Chapter 2: Brownian motion as a Markov process

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From now on, a BM is a Gaussian process $(B_t)_{t\geq 0}$ with continuous paths and independent Gaussian homogenous increments; in particular

$$B_{t_1},\ B_{t_2}-B_{t_1},\ \ldots,\ B_{t_n}-B_{t_{n-1}}$$

are independent Gaussian r.v., centred, with variance $t_i - t_{i-1}$.

1 The Markov property

Informally, $\mathcal{F}_t^- = \sigma(B_s,\, s\leq t)$, $\mathcal{F}_t^+ = \sigma(B_s,\, s\geq t)$. The law of a $\mathcal{F}_t^+$-measurable r.v., conditional on $\mathcal{F}_t^-$, only depends on the r.v. B_t.

Let us make this statement more precise. For any $t<t_1<\cdots<t_n$, we have

$$\begin{aligned}&\mathbb{E}\big[f(B_{t_1},B_{t_2},\ldots,B_{t_n})\mid \mathcal{F}_t^-\big]\\&=\mathbb{E}\big[f\big(B_t+(B_{t_1}-B_t),\,(B_{t_2}-B_t)+B_t,\ldots,(B_{t_n}-B_t)+B_t\big)\mid \mathcal{F}_t^-\big].\end{aligned}$$

Observe first that $(B_{t_i} - B_t)_{i\le n}$ is independent of $\mathcal{F}_t^-$. This follows from the fact that, for every $0 \le s_1 \le \cdots \le s_m \le t$,

$$(B_{s_1}, B_{s_2}, \dots, B_{s_m}) \text{ is independent of } (B_{t_1} - B_t,\, B_{t_2} - B_t, \dots, B_{t_n} - B_t),$$

as a consequence of the basic property $\mathbb{E}[B_{s_i} \cdot (B_{t_j} - B_t)] = 0$, that carries over to the concatenated vector is Gaussian, so that uncorrelation implies independence in a Gaussian space. As a consequence

$$\mathbb{E}\big[f(B_{t_1}, B_{t_2}, \dots, B_{t_n}) \mid \mathcal{F}_t^-\big] = \varphi(B_t) \text{ a.s.},$$

where

$$\begin{aligned}\varphi(x) &= \mathbb{E}\big[f(x + B_{t_1} - B_t,\, x + B_{t_2} - B_t, \dots, x + B_{t_n} - B_t)\big]\\ &= \mathbb{E}\big[f(x + B_{t_1-t},\, x + B_{t_2-t}, \dots, x + B_{t_n-t})\big],\end{aligned}$$

the last equality using the stationarity of the increments. We end the proof by a monotone-class argument: the family of bounded $\mathcal{F}_t^+$-measurable Z such that $\mathbb{E}[Z \mid \mathcal{F}_t^+] = \mathbb{E}[Z \mid \sigma(B_t)]$ is a vector space, stable under increasing limits, and contains all bounded $f(B_{t_1}, \dots, B_{t_n})$; this proves the general statement.

We say Brownian motion is thus a homogeneous Markov process. More generally:

$$\mathbb{E}\big[F(B_{t+u},\, u \ge 0) \mid \mathcal{F}_t^-\big] = \mathbb{E}_{\mathbb{W}_{B_t}}\big[F(B_u,\, u \ge 0)\big],$$

where W_x denotes the law of a Brownian motion started at x, i.e. the law of $\widetilde{B}_t = x + B_t$ for B a standard BM. As a probability measure on $(\mathcal{C}(\mathbb{R}_+, \mathbb{R}), \mathfrak{C})$, we can see W_x as the push-forward of the Wiener measure $\mathbb{W}$ under the shift map

$$\tau_x : \mathcal{C}(\mathbb{R}_+, \mathbb{R}) \to \mathcal{C}(\mathbb{R}_+, \mathbb{R}), \quad \omega \mapsto x + \omega :$$

$$\mathbb{W}_x = \tau_x \,\#\, \mathbb{W}.$$

We wrap-up this discussion ion the following

Proposition 1 (Simple Markov property). *For $s \ge 0$, the process $(\widetilde{B}_t = B_{t+s} - B_s)_{t\ge 0}$ is a Brownian motion, independent of $\mathcal{F}_s$.*

