline{\rho}$ and nonincreasing in $\bar{\rho}$.
Now this procedure can be iterated as many times as necessary to reach $T(n)=T$ as long as there is a bound on u_n independent of $T(n)$. The existence of such a bound will follow from estimates derived in the next section.

Global existence by energy estimate

In this section, we shall show that the a priori estimates obtained for exact solutions are compatible with our approximation scheme. Two types of bounds will be obtained :

1. estimates that are independent of time but may depend on the dimension n of X_n
2. estimates independent of n.

We start with the energy estimates. Before to differentiate in time (2.29) we recall that :

$$u_n\in C^0(\bar{I},X_n), \tag{2.58}$$

moreover we need to improve the regularity of u_n in the variable t, due to (2.43) we have :

$$\partial_t u_n=\mathcal{M}_\rho^{-1}\mathcal{M}_{\partial_t\rho_n}\mathcal{M}_{\rho_n}^{-1}\big(Pm_{0,n}+\mathcal{M}_\rho^{-1}P[\mathcal{N}(\rho_n,u_n)]. \tag{2.59}$$

Using (2.41) and (2.46) together with (1.20), (1.21) to evaluate the last expression, we get that :

$$\partial_t u_n\in L^2((0,T(n)),X_n). \tag{2.60}$$

Now differentiating (2.26) with respect to t and integrating by parts, one gets :

$$\int_\Omega \frac{\partial}{\partial t}(\rho_n u_n)\cdot\eta\,dx - \int_\Omega \rho_n u_n^i u_n^j \partial_j \eta^i dx + \mu\int_\Omega \partial_i u_n^j \partial_i \eta^j dx + (\mu+\lambda)\int_\Omega \operatorname{div} u_n \operatorname{div}\eta\,dx$$

$$= \int_\Omega [P(\rho_n)+\delta\rho_n^\beta]\operatorname{div}\eta\,dx - \varepsilon\int_\Omega (\nabla u_n\cdot\nabla\rho_n)\cdot\eta\,dx - \kappa\int_\Omega (\phi*\rho_n-\rho_n)\operatorname{div}\eta\,dx,$$

holds on $(0,T(n))$ for any $\eta\in X_n$. Moreover :

$$\int_\Omega \rho_{0,\delta}u_{0,\delta,n}(0)\cdot\eta dx = \int_\Omega m_{0,\delta}\cdot\eta dx \;\text{ for any }\; \eta\in X_n. \tag{2.61}$$

We now set :

$$\Pi(\rho) = \int_1^\rho \frac{P(z)}{z^2}dz.$$

Using integration by parts, employing (1.1), thanks to regularity (1.20), (2.58) and (2.60) for (ρ_n,u_n), we can jusitfy to take $\eta = u_n(t)$ and we obtain :

$$\frac{d}{dt}\int_\Omega \rho_n\big(\frac{1}{2}|u_n|^2 + E_{global}[\rho_n(\cdot,t)]\big)(x) + \Pi(\rho_n) + \frac{\delta}{\beta-1}\rho_n^{\beta-1}\big)dx$$

$$+\frac{\varepsilon}{2}\int_\Omega \Delta\rho_n|u_n|^2dx + \varepsilon\int_\Omega |\nabla\rho_n|^2\big(\frac{P'(\rho_n)}{\rho_n} + \delta\beta\rho_n^{\beta-2}+1\big)dx = \int_\Omega P(\rho_n)\operatorname{div}u_n\,dx \tag{2.62}$$

$$-\int_\Omega \big(S_n:\nabla u_n + \varepsilon(\nabla u_n\nabla\rho)\cdot u_n\big)dx - \varepsilon\int_{\mathbb{R}^N}\rho_n\Delta\phi*\rho_n dx.$$

Let us recall that :

$$P(\rho_n)\operatorname{div}u_n = -\operatorname{div}(\rho_n\Pi(\rho_n)u_n) - \partial_t(\rho_n\Pi(\rho_n)) + \varepsilon\Delta\rho_n\left(\Pi(\rho_n)+\frac{P(\rho_n)}{\rho_n}\right),$$

where :

$$\int_\Omega \Delta\rho_n\left(\Pi(\rho_n)+\frac{P(\rho_n)}{\rho_n}\right)dx = -\int_\Omega \frac{P'(\rho_n)}{\rho_n}|\nabla\rho_n|^2dx,$$

and :

$$\int_\Omega \partial_t(\rho_n u_n) - (\rho_n u_n\otimes u_n):\nabla u_n dx = \frac{d}{dt}\int_\Omega \frac{1}{2}\rho_n|u_n|^2dx + \frac{1}{2}\int_\Omega \big(\frac{d}{dt}\rho_n + \operatorname{div}(\rho_n u_n)\big)|u_n|^2dx$$

$$= \frac{d}{dt}\int_\Omega \frac{1}{2}\rho_n|u_n|^2dx + \frac{\varepsilon}{2}\int_\Omega \Delta\rho_n|u_n|^2dx$$

$$\begin{aligned}
&\kappa\int_{\mathbb{R}^N} u_n(t,x)\rho_n(t,x)\cdot\nabla([\phi*\rho_n(t,\cdot)](x)-\rho_n(t,x))dx\\
&\quad= -\kappa\int_{\mathbb{R}^N}\operatorname{div}(u_n(t,x)\rho_n(t,x))([\phi*\rho_n(t,\cdot)](x)-\rho_n(t,x))dx,\\
&\quad= \kappa\int_{\mathbb{R}^N}(\frac{\partial}{\partial t}\rho_n(t,x)-\varepsilon\Delta\rho_n)([\phi*\rho_n(t,\cdot))](x)-\rho_n(t,x))dx,\\
&\quad= -\frac{d}{dt}\int_{\mathbb{R}^N} E_{global}[\rho_n(t,\cdot)](x)dx - \varepsilon\int_{\mathbb{R}^N}\Delta\rho_n([\phi*\rho_n(t,\cdot))](x)-\rho_n(t,x))dx\,.
\end{aligned}$$

And we have :

$$\int_{\mathbb{R}^N}\Delta\rho_n([\phi*\rho_n(t,\cdot))](x)-\rho_n(t,x))dx = -\int_{\mathbb{R}^N}\partial_i\rho_n([\partial_i\phi*\rho_n(t,\cdot))](x)-\partial_i\rho_n(t,x))dx$$

$$= \int_{\mathbb{R}^N}\rho_n\Delta\phi*\rho_n dx + \int_{\mathbb{R}^N}|\nabla\rho_n|^2dx$$

To derive the last equality we use the relation :

