

Chapter 4: Continuous semimartingales

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1 Functions with bounded variation

Throughout this section, $T > 0$ is fixed and all functions are deterministic.

Proposition 1. *Let $F : [0, T] \rightarrow \mathbb{R}_+$ be non-decreasing and continuous, with $F(0) = 0$. There exists a unique positive measure dF on $([0, T], \mathcal{B}([0, T]))$, called the Stieltjes measure associated with F , such that*

$$\forall 0 \leq s < t \leq T, \quad dF((s, t]) = F(t) - F(s).$$

Moreover $dF(\{s\}) = 0$ for every $s \in [0, T]$, and $dF([0, t]) = F(t)$.

Note that the absence of atoms is a consequence of the continuity of F : for $s > 0$, $dF(\{s\}) = \lim_n (F(s) - F(s - \frac{1}{n})) = 0$.

Proof. Uniqueness follows from the fact that two finite measures which coincide on the intervals $(s, t]$ (a π -system generating $\mathcal{B}((0, T])$), and give the same (zero) mass to $\{0\}$, are equal.

Existence: we proceed as for the cumulative distribution function of a random variable. If $F(T) = 0$ then $F \equiv 0$ and $dF = 0$ works. Otherwise, set

$$G(r) = \inf\{t \geq 0, F(t) > r\}, \quad r \in [0, F(T)),$$

which is well defined since $F(T) > r$. By continuity of F , $F(G(r)) = r$, and one checks that

$$\{r < F(t)\} \subset \{G(r) \leq t\} \subset \{r \leq F(t)\}.$$

Let U be uniformly distributed on $[0, F(T))$ and μ be the law of $G(U)$, a probability measure on $[0, T]$. Since $\mathbb{P}(U = F(t)) = 0$ for every t ,

$$\mu((s, t]) = \mathbb{P}(s < G(U) \leq t) = \mathbb{P}(F(s) < U \leq F(t)) = F(T)^{-1}(F(t) - F(s)).$$

Hence $dF := F(T) \mu$ is a solution to the existence problem. $\square$

Definition 2. (i) A signed measure on a measurable space $(E, \mathcal{E})$ is the difference of two positive finite measures on $(E, \mathcal{E})$.

(ii) A continuous function $a : [0, T] \rightarrow \mathbb{R}$ with $a(0) = 0$ has finite variation if there exists a signed measure da on $[0, T]$ such that

$$a(t) = da([0, t]), \quad \forall t \in [0, T].$$

Remark 3. (i) If a is C^1 (and $a(0) = 0$), then a has finite variation and $da(dt) = a'(t) dt$.

(ii) The measure da is unique (by uniqueness of measures that coincide on intervals). But a decomposition $da = \mu_1 - \mu_2$, with μ_1, μ_2 positive finite measures, is not unique.

(iii) The decomposition becomes unique if we moreover impose that $\mu_1 \perp \mu_2$, i.e. that there exists $A \in \mathcal{B}([0, T])$ such that $\mu_1(A) = 0 = \mu_2(A^c)$.

Proposition 4. If a has finite variation, there exists a unique couple (da_+, da_-) of positive finite measures on $[0, T]$, mutually singular, such that

$$da = da_+ - da_-.$$

Proof. Existence. Let $\mu_1, \mu_2 \geq 0$ be such that $da = \mu_1 - \mu_2$, and set $\nu := \mu_1 + \mu_2$, a positive finite measure with $\mu_1 \ll \nu$ and $\mu_2 \ll \nu$. By the Radon–Nikodym theorem, there exist $h_i \geq 0$, $i = 1, 2$, such that $d\mu_i = h_i d\nu$. Let $h := h_1 - h_2$; then, with $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$,

$$da = h d\nu = h_+ d\nu - h_- d\nu =: da_+ - da_-.$$

These two measures are mutually singular: with $A := \{t \in [0, T], h(t) \geq 0\}$, we have $da_+(A^c) = 0 = da_-(A)$.

Uniqueness. Let $da = \mu_1 - \mu_2$ with $\mu_i \geq 0$ and $\mu_1(A) = \mu_2(A^c) = 0$ for some Borel set A . We claim that, necessarily, for every Borel set F ,

$$\mu_1(F) = \sup \{da(C), C \subseteq F, C \in \mathcal{B}([0, T])\},$$

so that μ_1 , and then $\mu_2 = \mu_1 - da$, are determined by da . Indeed, for $C \subseteq F$, $da(C) = \mu_1(C) - \mu_2(C) \leq \mu_1(F)$; and for $C = A^c \cap F$,

$$da(A^c \cap F) = \mu_1(A^c \cap F) - \underbrace{\mu_2(A^c \cap F)}_{=0} = \mu_1(F) - \underbrace{\mu_1(F \cap A)}_{=0} = \mu_1(F). \quad \square$$

Remark 5. The positive finite measure $|da| := da_+ + da_-$ is called the total variation of da . One has $|da|(A) \geq |da(A)|$ for every Borel set A , and

$$da = (\mathbf{1}_{D_+} - \mathbf{1}_{D_-}) |da|,$$

where D_+ is a Borel set such that $da_+(D_+^c) = 0 = da_-(D_+)$, and $D_- := D_+^c$. Indeed, for every Borel set F ,

$$\int_F (\mathbf{1}_{D_+} - \mathbf{1}_{D_-}) |da| = |da|(F \cap D_+) - |da|(F \cap D_-) = da(F),$$

since $|da|(F \cap D_+) = da_+(F \cap D_+) + da_-(F \cap D_+) = da_+(F)$, and similarly $|da|(F \cap D_-) = da_-(F)$.

Note also that $da_{\pm}$, hence $|da|$, have no atoms when a is continuous: by the formula of the uniqueness part of the lemma, $da_+(\{s\}) = \max(da(\{s\}), 0) = 0$, since $da(\{s\}) = a(s) - a(s-) = 0$ (and similarly for da_-).

Remark 6. Let $a : [0, T] \rightarrow \mathbb{R}$ be continuous with finite variation.

(i) For $f : [0, T] \rightarrow \mathbb{R}$ measurable such that $\int_{[0, T]} |f(t)| |da|(dt) < \infty$, we set

$$\int_0^T f da := \int_0^T f(s) da_+(ds) - \int_0^T f(s) da_-(ds)$$

and

$$\int_0^T f |da| := \int_0^T f(s) da_+(ds) + \int_0^T f(s) da_-(ds).$$

(ii) We then have

$$\left| \int_0^T f da \right| \leq \int_0^T |f| |da|.$$

(iii) Moreover, the function

$$g(t) = \int_0^t f \, da = \int_0^T f \mathbf{1}_{[0,t]} \, da$$

is also continuous with finite variation (continuity follows by dominated convergence, since $|da|$ has no atoms), and

$$dg = f \, da, \quad dg_+ = f_+ \, da_+ + f_- \, da_-, \quad dg_- = f_- \, da_+ + f_+ \, da_-.$$

(The two measures on the right are carried respectively by $(D_+ \cap \{f \geq 0\}) \cup (D_- \cap \{f < 0\})$ and by its complement, hence are mutually singular.)