Proof. $\widetilde{B}$ is Gaussian, centered, with continuous paths; $\widetilde{B}_0 = 0$ and $\mathbb{E}[\widetilde{B}_t \widetilde{B}_{t'}] = t \wedge t'$. Hence $\widetilde{B}$ is a BM. It remains to check independence: for this; it suffices to show that, for $0 \le t_1 \le \cdots \le t_n$ and $0 \le s_1 \le \cdots \le s_m \le s$,

$$(\widetilde{B}_{t_1}, \dots, \widetilde{B}_{t_n}) \text{ independent of } (B_{s_1}, B_{s_2}, \dots, B_{s_m}),$$

which holds since $\mathbb{E}\big[(B_{s+t_i} - B_s) B_{s_j}\big] = 0$ for $s \ge s_j$ (the concatenated Gaussian vector being uncorrelated is thus independent, componentwise). $\square$

2 The semigroup of Brownian motion

We address the following problem: how can we compute $\mathbb{W}_x(\Gamma)$ for Γ as general as possible. We abbreviate $\mathbb{E}_{\mathbb{W}_x}$ by $\mathbb{E}_x$ and start with

$$\mathbb{E}_x\big[f(X_{t_1},\dots,X_{t_n})\big],$$

(with f bounded for instance) and $(X_t)_{t\geq 0}$ denotes the canonical process on the Wiener space.

Start with $n=1$.

$$\begin{aligned}
\mathbb{E}_x\big[f(X_t)\big] &= \mathbb{E}_0\big[f(x+X_t)\big]\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} dy\, e^{-\frac{1}{2}y^2/t}\, f(x+y)\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} dy\, e^{-(y-x)^2/(2t)}\, f(y)\\
&= \int_{\mathbb{R}} p_t(x,dy)\, f(y) = P_t f(x),
\end{aligned}$$

say, where

$$p_t(x,dy) = \frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{1}{2t}(y-x)^2\Big)\, dy$$

is the kernel of the linear operator P_t that acts on bounded functions f (at least).

The case $n=2$. For $s<t$,

$$\begin{aligned}
\mathbb{E}_x\big[f(X_s)g(X_t)\big] &= \mathbb{E}_x\big[f(X_s)\, g(X_s+(X_t-X_s))\big]\\
&\overset{\text{Markov property}}{=} \mathbb{E}_x\big[f(X_s)\,\mathbb{E}_{X_s}[g(X_{t-s})]\big]\\
&= \mathbb{E}_x\big[f(X_s)\, P_{t-s}g(X_s)\big]\\
&= \int_{\mathbb{R}} P_s(x,dy)\, f(y)\int_{\mathbb{R}} P_{t-s}(y,dz)\, g(z)\\
&= \int_{\mathbb{R}} \frac{dy}{\sqrt{2\pi s}}\, e^{-(y-x)^2/(2s)}\, f(y)\int_{\mathbb{R}}\frac{dz}{\sqrt{2\pi(t-s)}}\, e^{-(z-y)^2/(2(t-s))}\, g(z).
\end{aligned}$$

The general case. We can iterate the above computation, applying repeatedly the Markov property to obtain

$$\mathbb{E}_x\big[f_1(X_{t_1})\cdots f_n(X_{t_n})\big] = \mathbb{E}_x\Big[f_1(X_{t_1})\cdots f_{n-1}(X_{t_{n-1}})\,\underbrace{\int_{\mathbb{R}} P_{t_n-t_{n-1}}(X_{t_{n-1}},dy_n)\, f_n(y_n)}_{=\mathbb{E}_{X_{t_{n-1}}}[f_n(X_{t_n-t_{n-1}})]}\Big]$$

to finally obtain the formula

$$\begin{aligned}
&\mathbb{E}_x\big[f_1(X_{t_1})\cdots f_n(X_{t_n})\big]\\
&= \int_{\mathbb{R}} P_{t_1}(x,dy_1)\, f_1(y_1) \int_{\mathbb{R}} P_{t_2-t_1}(y_1,dy_2)\, f_2(y_2)\cdots \int_{\mathbb{R}} P_{t_n-t_{n-1}}(y_{n-1},dy_n)\, f_n(y_n).
\end{aligned}$$

The formula is actually valid in a general Markovian context and is related to the so-called

Lemma 2 (The Chapman–Kolmogorov equation)**.** *We have, for any $s,t>0$.*

$$p_{t+s}(x,dy) = \int_{\mathbb{R}} p_s(x,dz)\, p_t(z,dy).$$

At the operator level, this expresses the semigroup property

$$P_{t+s} = P_t P_s = P_s P_t.$$

Proof. For a bounded function f, we have, from the preceding computations

$$\begin{aligned}
P_{t+s}f(x) &= \int P_{t+s}(x,dy)\, f(y) = \mathbb{E}_x\big[f(X_{t+s})\big] = \mathbb{E}_x\big[f\big(X_s + (X_{t+s}-X_s)\big)\big]\\
&= \mathbb{E}_x\big[\mathbb{E}_{X_s}[f(X_t)]\big] = \mathbb{E}_x\big[P_t f(X_s)\big] = P_s\big(P_t f\big)(x).
\end{aligned}$$