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} E_{global}[\rho_n(t,\cdot)](x)dx &= \frac{\kappa}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(x-y)(\rho_n(t,y)-\rho_n(t,x))\frac{\partial}{\partial t}\rho_n(t,y)dydx \\ &+\frac{\kappa}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(y-x)(\rho_n(t,x)-\rho_n(t,y))\frac{\partial}{\partial t}\rho_n(t,x)dydx, \\ &=\kappa\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\phi(x-y)(\rho_n(t,y)-\rho_n(t,x))\frac{\partial}{\partial t}\rho_n(t,y)dydx, \\ &=-\kappa\int_{\mathbb{R}^N}([\phi*\rho_n(t,\cdot)](x)-\rho_n(t,x))\frac{\partial}{\partial t}\rho_n(t,x)dx\ .\end{aligned}$$

Now, equation (2.62) reads :

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \rho_n\big(\frac{1}{2}|u_n|^2+\Pi(\rho_n)+\frac{\delta}{\beta-1}\rho_n^{\beta-1}\big)dx+\delta\int_\Omega S_n:\nabla u_n dx \\ &\quad+\int_\Omega \varepsilon|\nabla\rho_n|^2\Big(\frac{P'(\rho_n)}{\rho_n}+\delta\beta\rho_n^{\beta-2}+1\Big)dx=-\varepsilon\int_{\mathbb{R}^N}\rho_n\Delta\phi*\rho_n dx.\end{aligned}\tag{2.63}$$

Note that we got rid of the integral $\varepsilon\int_\Omega \Delta\rho_n|u_n|^2dx$ thanks to the extra term $\varepsilon\nabla u_n\cdot\nabla\rho_n$ in (1.4).

Moreover, one can integrate (1.1) to recover the principle of total mass conservation :

$$\int_\Omega \rho(t)dx=\int_\Omega \rho_{0,\delta}dx \quad \text{for any} \ \ t\geq 0.$$

Additionally, integration of (2.63) in time reveals a modified energy equality :

$$\begin{aligned}&\int_\Omega \rho_n(\tau)\big(\frac{1}{2}|u_n|^2+E_{global}[\rho_n(\cdot,t)]\big)+\Pi(\rho_n)+\frac{\delta}{\beta-1}\rho_n^{\beta-1}\big)dx \\ &+\delta\int_0^\tau\int_\Omega S_n:\nabla u_n dx\,dt+\varepsilon\int_0^\tau\int_\Omega|\nabla\rho_n|^2\big(\frac{P'(\rho_n)}{\rho_n}+\delta\beta\rho_n^{\beta-2}+1\big)dx\,dt \\ &\leq\int_\Omega \frac{1}{2}m_{0,\delta}\cdot u_n(0)+\rho_{0,\delta}\Pi(\rho_{0,\delta})+\int_\Omega E_{global}[\rho_{0,\delta}]+\frac{\delta}{\beta-1}\rho_{0,\delta}^\beta dx,\end{aligned}\tag{2.64}$$

for any $\tau\in[0,T(n)]$.

By virtue of (1.12), (2.61), we have :

$$\int_\Omega m_{0,\delta}\cdot u_{0,\delta,n}(0)dx\leq\frac{1}{2}\int_\Omega\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}}+\rho_{0,\delta}|u_{0,\delta}|^2dx=\frac{1}{2}\int_\Omega\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}}+m_{0,\delta}\cdot u_{0,\delta}(0)dx,$$

We deduce that :

$$u_n \ \text{ is bounded in } \ L^2(0,T_n;W_0^{1,2}(\Omega,\mathbb{R}^N)),$$

by a constant that is independent of n and $T(n)\leq T$. Since all norms are equivalent on X_n, this implies that the approximate u_n are bounded in $L^1(0,T(n);W^{1,\infty}(\Omega,\mathbb{R}^N))$, in particular, by virtue of (1.21), the density ρ_n is bounded both from below and from above by a constant independent of $T(n)\leq T$.

Since ρ_n is bounded from below, one can use (2.64) to deduce uniform boundedness in t of u_n in the space $L^2(\Omega,\mathbb{R}^N)$. Consequently by equivalence of the norm, the functions $u_n(t)$

remain bounded in X_n for any t independently of $T(n) \leq T$.

Thus we are allowed to iterate the previous local existence result to construct a solution defined on the whole time interval $[0,T]$ and we have shown Proposition 2.33.

2.2 Proof of Proposition 2.34

Estimates independent of n

Our goal now is to identify a limit for $n \to +\infty$ of the approximate solutions ρ_n, u_n as a solution of problem (1.1)-(1.6). In order to achieve this, additionnal estimates are needed.

Proposition 2.36. *Assume that $\beta > \max\{4, \frac{N}{2}\}$.*
Then the approximate solutions constructed in proposition 2.33 satisfy the following estimates :

$$\begin{cases} \|\rho_n\|_{L^\infty(0,T,L^\gamma)} \leq \mathcal{E}(\delta), \\ \|\rho_n\|_{L^\infty(0,T,L^\beta)} \leq \mathcal{E}(\delta), \\ \sqrt{\varepsilon}\|\nabla\rho_n\|_{L^2(0,T\times\Omega)} \leq \mathcal{E}(\delta), \\ \|\rho_n\|_{L^{\beta+1}((0,T)\times\Omega} \leq \mathcal{E}(\delta,\varepsilon), \\ \|u_n\|_{L^2(0,T,W_0^{1,2}(\Omega,\mathbb{R}^N))} \leq \mathcal{E}(\delta) \end{cases} \tag{2.65}$$

and :

$$\|\sqrt{\rho_n}u_n\|_{L^\infty(0,T,L^2)} \leq c(\delta), \tag{2.66}$$

where all the constant are independent of n.

Proof :

To begin with, it is easy to see that the energy equation (2.64) yields :

$$\sqrt{\varepsilon}\rho_n^{\frac{\beta}{2}} \text{ bounded in } L^2(0,T;W^{1,2}(\Omega)).$$

Evoking the embedding $W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$ we get :
- ρ_n^β bounded in $L^1(0,T;L^{\frac{N}{N-2}}(\Omega))$ if $N \geq 3$,
- ρ_n^β bounded in $L^1(0,T;L^q(\Omega))$, $q > 1$ arbitrary finite if $N = 2$,

where these bounds depend only on δ and ε. Moreover we have :

$$\rho_n^\beta \text{ is bounded in } L^\infty(0,T;L^1(\Omega)).$$

Consequently as $\beta > \frac{N}{2}$ by interpolation, we have :

$$\|\rho_n\|_{L^{\beta+1}((0,T)\times\Omega)} \leq c(\varepsilon,\delta) \quad \text{independent of } n.$$

