

Chapter 5: Stochastic integral

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1 Integration with respect to a square integrable martingale

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions ($\mathcal{F}_t = \mathcal{F}_{t+}$, and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets).

Let $T > 0$. We construct the stochastic integral with respect to a continuous martingale $M = (M_t)_{t \in [0, T]}$ in L^2 . Note that $M_t \in L^2$ for every $t \in [0, T]$ means $M_T \in L^2$, and implies, by Doob's L^2 inequality,

$$\mathbb{E} \left[\sup_{t \in [0, T]} M_t^2 \right] \leq 4 \mathbb{E} [M_T^2] < \infty.$$

Definition 1. We denote by $\mathcal{M}_T^2$ the space of continuous martingales $M = (M_t)_{t \in [0, T]}$ in L^2 such that $M_0 = 0$. It is endowed with the scalar product

$$(M, N)_{\mathcal{M}_T^2} = \mathbb{E} [M_T N_T] = \mathbb{E} [\langle M, N \rangle_T].$$

The second equality holds since $MN - \langle M, N \rangle$ is a true martingale, null at 0 (by polarization of the corresponding result for $M^2 - \langle M \rangle$ in Chapter 4). This is a

bona fide scalar product: in particular, if $\|M\|_{\mathcal{M}_T^2}^2 = 0$, then M is indistinguishable from 0, since

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} M_t^2 \right] \leq 4 \mathbb{E} [M_T^2] = 4 \|M\|_{\mathcal{M}_T^2}^2 = 0.$$

Proposition 2. $(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$ is a Hilbert space.

Proof. Let $(M^n)_{n \geq 0}$ be a Cauchy sequence for the norm $\|\cdot\|_{\mathcal{M}_T^2}$. By Doob's L^2 inequality applied to the martingale $M^n - M^m$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |M_t^n - M_t^m|^2 \right] \leq 4 \mathbb{E} [|M_T^n - M_T^m|^2] = 4 \|M^n - M^m\|_{\mathcal{M}_T^2}^2 \xrightarrow{n, m \rightarrow \infty} 0.$$

From the fundamental lemma of Chapter 4 (proved in the context of the quadratic variation), there exists a continuous adapted process M^∞ , in L^2 , such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |M_t^n - M_t^\infty|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

In particular $M_0^\infty = \lim_n M_0^n = 0$ and $\|M^n - M^\infty\|_{\mathcal{M}_T^2} \leq \mathbb{E} [\sup_t |M_t^n - M_t^\infty|^2]^{1/2} \rightarrow 0$. It remains to show that M^∞ is a martingale. Let $0 \leq s < t \leq T$. We have $\mathbb{E}[M_t^n | \mathcal{F}_s] = M_s^n$, where $M_s^n \rightarrow M_s^\infty$ in L^2 , and $\mathbb{E}[M_t^n | \mathcal{F}_s] \rightarrow \mathbb{E}[M_t^\infty | \mathcal{F}_s]$ in L^2 . Indeed,

$$\mathbb{E} \left[(\mathbb{E}[M_t^n | \mathcal{F}_s] - \mathbb{E}[M_t^\infty | \mathcal{F}_s])^2 \right] = \mathbb{E} \left[\mathbb{E} [M_t^n - M_t^\infty | \mathcal{F}_s]^2 \right] \leq \mathbb{E} [(M_t^n - M_t^\infty)^2] \xrightarrow{n \rightarrow \infty} 0.$$

Hence $\mathbb{E}[M_t^\infty | \mathcal{F}_s] = M_s^\infty$, and $M^\infty \in \mathcal{M}_T^2$. $\square$

Definition 3. For $M \in \mathcal{M}_T^2$, we let

$$L_T^2(M) = \left\{ \text{progressive processes } (H_s)_{0 \leq s \leq T} \text{ such that } \mathbb{E} \left[\int_0^T H_s^2 d\langle M \rangle_s \right] < \infty \right\}.$$

Remark 4 (Important). Let $\mathcal{P}$ be the progressive σ -field on $[0, T] \times \Omega$, i.e. the σ -field of the sets A such that the process $\mathbf{1}_A$ is progressive. Then

$$L_T^2(M) = L^2([0, T] \times \Omega, \mathcal{P}, \nu_M),$$

where

$$\nu_M(A) = \int_\Omega \int_0^T \mathbf{1}_{\{(s, \omega) \in A\}} d\langle M \rangle_s(\omega) \mathbb{P}(d\omega).$$

Hence $L_T^2(M)$ is an L^2 space (up to the identification of processes which coincide $\nu_M(d\omega, ds)$ -a.e.), equipped with the scalar product

$$(H, K)_{L_T^2(M)} = \mathbb{E} \left[\int_0^T H_s K_s d\langle M \rangle_s \right].$$

It is (of course) a Hilbert space. In particular, for a Brownian motion B (for which $\langle B \rangle_s = s$),

$$(H, K)_{L_T^2(B)} = \int_0^T \mathbb{E}[H_s K_s] ds, \quad \|H\|_{L_T^2(B)}^2 = \mathbb{E} \left[\int_0^T H_s^2 ds \right].$$

The strategy for constructing a stochastic integral in this context is the following:

- (i) the space $\mathcal{E}_T$ of simple processes (defined below) is dense in $L_T^2(M)$;
- (ii) for $H \in \mathcal{E}_T$, the stochastic integral $H \cdot M$ is defined explicitly; it belongs to $\mathcal{M}_T^2$, and the only point to prove is that $\langle H \cdot M \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s$, which yields an isometry property;
- (iii) for $H \in L_T^2(M)$, we take $H^n \in \mathcal{E}_T$ with $H^n \rightarrow H$ in $L_T^2(M)$, and define $\int_0^t H_s dM_s$ as the limit in $\mathcal{M}_T^2$ of $\int_0^t H_s^n dM_s$.

Definition 5. A process $H : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is an elementary (or simple) process if

$$H_t(\omega) = \sum_{i=1}^p H^{(i)}(\omega) \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad t \geq 0,$$

where $p \geq 0$ is an integer, $0 \leq t_0 \leq t_1 \leq \dots \leq t_p$ are real numbers, and each $H^{(i)}$ is a bounded real-valued $\mathcal{F}_{t_{i-1}}$ -measurable random variable. We denote by $\mathcal{E}$ the (vector) span of simple processes, and by $\mathcal{E}_T$ its restriction to $[0, T]$.

