

Chapter 3: Brownian motion as a martingale

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1 Filtrations and usual conditions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration $(\mathcal{F}_t)_{t \geq 0}$ is a non-decreasing family of sub- σ -fields of $\mathcal{F}$: $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$. The quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is called a filtered probability space. We write

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right) = \bigvee_{t \geq 0} \mathcal{F}_t.$$

Definition 1. A map $\tau : \Omega \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is a stopping time if

$$\forall t < \infty, \quad \{\tau \leq t\} \in \mathcal{F}_t.$$

The σ -field $\mathcal{F}_\tau$ associated to a stopping time τ is

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : \forall t \geq 0, A \cap \{\tau \leq t\} \in \mathcal{F}_t\}.$$

For $t \geq 0$, set $\mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$. One checks readily that $(\mathcal{F}_{t+})_{t \geq 0}$ is again a filtration.

Example 2. Let Z be a random variable and $X_t = Z \mathbf{1}_{\{t \geq 1\}}$, with $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Then $\mathcal{F}_1 = \{\emptyset, \Omega\}$, while $\mathcal{F}_{1+} = \sigma(Z)$: these two σ -fields can be very different!

Exercise. A map $\sigma : \Omega \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is a $(\mathcal{F}_{t+})$ -stopping time if and only if

$$\forall t \geq 0, \quad \{\sigma < t\} \in \mathcal{F}_t.$$

Proof. Assume $\{\sigma < t\} \in \mathcal{F}_t$ for every $t \geq 0$. Then for every $t \geq 0$ and $n \geq 1$,

$$\{\sigma \leq t\} = \bigcap_{k \geq n} \left\{ \sigma < t + \frac{1}{k} \right\} \in \mathcal{F}_{t + \frac{1}{n}},$$

so $\{\sigma \leq t\} \in \mathcal{F}_{t+}$ for every t , i.e. σ is an $(\mathcal{F}_{t+})$ -stopping time.

Conversely, assume $\{\sigma \leq t\} \in \mathcal{F}_{t+}$ for every $t \geq 0$. Then

$$\{\sigma < t\} = \bigcup_{n \geq 1} \left\{ \sigma \leq t - \frac{1}{n} \right\},$$

and $\{\sigma \leq t - \frac{1}{n}\} \in \mathcal{F}_{(t - \frac{1}{n})+} \subset \mathcal{F}_t$ for every n . Hence $\{\sigma < t\}$, being a countable union of elements of $\mathcal{F}_t$, belongs to $\mathcal{F}_t$. $\square$

Example 3. Let $(B_t)_{t \geq 0}$ be a Brownian motion with canonical filtration $(\mathcal{F}_t)_{t \geq 0}$, and let $a \geq 0$. Set

$$T_a = \inf\{t \geq 0, B_t = a\}.$$

This is an $(\mathcal{F}_t)$ -stopping time: by continuity of B ,

$$\{T_a \leq t\} = \left\{ \sup_{s \in [0, t]} B_s \geq a \right\} = \left\{ \sup_{s \in [0, t] \cap \mathbb{Q}} B_s \geq a \right\} \in \mathcal{F}_t,$$

the (uncountable) supremum over $[0, t]$ being replaced by a countable one thanks to continuity.

Consider also

$$\sigma_a = \inf\{t \geq 0, B_t > a\}.$$

Then a.s. $T_a = \sigma_a$: clearly $T_a \leq \sigma_a$; and, writing $\tilde{B}_t = B_{T_a+t} - B_{T_a}$ (which, by the strong Markov property, is a Brownian motion independent of $\mathcal{F}_{T_a}$ on $\{T_a < \infty\}$),

$$\{T_a < \sigma_a\} \subset \left\{ \exists \varepsilon > 0, \sup_{s \in [0, \varepsilon]} \tilde{B}_s = 0 \right\},$$

an event of probability 0, since a Brownian path takes positive values in every right neighborhood of 0 almost surely (this follows, e.g., from $\limsup_{t \rightarrow 0} B_t/\sqrt{t} = +\infty$ a.s. which was proved in the last chapter by means of Blumenthal's 0-1 law).

Remark 4. The stopping time σ_a above is a $(\mathcal{F}_{t+})$ -stopping time (since $\{\sigma_a < t\} = \{\sup_{s \in [0, t) \cap \mathbb{Q}} B_s > a\} \in \mathcal{F}_t$), but it is not an $(\mathcal{F}_t)$ -stopping time. A sketch of the argument is the following: one has

$$\{\sigma_a \leq t\} = \{S_t > a\} \cup \left\{ S_t = B_t = a, \inf\{s > 0, B_{t+s} > a\} = 0 \right\} \in \mathcal{F}_{t+} \setminus \mathcal{F}_t.$$

For the interested reader, we give a rigorous and complete proof at the end of the chapter (that can be skipped in a first reading of these notes). Thus the property of being a stopping time with respect to the canonical (raw) filtration is not stable under the operations one would like to perform, unless null sets are incorporated into the filtration.

Definition 5. A filtration $(\mathcal{F}_t)_{t \geq 0}$ is

- right-continuous (càd) if $\mathcal{F}_{t+} = \mathcal{F}_t$ for every $t \geq 0$;
- complete if $\mathcal{F}_0$ (hence every $\mathcal{F}_t$) contains all $\mathbb{P}$ -null sets of $\mathcal{F}$.

Note that $(\mathcal{F}_{t+})_{t \geq 0}$ is always right-continuous, since $\mathcal{F}_{(t+)+} = \mathcal{F}_{t+}$.

Definition 6. (i) A filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions if it is right-continuous and complete.

(ii) Given a filtration $(\mathcal{F}_t)_{t \geq 0}$, its natural augmentation is $(\overline{\mathcal{F}}_t)_{t \geq 0}$, where

$$\overline{\mathcal{F}}_t := \mathcal{F}_{t+} \vee \mathcal{N},$$

$\mathcal{N}$ denoting the collection of $\mathbb{P}$ -null sets of $\mathcal{F}_\infty$. The augmented filtration always satisfies the usual conditions.