Proposition 7. If a has finite variation, then for every $t \in (0, T]$,

$$|da|([0, t]) = \sup \left\{ \sum_{i=1}^p |a(t_i) - a(t_{i-1})|, \, p \geq 1, \, 0 = t_0 < t_1 < \dots < t_p = t \right\}.$$

Moreover, the supremum is the limit of the dyadic sums

$$S_n := \sum_{i=1}^{2^n} |a(i2^{-n}t) - a((i-1)2^{-n}t)|.$$

Proof. The lower bound is easy: for $0 = t_0 < \dots < t_p = t$,

$$\begin{aligned} \sum_{i=1}^p |a(t_i) - a(t_{i-1})| &= \sum_{i=1}^p |da((t_{i-1}, t_i])| \\ &\leq \sum_{i=1}^p |da|((t_{i-1}, t_i]) \\ &= |da|((0, t]) \leq |da|([0, t]). \end{aligned}$$

For the upper bound, we use a martingale argument, and show that $S_n \rightarrow |da|([0, t])$. We may assume $|da|([0, t]) > 0$. On $\Omega = (0, t]$, endowed with $\mathcal{B}((0, t])$ and the probability measure

$$\mathbb{P} = \frac{|da|}{|da|([0, t])}$$

(recall that $|da|(\{0\}) = 0$), consider the random variable $X := \mathbf{1}_{D_+} - \mathbf{1}_{D_-}$, so that $|X| = 1$ and $\mathbb{E}[X \mathbf{1}_I] = da(I)/|da|([0, t])$ for every Borel set I . Let $\mathcal{D}_n$ be the σ -field generated by the dyadic intervals $I_i^n = ((i-1)2^{-n}t, i2^{-n}t]$, $1 \leq i \leq 2^n$. Then

$$X_n := \mathbb{E}[X \mid \mathcal{D}_n] = \sum_{i=1}^{2^n} \frac{da(I_i^n)}{|da|(I_i^n)} \mathbf{1}_{I_i^n} \quad (\text{with the convention } 0/0 = 0)$$

is a bounded martingale, which converges a.s. and in L^1 to $\mathbb{E}[X \mid \mathcal{D}_\infty] = X$, since $\mathcal{D}_\infty = \sigma(\bigcup_n \mathcal{D}_n) = \mathcal{B}((0, t])$. Therefore

$$\frac{S_n}{|da|([0, t])} = \sum_{i=1}^{2^n} \frac{|da(I_i^n)|}{|da|([0, t])} = \mathbb{E}[|X_n|] \xrightarrow{n \rightarrow \infty} \mathbb{E}[|X|] = 1. \quad \square$$

Lemma 8. *Let $a : [0, T] \rightarrow \mathbb{R}$ be continuous with finite variation, and $f : [0, T] \rightarrow \mathbb{R}$ be continuous. Let $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = T)$ be a sequence of subdivisions of $[0, T]$ whose mesh $|\Delta_n| := \max_i(t_i^n - t_{i-1}^n)$ tends to 0. Then*

$$\int_0^T f(s) da(s) = \lim_{n \rightarrow \infty} \sum_{i=1}^{p_n} f(t_{i-1}^n)(a(t_i^n) - a(t_{i-1}^n)).$$

Proof. Let $f_n(s) = \sum_{i=1}^{p_n} f(t_{i-1}^n) \mathbf{1}_{\{s \in (t_{i-1}^n, t_i^n]\}}$. The formula holds for f_n , since $\int f_n da = \sum_i f(t_{i-1}^n) da((t_{i-1}^n, t_i^n])$. Moreover $f_n \rightarrow f$ pointwise on $(0, T]$ (by uniform continuity of f), and $|f_n| \leq \sup |f|$, with $|da|$ finite and $|da|(\{0\}) = 0$. Dominated convergence concludes. $\square$

2 Finite variation processes

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

Definition 9. *A finite variation process $A = (A_t)_{t \geq 0}$ is an adapted process whose paths are a.s. continuous with finite variation on $[0, T]$ for every $T > 0$, and such that $A_0 = 0$. The process A is non-decreasing if its paths are a.s. non-decreasing.*

Thanks to the completeness of the filtration, we may and do assume that all paths are continuous with finite variation (set $A \equiv 0$ on the exceptional null set).

Remark 10. *Let A be a finite variation process, and for every ω let $A_t^+(\omega) = dA_+([0, t])(\omega)$, $A_t^-(\omega) = dA_-([0, t])(\omega)$, where $dA(\omega) = dA_+(\omega) - dA_-(\omega)$ is the decomposition of the lemma above (applied on every $[0, T]$). Then A^+ and A^- are continuous non-decreasing adapted processes, and $A = A^+ - A^-$. Indeed,*

$$A_t^\pm = \frac{1}{2}(|dA|([0, t]) \pm A_t),$$

and $|dA|([0, t]) = \lim_n \sum_{i=1}^{2^n} |A_{i2^{-n}t} - A_{(i-1)2^{-n}t}|$ is $\mathcal{F}_t$ -measurable by Proposition 7; continuity follows from the absence of atoms of $|dA|$.

Proposition 11. *Let A be a finite variation process and H a progressively measurable process such that, for every $t \geq 0$,*

$$\int_0^t |H_s| |dA_s| < \infty \quad a.s.$$

Then the process

$$(H \cdot A)_t := \int_0^t H_s dA_s \quad (\text{defined } \omega \text{ by } \omega)$$

is also a finite variation process. In particular it is adapted (and, being continuous, progressively measurable).

Proof. Continuity and finite variation of the paths follow from the previous remark on $g(t) = \int_0^t f da$, applied ω by ω . It remains to prove adaptedness. We write

$$\int_0^t H_s dA_s = \int_0^t H_+(s) dA_s^+ - \int_0^t H_-(s) dA_s^+ - \int_0^t H_+(s) dA_s^- + \int_0^t H_-(s) dA_s^-,$$

where $H_{\pm}$ are progressively measurable and $A^{\pm}$ are adapted by Remark 10 (this is where the progressive measurability, and the adaptedness of $A^{\pm}$, matter). So, without loss of generality, we may take $H \geq 0$ and A non-decreasing.