Note that a simple process is left-continuous and adapted, hence progressive. Moreover, $\mathcal{E}$ is in fact equal to the set of simple processes: a linear combination of simple processes can be written as a simple process, using a common refinement of the subdivisions.

Proposition 6. $\mathcal{E}_T \subset L_T^2(M)$, and $\mathcal{E}_T$ is dense in $L_T^2(M)$.

Proof. The inclusion is clear: if $H \in \mathcal{E}_T$ is bounded by C , then $\mathbb{E}[\int_0^T H_s^2 d\langle M \rangle_s] \leq C^2 \mathbb{E}[\langle M \rangle_T] = C^2 \mathbb{E}[M_T^2] < \infty$.

Recall that a subspace F of a Hilbert space $\mathbb{H}$ is dense in $\mathbb{H}$ if and only if $F^\perp = \{0\}$. Let $K \in L_T^2(M)$ be orthogonal to $\mathcal{E}_T$. We need to prove that $K = 0$ in $L_T^2(M)$, i.e.

$$\mathbb{E} \left[\int_0^T K_s^2 d\langle M \rangle_s \right] = 0.$$

Let $X_t = \int_0^t K_s d\langle M \rangle_s$, $t \in [0, T]$. Then X is a finite variation process (Chapter 4), with $X_0 = 0$. Also $X_t \in L^1$:

$$\begin{aligned} \mathbb{E}[|X_t|] &\leq \mathbb{E} \left[\int_0^t |K_s| d\langle M \rangle_s \right] \leq \mathbb{E} \left[\int_0^t (1 + K_s^2) d\langle M \rangle_s \right] \\ &= \mathbb{E} \left[\langle M \rangle_t + \int_0^t K_s^2 d\langle M \rangle_s \right] < \infty. \end{aligned}$$

For $0 \leq u < v \leq T$ and $A \in \mathcal{F}_u$, the process $H_s := \mathbf{1}_A \mathbf{1}_{\{s \in (u, v]\}}$ belongs to $\mathcal{E}_T$, hence $(H, K)_{L_T^2(M)} = 0$, i.e.

$$0 = \mathbb{E} \left[\int_0^T H_s K_s d\langle M \rangle_s \right] = \mathbb{E}[\mathbf{1}_A (X_v - X_u)] \quad \text{for all such } A,$$

i.e. $\mathbb{E}[X_v - X_u | \mathcal{F}_u] = 0$, and X is a martingale. Since it has finite variation and takes value 0 at $t = 0$, it is 0 (Chapter 4): almost surely,

$$\forall t \in [0, T], \quad \int_0^t K_s d\langle M \rangle_s = 0.$$

So, a.s., $K_s d\langle M \rangle_s$ is the zero measure on $[0, T]$, hence¹ $\int_0^T K_s^2 d\langle M \rangle_s = 0$. $\square$

We recall the following simple yet fundamental lemma (that we already used in the previous chapter) :

¹Indeed, fix ω such that $\int_0^T |K_s| d\langle M \rangle_s < \infty$ and the above holds. The finite positive measures $K_s^+ d\langle M \rangle_s$ and $K_s^- d\langle M \rangle_s$ coincide on the intervals $(0, t]$, and give no mass to $\{0\}$, hence they coincide on $\mathcal{B}([0, T])$:

$$\int_A K_s^+ d\langle M \rangle_s = \int_A K_s^- d\langle M \rangle_s \quad \forall A \in \mathcal{B}([0, T]) \implies K_s^+ = K_s^- d\langle M \rangle_s\text{-a.e.}$$

This implies $K_s = 0 d\langle M \rangle_s\text{-a.e.}$ since $K^+ - K^- = 0$ and $\mathbb{E}[\int_0^T K_s^2 d\langle M \rangle_s] = 0$ by taking expectations.

Lemma 7. Let $N = (N_t)_{t \in [0, T]}$ be a continuous martingale, $0 \leq a \leq b \leq T$, and ξ an $\mathcal{F}_a$ -measurable random variable such that $\xi(N_b - N_a) \in L^1$ and $\xi(N_{t \wedge b} - N_{t \wedge a}) \in L^1$ for every $t \in [0, T]$ (for instance, ξ bounded; or $\xi \in L^2$ and N in L^2). Then

$$U_t := \xi(N_{t \wedge b} - N_{t \wedge a}), \quad t \in [0, T],$$

is a continuous martingale.

Proof. Continuity and adaptedness are clear ($U_t = 0$ for $t \leq a$, and U_t is $\mathcal{F}_t$ -measurable for $t \geq a$). Recall that, for $Y \in L^1$ and ξ $\mathcal{G}$ -measurable with $\xi Y \in L^1$, $\mathbb{E}[\xi Y | \mathcal{G}] = \xi \mathbb{E}[Y | \mathcal{G}]$. Let $s \leq u$.

- If $u \leq a$: $\mathbb{E}[U_u | \mathcal{F}_s] = 0 = U_s$.
- If $s \leq a \leq u$: $U_s = 0$ and $\mathbb{E}[U_u | \mathcal{F}_s] = \mathbb{E}[\xi \mathbb{E}[N_{u \wedge b} - N_a | \mathcal{F}_a] | \mathcal{F}_s] = 0$.
- If $a \leq s \leq u$: $\mathbb{E}[U_u | \mathcal{F}_s] = \xi \mathbb{E}[N_{u \wedge b} - N_a | \mathcal{F}_s] = \xi(N_{s \wedge b} - N_a) = U_s$. □

Theorem 8. Let $M \in \mathcal{M}_T^2$ and $H \in \mathcal{E}_T$, $H_t(\omega) = \sum_{i=1}^p H^{(i)}(\omega) \mathbf{1}_{(t_{i-1}, t_i]}(t)$. We define

$$\int_0^t H_s dM_s = (H \cdot M)_t := \sum_{i=1}^p H^{(i)}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}), \quad t \in [0, T].$$

Then:

- (i) $((H \cdot M)_t)_{0 \leq t \leq T} \in \mathcal{M}_T^2$ and $\langle H \cdot M \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s$;
- (ii) $\mathbb{E}\left[\left(\int_0^t H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 d\langle M \rangle_s\right]$ for every $t \in [0, T]$;
- (iii) $H \mapsto H \cdot M$ is an isometry of $(\mathcal{E}_T, \|\cdot\|_{L_T^2(M)})$ into $(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$.