From now on, unless stated otherwise, all filtrations under consideration are assumed to satisfy the usual conditions.

2 Measurability conditions

Definition 7. A process $(X_t)_{t \geq 0}$ is

- (i) measurable if $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ -measurable;
- (ii) adapted to $(\mathcal{F}_t)_{t \geq 0}$ if X_t is $\mathcal{F}_t$ -measurable for every $t \geq 0$;
- (iii) progressively measurable with respect to $(\mathcal{F}_t)_{t \geq 0}$ if, for every $t \geq 0$, the map

$$[0, t] \times \Omega \longrightarrow \mathbb{R}^d, \quad (s, \omega) \longmapsto X_s(\omega)$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable.

Remark 8. • If X is càdlàg (or càglàd) then it is measurable. Simply write (for $t \in [0, 1]$ say) $X_t(\omega) = \lim_{n \rightarrow \infty} \sum_{k=1}^n X_{(k-1)/n} \mathbf{1}_{\{(k-1)/n, k/n\}}(t)$. We have that $X_{(k-1)/n} \in \mathcal{F}$ and $\mathbf{1}_{\{(k-1)/n, k/n\}} \in \mathcal{B}([0, 1])$ hence the product

$$(t, \omega) \mapsto X_{(k-1)/n} \mathbf{1}_{\{(k-1)/n, k/n\}}(t)$$

belongs to $\mathcal{B}([0, 1]) \otimes \mathcal{F}$, so that $\sum_{k=1}^n X_{(k-1)/n} \mathbf{1}_{\{(k-1)/n, k/n\}}(t)$ and its limit as $n \rightarrow \infty$.

- If X is progressively measurable then it is measurable.
- If X is càdlàg (or càglàd) and adapted, then it is progressively measurable.

Definition 9. A set $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ is progressive if $X_t(\omega) := \mathbf{1}_A(t, \omega)$ is progressively measurable. The collection of progressive sets forms a σ -field, called the progressive σ -field, denoted $\mathcal{P}$.

Remark 10. $(X_t)_{t \geq 0}$ is progressively measurable if and only if $(t, \omega) \mapsto X_t(\omega)$ is measurable from $(\mathbb{R}_+ \times \Omega, \mathcal{P})$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Remark 11. A process need not be càdlàg or càglàd to be progressively measurable: for instance $X_t(\omega) = f(t)$, for $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ deterministic and measurable, is progressively measurable.

3 Stopping times

Exercise. Any stopping time τ is $\mathcal{F}_\tau$ -measurable. If $\sigma \leq \tau$ are two stopping times, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

Proposition 12. Let τ be a stopping time and, for $n \geq 1$, set

$$\tau_n := \frac{\lceil \tau 2^n \rceil}{2^n} \quad (\text{with } \tau_n = \infty \text{ if } \tau = \infty).$$

Then $(\tau_n)_{n \geq 1}$ is a non-increasing sequence of stopping times, and $\tau_n \downarrow \tau$.

Proof. That $\tau_n \downarrow \tau$ is immediate from the definition; it remains to check that each τ_n is a stopping time. Since $\tau_n \geq \tau$ always takes values in $2^{-n}\mathbb{N} \cup \{\infty\}$, for $t = k2^{-n}$ we have $\{\tau_n \leq t\} = \{\tau \leq t\} \in \mathcal{F}_t$. For general $t \geq 0$, letting $k = \lfloor 2^n t \rfloor$,

$$\{\tau_n \leq t\} = \{\tau_n \leq k2^{-n}\} = \{\tau \leq k2^{-n}\} \in \mathcal{F}_{k2^{-n}} \subset \mathcal{F}_t,$$

so $\{\tau_n \leq t\} \in \mathcal{F}_t$ for every t , i.e. τ_n is a stopping time. □

The main motivation for introducing progressively measurable processes is the following.

Theorem 13. *Let $(X_t)_{t \geq 0}$ be an $\mathbb{R}^d$ -valued progressively measurable process and τ a stopping time. Then $X_\tau \mathbf{1}_{\{\tau < \infty\}}$ is $\mathcal{F}_\tau$ -measurable.*

Proof. Fix $t \geq 0$. It suffices to show that $X_{\tau \wedge t}$ is $\mathcal{F}_t$ -measurable: indeed, for $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$\{X_\tau \mathbf{1}_{\{\tau < \infty\}} \in \Gamma\} \cap \{\tau \leq t\} = \{X_{\tau \wedge t} \in \Gamma\} \cap \{\tau \leq t\} \in \mathcal{F}_t,$$

which shows $\{X_\tau \mathbf{1}_{\{\tau < \infty\}} \in \Gamma\} \in \mathcal{F}_\tau$. Now, $X_{\tau \wedge t}(\omega) = \Phi(f(\omega))$, where

$$f(\omega) = (\tau(\omega) \wedge t, \omega), \quad \Phi(s, \omega) = X_s(\omega).$$

The map f is measurable from $(\Omega, \mathcal{F}_t)$ to $([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_t)$: indeed $\omega \mapsto \omega$ is $(\mathcal{F}_t, \mathcal{F}_t)$ -measurable, and $\omega \mapsto \tau(\omega) \wedge t$ is $(\mathcal{F}_t, \mathcal{B}([0, t]))$ -measurable since τ is a stopping time (so that, for every $s \in [0, t]$, $\{\tau \wedge t \leq s\} = \{\tau \leq s\} \in \mathcal{F}_s \subset \mathcal{F}_t$). On the other hand Φ , restricted to $[0, t] \times \Omega$, is measurable from $([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_t)$ to $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by progressive measurability of X . Composing gives the result. $\square$

Example 14. (i) Let $(X_t)_{t \geq 0}$ be adapted and continuous, taking values in a metric space (E, d) . For every open $G \subset E$,

$$\tau_G := \inf\{t \geq 0, X_t \in G\}$$

is an $(\mathcal{F}_{t+})$ -stopping time, since

$$\{\tau_G < t\} = \{\exists s \in [0, t), X_s \in G\} = \bigcup_{s \in [0, t) \cap \mathbb{Q}} \{X_s \in G\} \in \mathcal{F}_t,$$

using that G is open and X is continuous.