(i) Fix $t \geq 0$. For $0 \leq u < v \leq t$ and $\Gamma \in \mathcal{F}_t$,

$$(\mathbf{1}_{(u,v]} \mathbf{1}_{\Gamma} \cdot A)_t = \mathbf{1}_{\Gamma} (A_v - A_u) \in \mathcal{F}_t.$$

The class $\mathcal{S} = \{\Gamma \times (u, v], \Gamma \in \mathcal{F}_t, 0 \leq u < v \leq t\}$ (completed by the sets $\Gamma \times \{0\}$, for which $(\mathbf{1}_{\Gamma \times \{0\}} \cdot A)_t = 0$) is a π -system generating $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$, and the class of sets G such that $(\mathbf{1}_G \cdot A)_t$ is $\mathcal{F}_t$ -measurable is a monotone class (use monotone convergence for increasing unions). By the monotone class theorem,

$$(\mathbf{1}_G \cdot A)_t \in \mathcal{F}_t \quad \forall G \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t.$$

(ii) Since H is progressive, $H \mathbf{1}_{[0,t]}$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ -measurable, hence $H \mathbf{1}_{[0,t]} = \lim_n \uparrow H^n$, with $H^n = \sum_i a_i \mathbf{1}_{G_i^n}$, $a_i > 0$, $G_i^n \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t$. By (i), $(H^n \cdot A)_t \in \mathcal{F}_t$, and by monotone convergence $(H \cdot A)_t = \lim_n (H^n \cdot A)_t$. Hence $(H \cdot A)_t$ is $\mathcal{F}_t$ -measurable. $\square$

3 Local martingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

Notation 12. For a stopping time τ and a continuous process $X = (X_t)_{t \geq 0}$, we denote by

$$X^\tau = (X_t^\tau := X_{t \wedge \tau})_{t \geq 0}$$

the stopped process.

Definition 13. An adapted continuous process $(M_t)_{t \geq 0}$ is a (continuous) local martingale if there exists a non-decreasing sequence of stopping times $(\tau_n)_{n \geq 0}$ with $\tau_n \uparrow \infty$ a.s., such that for every n , $(M^{\tau_n} - M_0)$ is a uniformly integrable martingale. The sequence $(\tau_n)_{n \geq 0}$ is called a localizing sequence.

Remark 14. If M is a local martingale, M_t is not necessarily in $L^1(\mathbb{P})$; and there is no integrability assumption on M_0 either (except that M_0 is $\mathcal{F}_0$ -measurable). If X is any $\mathcal{F}_0$ -measurable random variable and $(M_t)_{t \geq 0}$ is a martingale, then $(X + M_t)_{t \geq 0}$ and $(XM_t)_{t \geq 0}$ are local martingales: the sequence $\tau_n := n \mathbf{1}_{\{|X| \leq n\}}$ is localizing. More generally, if M is a local martingale with $M_0 = 0$ and localizing sequence (τ_n) , then XM is a local martingale, localized by $\tau_n \mathbf{1}_{\{|X| \leq n\}}$.

Remark 15. (i) A continuous martingale is a local martingale ($\tau_n = n$ is localizing).

(ii) “Uniformly integrable martingale” can be replaced by “martingale” in the definition (replace τ_n by $\tau_n \wedge n$). [Exercise]

(iii) If (τ_n) is localizing for M and (σ_n) is a sequence of stopping times with $\sigma_n \uparrow \infty$, then $(\sigma_n \wedge \tau_n)$ is still localizing.

(iv) Local martingales form a vector space.

(v) If M is a local martingale and τ is a stopping time, then M^τ is a local martingale (with the same localizing sequence, by the optional stopping theorem).

Careful! Results that are valid for martingales are not obvious for local martingales.

Proposition 16. Let M be a (continuous) local martingale.

(i) If $M_0 \in L^1$ and $M \geq 0$, then M is a supermartingale.

(ii) If $\mathbb{E}[\sup_{s \in [0, t]} |M_s|] < \infty$ for every $t \geq 0$, then M is a martingale.

(iii) The sequence $\tau_n = \inf\{t \geq 0, |M_t - M_0| = n\}$ is localizing.

Proof. (i) Let (τ_n) be localizing. Since $M_0 \in L^1$, M^{τ_n} is a martingale (by definition, $(M - M_0)^{\tau_n}$ is). Hence

$$\mathbb{E}[M_{t \wedge \tau_n} | \mathcal{F}_s] = M_{s \wedge \tau_n} \quad \text{a.s.}, \quad s \leq t,$$

which is simply the martingale property of $M_{\cdot \wedge \tau_n}$. By Fatou's lemma (conditionally on $\mathcal{F}_s$), since $\lim_n M_{t \wedge \tau_n} = M_t$,

$$\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s \quad \text{a.s.}$$

Taking $s = 0$ and expectations gives $\mathbb{E}[M_t] \leq \mathbb{E}[M_0] < \infty$, i.e. $M_t \in L^1$. Hence (i) (integrability and supermartingale property).

(ii) We have $M_0 \in L^1$ too, $M_{t \wedge \tau_n} \rightarrow M_t$ a.s., and

$$|M_{t \wedge \tau_n}| \leq Y := \sup_{s \leq t} |M_s| \in L^1.$$

Conditional dominated convergence allows to pass to the limit in $\mathbb{E}[M_{t \wedge \tau_n} | \mathcal{F}_s] = M_{s \wedge \tau_n}$.

(iii) This is a consequence of (ii): $(M - M_0)^{\tau_n}$ is a bounded local martingale (Remark 15(v)), hence a bounded, thus uniformly integrable, martingale. It suffices to check that $\tau_n \rightarrow \infty$ a.s., and this follows from the continuity of the paths of M . $\square$

Remark 17 (Caution). *If M is a local martingale and if, for every t , the family $(M_s)_{0 \leq s \leq t}$ is uniformly integrable, then M is **not necessarily** a martingale. We will see later an example of a local martingale which is bounded in L^2 on every $[0, t]$ but is not a martingale.*

Theorem 18. *Let M be a local martingale with finite variation. Then, almost surely,*

$$\forall t \geq 0, \quad M_t = M_0.$$

Proof. Replacing M by $M - M_0$, we may assume $M_0 = 0$ without loss of generality.

(i) Assume first that M is a true martingale, bounded by some constant K , and that its total variation is bounded: $\int_0^\infty |dM_s| \leq K$. Let $t > 0$ and $0 = t_0 < t_1 < \dots < t_p = t$. Using the martingale property (orthogonality of increments),

$$\begin{aligned} \mathbb{E}[M_t^2] &= \mathbb{E}\left[\left(\sum_{i=1}^p (M_{t_i} - M_{t_{i-1}})\right)^2\right] = \sum_{i=1}^p \mathbb{E}[(M_{t_i} - M_{t_{i-1}})^2] \\ &\leq \mathbb{E}\left[\max_{1 \leq i \leq p} |M_{t_i} - M_{t_{i-1}}| \underbrace{\sum_{i=1}^p |M_{t_i} - M_{t_{i-1}}|}_{\leq |dM|([0, t]) \leq K}\right] \leq K \mathbb{E}\left[\max_{1 \leq i \leq p} |M_{t_i} - M_{t_{i-1}}|\right], \end{aligned}$$

where we used the characterization of $|dM|$ via subdivisions (Proposition 7). By continuity and boundedness of M , and dominated convergence, the right-hand side tends to 0 as the mesh of the subdivision tends to 0. Hence $\mathbb{E}[M_t^2] = 0$, i.e. $M_t = 0$ a.s. for every t , and by continuity again, a.s. $M_t = 0$ for every t .