Indeed we have :

$$\int_\Omega \rho_n^{\beta+1}dx = \|\rho^\beta\|_{L^{\frac{\beta+1}{\beta}}(\Omega)}^{\frac{\beta+1}{\beta}} \leq \|\rho^\beta\|_{L^{\frac{N}{N-2}}(\Omega)}^{\frac{N}{2\beta}} \|\rho^\beta\|_{L^1(\Omega)}^{\frac{2\beta+2-N}{2\beta}} \quad \text{if } N = 3,$$

where we need $\frac{N}{2\beta} \leq 1$. The case $N = 2$ is similar.
Next, equation (1.1) multiplied by ρ_n yields :

$$\varepsilon \int_0^T \int_\Omega |\nabla \rho_n|^2 dxdt \leq C(T)(\|\rho_{0,\delta}\|^2_{L^2(\Omega)} + \|\rho_n\|^2_{L^\infty(0,T;L^4(\Omega))})\Big(\int_0^T \int_\Omega |\nabla \rho_n|^2 dxdt\Big)^{\frac{1}{2}}.$$

Consequently we deduce the estimate :

$$\sqrt{\varepsilon}\|\nabla \rho_n\|_{L^2((0,T)\times\Omega)} \leq c(\varepsilon, \delta),$$

provided $\beta \geq 4$.

The first level approximate solutions

At this stage we are ready to pass to the limit for $n \to +\infty$ in the sequence of approximate solutions $(\rho_n, u_n)_{n\in\mathbb{N}}$ in order to obtain a solution of the system (1.1)-(1.6). To this end, we observe that the system of test functions $(\varphi_j)_{j=1,\cdots,+\infty}$ forms a dense set in the space $C_0^1(\bar{\Omega}, \mathbb{R}^N)$.
It follows from equation (1.1) and the estimates obtained in proposition 2.36 that the time derivative $\partial_t \rho_n$ is bounded in the space $L^2(0,T;W^{-1,2}(\Omega))$ provided $\beta \geq N$. Consequently, one can use the Aubin-Lions lemma to deduce that the sequence $(\rho_n)_{n\in\mathbb{N}}$ contains a subsequence such that :

$$\rho_n \to \rho \quad \text{in} \quad L^\beta((0,T)\times\Omega), \tag{2.67}$$

where ρ is a non-negative function.
Moreover we have :

$$u_n \to u \quad \text{in} \quad L^2((0,T); W_0^{1,2}(\Omega, \mathbb{R}^N)), \tag{2.68}$$

where the limit velocity u satisfies the no-slip boundary condition (1.5) in the sense of traces.
Since the convergence in (2.67) is strong, we also have :

$$\rho_n u_n \to \rho u \ \text{ weakly in } L^\infty(0,T; L^{m_\infty}(\Omega, \mathbb{R}^N)) \quad \text{with} \ \ m_\infty = \frac{2\gamma}{\gamma+1}. \tag{2.69}$$

Lemma 8. *There exists $r > 1$ and $p > 2$ such that :*

- *$\partial_t \rho_n$, $\Delta \rho_n$ are bounded in $L^r((0,T)\times\Omega)$,*
- *$\nabla \rho_n$ is bounded in $L^p((0,T); L^2(\Omega, \mathbb{R}^N))$,*

independently of n. Accordingly, the function ρ belongs to the same class, satisfies equation (1.1) for a.a $(t,x) \in (0,T)\times\Omega$ together with the homogeneous Neumann boundary conditions in the sense of trace.

Proof :

We can write :

$$\operatorname{div}(\rho_n u_n) = \nabla \rho_n \cdot u_n + \rho_n \operatorname{div} u_n, \tag{2.70}$$

where, by virtue of estimate (2.65),

– $\nabla\rho_n \cdot u_n$ is bounded in $L^1(0,T;L^{\frac{N}{N-1}}(\Omega))$, for $N \geq 3$,
– $\nabla\rho_n \cdot u_n$ is bounded in $L^1(0,T;L^q(\Omega))$, for any $q < 2$ if $N = 2$,

and :

$$\rho_n \mathrm{div} u_n \ \text{ is bounded in } \ L^2(0,T;L^{\frac{2\beta}{\beta+2}}(\Omega,\mathbb{R}^N)).$$

The idea is to apply the $L^p - L^q$ estimates stated in proposition 1.30. Due to this, however, we need an extra bit of information concerning integrability of the first term of (2.70) in time.

Since $\rho_n u_n$ is bounded in $L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^N)) \cap L^2(0,T;L^{\frac{2N\beta}{2N+\beta(N-2)}}(\Omega,\mathbb{R}^N))$ for $N \geq 3$ and $\rho_n u_n$ is bounded in $L^\infty(0,T;L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^N)) \cap L^2(0,T;L^q(\Omega,\mathbb{R}^N))$ for any $q < 2$ if $N = 2$, one can take $\beta > N$ to obtain :

– $\rho_n u_n$ is bounded in $L^p(0,T;L^2(\Omega,\mathbb{R}^N))$ for a certain $p > 2$.

Indeed the interpolation inequality (for $N \geq 3$) yields :

$$\|\rho_n u_n\|_{L^2(\Omega,\mathbb{R}^N)} \leq \|\rho_n u_n\|^{\frac{2(\beta-N)}{2\beta-N}}_{L^{\frac{2\beta}{\beta+1}}(\Omega,\mathbb{R}^N)} \|\rho_n u_n\|^{\frac{N}{2\beta-N}}_{L^{\frac{2N\beta}{2N+\beta(N-2)}}(\Omega,\mathbb{R}^N)},$$

so we need $\frac{2\beta-N}{N} > 1$.

Now in using (L^p-L^q) estimates ρ_n is bounded in $L^p(0,T,W^{1,2}(\Omega))$. In particular, $\nabla\rho_n$ belongs to the space $L^q(0,T;L^q(\Omega,\mathbb{R}^N))$ for a certain $q > 2$.

Thus we have :

– $\mathrm{div}(\rho_n u_n)$ bounded in $L^r((0,T)\times\Omega)$, with a certain $r > 1$,

and the rest of the proof follows from the standart (L^p-L^q) estimates (1.15). □

The estimates obtained in lemma 8 together with those of proposition 2.36 can be used to deduce from (1.1)-(1.3) that the integral mean functions :

$$t \rightarrow \int_\Omega (\rho_n u_n.\eta_j)(t)dx \quad \text{forms a precompact system in } \ C([0,T])$$

for any fixed j. This implies that :

$$\rho_n u_n \rightarrow \rho u \ \text{ in } \ C([0,T];L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega;\mathbb{R}^N)).$$