(ii) In the same setting, for every closed $F \subset E$,

$$\tau_F := \inf\{t \geq 0, X_t \in F\}$$

is an $(\mathcal{F}_t)$ -stopping time, since

$$\{\tau_F \leq t\} = \left\{ \inf_{s \in [0, t]} d(X_s, F) = 0 \right\} = \left\{ \inf_{s \in [0, t] \cap \mathbb{Q}} d(X_s, F) = 0 \right\} \in \mathcal{F}_t.$$

4 Recap on discrete-time martingales

Let $(M_n)_{n \geq 0}$, $n \in \{0, 1, \dots\}$, be a process on a (discrete) filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ such that $\mathbb{E}[|M_n|] < \infty$ for every $n \geq 0$. Recall the following facts.

- $(M_n)_{n \geq 0}$ is a martingale if $\mathbb{E}[M_m | \mathcal{F}_n] = M_n$ a.s. for every $m \geq n \geq 0$ (a supermartingale if $\leq$, a submartingale if $\geq$).
- If $(M_n)_{n \geq 0}$ is a martingale and there exists $M_\infty \in L^1(\mathbb{P})$ such that $M_n = \mathbb{E}[M_\infty | \mathcal{F}_n]$ for every n , we say $(M_n)_{n \geq 0}$ is closed.
- If $(M_n)_{n \geq 0}$ is a submartingale and τ is an $(\mathcal{F}_n)$ -stopping time, then $(M_{n \wedge \tau})_{n \geq 0}$ is a submartingale (note that $\{\tau \leq n\} \in \mathcal{F}_n$). Indeed,

$$M_{\tau \wedge (n+1)} = M_{n+1} \mathbf{1}_{\{\tau > n+1\}} + \sum_{k=0}^n M_k \mathbf{1}_{\{\tau = k\}},$$

and one takes conditional expectations with respect to $\mathcal{F}_n$.

- If $(M_n)_{n \geq 0}$ is a martingale and $S_n := \max_{0 \leq k \leq n} |M_k|$, then (Doob's maximal inequality)

$$\mathbb{P}(S_n > a) \leq \frac{1}{a} \mathbb{E}[|M_n| \mathbf{1}_{\{S_n > a\}}].$$

Proof. Let $\tau_a = \inf\{k \geq 0, |M_k| \geq a\}$. For $0 \leq k \leq n$,

$$a \mathbf{1}_{\{\tau_a = k\}} \leq |M_k| \mathbf{1}_{\{\tau_a = k\}} \leq \mathbb{E}[|M_n| | \mathcal{F}_k] \mathbf{1}_{\{\tau_a = k\}} = \mathbb{E}[|M_n| \mathbf{1}_{\{\tau_a = k\}} | \mathcal{F}_k],$$

using that $\{\tau_a = k\} \in \mathcal{F}_k$ and Jensen's inequality applied to the martingale (M_k) . Taking expectations and summing over $k = 0, \dots, n$ (noting $\{S_n > a\} = \{\tau_a \leq n\}$) gives the result. $\square$

Proposition 15 (Doob's L^p inequality, discrete case). *Let $(M_n)_{n \geq 0}$ be a martingale and $p > 1$, such that $\mathbb{E}[|M_k|^p] < \infty$ for every $k = 0, \dots, n$. Let q be the conjugate exponent of p , i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\mathbb{E}\left[\max_{0 \leq k \leq n} |M_k|^p\right] \leq q^p \mathbb{E}[|M_n|^p].$$

Proof. Using $x^p = p \int_0^x a^{p-1} da$ and Fubini,

$$\mathbb{E}[S_n^p] = p \int_0^\infty \mathbb{P}(S_n > a) a^{p-1} da \leq p \int_0^\infty \mathbb{E}[|M_n| \mathbf{1}_{\{S_n > a\}}] a^{p-2} da = \frac{p}{p-1} \mathbb{E}[|M_n| S_n^{p-1}],$$

by the maximal inequality above. Hölder's inequality then gives

$$\mathbb{E}[S_n^p] \leq \frac{p}{p-1} \mathbb{E}[|M_n|^p]^{1/p} \mathbb{E}[S_n^{p(p-1)/p}]^{(p-1)/p},$$

and dividing both sides by $\mathbb{E}[S_n^{p(p-1)/p}]^{(p-1)/p}$ (finite, since $S_n \leq \sum_{k \leq n} |M_k|$) yields the claim, since $q = p/(p-1)$. $\square$

Proposition 16 (Optional stopping, discrete case). *Let $(M_n)_{n \geq 0}$ be a closed martingale, and let $\sigma \leq \tau$ a.s. be two stopping times. Then*

$$\mathbb{E}[M_\tau | \mathcal{F}_\sigma] = M_\sigma.$$

Proof. It suffices to show $M_\tau = \mathbb{E}[M_\infty | \mathcal{F}_\tau]$: the general statement then follows since $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ gives

$$\mathbb{E}[M_\tau | \mathcal{F}_\sigma] = \mathbb{E}[\mathbb{E}[M_\infty | \mathcal{F}_\tau] | \mathcal{F}_\sigma] = \mathbb{E}[M_\infty | \mathcal{F}_\sigma] = M_\sigma.$$