Remark 9. The definition of $H \cdot M$ does not depend on the chosen subdivision: if

$$\sum_{i=1}^p H^{(i)} \mathbf{1}_{(t_{i-1}, t_i]} = \sum_{j=1}^q K^{(j)} \mathbf{1}_{(s_{j-1}, s_j]},$$

then

$$\sum_{i=1}^p H^{(i)}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) = \sum_{j=1}^q K^{(j)}(M_{s_j \wedge t} - M_{s_{j-1} \wedge t}).$$

Indeed, considering a common refinement, we may assume that $(t_0, \dots, t_p) \subset (s_0, \dots, s_q)$, with $t_i = s_{k_i}$, so that $K^{(j)} = H^{(i)}$ for $j = k_{i-1} + 1, \dots, k_i$ (on the non-empty intervals), and we conclude by splitting each increment $M_{t_i \wedge t} - M_{t_{i-1} \wedge t}$ into a telescopic sum. In particular $H \mapsto H \cdot M$ is linear on $\mathcal{E}_T$.

Proof. (iii) = (ii) with $t = T$ (together with the linearity of $H \mapsto H \cdot M$), and (ii) follows from (i) by the property $\mathbb{E}[N_t^2] = \mathbb{E}[\langle N \rangle_t]$ if N is a continuous martingale in L^2 such that $N_0 = 0$ a.s. (Chapter 4), applied to $N = H \cdot M$.

For (i), we write

$$(H \cdot M)_t = \sum_{i=1}^p L_t^{(i)}, \quad L_t^{(i)} = H^{(i)}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

By Lemma 7 (with $\xi = H^{(i)}$ bounded), $L^{(i)}$ is a continuous martingale, in L^2 since $H^{(i)}$ is bounded, and $L_0^{(i)} = 0$. Hence $H \cdot M \in \mathcal{M}_T^2$. It remains to compute its bracket. By bilinearity,

$$\langle H \cdot M \rangle_t = \sum_{i=1}^p \langle L^{(i)} \rangle_t + 2 \sum_{i < j} \langle L^{(i)}, L^{(j)} \rangle_t.$$

• **Cross terms.** Let $i < j$, so that $t_i \leq t_{j-1}$. Since $L_t^{(i)} = L_{t_i}^{(i)}$ for $t \geq t_i$, and $L_t^{(j)} = 0$ for $t \leq t_{j-1}$,

$$L_t^{(i)} L_t^{(j)} = (H^{(j)} L_{t_i}^{(i)})(M_{t_j \wedge t} - M_{t_{j-1} \wedge t}),$$

where $H^{(j)} L_{t_i}^{(i)}$ is $\mathcal{F}_{t_{j-1}}$ -measurable and in L^2 . By Lemma 7, $L^{(i)} L^{(j)}$ is a martingale, hence $\langle L^{(i)}, L^{(j)} \rangle = 0$ by uniqueness of the bracket.

• **Diagonal terms.** Using $(y - x)^2 = y^2 - x^2 - 2x(y - x)$,

$$\begin{aligned} (L_t^{(i)})^2 &= (H^{(i)})^2 (M_{t_i \wedge t}^2 + M_{t_{i-1} \wedge t}^2 - 2M_{t_{i-1} \wedge t} M_{t_i \wedge t}) \\ &= (H^{(i)})^2 (M_{t_i \wedge t}^2 - M_{t_{i-1} \wedge t}^2) - 2(H^{(i)})^2 M_{t_{i-1}} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}), \end{aligned}$$

where we used that $M_{t_{i-1} \wedge t} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) = M_{t_{i-1}} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$. The last term is a martingale by Lemma 7 (with $\xi = (H^{(i)})^2 M_{t_{i-1}} \in L^2$, $\mathcal{F}_{t_{i-1}}$ -measurable). Moreover, with $Z = M^2 - \langle M \rangle$, which is a (true) martingale (Chapter 4),

$$(H^{(i)})^2 (M_{t_i \wedge t}^2 - M_{t_{i-1} \wedge t}^2) - (H^{(i)})^2 (\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t}) = (H^{(i)})^2 (Z_{t_i \wedge t} - Z_{t_{i-1} \wedge t})$$

is a martingale by Lemma 7 (with $\xi = (H^{(i)})^2$ bounded). Hence $(L^{(i)})^2 - (H^{(i)})^2 (\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t})$ is a martingale; the process $(H^{(i)})^2 (\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t})$ being continuous, adapted, non-decreasing and null at 0, we obtain by uniqueness

$$\langle L^{(i)} \rangle_t = (H^{(i)})^2 (\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t}).$$

- **Conclusion.** Since the intervals $(t_{i-1}, t_i]$ are pairwise disjoint,

$$H_s^2 = \sum_{i=1}^p (H^{(i)})^2 \mathbf{1}_{(t_{i-1}, t_i]}(s),$$

and

$$\begin{aligned} \langle H \cdot M \rangle_t &= \sum_{i=1}^p (H^{(i)})^2 \left(\langle M \rangle_{t_i \wedge t} - \langle M \rangle_{t_{i-1} \wedge t} \right) \\ &= \sum_{i=1}^p (H^{(i)})^2 d\langle M \rangle((t_{i-1}, t_i] \cap [0, t]) = \int_0^t H_s^2 d\langle M \rangle_s. \end{aligned} \quad \square$$

Hence, $M \in \mathcal{M}_T^2$ being fixed, we have an isometry

$$H \mapsto H \cdot M, \quad (\mathcal{E}_T, \|\cdot\|_{L_T^2(M)}) \longrightarrow (\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2}),$$

and moreover $\mathcal{E}_T$ is dense in $L_T^2(M)$.