As $\gamma > \frac{N}{2}$, the space $L^{\frac{2\gamma}{\gamma+1}}(\Omega)$ is compactly imbedded into $W^{-1,2}(\Omega)$, and consequently :

$$\rho_n u_n \otimes u_n \rightarrow \rho u \otimes u \ \text{ weakly in } \ L^2(0,T;L^{c_2}(\Omega,\mathbb{R}^{N^2}))$$

$$\text{whenever } \ 1 < c_2 \leq \frac{2N\gamma}{N+2\gamma(N-2)}$$

The functions ρ_n and ρ, being strong solutions of the problem (1.1), they satisfy the energy equality :

$$\|\rho_n(t)\|^2_{L^2} + 2\varepsilon\int_0^t \|\nabla\rho_n\|^2_{L^2}dt = -\int_0^t\int_\Omega \mathrm{div} u_n\,\rho_n^2 dxdt + \|\rho_{0,\delta}\|^2_{L^2},$$

and :

$$\|\rho(t)\|^2_{L^2} + 2\varepsilon\int_0^t \|\nabla\rho\|^2_{L^2}dt = -\int_0^t\int_\Omega \mathrm{div} u\,\rho^2 dxdt + \|\rho_{0,\delta}\|^2_{L^2}.$$

We deduce that :

$$\|\nabla\rho_n\|_{L^2}^2 \to \|\nabla\rho\|_{L^2}^2$$

and :

$$\|\rho_n(t)\|_{L^2}^2 \to \|\rho(t)\|_{L^2}^2 \quad \text{for any } t \in [0,T]$$

which yields strong convergence :

$$\nabla\rho_n \to \nabla\rho \ \text{ in } L^2((0,T)\times\Omega),$$

In particular,

$$\nabla u_n \cdot \nabla\rho_n \to \nabla u \cdot \nabla\rho \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

2.3 Proof of proposition 2.35

Our next goal is to let $\varepsilon \to 0$ in the modified continuity equation (1.1). To this end, let us denote by $\rho_\varepsilon, u_\varepsilon$ the corresponding solution of the approximate problems the existence of which was stated in proposition 2.34. At this stage of the proof, we definitely loose boundedness of $\nabla\rho_\varepsilon$ and, consequently, strong convergence of the sequence $(\rho_\varepsilon)_{\varepsilon>0}$ in $L^1((0,T)\times\Omega)$ becomes a central issue.

Local pressure estimates

We evoke the local pressure estimates discussed in chapter **??**. Since the data $\rho_{0,\delta}$, $m_{0,\delta}$ are fixed, the energy inequality (2.33) renders :

- ρ_ε is bounded in $L^\infty(0,T,L^\beta)$
- u_ε is bounded in $L^2(0,T,W_0^{1,2}(\Omega,\mathbb{R}^N))$

which, together with Sobolev embedding and Hölder's inequality, yields :

- $\rho_\varepsilon u_\varepsilon$ is bounded in $L^2(0,T,L^q(\Omega,\mathbb{R}^N))$ with $q > 2$

provided $\beta > N$.

Lemma 9. *For any compact set $K \subset ((0,T)\times\Omega)$, there is a constant $c = c(\delta,K)$ independent of ε such that :*

$$\delta \int_K \rho_\varepsilon^{\beta+1} dx\, dt \leq c(\delta,K) \tag{2.71}$$

Remark 3. *We notice that the estimate (2.71) is independent of ε.*

Proof :

Although the heuristic principle is the same as in chapter 4, the proof of (2.71) requires a slight modification to accommodate the extra terms in (1.1) and (1.4).
Set :

$$B_\omega = \rho_\varepsilon^\omega$$

In accordance with lemma 8, the functions ρ_ε, u_ε satisfy (1.1) a.e on $(0,T)\times\Omega$ together with the boundary conditions (1.3), in particular,

$$\partial_t\rho_\varepsilon + \mathrm{div}(\rho_\varepsilon u_\varepsilon) = \varepsilon \mathrm{div}(1_\Omega \nabla\rho_\varepsilon) \ \text{ in } \ \mathcal{D}'((0,T)\times\mathbb{R}^N), \tag{2.72}$$

provided ρ_ε, u_ε were extended to be zero outside of Ω. Consequently we have :

$$\partial_t B_\omega + \mathrm{div}(B_\omega u_\varepsilon) = h_\omega \quad \text{in } \mathcal{D}'(0,T)\times\mathbb{R}^N),$$

with :

$$h_\omega = \mathrm{div}(B_\omega u_\varepsilon) - \mathrm{div}([\rho_\varepsilon u_\varepsilon]^\omega) + \varepsilon\mathrm{div}(1_\Omega\nabla\rho_\varepsilon)^\omega$$

As a matter of fact, equation (1.4) contains the extra term $\varepsilon\nabla u_\varepsilon.\nabla\rho_\varepsilon$ which is however bounded in $L^1((0,T)\times\Omega)$ and does not cause any additional problem in the proof of theorem 2.13.

Vanishing viscosity limit

In accordance with the estimates established in proposition 2.34, specifially the energy inequality (2.33), we can assume that :

$$\begin{aligned} &\rho_\varepsilon \to \rho \ \text{ weakly in } \ L^\infty(0,T;L^\beta(\Omega)),\\ &u_\varepsilon \to u \ \text{ weakly in } \ L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^N)). \end{aligned} \tag{2.73}$$

Next we have :

$$\varepsilon\mathrm{div}(1_\Omega\nabla\rho_\varepsilon) \to 0 \ \text{ in } L^2(0,T,W^{-1,2}(\mathbb{R}^N))$$

Therefore :

$$\rho_\varepsilon \to \rho \ \text{ in } C([0,T],L^\beta_{weak}(\Omega))$$

Moreover :

$$\varepsilon\nabla u_\varepsilon.\nabla\rho_\varepsilon \to 0 \ \text{ in } L^1(0,T;L^1(\Omega,\mathbb{R}^N)).$$

As ρ_ε, u_ε satisfy the energy inequality (2.33), the momentum $(\rho u)_\varepsilon$ is bounded in $L^\infty(0,T,L^{\frac{2\beta}{\beta+1}}(\Omega;\mathbb{R}^N))$, whence :

$$(\rho u)_\varepsilon \to \rho u \ \text{ in } C([0,T],L^{\frac{2\beta}{\beta+1}}_{weak}(\Omega;\mathbb{R}^N))$$

provided ρ_ε, u_ε were extended to be zero outside Ω. In particular we have shown that the limit functions ρ, u satisfy the continuity equation :

$$\partial_t\rho + \mathrm{div}(\rho u) = 0 \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N).$$

Moreover as $L^{\frac{2\beta}{\beta+1}}$ is compactly imbedded into $W^{-1,2}(\Omega)$ as soon $\beta > \frac{N}{2}$, we infer that :

$$\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \rho u\otimes u \ \text{ weakly in } L^2(0,T;L^{c_2}(\Omega,\mathbb{R}^N)),$$

with $c_2 > 1$.
So we recover the momentum equation in the form :

$$\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla\overline{P(\rho)+\kappa\rho^2+\delta\rho^\beta} = \mathrm{div}\mathcal{S} + \kappa\nabla(\phi*\rho).$$