For a stopping time τ and $Z \in L^1$ (or $Z \geq 0$), one has $\mathbb{E}[Z | \mathcal{F}_\tau] \mathbf{1}_{\{\tau=n\}} = \mathbb{E}[Z | \mathcal{F}_n] \mathbf{1}_{\{\tau=n\}}$ for every $n \in \mathbb{N} \cup \{\infty\}$. (Check that!.) Hence

$$\mathbb{E}[M_\infty | \mathcal{F}_\tau] = \sum_{n \in \mathbb{N} \cup \{\infty\}} \mathbb{E}[M_\infty | \mathcal{F}_\tau] \mathbf{1}_{\{\tau=n\}} = \sum_n \mathbb{E}[M_\infty | \mathcal{F}_n] \mathbf{1}_{\{\tau=n\}} = \sum_n M_n \mathbf{1}_{\{\tau=n\}} = M_\tau. \quad \square$$

5 Continuous-time martingales

Let $M = (M_t)_{t \geq 0}$ be a process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Definition 17. *M is an $(\mathcal{F}_t)_{t \geq 0}$ -martingale if*

- (i) *M is $(\mathcal{F}_t)_{t \geq 0}$ -adapted;*
- (ii) *$\mathbb{E}[|M_t|] < \infty$ for every $t \geq 0$;*
- (iii) *$\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ a.s., for every $s \leq t$*

(a supermartingale if $\leq$ in (iii), a submartingale if $\geq$).

Example 18. *Let $(B_t)_{t \geq 0}$ be a Brownian motion in its canonical filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$.*

- *$(B_t)_{t \geq 0}$ is a martingale.*

- $(B_t^2 - t)_{t \geq 0}$ is a martingale: for $s \leq t$, writing $B_t = B_t - B_s + B_s$ and using that $B_t - B_s$ is independent of $\mathcal{F}_s$, centered, with variance $t - s$,

$$\mathbb{E}[B_t^2 | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + 2B_s \mathbb{E}[B_t - B_s | \mathcal{F}_s] + B_s^2 = (t - s) + B_s^2,$$

$$\text{so } \mathbb{E}[B_t^2 - t | \mathcal{F}_s] = B_s^2 - s.$$

- For $\theta \in \mathbb{R}$, $(\exp(\theta B_t - \frac{\theta^2}{2}t))_{t \geq 0}$ is a martingale: for $s \leq t$,

$$\begin{aligned} \mathbb{E}[e^{\theta B_t - \frac{\theta^2}{2}t} | \mathcal{F}_s] &= \mathbb{E}[e^{\theta(B_t - B_s)} | \mathcal{F}_s] e^{\theta B_s - \frac{\theta^2}{2}s} \quad (\text{since } B_t - B_s \perp \mathcal{F}_s) \\ &= e^{\frac{1}{2}\theta^2(t-s)} e^{\theta B_s - \frac{\theta^2}{2}s} = e^{\theta B_s - \frac{\theta^2}{2}s}. \end{aligned}$$

Exercise. (i) If $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex and M is a martingale with $\mathbb{E}[|f(M_t)|] < \infty$ for every t , then $(f(M_t))_{t \geq 0}$ is a submartingale.

(ii) If M is a submartingale, f is convex and non-decreasing, and $\mathbb{E}[f(M_t)] < \infty$ for every t , then $(f(M_t))_{t \geq 0}$ is a submartingale.

Theorem 19 (Doob's L^p inequality). Let $(M_t)_{t \geq 0}$ be a càdlàg martingale and $p > 1$. If $M_t \in L^p(\mathbb{P})$ for every t , then

$$\mathbb{E}\left[\sup_{s \in [0, t]} |M_s|^p\right] \leq q^p \mathbb{E}[|M_t|^p], \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(The case $p = 1$ holds with a different right-hand side, by BDG inequality; this is treated later.)

Remark 20. Note that formally, we need every M_t in $L^p(\mathbb{P})$. Otherwise, there might exist some $p_0 < p_1$ such that $\mathbb{E}[|M_t|^{p_0}] = \infty$ while $\mathbb{E}[|M_t|^{p_1}] < \infty$, in which case the previous bound would be meaning less. We will see later that this kind of phenomenon cannot happen for continuous martingales.

Proof. Fix $t \geq 0$ and let $D_n := \{k2^{-n}, k \geq 0\}$. For fixed n , $M_k^n := M_{k2^{-n}}$ is a discrete-time martingale for the filtration $\mathcal{F}_k^n := \mathcal{F}_{k2^{-n}}$. By the discrete Doob inequality,

$$\mathbb{E}\left[\sup_{s \in D_n \cap [0, t]} |M_s|^p\right] = \mathbb{E}\left[\max_{k=0, \dots, 2^n} |M_k^n|^p\right] \leq q^p \mathbb{E}[|M_{2^n}^n|^p] = q^p \mathbb{E}[|M_t|^p].$$

Since M is càdlàg, $\sup_{s \in D_n \cap [0, t]} |M_s|^p \uparrow \sup_{s \in [0, t]} |M_s|^p$ as $n \rightarrow \infty$, and the result follows by monotone convergence. $\square$

Theorem 21 (Optional stopping theorem). *Let $(M_t)_{t \geq 0}$ be a càdlàg, closed martingale, and let σ, τ be two stopping times with $\sigma \leq \tau$ a.s. Then*

$$\mathbb{E}[M_\tau | \mathcal{F}_\sigma] = M_\sigma \quad \text{a.s.}$$

Proof. As before, we discretize and use the càdlàg property to pass to the limit. It suffices to show $M_\tau = \mathbb{E}[M_\infty | \mathcal{F}_\tau]$: then, since $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$,

$$\mathbb{E}[M_\tau | \mathcal{F}_\sigma] = \mathbb{E}[\mathbb{E}[M_\infty | \mathcal{F}_\tau] | \mathcal{F}_\sigma] = \mathbb{E}[M_\infty | \mathcal{F}_\sigma] = M_\sigma.$$