Definition 10. Let $M \in \mathcal{M}_T^2$ and $H \in L_T^2(M)$. We set

$$((H \cdot M)_t)_{0 \leq t \leq T} := \lim_{\mathcal{M}_T^2} ((H^n \cdot M)_t)_{0 \leq t \leq T},$$

for $H^n \in \mathcal{E}_T$ such that $\|H^n - H\|_{L_T^2(M)} \rightarrow 0$. The continuous (and L^2) martingale $H \cdot M$ is called the stochastic integral of H with respect to M , with the notation

$$(H \cdot M)_t = \int_0^t H_s dM_s.$$

Remark 11. (i) The limit exists, since (H^n) is a Cauchy sequence in $L_T^2(M)$, hence $(H^n \cdot M)$ is a Cauchy sequence in the Hilbert space $\mathcal{M}_T^2$, by isometry. It does not depend on the choice of the approximating sequence: if $\|H^n - H\|_{L_T^2(M)} \rightarrow 0$ and $\|K^n - H\|_{L_T^2(M)} \rightarrow 0$, then

$$\|H^n \cdot M - K^n \cdot M\|_{\mathcal{M}_T^2} = \|H^n - K^n\|_{L_T^2(M)} \xrightarrow{n \rightarrow \infty} 0.$$

(ii) $H \mapsto H \cdot M$ is an isometry from $(L_T^2(M), \|\cdot\|_{L_T^2(M)})$ into $(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$, and we have

$$\mathbb{E}[(H \cdot M)_t] = 0 \quad \text{and} \quad \mathbb{E}[(H \cdot M)_t^2] = \mathbb{E}\left[\int_0^t H_s^2 d\langle M \rangle_s\right], \quad \forall 0 \leq t \leq T.$$

Indeed, $(H^n \cdot M)_t \rightarrow (H \cdot M)_t$ in L^2 (by Doob's inequality), $H^n \mathbf{1}_{[0, t]} \rightarrow H \mathbf{1}_{[0, t]}$ in $L_T^2(M)$, and both identities hold for H^n by Theorem 8.

Definition 12. We let

$$\mathcal{M}^2 = \left\{ \text{continuous martingales } M = (M_t)_{t \geq 0} \text{ such that } (M_t)_{0 \leq t \leq T} \in \mathcal{M}_T^2 \ \forall T \geq 0 \right\},$$

$$L^2(M) = \left\{ \text{progressive processes } (H_t)_{t \geq 0} \text{ such that } H \in L_T^2(M) \ \forall T \geq 0 \right\}.$$

Then, for $M \in \mathcal{M}^2$ and $H \in L^2(M)$, the process

$$((H \cdot M)_t)_{t \geq 0} = \left(\int_0^t H_s dM_s \right)_{t \geq 0}$$

is well defined and belongs to $\mathcal{M}^2$.

Indeed, for $T < T'$, the stochastic integral constructed on $[0, T']$, restricted to $[0, T]$, coincides with the one constructed on $[0, T]$: this is clear for simple processes, and passes to the limit since $\|N_{|[0, T]}\|_{\mathcal{M}_{T'}^2} \leq \|N\|_{\mathcal{M}_{T'}^2}$ and $\|K_{|[0, T]}\|_{L_{T'}^2(M)} \leq \|K\|_{L_{T'}^2(M)}$.

Remark 13. (i) $\int_0^t H_s dM_s$ is not a Stieltjes integral: since M has infinite variation (unless it is constant), dM is not a signed measure.

(ii) If H is adapted, continuous and bounded, then

$$\int_0^t H_s dM_s = \lim_{L^2} \sum_{i=1}^{2^n} H_{(i-1)t2^{-n}} \left(M_{it2^{-n}} - M_{(i-1)t2^{-n}} \right),$$

i.e. the stochastic integral is an L^2 -limit of Riemann sums. Proof: let

$$H_s^n = \sum_{i=1}^{2^n} H_{(i-1)t2^{-n}} \mathbf{1}_{((i-1)t2^{-n}, it2^{-n}]}(s) \in \mathcal{E}_t.$$

By continuity of H , $H_s^n \rightarrow H_s$ for every $s \in (0, t]$; since $|H^n| \leq \sup |H|$, $\mathbb{E}[\langle M \rangle_t] < \infty$ and $d\langle M \rangle$ has no atom, dominated convergence gives $H^n \rightarrow H$ in $L_t^2(M)$. We conclude by isometry, since the Riemann sum is $(H^n \cdot M)_t$.

(iii) The discretization (i.e. the evaluation of H at the left end point of each interval) is crucial: $\int_0^t H_s dM_s$ is not in general

$$\lim_{L^2} \sum_{i=1}^{2^n} H_{it2^{-n}} \left(M_{it2^{-n}} - M_{(i-1)t2^{-n}} \right).$$

For example, take $H = M = B$ a Brownian motion². Then

$$\begin{aligned} \sum_{i=1}^{2^n} H_{it2^{-n}} \left(M_{it2^{-n}} - M_{(i-1)t2^{-n}} \right) &= \sum_{i=1}^{2^n} H_{(i-1)t2^{-n}} \left(M_{it2^{-n}} - M_{(i-1)t2^{-n}} \right) \\ &= \sum_{i=1}^{2^n} \left(B_{it2^{-n}} - B_{(i-1)t2^{-n}} \right)^2 \xrightarrow[n \rightarrow \infty]{L^2} t, \end{aligned}$$

and not 0!

2 The quadratic variation of a stochastic integral

Theorem 14. Let $M \in \mathcal{M}^2$ and $H \in L^2(M)$. Then $H \cdot M$ is the unique element of $\mathcal{M}^2$ such that

$$\langle H \cdot M, N \rangle_t = \int_0^t H_s d\langle M, N \rangle_s, \quad \forall t \geq 0, \forall N \in \mathcal{M}^2, \quad (1)$$

i.e.

$$\left\langle \int_0^\cdot H_s dM_s, N \right\rangle_t = \int_0^t H_s d\langle M, N \rangle_s, \quad \forall N \in \mathcal{M}^2.$$

The right-hand side is well defined by the Kunita–Watanabe inequality: a.s., $\int_0^t |H_s| |d\langle M, N \rangle_s| \leq (\int_0^t H_s^2 d\langle M \rangle_s)^{1/2} \langle N \rangle_t^{1/2} < \infty$.

Proof. Uniqueness (up to indistinguishability). If $L, \tilde{L} \in \mathcal{M}^2$ are such that $\langle L - \tilde{L}, N \rangle = 0$ for every $N \in \mathcal{M}^2$, then, with $N = L - \tilde{L}$, we find $\langle L - \tilde{L} \rangle = 0$, and we have seen (Chapter 4) that a continuous local martingale with zero bracket is constant, hence 0 since $L_0 = \tilde{L}_0 = 0$ in $\mathcal{M}^2$.