Strong convergence of densities

The next step will be to show strong convergence of the sequence of densities $(\rho_\varepsilon)_{\varepsilon>0}$ in $L^1((0,T)\times\Omega)$. We have to adapt the proof of chapter 4 on the density oscillations. In accordance with the estimates (2.34), we obtain similarly to the proof of the theorem 2.13 in chapter 4

Lemma 10. *We have :*

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \psi\eta(P(\rho_\varepsilon)+\kappa\rho_\varepsilon^2+\delta\rho_\varepsilon^\beta-(\lambda+2\mu)\mathrm{div}u_\varepsilon)\rho_\varepsilon dx\,dt$$
$$=\int_0^T\int_\Omega \psi\eta(\overline{P(\rho)}+\kappa\overline{\rho^2}+\delta\overline{\rho^\beta}-(\lambda+2\mu)\mathrm{div}u)\rho dx\,dt,$$

for any $\psi\in\mathcal{D}(0,T)$ provided $\beta>max(N,4,\gamma)$.

Now, since the limit functions satisfy the continuity equation in $\mathcal{D}^{'}((0,T)\times\mathbb{R}^N)$ and ρ belongs to $L^2((0,T)\times\mathbb{R}^N)$, we deduce that ρ is renormalized solution of the mass equation. In particular :

$$\partial_t(\rho\log\rho)+\mathrm{div}(\rho\log\rho u)+\rho\mathrm{div}u=0\;\;\mathcal{D}^{'}((0,T)\times\mathbb{R}^N). \tag{2.74}$$

On the other hand, by virtue of lemma 8, ρ_ε satisfies (1.1) a.a on the set $(0,T)\times\Omega$. Thus we are allowed to multiply by $B^{'}(\rho_\varepsilon)$ to obtain :

$$\partial_t B(\rho_\varepsilon)+\mathrm{div}(B(\rho_\varepsilon)u_\varepsilon)+(B^{'}(\rho_\varepsilon)\rho_\varepsilon-B(\rho_\varepsilon))\mathrm{div}u_\varepsilon$$
$$=\varepsilon\Delta B(\rho_\varepsilon)-\varepsilon B^{''}(\rho_\varepsilon)|\nabla\rho_\varepsilon|^2,$$

for any function $B\in C^2[0,\infty)$ with $B^{'}$, $B^{''}$ uniformly bounded. Moreover since ρ_ε satisfies the homogeneous Neumann boundary conditions and u_ε vanishes on $\partial\Omega$ in the sense of the traces, we have :

$$\begin{aligned}\not{\partial}_t B(\rho_\varepsilon)+\mathrm{div}(B(\rho_\varepsilon)u_\varepsilon)+(B^{'}(\rho_\varepsilon)\rho_\varepsilon-B(\rho_\varepsilon))\mathrm{div}u_\varepsilon\\ =\varepsilon\mathrm{div}(1_\Omega\nabla B(\rho_\varepsilon))-\varepsilon 1_\Omega B^{''}(\rho)|\nabla\rho_\varepsilon|^2\end{aligned} \tag{2.75}$$

provided $B(0)=0$.
If in addition, B is convex, we deduce :

$$\int_0^T\int_\Omega\psi(B^{'}(\rho_\varepsilon)\rho_\varepsilon-B(\rho_\varepsilon))\mathrm{div}u_\varepsilon dx\,dt\le\int_\Omega B(\rho_{0,\delta})dx+\int_0^T\int_\Omega\psi_t B(\rho_\varepsilon)dx\,dt$$

for any $\psi\in C^\infty[0,T]$, $\psi\ge 0$, $\psi(0)=1$, $\psi(T)=0$. Consequently, approximating $z\mapsto z\log z$ by a sequence of smooth convex functions we get :

$$\int_0^T\int_\Omega\psi\rho_\varepsilon\mathrm{div}u_\varepsilon dx\,dt\le\int_\Omega\rho_{0,\delta}\log(\rho_{0,\delta})dx+\int_0^T\int_\Omega\psi_t\rho_\varepsilon\log(\rho_\varepsilon)dx\,dt.$$

Here, the approximation of $z\log(z)$ can be taken, for instance as :

$$L_k(\rho)=z\int_1^\rho\frac{T_k(z)}{z^2}dz.$$

Passing to the limit for $\varepsilon \to 0$ we obtain :

$$\int_0^T \int_\Omega \psi \overline{\rho \mathrm{div} u} dx\, dt \leq \int_\Omega \rho_{0,\delta} \log(\rho_{0,\delta}) dx + \int_0^T \int_\Omega \psi_t \overline{\rho \log(\rho)} dx\, dt$$

from which we discover :

$$\int_0^\tau \int_\Omega \overline{\rho \mathrm{div} u} dx\, dt \leq \int_\Omega \rho_{0,\delta} \log(\rho_{0,\delta}) dx - \int_\Omega \overline{\rho \log(\rho)}(\tau) dx \tag{2.76}$$

for any Lebesgue point τ of the function $\overline{\rho \log(\rho)}$ a weak limit of the sequence $(\rho_\varepsilon \log(\rho_\varepsilon))_{\varepsilon \geq 0}$ in $L^\infty(0, T, L^q(\Omega))$, $q < \beta$.
One can use :

$$\varphi(t,x) = \psi(t)\eta(x),\ \psi \in \mathcal{D}(0,T),\ \psi \geq 0, \eta \geq 0, \eta \in \mathcal{D}(\mathbb{R}^N),\ \eta_\Omega = 1,$$

as a test function to obtain with (2.74) :

$$\int_0^\tau \int_\Omega \rho \mathrm{div} u dx\, dt = \int_\Omega \rho_{0,\delta} \log(\rho_{0,\delta}) dx - \int_\Omega \rho \log(\rho)(\tau) dx \tag{2.77}$$

for any $\tau \in [0, T]$. Note that the function :

$$t \to \int_\Omega \rho \log(\rho)(t) dx$$

is continuous on $[0, T]$.
Taking the sum of (2.76), (2.77) we arrive at inequality :

$$\int_\Omega (\overline{\rho \log \rho} - \rho \log \rho)(\tau) dx \leq \int_0^\tau \int_\Omega (\rho \mathrm{div} u - \overline{\rho\, \mathrm{div} u})\, dx\, dt \quad \text{for a.a } \tau \in [0, T], \tag{2.78}$$

where, by virtue of lemma 10,

$$\int_0^T \int_\Omega (\overline{\rho \mathrm{div} u} - \rho \mathrm{div} u) dx\, dt \geq \frac{1}{\lambda + 2\mu} \liminf_{\varepsilon \to 0} \int_0 \big[(\overline{P(\rho)\rho} + \delta \rho_\varepsilon^{\beta+1} + \kappa \overline{\rho^3}) - (\overline{P(\rho)} + \delta \overline{\rho^\beta} + \kappa \overline{\rho^2})\, \overline{\rho} \big]\, dx dt,$$

for any compact set $O \subset ((0,T) \times \Omega)$.
Now, as the function $z \to \delta z^\beta$ is increasing, we get :

$$\liminf_{\varepsilon \to 0} \int_0^T \int_\Omega \rho_\varepsilon^{\beta+1} - \overline{\rho}^\beta \rho dx dt \geq 0. \tag{2.79}$$