Fix $n \geq 1$ and set $\mathcal{F}_k^n := \mathcal{F}_{k2^{-n}}$; then $M_k^n := M_{k2^{-n}}$ is an $(\mathcal{F}_k^n)$ -martingale closed by M_∞ . Let $\sigma_n := \lceil 2^n \tau \rceil$ (with $\lceil \infty \rceil = \infty$); this is an $(\mathcal{F}_k^n)_{k \geq 0}$ -stopping time, since

$$\{\sigma_n \leq k\} = \{2^n \tau \leq k\} = \{\tau \leq k2^{-n}\} \in \mathcal{F}_{k2^{-n}} = \mathcal{F}_k^n.$$

By the discrete-time optional stopping theorem,

$$\mathbb{E}[M_\infty | \mathcal{F}_{\sigma_n}^n] = M_{\sigma_n}^n = M_{\tau_n}, \quad \tau_n := \sigma_n 2^{-n}.$$

In particular, for every $A \in \mathcal{F}_{\sigma_n}^n$, $\mathbb{E}[M_\infty \mathbf{1}_A] = \mathbb{E}[M_{\tau_n} \mathbf{1}_A]$.

We claim $\mathcal{F}_\tau \subset \mathcal{F}_{\sigma_n}^n$: if $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$; taking $t = k2^{-n}$,

$$A \cap \{\sigma_n \leq k\} = A \cap \{\tau \leq k2^{-n}\} \in \mathcal{F}_{k2^{-n}} = \mathcal{F}_k^n,$$

which is exactly the statement that $A \in \mathcal{F}_{\sigma_n}^n$. Hence, for every $A \in \mathcal{F}_\tau$ and every $n \geq 0$,

$$\mathbb{E}[M_\infty \mathbf{1}_A] = \mathbb{E}[M_{\tau_n} \mathbf{1}_A].$$

Moreover, $(M_{\tau_n})_{n \geq 0}$ is uniformly integrable, since $M_{\tau_n} = \mathbb{E}[M_\infty | \mathcal{F}_{\tau_n}]$ with $M_\infty \in L^1$, and $M_{\tau_n} \rightarrow M_\tau$ a.s. as $n \rightarrow \infty$, since $\tau_n \downarrow \tau$ and M is càdlàg. Passing to the limit,

$$\mathbb{E}[M_\infty \mathbf{1}_A] = \mathbb{E}[M_\tau \mathbf{1}_A] \quad \forall A \in \mathcal{F}_\tau.$$

Since M_τ is $\mathcal{F}_\tau$ -measurable, this shows $\mathbb{E}[M_\infty | \mathcal{F}_\tau] = M_\tau$, as claimed. $\square$

Example 22. Let $a > 0$, $\theta > 0$, and $\tau_a = \inf\{t \geq 0, B_t = a\}$. We have seen that $(e^{\theta B_t - \frac{\theta^2}{2}t})_{t \geq 0}$ is a martingale, so

$$\left(e^{\theta B_{t \wedge \tau_a} - \frac{\theta^2}{2}(t \wedge \tau_a)} \right)_{t \geq 0}$$

is a martingale as well, bounded by $e^{\theta a}$, hence uniformly integrable; by its almost sure limit as $t \rightarrow \infty$ (using $\tau_a < \infty$ a.s.), its limit is $e^{\theta a - \frac{\theta^2}{2}\tau_a}$. By the optional stopping theorem,

$$\mathbb{E}[e^{\theta a - \frac{\theta^2}{2}\tau_a}] = 1, \quad \text{i.e.} \quad \forall \lambda > 0, \quad \mathbb{E}[e^{-\lambda \tau_a}] = e^{-a\sqrt{2\lambda}}.$$

Remark 23. (i) If $(M_t)_{t \geq 0}$ is càdlàg and σ, τ are two stopping times bounded by a deterministic constant K , then the optional stopping identity holds without assuming M to be closed: simply write $M_\tau = M_{\tau \wedge K}$, $M_\sigma = M_{\sigma \wedge K}$ and apply the result to $(M_{t \wedge K})_{t \geq 0}$, which is closed by M_K .

(ii) Caution: the result may fail if M is a non-closed martingale. Indeed, $\tau = \inf\{t \geq 0, B_t = 1\}$ is a.s. finite, and $B_\tau = 1$ a.s., yet

$$1 = \mathbb{E}[B_\tau | \mathcal{F}_0] \neq B_0 = 0.$$

Proposition 24. Let M be a càdlàg martingale and τ a stopping time. Then $(M_{t \wedge \tau})_{t \geq 0}$ is an $(\mathcal{F}_t)$ -martingale (in particular an $(\mathcal{F}_{t \wedge \tau})$ -martingale). as an immediate consequence of the optional stopping theorem).

Proof. $(M_{t \wedge \tau})_{t \geq 0}$ is clearly $(\mathcal{F}_t)$ -adapted. Let $0 \leq s \leq t$; we must show $M_{\tau \wedge t} \in L^1$ and $\mathbb{E}[M_{\tau \wedge t} | \mathcal{F}_s] = M_{\tau \wedge s}$.

Fix $n \geq 1$, set $\mathcal{F}_k^n = \mathcal{F}_{k2^{-n}}$, $\sigma_n = \lceil 2^n \tau \rceil$ (an $(\mathcal{F}_k^n)$ -stopping time), $\tau_n = \sigma_n 2^{-n}$, and let $t_n = 2^{-n} \lceil 2^n t \rceil \geq t$, $s_n = 2^{-n} \lceil 2^n s \rceil \geq s$. By the discrete-time result applied to the stopping times $\sigma_n \wedge 2^n t_n$ and $\sigma_n \wedge 2^n s_n$,

$$\mathbb{E}[M_{\tau_n \wedge t_n} | \mathcal{F}_{s_n}] = M_{\tau_n \wedge s_n}.$$

Hence, for every $A \in \mathcal{F}_s \subset \mathcal{F}_{s_n}$,

$$\mathbb{E}[\mathbf{1}_A M_{\tau_n \wedge t_n}] = \mathbb{E}[\mathbf{1}_A M_{\tau_n \wedge s_n}].$$