(ii) General case: M is a local martingale with $M_0 = 0$ and finite variation. Let

$$\tau_n = \inf \left\{ t \geq 0, |M_t| + \int_0^t |dM_s| \geq n \right\},$$

a stopping time (the process $t \mapsto \int_0^t |dM_s|$ is continuous and adapted, by Remark 10). We have $\tau_n \uparrow \infty$ a.s., since $\int_0^T |dM_s| < \infty$ for every T . Then M^{τ_n} is a bounded local martingale, hence a true martingale (Proposition 16(ii)), bounded by n and with total variation bounded by n . By (i), a.s. $M_t^{\tau_n} = 0$ for every t , i.e. $M_t = 0$ for every $t \in [0, \tau_n]$. Letting $n \rightarrow \infty$ concludes. $\square$

Remark 19. (i) *The proof works verbatim on a finite horizon: a local martingale on $[0, T]$ with finite variation is a.s. constant on $[0, T]$.*

(ii) *The continuity assumption is crucial: if N is a Poisson process with intensity λ , then $(N_t - \lambda t)_{t \geq 0}$ is a martingale with finite variation, which is (of course) not constant.*

4 The quadratic variation of a local martingale

In this section, we prove the most important (and somehow also the most difficult, or at least the most technical) result of these lectures.

For a subdivision $\Delta = (0 = t_0 < t_1 < \dots < t_p = t)$ of $[0, t]$, we denote by $|\Delta| = \max_{1 \leq i \leq p} (t_i - t_{i-1})$ its mesh.

Theorem 20. *Let M be a (continuous) local martingale. There exists a unique (up to indistinguishability) non-decreasing, adapted and continuous process, denoted $(\langle M \rangle_t)_{t \geq 0}$, with $\langle M \rangle_0 = 0$, such that*

$$(M_t^2 - \langle M \rangle_t)_{t \geq 0} \text{ is a local martingale.}$$

Moreover, for every $t > 0$, if $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t)$ is a sequence of subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$, then

$$\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle M \rangle_t.$$

The process $\langle M \rangle$ is called the quadratic variation of M . It does not depend on M_0 : $\langle M \rangle = \langle M - M_0 \rangle$.

Remark 21. (i) If $M = B$ is a Brownian motion, then $\langle B \rangle_t = t$, since $(B_t^2 - t)_{t \geq 0}$ is a martingale.

(ii) Discrete case: if $(M_n)_{n \geq 0}$ is a martingale in L^2 , then

$$A_n = \sum_{k=1}^n \mathbb{E}[(M_k - M_{k-1})^2 \mid \mathcal{F}_{k-1}]$$

is $\mathcal{F}_{n-1}$ -measurable (predictable), and $M_n^2 - A_n$ is a martingale. We do not insist here on predictability, which, in our continuous setting, is replaced by continuity. If M had jumps, the situation would be much more delicate.

The proof relies on two lemmas.

Lemma 22 (Fundamental lemma). *Let $t > 0$, $p \geq 1$, and for every n , let $(X_s^n)_{s \in [0, t]}$ be a continuous adapted process, such that*

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X_s^n - X_s^m|^p \right] \xrightarrow{n, m \rightarrow \infty} 0.$$

Then there exists a continuous adapted process $(X_s)_{s \in [0, t]}$ such that

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X_s^n - X_s|^p \right] \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We write $\|Y\|_p = \mathbb{E}[|Y|^p]^{1/p}$. Let $(n_k)_{k \geq 0}$ be an increasing sequence such that

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X_s^n - X_s^m|^p \right] \leq 2^{-k} \quad \forall n, m \geq n_k;$$

in particular $\mathbb{E}[\sup_{s \in [0, t]} |X_s^{n_{k+1}} - X_s^{n_k}|^p] \leq 2^{-k}$. Then, by Minkowski's inequality (and monotone convergence),

$$\left\| \sum_{k \geq 0} \sup_{s \in [0, t]} |X_s^{n_{k+1}} - X_s^{n_k}| \right\|_p \leq \sum_{k \geq 0} \left\| \sup_{s \in [0, t]} |X_s^{n_{k+1}} - X_s^{n_k}| \right\|_p \leq \sum_{k \geq 0} 2^{-k/p} < \infty.$$

It follows that

$$\sum_{k \geq 0} \sup_{s \in [0, t]} |X_s^{n_{k+1}} - X_s^{n_k}| < \infty \quad \text{a.s.,}$$

so that the series $\sum_{k \geq 0} (X_s^{n_{k+1}} - X_s^{n_k})$ converges normally, hence uniformly in $s \in [0, t]$, almost surely. That is,

$$X_s^{n_k} \xrightarrow[k \rightarrow \infty]{\text{a.s.}} X_s := X_s^{n_0} + \sum_{k \geq 0} (X_s^{n_{k+1}} - X_s^{n_k}), \quad \text{uniformly in } s \in [0, t],$$

to a limit which is continuous and adapted (on the negligible set where the series diverges, we set $X \equiv 0$; the filtration being complete, X is adapted). Moreover,

$$\left\| \sup_{s \in [0, t]} |X_s - X_s^{n_k}| \right\|_p \leq \sum_{\ell \geq k} \left\| \sup_{s \in [0, t]} |X_s^{n_{\ell+1}} - X_s^{n_\ell}| \right\|_p \leq \sum_{\ell \geq k} 2^{-\ell/p} \xrightarrow[k \rightarrow \infty]{} 0.$$

Conclusion: by triangle inequality,

$$\begin{aligned} & \limsup_n \left\| \sup_{s \in [0, t]} |X_s - X_s^n| \right\|_p \\ & \leq \limsup_n \limsup_k \left(\left\| \sup_{s \in [0, t]} |X_s - X_s^{n_k}| \right\|_p + \left\| \sup_{s \in [0, t]} |X_s^{n_k} - X_s^n| \right\|_p \right) = 0, \end{aligned}$$

and this completes the proof. $\square$

Lemma 23 (second fundamental lemma). *Let M be a bounded continuous martingale (say $|M| \leq K$) such that $M_0 = 0$. Let $t > 0$ and $\Delta_n = (0 = t_0^n < \dots < t_{p_n}^n = t)$ be subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$. Set*

$$X_s^n := \sum_{i=1}^{p_n} M_{t_{i-1}^n} (M_{t_i^n \wedge s} - M_{t_{i-1}^n \wedge s}), \quad s \in [0, t].$$

Then $(X_s^n)_{0 \leq s \leq t}$ is a bounded continuous martingale. Moreover,

$$\mathbb{E}[|X_t^n - X_t^m|^2] \xrightarrow[n, m \rightarrow \infty]{} 0. \quad (1)$$

Proof. X^n is a bounded continuous martingale. Continuity is clear. It suffices to show that, for $0 \leq a < b \leq t$,

$$U_s := M_{a \wedge s} (M_{s \wedge b} - M_{s \wedge a}), \quad s \in [0, t],$$

is a martingale (it is bounded by $2K^2$). Note that, for $s \geq a$, $U_s = M_a(M_{s \wedge b} - M_a)$, while $U_s = 0$ for $s \leq a$. Let $s \leq u$.