Futhermore, since non-linear composition of ρ_ε satisfy (2.75), together with estimate (2.33), yields :

$$B(\rho_\varepsilon) \to \overline{B(\rho)} \ \text{(strongly) in } L^2(0, T; W^{-1,2}(\Omega)), \tag{2.80}$$

for any function $B \in C^2[0, \infty)$ with B', B'' uniformly bounded.
Accordingly, relation (2.78) reduces to :

$$\int_\Omega \overline{\rho \log \rho} - \rho \log \rho)(\tau) dx \leq \frac{1}{2\mu + \lambda} \int_0^\tau \int_\Omega \overline{P(\rho)}\, \rho - \overline{P(\rho) \rho} dx.$$

The same argument as in the proof of theorem 1.12 can be used to conclude that :

$$\overline{\rho \log \rho} = \rho \log \rho \quad \text{a.a on } (0,T) \times \Omega,$$

which entails strong convergence :

$$\rho_\varepsilon \to \rho \ \text{ in } L^1((0,T) \times \Omega).$$

Consequently, the limit functions ρ and u satisfy the momentum equation (2.35) in $\mathcal{D}'((0,T)\times\Omega)$.

Proof of the theorem 2.28

Our ultimate goal in the proof of theorem 2.28 is to carry out the limit process when the parameter δ tends to zero. Denote ρ_δ, u_δ the corresponding approximate solutions constructed in proposition 2.35.

Energy estimates

In light of the energy inequality stated in proposition 2.35, we have :

$$\begin{aligned}
&\bullet\ \rho_\delta \ \text{ bounded in } \ L^\infty(0,T;L^\gamma(\Omega)),\\
&\bullet\ \sqrt{\rho_\delta}u_\delta \ \text{ bounded in } \ L^\infty(0,T;L^2(\Omega,\mathbb{R}^N)),\\
&\bullet\ u_\delta \ \text{ bounded in } \ L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^N)).
\end{aligned} \tag{2.81}$$

Strong convergence of densities

Having established all necessary estimates we adress the question of convergence. As the approximate sloutions ρ_δ, u_δ satisfy the equation of continuity (1.1), and the momentum equation (1.4) without any extra terms, the result obtained in chapter 4 can be used without modification.
It follows from estimates (2.81) that if $\omega > 0$ then :

$$\int_O P(\rho_\delta)\rho_\delta^\omega \, dxdt \le c((0,T)\times\Omega),$$

for any compact set $O \subset (0,T)\times\Omega$. This implies local estimates :

$$\|\rho_\delta\|_{L^{\gamma+\omega}(O)} \le c(O), \tag{2.82}$$

and :

$$\delta \int_O \rho_\delta^{\beta+\omega} \, dxdt \le c(O), \tag{2.83}$$

for any compact set $O \subset (0,T)\times\Omega$.
In view of the above estimates, we may assume that, up to a subsequence,

$$\rho_\delta \to \rho \ \text{ in } C([0,T], L^\gamma_{weak}(\Omega)),$$

$$u_\delta \to u \text{ weakly in } L^2(0,T,W_0^{1,2}(\Omega)).$$

Moreover, thanks to our choice of initial data $\rho_{0,\delta}$,

$$\rho(0,x) = \rho_0(x) \text{ a.a on } \Omega,$$

and

$$\delta \int_\Omega \frac{1}{\beta - 1}\rho_{0,\delta}^\beta dx \to 0 \text{ for } \delta \to 0.$$

As we have already observed, the space $L^\gamma(\Omega)$ is compactly embedded into $W^{-1,2}(\Omega)$ and, consequently :

$$\rho_\delta u_\delta \to \rho u \text{ weakly}(-^*) \text{ in } L^\infty(0,T;L^{m_\infty}(\Omega,\mathbb{R}^N)), \tag{2.84}$$

with :

$$m_\infty = \frac{2\gamma}{\gamma+1}.$$

The bound on $\partial_t(\rho_\delta u_\delta)$ resulting from the momentum equation in (2.35) can be used to strengthen (2.84) to :

$$\rho_\delta u_\delta \to \rho u \text{ in } C([0,T],L^{m_\infty}_{weak}(\Omega,\mathbb{R}^N)),$$

which yields, because of compact embedding L^{m_∞} into $W^{-1,2}$, convergence of the convective terms :

$$\rho_\delta u_\delta \otimes u_\delta \to \rho u \otimes u \text{ weakly in } L^2(0,T;L^{c_2}(\Omega,\mathbb{R}^{N^2})),$$

with $c_2 > 1$. Moreover,

$$\rho u(0,x) = m_0(x) \text{ a.a on } \Omega.$$

In order to establish strong convergence of the sequence $(\rho_\delta)_{\delta>0}$, we pursue the approach developped in chapter 4. More specially, a direct application of chapter 4 yields :

$$\begin{aligned}\lim_{\delta\to 0}\int_0^T\int_\Omega \psi\eta(P_\delta - (\lambda+2\mu)\text{div}u_\delta)T_k(\rho_\delta)dxdt \\ = \lim_{\delta\to 0}\int_0^T\int_\Omega \psi\eta(\overline{P_\delta} - (\lambda+2\mu)\text{div}u_\delta)\overline{T_k(\rho)}dxdt,\end{aligned} \tag{2.85}$$

where, as usual,

$$P(\rho_\delta) \to \overline{P(\rho)} \text{ weakly in } L^{\frac{\gamma+\omega}{\gamma}}(O),$$

and :

$$T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0,T];L^q_{weak}(\Omega)) \text{ for any } q \geq 1,$$

for any compact set $O \subset (0,T)\times\Omega$. Note that :

$$\delta\rho_\delta^\beta \to 0 \text{ in } L^1((0,T)\times\Omega)$$

as a consequence of (2.83).