Now $(M_{\tau_n \wedge t_n})_{n \geq 0}$ and $(M_{\tau_n \wedge s_n})_{n \geq 0}$ are uniformly integrable, since $t_n \leq t + 1$ and $s_n \leq s + 1$ for every n , so that $M_{\tau_n \wedge t_n} = \mathbb{E}[M_{t+1} | \mathcal{F}_{\tau_n \wedge t_n}]$ (and similarly for s_n). Moreover $M_{\tau_n \wedge t_n} \rightarrow M_{\tau \wedge t}$ a.s., since M is càdlàg and $\tau_n \downarrow \tau$. Hence $M_{\tau \wedge t} \in L^1$, and letting $n \rightarrow \infty$,

$$\mathbb{E}[\mathbf{1}_A M_{\tau \wedge t}] = \mathbb{E}[\mathbf{1}_A M_{\tau \wedge s}] \quad \forall A \in \mathcal{F}_s.$$

Since $M_{\tau \wedge s}$ is $\mathcal{F}_s$ -measurable, this shows $\mathbb{E}[M_{\tau \wedge t} | \mathcal{F}_s] = M_{\tau \wedge s}$. $\square$

Theorem 25 (Martingale convergence theorem). Let M be a càdlàg martingale bounded in L^1 , i.e. $\sup_{t \geq 0} \mathbb{E}[|M_t|] < \infty$. Then $M_\infty := \lim_{t \rightarrow \infty} M_t$ exists almost surely.

Proof. We proceed in five steps.

Step 1. A discrete-time martingale $(M_n)_{n \geq 0}$ bounded in L^2 converges almost surely. Indeed, writing $M_n - M_0 = \sum_{k=1}^n (M_k - M_{k-1})$ as a sum of orthogonal increments,

$$\mathbb{E}[(M_n - M_0)^2] = \sum_{k=1}^n \mathbb{E}[(M_k - M_{k-1})^2],$$

so $\sup_n \mathbb{E}[(M_n - M_0)^2] < \infty$ forces $\sum_{k \geq 0} \mathbb{E}[(M_{k+1} - M_k)^2] < \infty$. Hence

$$\sup_{i,j \geq n} \mathbb{E}[(M_i - M_j)^2] \leq 4 \sum_{k \geq n} \mathbb{E}[(M_{k+1} - M_k)^2] \xrightarrow{n \rightarrow \infty} 0,$$

i.e. $(M_n)_{n \geq 0}$ is Cauchy in L^2 , hence converges a.s. along a subsequence, and in fact a.s. (being L^2 -Cauchy along the full sequence).

Step 2. A continuous-time martingale $(M_t)_{t \geq 0}$ bounded in L^2 converges in L^2 along any sequence $t_n \rightarrow \infty$: indeed $(M_{t_n})_n$ is (after reindexing to be monotone) a discrete-time martingale bounded in L^2 , hence converges by Step 1; and the limit does not depend on the chosen sequence (t_n) , since for two sequences $(t_n), (t'_n) \rightarrow \infty$, the merged sequence $(s_n) = \{t_n\} \cup \{t'_n\}$ (reordered) also gives a convergent sequence M_{s_n} , whose limit must agree with both subsequential limits.

Step 3. A càdlàg martingale $(M_t)_{t \geq 0}$ bounded in L^2 converges almost surely. Let

$$\Delta_k := \sup_{s, t \geq k} |M_t - M_s|,$$

a non-increasing sequence, hence convergent a.s. By Doob's inequality,

$$\mathbb{E}[\Delta_k^2] \leq 4 \mathbb{E} \left[\sup_{s \geq k} |M_s - M_k|^2 \right] \leq 16 \sup_{s \geq k} \mathbb{E}[(M_s - M_k)^2],$$

using $(M_t - M_s)^2 \leq 2(|M_t - M_k|^2 + |M_s - M_k|^2) \leq 4 \sup_{s \geq k} |M_s - M_k|^2$; the right-hand side tends to 0 as $k \rightarrow \infty$ by Step 2. Hence $\Delta_k \rightarrow 0$ a.s., and in particular $(M_{t_n})_n$ is Cauchy (hence convergent) for every sequence $t_n \rightarrow \infty$, possibly random.

Step 4. A càdlàg martingale $(M_t)_{t \geq 0}$ bounded in L^1 satisfies $M^* := \sup_{t \geq 0} |M_t| < \infty$ a.s. Indeed, by the discrete maximal inequality applied to $(M_{k2^{-n}})_k$,

$$\mathbb{P} \left(\sup_{k \geq 0} |M_{k2^{-n}}| > a \right) \leq a^{-1} \sup_{k \geq 0} \mathbb{E}[|M_{k2^{-n}}|] \leq Ca^{-1},$$

with $C = \sup_t \mathbb{E}[|M_t|] < \infty$; letting $n \rightarrow \infty$ and using that M is càdlàg,

$$\mathbb{P} \left(\sup_{t \geq 0} |M_t| > a \right) \leq Ca^{-1}.$$

Step 5. For $A \in \mathbb{N}$, let $\tau_A := \inf\{t \geq 0, |M_t| > A\}$. The stopped martingale $(M_{t \wedge \tau_A})_{t \geq 0}$ is bounded (by A , plus one jump), hence bounded in L^2 , so by Step 3, $M_{t \wedge \tau_A} \rightarrow Z_A$ a.s. as $t \rightarrow \infty$, for every A . By Step 4, $\tau_A = \infty$ for A large enough, on a set of probability tending to 1 as $A \rightarrow \infty$: more precisely, almost surely there exists (random) $A_0 = A_0(\omega) < \infty$ such that $\tau_A = \infty$ for every $A \geq A_0$ (since $M^* < \infty$ a.s.). On this event, $M_t = M_{t \wedge \tau_{A_0}} \rightarrow Z_{A_0}$ as $t \rightarrow \infty$. Since this holds almost surely, M_t converges almost surely as $t \rightarrow \infty$. $\square$

Remark 26. For $x : \mathbb{R}_+ \rightarrow E$ (a metric space) and $\ell \in E$, consider the properties

- (i) $x(t) \rightarrow \ell$ as $t \rightarrow \infty$;
- (ii) for every sequence $t_n \rightarrow \infty$, $x(t_n) \rightarrow \ell$;
- (iii) for every sequence $t_n \rightarrow \infty$, there is a subsequence t_{n_k} with $x(t_{n_k}) \rightarrow \ell$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) trivially. Whether (iii) implies (ii) (hence (i)) in general requires additional structure (e.g. that every subsequential limit equals ℓ , as is the case in the martingale convergence argument above); this converse implication is not automatic for an arbitrary function x .