□

We end this section by gathering some useful (and deep) results about the Brownian semigroup P_t, that are quite easy to obtain in our setting. Introduce the functions spaces

$$\begin{aligned}
C_b(\mathbb{R}) &= \{f:\mathbb{R}\to\mathbb{R},\ f \text{ continuous and bounded}\},\\
C_0(\mathbb{R}) &= \Big\{f\in C_b(\mathbb{R}):\ \lim_{|x|\to\infty} f(x)=0\Big\},\\
C_c(\mathbb{R}) &= \{f\in C^2(\mathbb{R}):\ f \text{ has compact support}\},
\end{aligned}$$

where for integer $k\geq 0$, $C^k(\mathbb{R})$ denotes the space of functions that are k-fold continuously differentiable. As, usual we set[1] as usual $\|f\|_\infty = \sup_{x\in\mathbb{R}}|f(x)|$.

[1] No need to take essential supremum if we work with continuous functions, we still have a valid norm on $C^k(\mathbb{R})$ (possibly taking the value ∞).

Proposition 3. *(i) For every $f \in C_0(\mathbb{R})$, $P_t f \in C_0(\mathbb{R})$, $\|P_t f\|_\infty \le \|f\|_\infty$, and $P_t(P_s f) = P_{t+s} f$.*

(ii) ***(Feller property)*** *$f \in C_0(\mathbb{R}) \implies P_t f \in C_0(\mathbb{R})$, and $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$.*

(iii) ***(Infinitesimal generator)*** *If $f \in C_c^2$, then*

$$\lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} = \tfrac{1}{2} f''(x).$$

(iv) ***(Heat equation)*** *If $f \in C_b(\mathbb{R})$, then $u(t,x) = P_t f(x)$ is continuous on $[0,\infty) \times \mathbb{R}$ and C^∞ on $(0,\infty) \times \mathbb{R}$; moreover $u(0,x) = f(x)$ and*

$$\partial_t u = \frac{1}{2} \partial_x^2 u, \qquad t > 0.$$

Proof. (i) From $P_t f(x) = \mathbb{E}_x[f(x + B_t)]$, we obtain continuity by Lebesgue's dominated convergence theorem.

(ii) For $f \in C_0(\mathbb{R})$, $P_t f \in C_0(\mathbb{R})$: indeed $\lim_{|x|\to\infty} \mathbb{E}[f(x+B_t)] = 0$ by dominated convergence. Moreover, $f \in C_0(\mathbb{R}) \implies f$ is uniformly continuous on $\mathbb{R}$, hence

$$\|P_t f - f\|_\infty \le \mathbb{E}\Big[\sup_x |f(x + B_t) - f(x)|\Big] \xrightarrow[t\to 0]{} 0$$

by dominated convergence (using uniform continuity of f and $B_t \to 0$ a.s. along a suitable sequence, or directly since the modulus of continuity of f evaluated at $|B_t|$ tends to 0 a.s. and is bounded).

(iii) Let $f \in C_c^2(\mathbb{R})$; since f'' is bounded, say by $K > 0$, Taylor's formula with Lagrange remainder gives, for $x, z \in \mathbb{R}$,

$$f(x+z) = f(x) + z f'(x) + \frac{z^2}{2} f''(\xi) \qquad \text{for some } \xi \text{ between } x \text{ and } x+z,$$

so that $\big|f(x+z) - f(x) - z f'(x)\big| \le \frac{K}{2} z^2$. Using $\mathbb{E}[B_t] = 0$, it follows that

$$\begin{aligned}\frac{P_t f(x) - f(x)}{t} &= \frac{1}{t}\, \mathbb{E}\big[f(x+B_t) - f(x)\big] = \frac{1}{t}\, \mathbb{E}\big[f(x+B_t) - f(x) - f'(x) B_t\big] \\ &= \mathbb{E}\Big[\underbrace{\frac{f(x+\sqrt{t}\,B_1) - f(x) - f'(x)\sqrt{t}\,B_1}{t}}_{|\cdot| \,\le\, \frac{K}{2} B_1^2 \,\in\, L^1(\mathbb{R})}\Big],\end{aligned}$$

using $B_t \overset{d}{=} \sqrt{t}\, B_1$. By dominated convergence (the integrand converging pointwise to $\frac{1}{2} f''(x)$ by Taylor's formula, and being dominated by the integrable $\frac{K}{2} B_1^2$), this tends to $\frac{1}{2} f''(x)$ as $t \to 0$.