- If $u \leq a$: $\mathbb{E}[U_u | \mathcal{F}_s] = 0 = U_s$.

- If $s \leq a \leq u$: $U_s = 0$ and

$$\mathbb{E}[U_u | \mathcal{F}_s] = \mathbb{E}\left[M_a \mathbb{E}[M_{u \wedge b} - M_{u \wedge a} | \mathcal{F}_a] \mid \mathcal{F}_s\right] = 0,$$

since $\mathbb{E}[M_{u \wedge b} | \mathcal{F}_a] = M_a = \mathbb{E}[M_{u \wedge a} | \mathcal{F}_a]$ (optional stopping, or simply the martingale property of M^b and M^a).

- If $a \leq s \leq u$: $\mathbb{E}[U_u | \mathcal{F}_s] = M_a \mathbb{E}[M_{u \wedge b} - M_{u \wedge a} | \mathcal{F}_s] = M_a(M_{s \wedge b} - M_a) = U_s$.

Proof of (1). Fix n, m , and let $0 = r_0 < r_1 < \dots < r_q = t$ be the points of the common refinement $\Delta_n \cup \Delta_m$. For $r \in [0, t]$, let $\pi_n(r) := \max\{t_i^n \in \Delta_n, t_i^n \leq r\}$ and similarly for π_m . Each interval $(r_{j-1}, r_j]$ is contained in a unique interval $(t_{i-1}^n, t_i^n]$ of Δ_n , namely the one with $t_{i-1}^n = \pi_n(r_{j-1})$. Splitting each increment $M_{t_i^n}^n - M_{t_{i-1}^n}^n$ into a telescopic sum along the refinement, we get

$$X_t^n = \sum_{j=1}^q M_{\pi_n(r_{j-1})} (M_{r_j} - M_{r_{j-1}}), \quad X_t^m = \sum_{j=1}^q M_{\pi_m(r_{j-1})} (M_{r_j} - M_{r_{j-1}}).$$

Therefore

$$X_t^n - X_t^m = \sum_{j=1}^q H_{j-1} (M_{r_j} - M_{r_{j-1}}), \quad \text{where } H_{j-1} := M_{\pi_n(r_{j-1})} - M_{\pi_m(r_{j-1})}.$$

(When $\Delta_n \subset \Delta_m$, the refinement is Δ_m itself and $\pi_m(r_{j-1}) = r_{j-1}$.) Using that $\mathbb{E}[M_{r_j} - M_{r_{j-1}} | \mathcal{F}_{r_{j-1}}] = 0$ and that H_{j-1} is $\mathcal{F}_{r_{j-1}}$ -measurable (the cross terms vanish), we obtain

$$\mathbb{E}[(X_t^n - X_t^m)^2] = \mathbb{E}\left[\sum_{j=1}^q H_{j-1}^2 (M_{r_j} - M_{r_{j-1}})^2\right] \leq \mathbb{E}\left[\sum_{j=1}^q \bar{H}_{j-1}^2 (M_{r_j} - M_{r_{j-1}})^2\right],$$

with $\bar{H}_j := \max_{0 \leq k \leq j} |H_k|$ for $0 \leq j \leq q-1$, and $\bar{H}_q := \bar{H}_{q-1}$. Note that $\bar{H}_{j-1}$ is $\mathcal{F}_{r_{j-1}}$ -measurable, and non-decreasing in j . We also have

$$\mathbb{E}[(M_{r_j} - M_{r_{j-1}})^2 | \mathcal{F}_{r_{j-1}}] = \mathbb{E}[M_{r_j}^2 - M_{r_{j-1}}^2 | \mathcal{F}_{r_{j-1}}],$$

therefore

$$\begin{aligned} \mathbb{E}[(X_t^n - X_t^m)^2] &\leq \mathbb{E}\left[\sum_{j=1}^q \bar{H}_{j-1}^2 (M_{r_j}^2 - M_{r_{j-1}}^2)\right] \leq \mathbb{E}\left[\sum_{j=1}^q \left(\bar{H}_j^2 M_{r_j}^2 - \bar{H}_{j-1}^2 M_{r_{j-1}}^2\right)\right] \\ &= \mathbb{E}[\bar{H}_q^2 M_t^2], \end{aligned}$$

by a telescopic sum, and using $M_0 = 0$. The second inequality is where the monotonicity of $\bar{H}$ is used (this is why we introduced $\bar{H}$): indeed $\bar{H}_j^2 M_{r_j}^2 - \bar{H}_{j-1}^2 M_{r_{j-1}}^2 - \bar{H}_{j-1}^2 (M_{r_j}^2 - M_{r_{j-1}}^2) = (\bar{H}_j^2 - \bar{H}_{j-1}^2) M_{r_j}^2 \geq 0$. Since $|M_t| \leq K$, we derive

$$\mathbb{E}[(X_t^n - X_t^m)^2] \leq K^2 \mathbb{E}[\bar{H}_q^2].$$

Now, $\bar{H}_q^2 \leq 4K^2$ and, since $r - \pi_n(r) < |\Delta_n|$ and $r - \pi_m(r) < |\Delta_m|$,

$$\bar{H}_q \leq \omega_M(|\Delta_n| \vee |\Delta_m|), \quad \text{where } \omega_M(\delta) := \max \{|M_u - M_s|, u, s \in [0, t], |u - s| \leq \delta\}.$$

By (uniform) continuity of the trajectories of M on $[0, t]$, $\omega_M(\delta) \rightarrow 0$ a.s. as $\delta \rightarrow 0$, so that $\bar{H}_q \rightarrow 0$ a.s. as $n, m \rightarrow \infty$. Dominated convergence ends the proof. $\square$

Proof of Theorem 20. Uniqueness. If $M^2 - A$ and $M^2 - \tilde{A}$ are two local martingales, with $A, \tilde{A}$ as in the statement, then so is $A - \tilde{A} = (M^2 - \tilde{A}) - (M^2 - A)$, which moreover has finite variation, and value 0 at $t = 0$. By Theorem 18, $A - \tilde{A} = 0$. The same argument works on a finite horizon $[0, t]$.

Step 1: M bounded (hence a true martingale) and $M_0 = 0$. Let $|M| \leq K$. Fix $t > 0$ and a sequence Δ_n of subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$, and let X^n be as in Lemma 23.

• **Fundamental decomposition.** Writing $M_s^2 = \sum_{i=1}^{p_n} (M_{t_i^n \wedge s}^2 - M_{t_{i-1}^n \wedge s}^2)$ and $b^2 - a^2 = 2a(b - a) + (b - a)^2$, we get, for every $s \in [0, t]$,

$$A_s^n := M_s^2 - 2X_s^n = \sum_{i=1}^{p_n} (M_{t_i^n \wedge s} - M_{t_{i-1}^n \wedge s})^2.$$

In particular, for $s = t_k^n \in \Delta_n$, $A_{t_k^n}^n = \sum_{i=1}^k (M_{t_i^n} - M_{t_{i-1}^n})^2$, so that A^n is non-decreasing on the points of Δ_n , and $A_t^n = \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2$.