Theorem 27. Let M be a càdlàg martingale bounded in L^1 , and $M_\infty = \lim_{t \rightarrow \infty} M_t$ (which exists a.s. by the previous theorem). The following are equivalent:

- (i) $M_t \xrightarrow{L^1} M_\infty$;
- (ii) $(M_t)_{t \geq 0}$ is uniformly integrable;
- (iii) $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$ for every $t \geq 0$.

Proof. As in the discrete-time case:

(i) $\Rightarrow$ (iii): for fixed t and $n \rightarrow \infty$, $\mathbb{E}[M_{t+n} | \mathcal{F}_t] = M_t$; by (i), $M_{t+n} \rightarrow M_\infty$ in L^1 , so $\mathbb{E}[M_{t+n} | \mathcal{F}_t] \rightarrow \mathbb{E}[M_\infty | \mathcal{F}_t]$ in L^1 as well, while the right-hand side is constantly M_t . Hence $\mathbb{E}[M_\infty | \mathcal{F}_t] = M_t$.

(iii) $\Rightarrow$ (ii): the family $\{\mathbb{E}[Z | \mathcal{G}], \mathcal{G} \subset \mathcal{F}\}$, for $Z \in L^1$ fixed, is uniformly integrable; apply this with $Z = M_\infty$.

(ii) $\Rightarrow$ (i): this was already seen (a form of the “optional” dominated convergence theorem: a.s. convergence plus uniform integrability implies L^1 convergence). $\square$

6 Brownian motion as a martingale

Definition 28. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space. A process $B = (B_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion if

- (i) $(B_t)_{t \geq 0}$ is $(\mathcal{F}_t)_{t \geq 0}$ -adapted;
- (ii) for every $s \geq 0$, $(B_{t+s} - B_s)_{t \geq 0}$ is a Brownian motion independent of $\mathcal{F}_s$.

Proposition 29. *A Brownian motion B is a Brownian motion in its canonical filtration. It is also a Brownian motion in its naturally augmented filtration $\bar{\mathcal{F}}_t = \mathcal{F}_{t+} \vee \mathcal{N}$ (this uses Blumenthal's 0-1 law). More generally, it is a Brownian motion in $\mathcal{F}_t \vee \mathcal{G}_t$, for any filtration $(\mathcal{G}_t)_{t \geq 0}$ independent of $(\mathcal{F}_t)_{t \geq 0}$.*

Exercise. *The strong Markov property still holds in this setting: if $(B_t)_{t \geq 0}$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion and τ is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time, then, on $\{\tau < \infty\}$, $(B_{t+\tau} - B_\tau)_{t \geq 0}$ is a Brownian motion independent of $\mathcal{F}_\tau$.*

Proposition 30 (Wald's identities). *Let τ be a stopping time with $\mathbb{E}[\tau] < \infty$. Then*

$$B_\tau \in L^2, \quad \mathbb{E}[B_\tau] = 0, \quad \mathbb{E}[B_\tau^2] = \mathbb{E}[\tau].$$

We first note that $(B_t)_{t \geq 0}$ is not a closed martingale: otherwise, taking $\tau_1 = \inf\{t \geq 0, B_t = 1\}$, which is a.s. finite with $B_{\tau_1} = 1$ a.s., the optional stopping theorem would give

$$1 = \mathbb{E}[B_{\tau_1} | \mathcal{F}_0] = B_0 = 0,$$

a contradiction. (Consistently, we already know $\mathbb{E}[\tau_1] = \infty$.) Thus Wald's identities are not entirely automatic and require a localization argument.

Proof. The processes $(B_{t \wedge \tau})_{t \geq 0}$ and $(B_{t \wedge \tau}^2 - t \wedge \tau)_{t \geq 0}$ are martingales, by the stopping proposition of the previous section. Hence

$$\mathbb{E}[B_{t \wedge \tau}] = 0 = \mathbb{E}[B_{t \wedge \tau}^2 - t \wedge \tau],$$

so that, for every $t \geq 0$,

$$\mathbb{E}[B_{t \wedge \tau}^2] = \mathbb{E}[t \wedge \tau] \leq \mathbb{E}[\tau] < \infty.$$

Thus $\sup_{t \geq 0} \mathbb{E}[B_{t \wedge \tau}^2] < \infty$, i.e. $(B_{t \wedge \tau})_{t \geq 0}$ is a continuous martingale bounded in L^2 ; in particular it is uniformly integrable, and converges a.s. (as $t \rightarrow \infty$) to B_τ (using $\tau < \infty$ a.s., which follows since $\mathbb{E}[\tau] < \infty$). Applying the optional stopping theorem to the closed martingale $(B_{t \wedge \tau})_{t \geq 0}$,

$$\mathbb{E}[B_\tau] = \lim_{t \rightarrow \infty} \mathbb{E}[B_{t \wedge \tau}] = \mathbb{E}[B_0] = 0.$$