(iv) $u(t,x) = \mathbb{E}[f(x + B_t)]$ is continuous on $[0,\infty) \times \mathbb{R}$, since $(t,x) \mapsto f(x + B_t)$ is continuous and bounded on $[0,\infty) \times \mathbb{R}$ (dominated convergence). The rest of the proof is a direct, if tedious, computation: writing

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y)\, \exp\Big(- \frac{1}{2t} (y-x)^2 \Big)\, dy,$$

one checks that u is C^∞ on $(0,\infty) \times \mathbb{R}$, and that differentiation under the integral sign (justified since f is bounded) gives

$$\partial_t u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y) \Big(- \frac{1}{2t^{3/2}} + \frac{(y-x)^2}{2t^{5/2}} \Big) \exp\Big(- \frac{(y-x)^2}{2t} \Big)\, dy,$$

$$\partial_x^2 u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(y)\, \frac{1}{t} \Big(-1 + \frac{(y-x)^2}{t} \Big) \exp\Big(- \frac{(y-x)^2}{2t} \Big)\, dy,$$

and the result $\partial_t u = \frac{1}{2} \partial_x^2 u$ follows by comparing the two expressions. □

3 The 0–1 law of Blumenthal

Proposition 4. *For $s \geq 0$, define $\mathcal{F}_{s+} = \bigcap_{u>s} \mathcal{F}_u$. Then the process $(\widetilde{B}_t = B_{t+s} - B_s)_{t \geq 0}$ is independent of $\mathcal{F}_{s+}$.*

Proof. It suffices to show that, for $A \in \mathcal{F}_{s+}$, $0 \leq t_1 < t_2 < \cdots < t_n$, and $F : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded,

$$\mathbb{E}\big[\mathbf{1}_A\, F(\widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_n})\big] = \mathbb{P}(A)\, \mathbb{E}\big[F(\widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_n})\big].$$

For $\varepsilon > 0$, the process $t \mapsto B_{t+s+\varepsilon} - B_{s+\varepsilon}$ is independent of $\mathcal{F}_{s+\varepsilon}$, by the (classical) simple Markov property, and hence in particular independent of $\mathcal{F}_{s+} \subset \mathcal{F}_{s+\varepsilon}$:

$$\begin{aligned}&\mathbb{E}\big[\mathbf{1}_A\, F(B_{t_1+s+\varepsilon} - B_{s+\varepsilon}, \ldots, B_{t_n+s+\varepsilon} - B_{s+\varepsilon})\big] \\ &\quad = \mathbb{P}(A)\, \mathbb{E}\big[F(B_{t_1+s+\varepsilon} - B_{s+\varepsilon}, \ldots, B_{t_n+s+\varepsilon} - B_{s+\varepsilon})\big] = \mathbb{P}(A)\, \mathbb{E}\big[F(B_{t_1}, \ldots, B_{t_n})\big].\end{aligned}$$

We conclude the proof by letting $\varepsilon \to 0^+$, using continuity of the Brownian paths and F, and dominated convergence. □

Theorem 5 (Blumenthal's 0–1 law). *$\mathcal{F}_{0+}$ is trivial: for every $A \in \mathcal{F}_{0+}$, $\mathbb{P}(A) = 0$ or 1.*

This will be seen to be of later importance for general filtrations.

Proof. By the preceding Proposition, $\mathcal{F}_{0+}$ is independent of $\sigma(B_t,\, t \geq 0)$. Moreover $\mathcal{F}_{0+} \subset \sigma(B_t,\, t \geq 0)$, hence $\mathcal{F}_{0+}$ is independent of itself, and is therefore trivial. $\square$

An application. We almost-surely have

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{t}} = +\infty, \qquad \liminf_{t\to 0} \frac{B_t}{\sqrt{t}} = -\infty,$$

$$\limsup_{t\to \infty} \frac{B_t}{\sqrt{t}} = +\infty, \qquad \liminf_{t\to \infty} \frac{B_t}{\sqrt{t}} = -\infty.$$