Proof of the characterization (1) for H simple. Take $H = \sum_{i=1}^p H^{(i)} \mathbf{1}_{(t_{i-1}, t_i]} \in \mathcal{E}$. Then

$$(H \cdot M)_t = \sum_{i=1}^p L_t^{(i)}, \quad L_t^{(i)} = H^{(i)} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}) \text{ (a martingale)}.$$

Fix $t > 0$, and let $\Delta_n = (0 = s_0^n < \dots < s_{p_n}^n = t)$ be subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$. By the characterization of the bracket as a limit in probability (Chapter 4),

$$\langle L^{(i)}, N \rangle_t = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \sum_{k=1}^{p_n} (L_{s_k^n}^{(i)} - L_{s_{k-1}^n}^{(i)}) (N_{s_k^n} - N_{s_{k-1}^n})$$

²This is not a completely valid counter-example in our setting since H is not bounded; complete the argument by localization as an exercise!

$$\begin{aligned}
&= H^{(i)} \mathbb{P}\text{-}\lim_{n \rightarrow \infty} \sum_{k=1}^{p_n} \left[\left(M_{t_i \wedge s_k^n} - M_{t_i \wedge s_{k-1}^n} \right) - \left(M_{t_{i-1} \wedge s_k^n} - M_{t_{i-1} \wedge s_{k-1}^n} \right) \right] \\
&\quad \times \left(N_{s_k^n} - N_{s_{k-1}^n} \right) \\
&= H^{(i)} (\langle M^{t_i}, N \rangle_t - \langle M^{t_{i-1}}, N \rangle_t) = H^{(i)} (\langle M, N \rangle_{t \wedge t_i} - \langle M, N \rangle_{t \wedge t_{i-1}}) \\
&= H^{(i)} d\langle M, N \rangle([0, t] \cap (t_{i-1}, t_i]),
\end{aligned}$$

where we used $\langle M^\tau, N \rangle = \langle M, N \rangle^\tau$ (Chapter 4). So that

$$\langle H \cdot M, N \rangle_t = \sum_{i=1}^p \langle L^{(i)}, N \rangle_t = \sum_{i=1}^p H^{(i)} d\langle M, N \rangle([0, t] \cap (t_{i-1}, t_i]) = \int_0^t H_s d\langle M, N \rangle_s.$$

This holds a.s. for every fixed t , hence, both sides being continuous in t , up to indistinguishability.

The case of a general integrand H . Let $T > 0$ and take $(H^n)_{n \geq 1} \subset \mathcal{E}_T$ such that $H^n \rightarrow H$ in $L_T^2(M)$; by definition, $H^n \cdot M \rightarrow H \cdot M$ in $\mathcal{M}_T^2$, and, for $t \leq T$,

$$\langle H^n \cdot M, N \rangle_t = \int_0^t H_s^n d\langle M, N \rangle_s \quad \forall n. \tag{2}$$

We let $n \rightarrow \infty$, using the following. For the right-hand side, by the Kunita–Watanabe and Cauchy–Schwarz inequalities,

$$\begin{aligned}
\mathbb{E} \left[\left| \int_0^t (H_s^n - H_s) d\langle M, N \rangle_s \right| \right] &\leq \mathbb{E} \left[\left(\int_0^t (H_s^n - H_s)^2 d\langle M \rangle_s \right)^{1/2} \left(\int_0^t d\langle N \rangle_s \right)^{1/2} \right] \\
&\leq \|H^n - H\|_{L_T^2(M)} \|N\|_{\mathcal{M}_T^2} \xrightarrow{n \rightarrow \infty} 0,
\end{aligned}$$

since $\mathbb{E}[\langle N \rangle_t] \leq \mathbb{E}[\langle N \rangle_T] = \mathbb{E}[N_T^2] = \|N\|_{\mathcal{M}_T^2}^2$. For the left-hand side of (2), recall that $|\langle X, N \rangle_t| \leq \langle X \rangle_t^{1/2} \langle N \rangle_t^{1/2}$ (Kunita–Watanabe again), so that

$$\mathbb{E} \left[|\langle H^n \cdot M - H \cdot M, N \rangle_t| \right] \leq \mathbb{E}[\langle H^n \cdot M - H \cdot M \rangle_t]^{1/2} \mathbb{E}[\langle N \rangle_t]^{1/2} \xrightarrow{n \rightarrow \infty} 0,$$

since, by linearity and Itô's isometry,

$$\mathbb{E}[\langle H^n \cdot M - H \cdot M \rangle_t] = \mathbb{E}[\langle (H^n - H) \cdot M \rangle_t] \leq \|H^n - H\|_{L_T^2(M)}^2 \xrightarrow{n \rightarrow \infty} 0$$

(we used $\mathbb{E}[X_t^2] = \mathbb{E}[\langle X \rangle_t]$ for $X \in \mathcal{M}^2$). Both sides of (2) converge in L^1 , which gives (1) a.s. for every t , and, by continuity, up to indistinguishability. $\square$

Corollary 15. (i) If $M \in \mathcal{M}^2$ and $H \in L^2(M)$, then $\langle H \cdot M \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s$.

(ii) If $M, N \in \mathcal{M}^2$, $H \in L^2(M)$ and $K \in L^2(N)$, then

$$\langle H \cdot M, K \cdot N \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s.$$

Proof. (ii) implies (i). For (ii), the preceding result, applied twice, implies

$$\langle H \cdot M, K \cdot N \rangle_t = \int_0^t H_s d\langle M, K \cdot N \rangle_s,$$

$$\langle M, K \cdot N \rangle_t = \langle K \cdot N, M \rangle_t = \int_0^t K_s d\langle M, N \rangle_s,$$

i.e. $d\langle M, K \cdot N \rangle_s = K_s d\langle M, N \rangle_s$, and the result follows³. $\square$

Remark 16. Let B be a Brownian motion, and $H \in L^2(B)$, i.e. $\int_0^T \mathbb{E}[H_s^2] ds < \infty$ for every $T \geq 0$. Then $(H \cdot B)_t = \int_0^t H_s dB_s$ is a martingale with bracket $\int_0^t H_s^2 ds$; in particular,

$$\mathbb{E}\left[\int_0^t H_s dB_s\right] = 0, \quad \mathbb{E}\left[\left(\int_0^t H_s dB_s\right)^2\right] = \int_0^t \mathbb{E}[H_s^2] ds.$$