For the second moment, apply Doob's L^2 inequality to $(B_{t \wedge \tau})_{t \geq 0}$:

$$\mathbb{E}\left[\sup_{t \geq 0} B_{t \wedge \tau}^2\right] \leq 4 \sup_{t \geq 0} \mathbb{E}[B_{t \wedge \tau}^2] \leq 4 \mathbb{E}[\tau] < \infty.$$

This shows that $(B_{t \wedge \tau}^2)_{t \geq 0}$ is dominated by an integrable random variable, hence uniformly integrable; and since $t \wedge \tau \leq \tau \in L^1$, $(t \wedge \tau)_{t \geq 0}$ is uniformly integrable as

well (dominated by $\tau \in L^1$). Consequently $(B_{t \wedge \tau}^2 - t \wedge \tau)_{t \geq 0}$ is uniformly integrable, hence closed by its a.s. limit $B_\tau^2 - \tau$. Applying the optional stopping theorem once more,

$$\mathbb{E}[B_\tau^2 - \tau] = \mathbb{E}[B_{0 \wedge \tau}^2 - 0 \wedge \tau] = 0,$$

i.e. $\mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$. □

7 Miscellaneous

σ_a is not an $(\mathcal{F}_t)$ -stopping time: a proof on Wiener space

We come back to Example 3 and Remark 4, where it was claimed that, for $a \geq 0$,

$$\sigma_a = \inf\{t \geq 0, B_t > a\}$$

is an $(\mathcal{F}_{t+})$ -stopping time but not an $(\mathcal{F}_t)$ -stopping time, $(\mathcal{F}_t)_{t \geq 0}$ being the canonical (raw, non-completed) filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$. The first claim was checked in Remark 4. We now prove the second one rigorously, in a specific setting: the canonical Wiener space¹. We take

$$\Omega = C(\mathbb{R}_+, \mathbb{R}), \quad B_t(\omega) = \omega(t), \quad \mathcal{F}_t = \sigma(B_s, s \leq t), \quad \mathcal{F} = \mathcal{F}_\infty,$$

and $\mathbb{P}$ is the Wiener measure, i.e. the law of a Brownian motion started at 0. The key feature of this space is that *every* continuous path is an element of Ω .

Lemma 31. *Let $t \geq 0$ and $A \in \mathcal{F}_t$. If $\omega \in A$ and $\omega' \in \Omega$ satisfies $\omega'(s) = \omega(s)$ for every $s \in [0, t]$, then $\omega' \in A$.*

In words: an event of $\mathcal{F}_t$ is determined by the path up to time t . Consequently, to show that an event E does *not* belong to $\mathcal{F}_t$, it suffices to exhibit two paths that coincide on $[0, t]$, one in E and the other one not.

Proof. Let $\mathcal{G}$ be the collection of sets $A \subseteq \Omega$ such that $\mathbf{1}_A(\omega) = \mathbf{1}_A(\omega')$ for all $\omega, \omega' \in \Omega$ that coincide on $[0, t]$.

- $\mathcal{G}$ is a σ -field: it contains Ω ; it is stable under complementation, since the defining property only involves $\mathbf{1}_A$; and it is stable under countable unions, since if $\omega \in \bigcup_n A_n$ with $A_n \in \mathcal{G}$, then $\omega \in A_k$ for some k , hence $\omega' \in A_k \subseteq \bigcup_n A_n$ (and symmetrically).
- $\mathcal{G}$ generates $\mathcal{F}_t$: for $s \leq t$ and $\Gamma \in \mathcal{B}(\mathbb{R})$, the set $\{B_s \in \Gamma\}$ belongs to $\mathcal{G}$, because $B_s(\omega) = \omega(s) = \omega'(s) = B_s(\omega')$.

¹As explained in Remark ?? below, the statement is genuinely less clear on an arbitrary probability space, and may even fail.

Hence $\mathcal{F}_t = \sigma(B_s, s \leq t) \subseteq \mathcal{G}$. □

We can now prove that on Wiener space, for every $a \geq 0$ and every $t > 0$,

$$\{\sigma_a \leq t\} \in \mathcal{F}_{t+} \setminus \mathcal{F}_t.$$

The fact that $\{\sigma_a \leq t\} \in \mathcal{F}_{t+}$ follows from Remark 4 and the Exercise characterising $(\mathcal{F}_{t+})$ -stopping times. Fix $t > 0$ and define two continuous paths:

$$\omega(s) = \begin{cases} \frac{as}{t}, & s \leq t, \\ a + (s - t), & s > t, \end{cases} \quad \omega'(s) = \begin{cases} \frac{as}{t}, & s \leq t, \\ a - (s - t), & s > t. \end{cases}$$

Both belong to Ω (here we use that Ω is the whole of $C(\mathbb{R}_+, \mathbb{R})$), and they coincide on $[0, t]$: both reach the level a exactly at time t without exceeding it before.

- For ω : $\omega(s) \leq a$ for $s \leq t$, and $\omega(s) > a$ for every $s > t$. Hence $\sigma_a(\omega) = t$, so $\omega \in \{\sigma_a \leq t\}$.
- For ω' : $\omega'(s) \leq a$ for every $s \geq 0$, so $\{s \geq 0, \omega'(s) > a\} = \emptyset$ and $\sigma_a(\omega') = +\infty$. Thus $\omega' \notin \{\sigma_a \leq t\}$.

By Lemma 31, $\{\sigma_a \leq t\} \notin \mathcal{F}_t$.

Remark 32 (The phenomenon is invisible under the usual conditions). *The bad paths such as ω' form a negligible set: by Example 3, $T_a = \sigma_a$ almost surely. Moreover, for the naturally augmented filtration $\bar{\mathcal{F}}_t = \mathcal{F}_{t+} \vee \mathcal{N}$, one has $\{\sigma_a \leq t\} \in \mathcal{F}_{t+} \subseteq \bar{\mathcal{F}}_t$, so σ_a is an $(\bar{\mathcal{F}}_t)$ -stopping time. This is one of the reasons for working under the usual conditions, as we do from Section 2 on.*