Proof. By symmetry and time-inversion, only the first claim needs to be proved. For $k > 0$, define $A_n = \{\sqrt{n}\, B_{1/n} > k\}$. We claim that $\limsup_n A_n \in \mathcal{F}_{0+}$. Indeed, for every $k \geq 1$,

$$\limsup_n A_n = \bigcap_{n\geq 0} \bigcup_{\ell \geq n} A_\ell = \bigcap_{n\geq 0} \bigcup_{\ell \geq n} A_\ell \in \mathcal{F}_{1/k} \qquad \Big(\text{since } \bigcup_{\ell\geq n} A_\ell \downarrow \text{ in } n\Big),$$

hence $\limsup_n A_n \in \bigcap_{k\geq 1} \mathcal{F}_{1/k} = \mathcal{F}_{0+}$. Moreover

$$\mathbb{P}(\limsup_n A_n) \geq \limsup_N \mathbb{P}(A_N) \geq \mathbb{P}(A_1) > 0,$$

hence, by the 0–1 law, $\mathbb{P}(\limsup_n A_n) = 1$. It follows that $\limsup_{t\to 0} B_t/\sqrt{t} \geq k$ a.s., for every k, which proves the first claim. $\square$

This entails the following interesting properties of Brownian paths:

(i) a.s. $t \mapsto B_t$ is not $\frac{1}{2}$-Hölder;

(ii) a.s. there exist $t_n \downarrow 0$ such that $B_{t_n} > 0$ for infinitely many n, and $B_{t_n} < 0$ for infinitely many n;

(iii) a.s. $\forall a \in \mathbb{R}$: $\tau_a = \inf\{t \geq 0,\, B_t = a\} < \infty$;

(iv) a.s. $\{t \geq 0,\, B_t = 0\}$ is unbounded.

4 The strong Markov property

In the following, $\mathcal{F}_t = \sigma(B_s,\, 0 \le s \le t)$ and $\mathcal{F}_\infty = \sigma(B_s,\, s \ge 0)$ and $(B_t)_{t\ge 0}$ is a BM.

Definition 6. *A map $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a stopping time if $\{\tau \le t\} \in \mathcal{F}_t$ for every $t \ge 0$. One then defines*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty :\ \forall t \ge 0,\ A \cap \{\tau \le t\} \in \mathcal{F}_t\}.$$

Example 7.

(i) *$\tau \equiv t_0$ is a stopping time.*

(ii) *$\tau_a = \inf\{t > 0,\, B_t = a\}$ (with $\inf \emptyset = \infty$) is a stopping time. Indeed, for $a \ge 0$,*

$$\{\tau_a \le t\} = \Big\{ \sup_{s\in[0,t]} B_s \ge a \Big\} = \Big\{ \sup_{s\in[0,t]\cap\mathbb{Q}} B_s \ge a \Big\} \in \mathcal{F}_t,$$

using the continuity of B to replace the (uncountable) supremum over $[0,t]$ by a countable one.

(iii) *$\tau = \sup\{s \le 1,\, B_s = 0\}$ is not a stopping time (exercise; see later, in connection with the strong Markov property).*

Exercise. *$\mathcal{F}_\tau$ is a σ-field, and both τ and $B_\tau \mathbf{1}_{\tau<\infty}$ are $\mathcal{F}_\tau$-measurable random variables.*

Indeed, $\{\tau \le u\} \cap \{\tau \le t\} = \{\tau \le u \wedge t\} \in \mathcal{F}_t$ for all $u, t \ge 0$, which shows τ is $\mathcal{F}_\tau$-measurable; and

$$B_\tau \mathbf{1}_{\tau<\infty} = \lim_{n\to\infty} \sum_{i=0}^{\infty} \mathbf{1}_{\{i2^{-n} \le \tau < (i+1)2^{-n}\}}\, B_{i2^{-n}} = \lim_{n\to\infty} \sum_{i=0}^{\infty} \big(\mathbf{1}_{\{i2^{-n} \le \tau\}} - \mathbf{1}_{\{(i+1)2^{-n} \le \tau\}}\big)\, B_{i2^{-n}},$$

each term being $\mathcal{F}_\tau$-measurable since, for $u \le s$, $\mathbf{1}_{\{s<\tau\}} B_u$ is $\mathcal{F}_\tau$-measurable, as follows from

$$\{\mathbf{1}_{\{s<\tau\}} B_u \le a\} \cap \{\tau \le t\} = \big(\{B_u \in a,\, s < \tau \le t\}\big) \cup \big(\{0 \le a,\, \tau \le s \wedge t\}\big) \in \mathcal{F}_t.$$

We are ready to state the main result of the chapter:

Theorem 8 (Strong Markov property)**.** *Let τ be a stopping time. Conditionally on $\{\tau < \infty\}$, $(\widetilde{B}_t = B_{t+\tau} - B_\tau)_{t\ge 0}$ is a Brownian motion, independent of $\mathcal{F}_\tau$.*