Proposition 17 (Associativity). Let $M \in \mathcal{M}^2$, $K \in L^2(M)$ and $H \in L^2(K \cdot M)$. Then $HK \in L^2(M)$ and

$$(HK) \cdot M = H \cdot (K \cdot M), \quad \text{i.e.} \quad \int_0^t H_s K_s dM_s = \int_0^t H_s d\left(\int_0^s K_u dM_u\right).$$

Proof. Exercise. Hint: $\int_0^T H_s^2 K_s^2 d\langle M \rangle_s = \int_0^T H_s^2 d\langle K \cdot M \rangle_s$ by Corollary 15, and for $N \in \mathcal{M}^2$, $\langle H \cdot (K \cdot M), N \rangle = \int_0^t H_s d\langle K \cdot M, N \rangle_s = \int_0^t H_s K_s d\langle M, N \rangle_s$; conclude by the uniqueness part of Theorem 14. $\square$

Proposition 18 (Crucial for the extension to local martingales). Let $M \in \mathcal{M}^2$, $H \in L^2(M)$, and τ a stopping time. Then

$$H \cdot M^\tau = (H \mathbf{1}_{[0, \tau]}) \cdot M = (H \cdot M)^\tau.$$

³the integral $\int_0^t |H_s K_s| |d\langle M, N \rangle_s|$ being finite by Kunita–Watanabe.

Proof. For every $N \in \mathcal{M}^2$, $\langle M^\tau, N \rangle_t = \langle M, N \rangle_{\tau \wedge t}$ (Chapter 4), hence

$$\langle M^\tau, N \rangle_t = \int_0^{\tau \wedge t} d\langle M, N \rangle_s = \int_0^t \mathbf{1}_{[0, \tau]}(s) d\langle M, N \rangle_s \quad \forall N \in \mathcal{M}^2.$$

Since $M^\tau \in \mathcal{M}^2$ and $\mathbf{1}_{[0, \tau]} \in L^2(M)$ (it is adapted and left-continuous, hence progressive, and bounded), Theorem 14 gives

$$M_t^\tau = \int_0^t \mathbf{1}_{[0, \tau]}(s) dM_s = (\mathbf{1}_{[0, \tau]} \cdot M)_t. \quad (3)$$

By associativity (Proposition 17), it follows that

$$H \cdot M^\tau = H \cdot (\mathbf{1}_{[0, \tau]} \cdot M) = (H \mathbf{1}_{[0, \tau]}) \cdot M.$$

Apply now (3) to $H \cdot M$ instead of M :

$$(H \cdot M)^\tau = \mathbf{1}_{[0, \tau]} \cdot (H \cdot M) = (H \mathbf{1}_{[0, \tau]}) \cdot M. \quad \square$$

3 Stochastic integration with respect to local martingales

Definition 19. Let M be a local martingale with $M_0 = 0$. We denote by $L_{\text{loc}}^2(M)$ the set of progressive processes H such that, for every $t \geq 0$,

$$\int_0^t H_s^2 d\langle M \rangle_s < \infty \quad a.s.$$

Theorem 20. (i) Let M be a local martingale with $M_0 = 0$. For every $H \in L_{\text{loc}}^2(M)$, there exists a unique local martingale with value 0 at $t = 0$, denoted by $H \cdot M = \int_0^\cdot H_s dM_s$, such that, for every local martingale N ,

$$\langle H \cdot M, N \rangle_t = \int_0^t H_s d\langle M, N \rangle_s, \quad i.e. \quad \left\langle \int_0^\cdot H_s dM_s, N \right\rangle_t = \int_0^t H_s d\langle M, N \rangle_s.$$

(ii) If M, N are local martingales (null at 0), $H \in L_{\text{loc}}^2(M)$ and $K \in L_{\text{loc}}^2(N)$, then

$$\langle H \cdot M, K \cdot N \rangle_t = \int_0^t H_s K_s d\langle M, N \rangle_s.$$

(iii) If $M \in \mathcal{M}^2$ and $H \in L^2(M)$, then $H \cdot M$ is a true martingale (in L^2), and this definition is an extension of the previous one.

(iv) If M is a local martingale, $K \in L^2_{\text{loc}}(M)$ and $H \in L^2_{\text{loc}}(K \cdot M)$, then $HK \in L^2_{\text{loc}}(M)$ and $H \cdot (K \cdot M) = (HK) \cdot M$.

(v) If τ is a stopping time, M a local martingale and $H \in L^2_{\text{loc}}(M)$, then

$$H \cdot M^\tau = (H \mathbf{1}_{[0, \tau]}) \cdot M = (H \cdot M)^\tau.$$

Proof. (i) *Uniqueness.* Let $L, \tilde{L}$ be local martingales, null at 0, such that $\langle L, N \rangle = \langle \tilde{L}, N \rangle$ for every local martingale N . Then, with $N = L - \tilde{L}$, $\langle L - \tilde{L}, N \rangle = 0$, hence $L - \tilde{L} = 0$ up to indistinguishability.

Existence. Let

$$\tau_n = \inf \left\{ t \geq 0, |M_t| + \int_0^t H_s^2 d\langle M \rangle_s \geq n \right\}.$$

Then $\tau_n \rightarrow \infty$ a.s., since $H \in L^2_{\text{loc}}(M)$ (hence the continuous process $t \mapsto \int_0^t H_s^2 d\langle M \rangle_s$ is finite a.s.).

(α) The process M^{τ_n} is a local martingale bounded by n , hence a bounded martingale, so that $M^{\tau_n} \in \mathcal{M}^2$ (as previously seen). Also,

$$\int_0^\infty H_s^2 d\langle M^{\tau_n} \rangle_s = \int_0^{\tau_n} H_s^2 d\langle M \rangle_s \leq n,$$

so that $H \in L^2(M^{\tau_n})$, and we can define $H \cdot M^{\tau_n}$ with the L^2 theory.

(β) For $m \geq n$, since $\tau_n \leq \tau_m$, Proposition 18 gives

$$(H \cdot M^{\tau_m})^{\tau_n} = H \cdot (M^{\tau_m})^{\tau_n} = H \cdot M^{\tau_n}.$$

Hence, a.s., for every $m \geq n$: $(H \cdot M^{\tau_m})_s = (H \cdot M^{\tau_n})_s$ for $s \in [0, \tau_n]$.