• By Lemma 23, $X^n - X^m$ is a continuous martingale with $\mathbb{E}[|X_t^n - X_t^m|^2] \rightarrow 0$. Doob's L^2 inequality gives

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X_s^n - X_s^m|^2 \right] \leq 4 \mathbb{E}[|X_t^n - X_t^m|^2] \xrightarrow{n, m \rightarrow \infty} 0.$$

By the fundamental lemma (Lemma 22, with $p = 2$), there exists a continuous adapted process $(X_s)_{0 \leq s \leq t}$ such that

$$\mathbb{E} \left[\sup_{s \in [0, t]} |X_s^n - X_s|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

- X is a martingale on $[0, t]$: for $u \leq s \leq t$, $\mathbb{E}[X_s^n | \mathcal{F}_u] = X_u^n$, and both sides converge in L^2 (conditional expectation is a contraction in L^2), so that $\mathbb{E}[X_s | \mathcal{F}_u] = X_u$.

- Set $A_s := M_s^2 - 2X_s$, $s \in [0, t]$. Then A is continuous and adapted, $A_0 = 0$, $M^2 - A = 2X$ is a martingale on $[0, t]$, and

$$\mathbb{E} \left[\sup_{s \in [0, t]} |A_s^n - A_s|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

In particular $\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 = A_t^n \rightarrow A_t$ in L^2 .

- A is non-decreasing. Let $0 \leq s' \leq s \leq t$. With $\pi_n(r) = \max\{t_i^n \in \Delta_n, t_i^n \leq r\}$, we have $\pi_n(s') \leq \pi_n(s)$ and, by the fundamental decomposition,

$$A_s^n \geq A_{\pi_n(s)}^n \geq A_{\pi_n(s')}^n = A_{s'}^n - (M_{s'} - M_{\pi_n(s')})^2 \geq A_{s'}^n - \omega_M(|\Delta_n|)^2,$$

with ω_M the modulus of continuity of M on $[0, t]$ introduced in the proof of Lemma 23. Take a subsequence along which $\sup_{s \in [0, t]} |A_s^n - A_s| \rightarrow 0$ a.s.; since $\omega_M(|\Delta_n|) \rightarrow 0$ a.s., we obtain, almost surely, $A_s - A_{s'} \geq 0$ for all $0 \leq s' \leq s \leq t$.

- Extension to $\mathbb{R}_+$. The construction above, performed on $[0, N]$ for every $N \in \mathbb{N}$, provides processes $A^{(N)}$ on $[0, N]$. By uniqueness on $[0, N]$, $A^{(N+1)} = A^{(N)}$ on $[0, N]$ up to indistinguishability, so that there is a continuous adapted non-decreasing process A on $\mathbb{R}_+ = \bigcup_N [0, N]$ such that $M^2 - A$ is a martingale. By uniqueness again, the process constructed on $[0, t]$ from any sequence (Δ_n) coincides with A on $[0, t]$: $\langle M \rangle := A$ depends neither on t nor on (Δ_n) , and

$$\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{L^2} \langle M \rangle_t.$$

Step 2: M local martingale with $M_0 = 0$. Let

$$\tau_n = \inf\{s \geq 0, |M_s| \geq n\},$$

which is localizing (Proposition 16(iii)). The process M^{τ_n} is a bounded martingale, hence, by Step 1, $A^n := \langle M^{\tau_n} \rangle$ exists.

Let $m \geq n$. The process $(M_{s \wedge \tau_m}^2 - A_s^m)_{s \geq 0}$ is a martingale, hence (by stopping)

$$M_{s \wedge \tau_n \wedge \tau_m}^2 - A_{s \wedge \tau_n}^m = M_{s \wedge \tau_n}^2 - A_{s \wedge \tau_n}^m$$

is a martingale. Since $(A_{s \wedge \tau_n}^m)_{s \geq 0}$ is continuous, adapted, non-decreasing and null at 0, by uniqueness it is the bracket of M^{τ_n} :

$$A_{s \wedge \tau_n}^m = A_s^n \quad (= \langle M^{\tau_n} \rangle_s).$$

Applied with $m = n$, this gives $A_{s \wedge \tau_n}^n = A_s^n$, so that

$$A_{s \wedge \tau_n}^m = A_{s \wedge \tau_n}^n, \quad m \geq n.$$

So there exists A continuous, adapted, non-decreasing, such that

$$\forall n \geq 1, \quad A_{s \wedge \tau_n} = A_{s \wedge \tau_n}^n :$$

it suffices to put

$$A_s = \sum_{n \geq 1} A_s^n \mathbf{1}_{\{s \in (\tau_{n-1}, \tau_n]\}} = \lim_{n \rightarrow \infty} A_s^n \quad (\tau_0 = 0, A_0 = 0).$$

Then $M^2 - A$ is a local martingale, since for every $n \geq 1$,

$$M_{s \wedge \tau_n}^2 - A_{s \wedge \tau_n} = M_{s \wedge \tau_n}^2 - A_{s \wedge \tau_n}^n = M_{s \wedge \tau_n}^2 - A_s^n$$

is a martingale; hence (τ_n) localizes $M^2 - A$ (Remark 15(ii)). It remains to show that

$$A_t = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2.$$

From Step 1, for every k ,

$$A_t^k = \lim_{n \rightarrow \infty} \sum_{i=1}^{p_n} (M_{t_i^n}^{\tau_k} - M_{t_{i-1}^n}^{\tau_k})^2 \quad \text{in } L^2.$$

Let $\Omega_k := \{\tau_k \geq t\}$; then $\lim_k \mathbb{P}(\Omega_k) = 1$, and

$$\Omega_k \subseteq \{A_t^k = A_t\} \cap \{M_s^{\tau_k} = M_s, \forall s \in [0, t]\}$$

(just the definition of stopping). It follows that, for every $\varepsilon > 0$,

$$\mathbb{P}\left(\left|A_t - \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2\right| \geq \varepsilon\right) \leq \mathbb{P}(\Omega_k^c) + \mathbb{P}\left(\left|\sum_{i=1}^{p_n} (M_{t_i^n}^{\tau_k} - M_{t_{i-1}^n}^{\tau_k})^2 - A_t^k\right| \geq \varepsilon\right),$$

and we let $n \rightarrow \infty$, then $k \rightarrow \infty$.