Proof. It suffices to show that, for every $A \in \mathcal{F}_\tau$, $0 \le t_1 < \cdots < t_n$, and $F : \mathbb{R}^n \to \mathbb{R}$ continuous and bounded,

$$\mathbb{E}\big[\mathbf{1}_{A\cap\{\tau<\infty\}}\, F(\widetilde{B}_{t_1}, \ldots, \widetilde{B}_{t_n})\big] = \mathbb{P}(A \cap \{\tau < \infty\})\, \mathbb{E}\big[F(B_{t_1}, \ldots, B_{t_n})\big].$$

Set $\tau_m = \lceil \tau 2^m \rceil / 2^m$ on $\{\tau < \infty\}$ (where $\lceil x \rceil = \min\{k \in \mathbb{Z},\ k \ge x\}$). By continuity of the paths and dominated convergence, the left-hand side equals $\lim_m \mathrm{LHS}_m$, where

$$\begin{aligned}\mathrm{LHS}_m &= \mathbb{E}\big[\mathbf{1}_{A\cap\{\tau<\infty\}}\, F(B_{\tau_m+t_1} - B_{\tau_m}, \ldots, B_{\tau_m+t_n} - B_{\tau_m})\big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{\infty} \mathbf{1}_{A\cap\{(k-1)2^{-m}<\tau\le k2^{-m}\}}\, F(B_{k2^{-m}+t_1} - B_{k2^{-m}}, \ldots, B_{k2^{-m}+t_n} - B_{k2^{-m}})\Big].\end{aligned}$$

For each k, $A \cap \{(k-1)2^{-m} < \tau \le k2^{-m}\} \in \mathcal{F}_{k2^{-m}}$. By the simple Markov property,

$$\begin{aligned}&\mathbb{E}\big[\mathbf{1}_{A\cap\{(k-1)2^{-m}<\tau\le k2^{-m}\}}\, F(B_{k2^{-m}+t_1} - B_{k2^{-m}}, \ldots)\big] \\ &= \mathbb{E}\big[\mathbf{1}_{A\cap\{(k-1)2^{-m}<\tau\le k2^{-m}\}}\big]\, \mathbb{E}\big[F(B_{t_1}, \ldots, B_{t_n})\big].\end{aligned}$$

Summing over k gives

$$\begin{aligned}\mathrm{LHS}_m &= \sum_{k=0}^{\infty} \mathbb{E}\big[\mathbf{1}_{A\cap\{(k-1)2^{-m}<\tau\le k2^{-m}\}}\big]\, \mathbb{E}\big[F(B_{t_1}, \ldots, B_{t_n})\big] \\ &= \mathbb{P}(A \cap \{\tau < \infty\})\, \mathbb{E}[F(B_{t_1}, \ldots, B_{t_n})],\end{aligned}$$

say, for every m; hence $\lim_m \mathrm{LHS}_m = \mathbb{P}(A \cap \{\tau < \infty\})\, \mathbb{E}[F(B_{t_1}, \ldots, B_{t_n})]$ and this proves the claim. □

Remark 9. *It is essential that τ be a stopping time: if $\tau = \sup\{s \le 1,\ B_s = 0\}$, then $(B_{\tau+t} - B_\tau)_{t\ge 0}$ is clearly not a Brownian motion, since it does not vanish on $(0, 1-\tau)$ (because $\tau < 1$ a.s., as $B_1 \ne 0$ a.s. and B is continuous).*

Proposition 10. *Let $\tau_a = \inf\{t > 0,\ B_t = a\}$. By the strong Markov property, $(\tau_a)_{a\ge 0}$ has independent and stationary increments (it is a Lévy process).*

Proof. For $0 < a < b$: $\tau_b - \tau_a \overset{d}{=} \tau_{b-a}$, since

$$\tau_b - \tau_a = \widetilde{\tau}_{b-a}, \qquad \widetilde{\tau}_x = \inf\{s > 0,\ \widetilde{B}_{\tau_a+s} - B_{\tau_a} = x\},$$

and $\widetilde{\tau}_{b-a}$ is independent of $\mathcal{F}_{\tau_a}$ (using that, for $\alpha \in [0, a]$, τ_α is $\mathcal{F}_{\tau_a}$-measurable, since τ_α is $\mathcal{F}_{\tau_\alpha}$-measurable and $\mathcal{F}_{\tau_\alpha} \subset \mathcal{F}_{\tau_a}$). □

We also have that $(\tau_a)_{a\geq 0}$ has non-decreasing paths. Moreover

$$(c^{-2}\tau_{ca})_{a\geq 0} \overset{d}{=} (\tau_a)_{a\geq 0},$$

since

$$\begin{aligned}(c^{-2}\tau_{ca})_{a\geq 0} &= \big(\inf\{c^{-2}t>0,\ B_t = ca\}\big)_{a\geq 0}\\ &= \big(\inf\{s>0,\ c^{-1}B_{c^2 s} = a\}\big)_{a\geq 0} \overset{d}{=} \big(\inf\{s>0,\ B_s = a\}\big)_{a\geq 0} = (\tau_a)_{a\geq 0}.\end{aligned}$$

Terminology: $(\tau_a)_{a\geq 0}$ is a stable subordinator of index $1/2$.