(γ) From (β), the limit

$$Z_t = \lim_{m \rightarrow \infty} (H \cdot M^{\tau_m})_t$$

exists a.s. (the sequence is stationary), and $Z^{\tau_n} = H \cdot M^{\tau_n}$ for every $n \geq 1$. Hence Z is continuous (since it is continuous on $[0, \tau_n]$ for every n) and adapted (the filtration being complete). Since $H \cdot M^{\tau_n}$ is an L^2 martingale, Z is a local martingale, with value 0 at 0, and (τ_n) is localizing.

(δ) Finally, let us show that if N is a local martingale (with $N_0 = 0$, without loss of generality, since $\langle \cdot, N \rangle = \langle \cdot, N - N_0 \rangle$), then

$$\langle Z, N \rangle_t = \int_0^t H_s d\langle M, N \rangle_s.$$

Let $\sigma_n := \inf\{t \geq 0, |N_t| \geq n\}$ and $S_n = \tau_n \wedge \sigma_n$. The processes M^{S_n} and N^{S_n} are bounded, hence in $\mathcal{M}^2$, and $H \in L^2(M^{S_n})$; hence, by Theorem 14,

$$\langle H \cdot M^{S_n}, N^{S_n} \rangle_t = \int_0^t H_s d\langle M^{S_n}, N^{S_n} \rangle_s = \int_0^{t \wedge S_n} H_s d\langle M, N \rangle_s.$$

Since $Z^{\tau_n} = H \cdot M^{\tau_n}$ and $\tau_n \geq S_n$, Proposition 18 gives $Z^{S_n} = (H \cdot M^{\tau_n})^{S_n} = H \cdot M^{S_n}$, so that

$$\langle Z, N \rangle^{S_n} = \langle Z^{S_n}, N^{S_n} \rangle = \langle H \cdot M^{S_n}, N^{S_n} \rangle = \int_0^{\cdot \wedge S_n} H_s d\langle M, N \rangle_s,$$

and we let $n \rightarrow \infty$ (note that $S_n \rightarrow \infty$ a.s.). We set $H \cdot M := Z$.

(ii) Same proof as for Corollary 15.

(iii) Let $M \in \mathcal{M}^2$ and $H \in L^2(M)$, and let $H \cdot M$ be the L^2 stochastic integral. For a local martingale N with $N_0 = 0$, let $\sigma_n = \inf\{t \geq 0, |N_t| \geq n\}$; then $N^{\sigma_n} \in \mathcal{M}^2$ and, by Theorem 14,

$$\langle H \cdot M, N \rangle^{\sigma_n} = \langle H \cdot M, N^{\sigma_n} \rangle = \int_0^{\cdot} H_s d\langle M, N^{\sigma_n} \rangle_s = \int_0^{\cdot \wedge \sigma_n} H_s d\langle M, N \rangle_s;$$

letting $n \rightarrow \infty$, the L^2 stochastic integral satisfies the characterization (i), hence coincides with the integral defined in (i). Moreover, it is a true martingale in L^2 , since

$$\langle H \cdot M \rangle_t = \int_0^t H_s^2 d\langle M \rangle_s \implies \mathbb{E}[\langle H \cdot M \rangle_t] < \infty \quad \forall t$$

$\implies H \cdot M$ is a true martingale, in L^2 (Chapter 4).

(iv) and (v): as before, using the characterization (i) and its uniqueness part.

For (iv): $\int_0^t H_s^2 K_s^2 d\langle M \rangle_s = \int_0^t H_s^2 d\langle K \cdot M \rangle_s < \infty$ a.s., and

$$\langle H \cdot (K \cdot M), N \rangle = \int_0^{\cdot} H_s d\langle K \cdot M, N \rangle_s = \int_0^{\cdot} H_s K_s d\langle M, N \rangle_s.$$

For (v):

$$\langle M^\tau, N \rangle = \langle M, N \rangle^\tau = \int_0^{\cdot} \mathbf{1}_{[0, \tau]}(s) d\langle M, N \rangle_s,$$

so that $M^\tau = \mathbf{1}_{[0, \tau]} \cdot M$, and we conclude by (iv) as in the proof of Proposition 18. $\square$

Exercise. Let M be a local martingale with $M_0 = 0$, $H \in L_{\text{loc}}^2(M)$, and τ a stopping time. If $\mathbb{E}[\langle H \cdot M \rangle_\tau] < \infty$, then

$$\mathbb{E}\left[\int_0^\tau H_s dM_s\right] = 0, \quad \mathbb{E}\left[\left(\int_0^\tau H_s dM_s\right)^2\right] = \mathbb{E}\left[\int_0^\tau H_s^2 d\langle M \rangle_s\right].$$

4 Integration with respect to semimartingales

Definition 21. A progressive process H is locally bounded if

$$a.s., \forall t \geq 0, \quad \sup_{0 \leq s \leq t} |H_s| < \infty.$$

Remark 22. (i) A continuous adapted process is locally bounded.

(ii) If H is locally bounded, then $\int_0^t H_s dV_s$ and $\int_0^t H_s dM_s$ are well defined, for any finite variation process V and any local martingale M (with $M_0 = 0$), since

$$a.s., \forall t \geq 0, \quad \int_0^t |H_s| |dV_s| < \infty \quad \text{and} \quad \int_0^t H_s^2 d\langle M \rangle_s < \infty.$$

Definition 23. Let $X = X_0 + M + V$ be a continuous semimartingale, and let H be a progressive locally bounded process. We define

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dV_s.$$

The resulting process is a continuous semimartingale, with local martingale part $H \cdot M$ and finite variation part $H \cdot V$.

Exercise. (i) $(H, X) \mapsto \int_0^\cdot H_s dX_s$ is bilinear.