The limit function ρ is a renormalized solution of the mass equation in sense of Diperna-Lions. Finally, the result on propagation of oscillation stated in chapter 4 can be now used in order to conclude that :

$$\rho_\delta \to \rho \text{ strongly in } L^1((0,T)\times\Omega), \tag{2.86}$$

which can be improved, similarly to chapter 4 to :

$$\rho_\delta \to \rho \ \text{ in } C([0,T], L^1(\Omega)). \tag{2.87}$$

Consequently, the limit function ρ, u satisfy the mass equation :

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N),$$

as well as in the sense of renormalized solutions. In other words, they represent a variational solution of equation (1.1)-(1.6) in the sense of definition of chapter 4.
Similarly, the momentum equation :

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\operatorname{div} u + \nabla P(\rho) = \kappa\rho\nabla(\phi * \rho - \rho),$$

is satisfied in $\mathcal{D}'((0,T)\times\Omega)$.
Finally, the limit quantities ρ, u satisfy the energy inequality :

$$\int_\Omega \rho(\tau)(\frac{1}{2}|u|^2 + P(\rho))(\tau)dx \leq \int_\Omega \big(\frac{|m_0|^2}{\rho_0} + \rho_0 P(\rho_0)\big)dx$$

for a.a. $\tau \in [0,T]$, which can easily be verified.

Passage to the case $\Omega = \mathbb{R}^N$

In order to treat the case $\Omega = \mathbb{R}^N$, we then consider (ρ_R, u_R) the solution of $(1.1)-(1.6)$ set in the ball $B(0,R)$ with Dirichlet boundary conditions on ∂B_R. We choose $R > R_0$ so that $\int_{B_R} \rho_0 dx > 0$. In particular, we have for almost all $t \in (0,T)$:

$$\int_{B_R} \big(\frac{1}{2}\rho_R|u_R|^2 + \frac{a}{\gamma-1}\rho_R^\gamma\big)\, dx + \int_0^t ds \int_{B_R} \big(\mu|Du_R|^2 + \xi(\operatorname{div} u_R)^2\big)\, dx \leq 0.$$

The conservation of mass and this inequality yield, as usual, bounds uniform in R on ρ_R in $L^\infty(0,T;L^1\cap L^\gamma(B_R))$, on $\rho_R|u_R|^2$ in $L^\infty(0,T;L^1(B_R))$ and on Du_R in $L^2((0,T)\times B_R)$. If $N=2$, we have a bound on u_R in $L^2(0,T;L^q(B_M))$ for all $1 \leq q < +\infty$, $M \in (0,+\infty)$ and in $L^2(0,T;BMO)$ considering u_R as a function on $\mathbb{R}^2$ by extending it to $\mathbb{R}^2$ by 0.
Next, as in the chapter 4 we show that ρ_R is bounded in $L^p((0,T)\times B_M)$ for any $M \in (0,\infty)$, with $p = \gamma + \frac{2}{N}\gamma - 1$, uniformly in $R \geq 1 + M$.
It only remains to apply the compactness analysis of chapter 4 in order to recover a solution of the Rohde system in whole space.

Bibliographie

[1] H. Amann, Linear and quasilinear parabolic problem, VolumeI, II. Birkhaüser, Basel (1995).

[2] E. Feireisl, Dynmamics of Viscous Compressible Fluids-Oxford Lecture Series in Mathematics and its Applications-26.

[3] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 1,

[4] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol 2, Compressible models, Oxford University Press (1996)

[5] A. Novotný and I. Straškraba. Introduction to the mathematical theory of compressible flow, Oxford lecture series in mathematics and its application, 27.

[6] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. ZAMMZ. Angew. Math. Mech. 85(2005), no. 12, 839-857.

Chapitre 7

Perspectives

Suite à de nombreux travaux très récents comme ceux de Bresch, Desjardins, Mellet, Varet, Vasseur et tant d'autres, de nombreuses avancées en mécanique des fluides ont été réalisées et ouvrent ainsi de nouvels horizons. Ainsi les problèmes liés à l'existence de solutions faibles pour le système de Navier-Stokes compressible ainsi que pour le système de Korteweg, ont abondamment progressé depuis deux trois ans et laissent place encore à de nombreuses questions ouvertes. Dans ce dernier chapitre, je vais donner quelques directions envisagées pour des travaux ultérieurs et exposer différents problèmes encore ouverts et faisant suite aux travaux de certains auteurs précités plus haut.

1 Problème des équations du système de Korteweg

Nous allons commencer par considérer le sysème de Korteweg. Rappelons la forme du système (voir [11]) dans le cas isotherme :

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho\, u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}\big(2\mu(\rho) D(u)\big)u - \nabla\big(\lambda(\rho)\mathrm{div} u\big) + \nabla(P(\rho)) = \kappa\rho\nabla\Delta\rho \\ (\rho_{/t=0}, u_{/t=0}) = (\rho_0, u_0) \end{cases}$$

Je vais exposer ici quelques projets d'étude qui font suite aux résultats obtenus dans cette thèse.

- A l'heure actuelle le problème reste complètement ouvert en ce qui concerne l'existence de solutions faibles pour des coefficients de viscosité constants même en dimension $N = 2$. Cependant nous avons vu qu'il suffit en dimension $N = 2$ de contrôler le vide (i.e $\frac{1}{\rho}$ en norme L^∞) pour obtenir des solutions faibles globales à condition de prendre des données initiales petites. Il peut être alors intéressant d'étudier le comportement asymptotique en temps du système de Korteweg. Effectivement le but serait de vérifier si le vide peut être contôler en temps grand, ceci en utilisant le gain de régularité sur la densité obtenu dans le troisième chapitre . En fait on s'attend à ce que la densité soit proche d'un état constant. Ainsi essayer d'avoir un comportement asymptotique en temps de nos solutions pourrait nous permettre d'obtenir l'existence globale de solutions faibles à partir d'un temps assez grand.

- D'autre part on a vu que dans le cas de coefficients de viscosité de type Saint-Venant, Bresch, Desjardins et Lin dans [1] obtiennent l'existence de solutions faibles pour des fonctions tests dépendant de la densité et s'écrivant sous la forme $\rho\varphi$ avec $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N)$. On a vu en utilisant les résultats du second chapitre que l'on peut obtenir de véritables solutions faibles en temps grand pour $N=2$ car l'on contrôle le vide, il peut être alors très intéressant de connaitre exactement le comportement asymptotique des solutions en déterminant précisément leur décroissance en temps toujours en dimension $N=2$ afin de savoir à partir de quel temps T les solutions de Bresch, Desjardins et Lin deviennent de véritables solutions faibles.

  On peut aussi plus généralement étudier en dimension quelconque le comportement asymptotique des solutions fortes avec données petites du système de Korteweg dans les espaces de Besov critiques. Ce genre de résultat est à lier aux résultats de Hoff et Zumbrum [7],[8] et ceux de T. Kobayashi et Y. Shibata [9], [10] dans le cas du système de Navier-Stokes compressible.
- Enfin je m'intéresse aussi actuellement à construire des solutions approchées globales du système de Korteweg qui puissent vérifier les inégalités d'énergie du troisième chapitre en dimension $N=1$ et 2.