Step 3: removing the assumption $M_0 = 0$. Let $N := M - M_0$, a local martingale with $N_0 = 0$, and set $\langle M \rangle := \langle N \rangle$. It suffices to show that $M^2 - \langle N \rangle$ is a local martingale. Indeed,

$$M^2 - \langle N \rangle = (N^2 - \langle N \rangle) + 2M_0N + M_0^2,$$

where $N^2 - \langle N \rangle$ is a local martingale by Step 2, $M_0 N$ is a local martingale (M_0 being $\mathcal{F}_0$ -measurable and $N_0 = 0$), and the constant process M_0^2 is trivially a local martingale. Also,

$$\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 = \sum_{i=1}^{p_n} (N_{t_i^n} - N_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle N \rangle_t = \langle M \rangle_t,$$

the convergence being that of Step 2 for N . Uniqueness holds by the general argument above. $\square$

Proposition 24. *Let M be a local martingale and τ a stopping time. Then*

$$\langle M^\tau \rangle = \langle M \rangle^\tau.$$

Proof. $M^2 - \langle M \rangle$ is a local martingale, so is its stopped process $(M_{t \wedge \tau}^2 - \langle M \rangle_{t \wedge \tau})_{t \geq 0}$ (Remark 15(v)). Since $\langle M \rangle^\tau$ is continuous, adapted, non-decreasing and null at 0, uniqueness in Theorem 20 gives the result. $\square$

Theorem 25. *Let M be a local martingale such that $M_0 = 0$ a.s. Then*

$$\mathbb{E}[\langle M \rangle_t] < \infty \quad \forall t \geq 0$$

if and only if

$$M \text{ is a (true) martingale, square integrable, i.e. } \mathbb{E}[M_t^2] < \infty \quad \forall t \geq 0.$$

In this case, $(M_t^2 - \langle M \rangle_t)_{t \geq 0}$ is a (true) martingale, and $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M \rangle_t]$.

Proof. Let $\tau_n = \inf\{t \geq 0, |M_t| + \langle M \rangle_t \geq n\}$; then $\tau_n \rightarrow \infty$ a.s. The process $(M_{t \wedge \tau_n}^2 - \langle M \rangle_{t \wedge \tau_n})_{t \geq 0}$ is a bounded local martingale, hence a martingale, and

$$\mathbb{E}[M_{t \wedge \tau_n}^2] = \mathbb{E}[\langle M \rangle_{t \wedge \tau_n}].$$

- If M is a (true) martingale in L^2 , then, by monotone convergence,

$$\mathbb{E}[\langle M \rangle_t] = \lim_n \mathbb{E}[\langle M \rangle_{t \wedge \tau_n}] = \lim_n \mathbb{E}[M_{t \wedge \tau_n}^2] = \mathbb{E}[M_t^2],$$

the last equality by dominated convergence: $M_{t \wedge \tau_n}^2 \leq \sup_{s \in [0, t]} M_s^2 \in L^1$ by Doob's inequality, and $M_{t \wedge \tau_n}^2 \rightarrow M_t^2$ a.s.

- Conversely, assume $\mathbb{E}[\langle M \rangle_t] < \infty$ for every $t \geq 0$. The process M^{τ_n} is a bounded martingale, so Doob's L^2 inequality gives

$$\mathbb{E}\left[\sup_{s \in [0, t]} M_{s \wedge \tau_n}^2\right] \leq 4 \mathbb{E}[M_{t \wedge \tau_n}^2] = 4 \mathbb{E}[\langle M \rangle_{t \wedge \tau_n}] \leq 4 \mathbb{E}[\langle M \rangle_t].$$

Hence, by monotone convergence,

$$\mathbb{E} \left[\sup_{s \in [0, t]} M_s^2 \right] \leq 4 \mathbb{E} [\langle M \rangle_t] < \infty \quad \forall t.$$

Therefore (Proposition 16(ii): a local martingale with $\mathbb{E}[\sup_{s \in [0, t]} |M_s|] < \infty$ for every t is a martingale), M is a martingale, which moreover is in L^2 .

• Finally, in both cases $\mathbb{E}[\sup_{s \in [0, t]} M_s^2] < \infty$, so

$$\mathbb{E} \left[\sup_{s \in [0, t]} |M_s^2 - \langle M \rangle_s| \right] \leq \mathbb{E} \left[\sup_{s \in [0, t]} M_s^2 \right] + \mathbb{E} [\langle M \rangle_t] < \infty,$$

hence $(M_t^2 - \langle M \rangle_t)_{t \geq 0}$ is a martingale likewise, and $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M \rangle_t]$. $\square$

Remark 26. We will see that if M is a local martingale such that $M_0 = 0$ and $\mathbb{E}[\langle M \rangle_t^{1/2}] < \infty$ for every t , then M is a (true) martingale.

Corollary 27. (i) If M is a martingale in L^2 such that $M_0 = 0$, then $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M \rangle_t]$ for every $t \geq 0$.

(ii) If M is a continuous local martingale such that $M_0 = 0$ and $\langle M \rangle_t = 0$ for every $t \geq 0$, then $M \equiv 0$ up to indistinguishability.

Proof. (i) is part of Theorem 25. For (ii), Theorem 25 shows that M is a martingale in L^2 with $\mathbb{E}[M_t^2] = 0$, so that $M_t = 0$ a.s. for every t , and we conclude by continuity. $\square$

Definition 28. Let M, N be two local martingales. We set

$$\langle M, N \rangle_t = \frac{1}{2} (\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t).$$

In particular $\langle M, M \rangle = \langle M \rangle$, since $\langle \lambda M \rangle = \lambda^2 \langle M \rangle$.

Proposition 29. Let M, N be two local martingales.

(i) $(M, N) \mapsto \langle M, N \rangle$ is bilinear and symmetric.

(ii) $\langle M, N \rangle$ is the unique (up to indistinguishability) continuous adapted process with finite variation such that $MN - \langle M, N \rangle$ is a local martingale.

(iii) If $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t)$ is a sequence of subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$, then

$$\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n}) (N_{t_i^n} - N_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle M, N \rangle_t. \quad (2)$$

(iv) For every stopping time τ ,

$$\langle M, N \rangle^\tau = \langle M^\tau, N^\tau \rangle = \langle M, N^\tau \rangle.$$

Proof. (ii) If $MN - A$ and $MN - \tilde{A}$ are two local martingales, then $A - \tilde{A} = (MN - \tilde{A}) - (MN - A)$ is a local martingale with finite variation and value 0 at $t = 0$; this implies $A - \tilde{A} = 0$ by Theorem 18. Moreover,

$$MN - \langle M, N \rangle = \frac{1}{2} \left(((M + N)^2 - \langle M + N \rangle) - (M^2 - \langle M \rangle) - (N^2 - \langle N \rangle) \right)$$

is a local martingale, and $\langle M, N \rangle$ has finite variation, as a difference of non-decreasing processes.

(iii) follows from the local martingale case (Theorem 20) by polarization, since $(a + b)^2 - a^2 - b^2 = 2ab$.

(i) follows from (ii) (or from (iii)).

(iv) Exercise. Hint: use (2), and the fact that stopping M (or N) at τ only changes the terms of the sum corresponding to the t_i^n such that $t_i^n > \tau$. $\square$

Definition 30. Two local martingales M, N are orthogonal if $\langle M, N \rangle = 0$.

Exercise. Two independent local martingales are orthogonal.