(ii) If H, K are locally bounded and $Y_t = \int_0^t K_s dX_s$, then

$$\int_0^t H_s dY_s = \int_0^t H_s K_s dX_s.$$

(iii) If τ is a stopping time, then

$$\int_0^{t \wedge \tau} H_s dX_s = \int_0^t H_s \mathbf{1}_{\{s \in [0, \tau]\}} dX_s = \int_0^t H_s dX_s^\tau.$$

Lemma 24. Let $a : [0, t] \rightarrow \mathbb{R}$ be continuous with finite variation, $f : [0, t] \rightarrow \mathbb{R}$ continuous, and $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t)$ subdivisions of $[0, t]$ such that $|\Delta_n| \rightarrow 0$. Then

$$\int_0^t f(s) da(s) = \lim_{n \rightarrow \infty} \sum_{i=1}^{p_n} f(t_{i-1}^n) (a(t_i^n) - a(t_{i-1}^n)).$$

Proof. This is the lemma on Riemann–Stieltjes sums of Chapter 4. Let $f_n(s) = \sum_{i=1}^{p_n} f(t_{i-1}^n) \mathbf{1}_{\{s \in (t_{i-1}^n, t_i^n]\}}$. Then

$$\sum_{i=1}^{p_n} f(t_{i-1}^n) (a(t_i^n) - a(t_{i-1}^n)) = \int_0^t f_n(s) da(s),$$

and the result follows by dominated convergence, since $\sup_n \|f_n\|_\infty \leq \|f\|_\infty < \infty$, $|da|$ is finite (without atoms), and $f_n \rightarrow f$ pointwise on $(0, t]$. $\square$

Proposition 25. *Let X be a continuous semimartingale and H a continuous adapted process. For every $t \geq 0$ and every sequence $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t)$ of subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$, we have*

$$\sum_{i=1}^{p_n} H_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t H_s dX_s.$$

Proof. We consider the local martingale part and the finite variation part separately.

- The finite variation part follows from Lemma 24, applied ω by ω with $f = H(\omega)$ and $a = V(\omega)$.
- So let $X = M$ be a local martingale with $M_0 = 0$. Then

$$\sum_{i=1}^{p_n} H_{t_{i-1}^n} (M_{t_i^n} - M_{t_{i-1}^n}) = \int_0^t H_s^n dM_s, \quad \text{where} \quad H_s^n = \sum_{i=1}^{p_n} H_{t_{i-1}^n} \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s).$$

This is the definition of the stochastic integral on simple processes, except that the $H_{t_{i-1}^n}$ are not necessarily bounded, so let us check it. The process H^n is locally bounded, hence in $L_{\text{loc}}^2(M)$. The process $Y_r := \sum_{i=1}^{p_n} H_{t_{i-1}^n} (M_{t_i^n \wedge r} - M_{t_{i-1}^n \wedge r})$ is a local martingale (for each i , localize by $\rho_k \wedge \rho'_k$, with $\rho_k = \inf\{r \geq 0, |M_r| \geq k\}$ and $\rho'_k = t_{i-1}^n$ on $\{|H_{t_{i-1}^n}| > k\}$, $\rho'_k = \infty$ otherwise, and use Lemma 7). For every local martingale N , using the bracket as a limit in probability, as in the proof of Theorem 14,

$$\begin{aligned} \langle Y, N \rangle_t &= \sum_{i=1}^{p_n} H_{t_{i-1}^n} \langle M_{t_i^n \wedge \cdot} - M_{t_{i-1}^n \wedge \cdot}, N \rangle_t = \sum_{i=1}^{p_n} H_{t_{i-1}^n} \left(\langle M, N \rangle_{t_i^n \wedge t} - \langle M, N \rangle_{t_{i-1}^n \wedge t} \right) \\ &= \int_0^t H_s^n d\langle M, N \rangle_s, \end{aligned}$$

so that $Y = H^n \cdot M$ on $[0, t]$, by uniqueness in Theorem 20.

Next, for $m \geq 1$, let $\tau_m = \inf\{s \geq 0, |H_s| + \langle M \rangle_s \geq m\}$; then $\tau_m \rightarrow \infty$ a.s. For fixed m , on $[0, \tau_m]$ we have $|H_s^n| \leq m$ and $|H_s| \leq m$, so that

$$\int_0^{t \wedge \tau_m} (H_s^n - H_s)^2 d\langle M \rangle_s \leq 4m^2 \langle M \rangle_{t \wedge \tau_m} \leq 4m^3.$$

In particular $M^{\tau_m} \in \mathcal{M}^2$ (its bracket is bounded by m) and $(H^n - H)\mathbf{1}_{[0, \tau_m]} \in L^2(M)$. Hence, by Theorem 20(v) and the isometry of the L^2 theory,

$$\mathbb{E}\left[\left(\int_0^{t \wedge \tau_m} (H_s^n - H_s) dM_s\right)^2\right] = \mathbb{E}\left[\int_0^{t \wedge \tau_m} (H_s^n - H_s)^2 d\langle M \rangle_s\right] \xrightarrow{n \rightarrow \infty} 0,$$

by dominated convergence and continuity of H (which gives $H_s^n \rightarrow H_s$ for every $s \in (0, t]$). Hence

$$\int_0^{t \wedge \tau_m} H_s^n dM_s \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^{t \wedge \tau_m} H_s dM_s;$$

we end with the usual argument: for $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(\left|\int_0^t H_s^n dM_s - \int_0^t H_s dM_s\right| > \varepsilon\right) \\ \leq \mathbb{P}(\tau_m < t) + \mathbb{P}\left(\left|\int_0^{t \wedge \tau_m} H_s^n dM_s - \int_0^{t \wedge \tau_m} H_s dM_s\right| > \varepsilon\right), \end{aligned}$$

and we let $n \rightarrow \infty$, and then $m \rightarrow \infty$. $\square$

We end by a particular case of Itô's formula, namely the integration by parts formula.

Proposition 26 (Integration by parts). *Let X, Y be two continuous semimartingales. Then*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

In particular,

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X \rangle_t.$$

Proof. Let $\Delta_n = (0 = t_0^n < \dots < t_{p_n}^n = t)$ be subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$. By Proposition 25 and the convergence of the brackets (Chapter 4),

$$\sum_{i=1}^{p_n} X_{t_{i-1}^n} (Y_{t_i^n} - Y_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t X_s dY_s,$$

$$\sum_{i=1}^{p_n} Y_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t Y_s dX_s,$$

$$\sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n}) (Y_{t_i^n} - Y_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle X, Y \rangle_t,$$

and, summing these three terms, the left-hand side is equal to $X_t Y_t - X_0 Y_0$, by a telescopic sum, since

$$x_{i-1}(y_i - y_{i-1}) + y_{i-1}(x_i - x_{i-1}) + (x_i - x_{i-1})(y_i - y_{i-1}) = x_i y_i - x_{i-1} y_{i-1}.$$

Hence the formula holds a.s. for every fixed t , and, both sides being continuous in t , up to indistinguishability. $\square$