Theorem 11 (Reflection principle). *Let $S_t = \sup_{0\leq s\leq t} B_s$, $t\geq 0$. Then, for $a\geq 0$ and $b\leq a$,*

$$\mathbb{P}(S_t \geq a,\ B_t \leq b) = \mathbb{P}(B_t \geq 2a-b).$$

In particular, for fixed $t>0$, $S_t \overset{d}{=} |B_t|$. However $(S_t)_{t\geq 0}$ and $(|B_t|)_{t\geq 0}$ do not have the same law as processes: indeed (S_t) has non-decreasing paths, while $\mathbb{P}(|B_1| < |B_2|) < 1 = \mathbb{P}(S_1 < S_2)$.

Proof. We apply the strong Markov property to the stopping time

$$T_a = \inf\{t\geq 0,\ B_t = a\}.$$

Conditional on $\{T_a \leq t\}$, the probability that the (unreflected) Brownian path remains below b after time T_a equals the probability that the reflected path is beyond $2a-b$ at time t.

We have already seen that $T_a < \infty$ a.s. We compute:

$$\mathbb{P}(S_t\geq a,\ B_t\leq b) = \mathbb{P}(T_a\leq t,\ B_t \leq b) = \mathbb{P}\Big(T_a\leq t,\ \underbrace{B_{(t-T_a)+T_a} - B_{T_a}}_{=:\,\widetilde{B}_{t-T_a}} \leq b-a\Big).$$

Here $\widetilde{B}$ is a BM, independent of $\mathcal{F}_{T_a}$, by the strong Markov property, with $\widetilde{B}_s = B_{s+T_a} - B_{T_a}$; in particular T_a is independent of $\widetilde{B}$. Since $-\widetilde{B} \overset{d}{=} \widetilde{B}$, we get $(T_a, -\widetilde{B}) \overset{d}{=} (T_a, \widetilde{B})$.

Now, consider the map $\psi : \mathbb{R}_+ \times \mathcal{C}(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}_+\times\mathbb{R}$, $(s,\omega)\mapsto (s, \omega(t-s)\mathbf{1}_{t>s})$: since ψ is measurable, it preserves the equality in distribution, so that

$$\big(T_a,\ \widetilde{B}_{t-T_a}\mathbf{1}_{t>T_a}\big) \overset{d}{=} \big(T_a,\ -\widetilde{B}_{t-T_a}\mathbf{1}_{t>T_a}\big).$$

It follows that

$$\begin{aligned}\mathbb{P}\big(T_a \le t,\, \widetilde{B}_{t-T_a} \le b-a\big) &= \mathbb{P}\big(T_a \le t,\, -\widetilde{B}_{t-T_a} \le b-a\big) = \mathbb{P}\big(T_a \le t,\, -B_t + a \le b-a\big)\\ &= \mathbb{P}\big(T_a \le t,\, B_t \ge 2a-b\big) = \mathbb{P}\big(B_t \ge 2a-b\big),\end{aligned}$$

the last equality because $B_t \ge 2a-b \ge a$ (as $b \le a$) already implies $T_a \le t$. To complete the proof, note that

$$\begin{aligned}&\mathbb{P}(S_t \ge a) = \mathbb{P}(S_t \ge a,\, B_t \ge a) + \mathbb{P}(S_t \ge a,\, B_t \le a)\\ &= \mathbb{P}(B_t \ge a) + \mathbb{P}(B_t \ge a) = 2\mathbb{P}(B_t \ge a) = \mathbb{P}(|B_t| \ge a),\end{aligned}$$

using the identity just proved with $b = a$ for the first term (where $\{S_t \ge a,\, B_t \ge a\} = \{B_t \ge a\}$ since $B_t \ge a \Rightarrow S_t \ge a$), and again with $b = a$ for the second. □

Exercise. *The r.v. τ_a has a density given by*

$$f_{\tau_a}(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\Big(-\frac{a^2}{2t}\Big)\, \mathbf{1}_{t>0}.$$

Proof. From the reflection principle, the joint density of (S_t, B_t) is

$$f_{(S_t,B_t)}(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(2a-b)^2}{2t}\Big)\, \mathbf{1}_{\{a>0,\, b<a\}}.$$