Lemma 31. Let F be a continuous function with finite variation, and assume that $F = G^+ - G^-$, where G^+, G^- are continuous and non-decreasing (but not necessarily the minimal decomposition). Then

$$|dF| \leq dG^+ + dG^- =: d\rho.$$

In particular $dF \ll \rho$, so that $\frac{dF}{d\rho}$ exists by the Radon–Nikodym theorem, as a real-valued function (with $|\frac{dF}{d\rho}| \leq 1$).

Proof. By uniqueness of dF , $dF = dG^+ - dG^-$. With $D_\pm$ as in the remark on the total variation, for every Borel set B ,

$$|dF|(B) = dF(B \cap D_+) - dF(B \cap D_-) \leq dG^+(B \cap D_+) + dG^-(B \cap D_-) \leq \rho(B). \quad \square$$

Theorem 32 (Kunita–Watanabe). Let M, N be two (continuous) local martingales and H, K two measurable processes. Then, almost surely, for every $t \geq 0$,

$$\int_0^t |H_s K_s| |d\langle M, N \rangle_s| \leq \left(\int_0^t H_s^2 d\langle M \rangle_s \right)^{1/2} \left(\int_0^t K_s^2 d\langle N \rangle_s \right)^{1/2}.$$

Proof. It is an “ ω -by- ω ” inequality. By bilinearity, for every $x \in \mathbb{R}$, $\langle M + xN \rangle = \langle M \rangle + 2x\langle M, N \rangle + x^2\langle N \rangle$ up to indistinguishability, and $\langle M + xN \rangle$ is non-decreasing. Fix ω outside a negligible set, such that these properties hold for every rational x (a countable family). We associate the following measures on $[0, t]$ to the following functions:

$$\begin{aligned} \langle M \rangle &\longrightarrow d\mu \geq 0, & \langle N \rangle &\longrightarrow d\nu \geq 0, \\ \langle M + N \rangle + \langle M - N \rangle &\longrightarrow d\eta \geq 0, & \langle M, N \rangle &\longrightarrow d\zeta \text{ (signed!)}. \end{aligned}$$

The Kunita–Watanabe inequality then reads

$$\int |h(s)k(s)| |\zeta|(ds) \leq \left(\int h(s)^2 \mu(ds) \right)^{1/2} \left(\int k(s)^2 \nu(ds) \right)^{1/2}$$

for all measurable functions h, k on $[0, t]$.

Since $\langle M + N \rangle + \langle M - N \rangle = 2(\langle M \rangle + \langle N \rangle)$, we have $\eta = 2(\mu + \nu)$, so that $\mu \ll \eta$ and $\nu \ll \eta$. Moreover, by bilinearity, $\langle M, N \rangle = \frac{1}{4}(\langle M + N \rangle - \langle M - N \rangle)$, so Lemma 31 yields

$$|d\zeta| \leq \frac{1}{4} d\eta.$$

It follows that μ, ν, ζ are all absolutely continuous with respect to η . Let

$$\alpha = \frac{d\mu}{d\eta}, \quad \beta = \frac{d\nu}{d\eta}, \quad \gamma = \frac{d\zeta}{d\eta} \text{ (signed),}$$

so that $|d\zeta| = |\gamma| d\eta$. Moreover, for every rational x , $\langle M + xN \rangle$ is non-decreasing, so the associated measure

$$\mu + 2x\zeta + x^2\nu \text{ is } \geq 0,$$

i.e. $\alpha + 2x\gamma + x^2\beta \geq 0$ η -a.e., for every rational x . The union of countably many η -null sets being η -null, and by continuity in x , this holds η -a.e. for every $x \in \mathbb{R}$. It follows (non-positive discriminant) that

$$\gamma^2 \leq \alpha\beta \quad \eta\text{-a.e.}$$

Finally, by the Cauchy–Schwarz inequality,

$$\int |h(s)k(s)| |\zeta|(ds) = \int |hk\gamma| d\eta \leq \int |hk| \sqrt{\alpha\beta} d\eta \leq \left(\int h^2 \alpha d\eta \right)^{1/2} \left(\int k^2 \beta d\eta \right)^{1/2},$$

and the right-hand side is $(\int h^2 d\mu)^{1/2} (\int k^2 d\nu)^{1/2}$. Applying this with $h(s) = H_s(\omega)$, $k(s) = K_s(\omega)$ concludes the proof. $\square$

5 Continuous semimartingales

Definition 33. A process $X = (X_t)_{t \geq 0}$ is a (continuous) semimartingale if

$$X_t = X_0 + M_t + V_t,$$

where M is a (continuous) local martingale with $M_0 = 0$, V is a finite variation process (so $V_0 = 0$), and X_0 is $\mathcal{F}_0$ -measurable.

The decomposition is unique, up to indistinguishability: if $X_0 + M + V = X_0 + M' + V'$, then $M - M' = V' - V$ is a local martingale with finite variation, null at 0, hence $M = M'$ and $V = V'$ by Theorem 18.

Definition 34. Let $X = X_0 + M + V$ and $\tilde{X} = \tilde{X}_0 + \tilde{M} + \tilde{V}$ be two continuous semimartingales. We set

$$\langle X \rangle_t = \langle M \rangle_t \quad \text{and} \quad \langle X, \tilde{X} \rangle_t = \langle M, \tilde{M} \rangle_t.$$

Note that $\langle X, \tilde{X} \rangle = \frac{1}{2}(\langle X + \tilde{X} \rangle - \langle X \rangle - \langle \tilde{X} \rangle)$, since the local martingale part of $X + \tilde{X}$ is $M + \tilde{M}$.

Proposition 35. Let $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t)$ be a sequence of subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$. Then

$$\sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})(\tilde{X}_{t_i^n} - \tilde{X}_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle X, \tilde{X} \rangle_t.$$

Proof. By polarization, only the case $X = \tilde{X}$ needs to be established. Then

$$\begin{aligned} \sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 &= \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 + \sum_{i=1}^{p_n} (V_{t_i^n} - V_{t_{i-1}^n})^2 \\ &\quad + 2 \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})(V_{t_i^n} - V_{t_{i-1}^n}) =: I_n + II_n + III_n. \end{aligned}$$

By Theorem 20, $I_n \rightarrow \langle M \rangle_t = \langle X \rangle_t$ in probability. Moreover,

$$II_n + |III_n| \leq \left(2 \max_{1 \leq i \leq p_n} |M_{t_i^n} - M_{t_{i-1}^n}| + \max_{1 \leq i \leq p_n} |V_{t_i^n} - V_{t_{i-1}^n}| \right) \times \underbrace{\sum_{i=1}^{p_n} |V_{t_i^n} - V_{t_{i-1}^n}|}_{\leq \int_0^t |dV_s| < \infty \text{ a.s.}},$$

and the first factor tends to 0 a.s., by (uniform) continuity of $s \mapsto M_s$ and $s \mapsto V_s$ on $[0, t]$. Hence $II_n + III_n \rightarrow 0$ a.s., and the result follows. $\square$