2 Problème des équations du système de Rohde

Nous allons maintenant donner quelques prolongements envisagés pour le système de Rohde (voir [63]) que nous rappelons :

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho\, u) + \mathrm{div}(\rho u\otimes u) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div}u) + \nabla(P(\rho)) = \kappa\rho\nabla D[\rho] \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0) \end{cases}$$

avec $D[\rho] = \phi * \rho - \rho$ représentant le terme de capillarité non local.
Je présente ici quelques travaux en cours.

- Dans un travail en cours j'étudie le lien entre le système de Rohde et celui de Korteweg, c'est à dire comment peut-on passer d'un système à l'autre en faisant varier la fonction ϕ. C'est en fait d'un point de vue physique expliciter les différences de comportement entre une méthodes à interfaces discontinues et une à interfaces diffuses. La difficulté pour relier ces deux systèmes est que les solutions fortes obtenues dans le cas du système de Rohde sont moins régulières que celles du système de Korteweg. Effectivement on perd l'effet régularisant dû à la capillarité en $\kappa\rho\nabla\Delta\rho$ en haute fréquence, et cela est dû au fait que dans le cas du système de Rohde $\hat{\phi}$ tends vers 0 à l'infini. Il s'agit donc de relier les deux système en fonction du support de $\hat{\phi}$.
- Dans le cas où la capillarité κ est nulle, et en choisissant des coefficients de viscosité

comme ceux de Bresch et Desjardins, il serait intéressant de prolonger les résultats de Bresch, Desjardins, Mellet et Vasseur à des données proches d'un état d'équilibre, comme c'est le cas dans le quatrième chapitre où l'on travaille dans des espaces de Orlicz.

– Il reste aussi à voir si on peut étendre les résultats de Bresch, Desjardins sur le système de Rohde et ainsi expliciter des solutions faibles avec des données initiales plus régulières qui se rapprocherainet plus des données initiales concernant les solutions fortes.
– De même, je vais m'intéresser également à trouver des solutions approchées globales pour le système de Rohde afin d'obtenir un résultat d'existence globale de solutions faibles, le quatrième chapitre étant un résultat de stabilité.
– Sinon il reste à voir si au niveau des solutions fortes on peut obtenir l'existence de solutions fortes dans les espaces de Besov critique pour le scaling sans imposer une condition de petitesse du type $\|\rho_0 - \bar{\rho}\|_{B^{\frac{N}{p}}} \leq \varepsilon$ comme c'est le cas dans le cinquième chapitre.
– Enfin comme pour le système de Korteweg je m'intéresse au comportement asymptotique en temps des solutions fortes avec données initiales petites.

3 Problèmes ouverts actuels

Je vais ici présenter de nombreux problèmes liés à ces équations restant encore ouvert et faisant suite aux travaux de Bresch, Desjardins, Mellet, Vasseur et autres, nous pouvons citer ainsi les cas suivants :

– Mellet et Vasseur ont monté un théorème de stabilité pour les coefficients de Bresch et Desjardins (voir [2]) incluant notamment le cas de Saint Venant, par contre l'existence de solutions approchées vérifiant leurs inégalités d'énergie reste ouvert. Effectivement ces inégalités étant fortement non linéaires, il est difficile de trouver des solutions régulières conservant les bornes uniformes sur la vélocié u. Ainsi les solutions approchées de Bresch et Desjardins dans [3] ne sont pas adaptées à l'inégalité d'énergie sur u, effectivement on ne peut contrôler le terme régularisant en $\varepsilon\rho_\varepsilon\nabla(\mu^{'}(\rho_\varepsilon\Delta^s\mu(\rho_\varepsilon))$ avec $s > 1$.
– Bresch et Desjardins montrent actuellement l'existence de solution forte en dimension $N = 2$ pour le système de Navier-Stokes compressible avec leur choix de coefficients de viscosité et de pression (voir [2]).
– Alazard, Bresch, Desjardins étudient le comportement du nombre de mach des solutions faibles globales de [2].
– Un problème restant ouvert est celui de l'unicité fort-faible, peut-on obtenir des résultats dans le cadre des solutions faibles de Bresch-Desjardins dans [2].
– Enfin un problème ouvert intéressant est celui de l'existence de solutions fortes avec des données initiales comportant du vide pour le système de Korteweg. Peut-on ainsi

avoir un pendant des résultats de Choe et Kim ainsi que Cho et Choe [4], [5], [6] qui sont dans le cas du système de Navier-Stokes compressibles.

– Un autre travail peut être d'étudier le comportement asymptotique des solutions faibles de Bresch et Desjardins dans [2] comme le font Novotný et Straškraba [12] pour le cas des solutions faibles de Feireisl.

Bibliographie

[1] D. Bresch, B. Desjardins and C.-K. Lin, On some compressible fluid models : Korteweg,lubrication and shallow water systems. Comm. Partial Differential Equations, 28(3-4) : 843-868, 2003.

[2] D. Bresch and B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, Journal de Mathématiques Pures et Appliqués, Volume 87, Issue 1, January 2007, Pages 57-90.

[3] D. Bresch and B. Desjardins, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models,Journal de Mathématiques Pures et Appliqués Volume 86, Issue 4, October 2006, Pages 362-368.

[4] H. J. Cho and H. Kim, Existence results for viscous polytropic fluids with vacuum, J. Differential Equations 228 (2006) 377-411.

[5] Y. Cho, H.J Choe and H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, Journal de Mathématiques pures. 83(2), (2004), 243-275(33).

[6] H. J. Choe and H. Kim, Strong solution of the Navier-Stokes equations for isentropic compressible fluids, J. Differential Equations 190 (2003), 504-523.

[7] David Hoff and Kevin Zumbrum. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow. Indiana University Maathematics Journal, Vol 44, No. 2 (1995), p603-677.

[8] David Hoff and Kevin Zumbrum. Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves. Zeitschrift für angewandte mathematik un physik, 48 (1997), 597-614.

[9] T. Kobayashi and Y. Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbb{R}^3$. Communications in Mathematical Physics, 1999, 200, 621-660.

[10] T. Kobayashi and Y. Shibata, Remark on the rate of decay of solutions to linearized compressible Navier-Stokes equations. Pacific journal of mathematics. Vol 207, No 1 (2002), p 199-234.

[11] D.J. Korteweg. Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité. Arch. Néer. Sci. Exactes Sér. II, 6 : 1-24, 1901.

[12] A. Novotný and I. Straškraba. Introduction to the mathematical theory of compressible flow, Oxford lecture series in mathematics and its application. 27 (2004)

[13] C. Rohde, On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. ZAMM Vol. 85, No. 12 (2005).