Now

$$\mathbb{P}(\tau_a \le t) = \mathbb{P}(S_t \ge a) = \mathbb{P}(|B_t| \ge a) = \mathbb{P}(\sqrt{t}\,|B_1| \ge a) = \mathbb{P}\Big(\frac{a^2}{B_1^2} \le t\Big),$$

i.e. $\tau_a \overset{d}{=} a^2/B_1^2$, and the stated density follows by a standard computation. In particular $\mathbb{E}[\tau_a] = \infty$ (and, if $a < 0$, $\tau_{-a} = \tau_{|a|}$ by symmetry). □

5 The quadratic variation of Brownian motion

Proposition 12. *Fix $t \ge 0$. For $n \ge 1$, let $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\}$ be a subdivision of $[0,t]$. Assume $|\Delta_n| = \max_{1\le i\le k_n}(t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$. Then*

$$\sum_{i=1}^{k_n} \big(B_{t_i^n} - B_{t_{i-1}^n}\big)^2 \xrightarrow[n\to\infty]{} t \qquad \text{in } L^2(\mathbb{P}).$$

If moreover $t_i^n = i2^{-n}$ (and $k_n = 2^n$), the convergence holds almost surely.

Proof. Set $Y_i^n = \big(B_{t_i^n} - B_{t_{i-1}^n}\big)^2 - (t_i^n - t_{i-1}^n)$; the $(Y_i^n)_i$ are i.i.d. (for fixed n, across i), with mean 0 and variance $2(t_i^n - t_{i-1}^n)^2$. It follows that

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{k_n}\big(B_{t_i^n} - B_{t_{i-1}^n}\big)^2 - t\Big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{i=1}^{k_n} Y_i^n\Big)^2\Big] = \sum_{i=1}^{k_n}\mathbb{E}\big[(Y_i^n)^2\big] = 2\sum_{i=1}^{k_n}(t_i^n - t_{i-1}^n)^2$$
$$\leq 2t\sup_{i\leq k_n}(t_i^n - t_{i-1}^n) \xrightarrow[n\to\infty]{} 0$$

(and equal to $2t^2 2^{-n}$ in the dyadic case). This proves the L^2 convergence. In the dyadic case, by Markov's inequality,

$$\mathbb{P}\Big(\Big|\sum_{i=1}^{2^n} Y_i^n\Big| > \frac{1}{n}\Big) \leq n^2 t^2 2^{-n},$$

a summable sequence in n; by Borel–Cantelli, there exists (a.s.) $n_0 \geq 1$ such that $\big|\sum_{i=1}^{2^n} Y_i^n\big| \leq \frac{1}{n}$ for all $n \geq n_0$, whence the almost sure convergence. $\square$

Corollary 13 (Important)**.** *Brownian motion has, almost surely, infinite variation on every interval: a.s., for every $0 \leq a < b$,*

$$|B|([a,b]) = \sup\Big\{\sum_i |B_{t_i} - B_{t_{i-1}}|\Big\} = \infty,$$

the supremum being taken over all subdivisions $a = t_0 < \cdots < t_n = b$ of $[a,b]$.

Proof. Take $a = 0, b = 1$. We have, almost surely,

$$1 = \lim_{n\to\infty}\sum_{i=1}^{2^n}\big(B_{i2^{-n}} - B_{(i-1)2^{-n}}\big)^2 \;\leq\; |B|([0,1]) \times \sup_{i=1,\dots,2^n}\big|B_{i2^{-n}} - B_{(i-1)2^{-n}}\big|.$$

Since B is continuous, hence uniformly continuous, on $[0,1]$, we have

$$\sup_{i=1,\dots,2^n}|B_{i2^{-n}} - B_{(i-1)2^{-n}}| \to 0 \text{ almost surely}.$$

Hence

$$\big\{|B|([0,1]) < \infty\big\} \subset \Big\{\lim_n \sum_{i=1}^{2^n}\big(B_{i2^{-n}} - B_{(i-1)2^{-n}}\big)^2 = 0\Big\},$$

an event of probability 0 since the left-hand side of the display above converges a.s. to $1 \neq 0$; hence $|B|([0,1]) = \infty$ a.s.

Consequently, a.s. $\forall 0 \leq a < b$ with $a, b \in \mathbb{Q}$, $|B|([a,b]) = \infty$ (by the same argument, rescaled); hence a.s. $\forall 0 \leq a < b$ (not necessarily rational), $|B|([a,b]) = \infty$, since $|B|([a,b]) \geq |B|([a',b'])$ whenever $a < a' < b' < b$ with $a', b' \in \mathbb{Q}$. $\square$