

Chapter 5: Stochastic integral

4. Integration with respect to semimartingales

Outline

- ▶ Locally bounded integrands
- ▶ Definition of $\int_0^t H_s dX_s$ for a continuous semimartingale X
- ▶ Riemann sums
- ▶ Integration by parts

Locally bounded processes

Definition 1

A progressive process H is **locally bounded** if

$$\text{a.s., } \forall t \geq 0, \quad \sup_{0 \leq s \leq t} |H_s| < \infty.$$

Remark 2

1. *A continuous adapted process is locally bounded.*
2. *If H is locally bounded, then $\int_0^t H_s dV_s$ and $\int_0^t H_s dM_s$ are well defined, for any finite variation process V and any local martingale M (with $M_0 = 0$), since a.s., $\forall t \geq 0$,*

$$\int_0^t |H_s| |dV_s| < \infty \quad \text{and} \quad \int_0^t H_s^2 d\langle M \rangle_s < \infty.$$

Integral with respect to a semimartingale

Definition 3

Let $X = X_0 + M + V$ be a continuous semimartingale, and let H be a progressive locally bounded process. We define

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dV_s.$$

The resulting process is a **continuous semimartingale**, with

- ▶ local martingale part $H \cdot M$,
- ▶ finite variation part $H \cdot V$.

Elementary properties

Exercise

Let X be a continuous semimartingale, H, K locally bounded.

1. $(H, X) \mapsto \int_0^\cdot H_s dX_s$ is bilinear.

2. If $Y_t = \int_0^t K_s dX_s$, then

$$\int_0^t H_s dY_s = \int_0^t H_s K_s dX_s.$$

3. If τ is a stopping time, then

$$\int_0^{t \wedge \tau} H_s dX_s = \int_0^t H_s \mathbf{1}_{\{s \in [0, \tau]\}} dX_s = \int_0^t H_s dX_s^\tau.$$

Riemann–Stieltjes sums

Lemma 4

Let $a : [0, t] \rightarrow \mathbb{R}$ be continuous with finite variation, $f : [0, t] \rightarrow \mathbb{R}$ continuous, and $\Delta_n = (0 = t_0^n < \cdots < t_{p_n}^n = t)$ subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$. Then

$$\int_0^t f(s) da(s) = \lim_{n \rightarrow \infty} \sum_{i=1}^{p_n} f(t_{i-1}^n) (a(t_i^n) - a(t_{i-1}^n)).$$

Proof.

Let $f_n(s) = \sum_{i=1}^{p_n} f(t_{i-1}^n) \mathbf{1}_{\{s \in (t_{i-1}^n, t_i^n]\}}$. The sum is $\int_0^t f_n(s) da(s)$.

Dominated convergence: $\sup_n \|f_n\|_\infty \leq \|f\|_\infty$, $|da|$ is finite without atoms, and $f_n \rightarrow f$ pointwise on $(0, t]$. □

Riemann sums for semimartingales

Proposition 5

Let X be a continuous semimartingale and H a continuous adapted process. For every $t \geq 0$ and every sequence $\Delta_n = (0 = t_0^n < \dots < t_{p_n}^n = t)$ of subdivisions of $[0, t]$ with $|\Delta_n| \rightarrow 0$,

$$\sum_{i=1}^{p_n} H_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t H_s dX_s.$$

- ▶ The integrand is evaluated at the **left end point** of each interval.
- ▶ No nested subdivisions needed.

Proof (1/4): the finite variation part

We consider separately the finite variation part and the local martingale part.

- **Finite variation part:** Lemma 4, applied ω by ω with $f = H(\omega)$ and $a = V(\omega)$.
- **Local martingale part:** let $X = M$, with $M_0 = 0$. Then

$$\sum_{i=1}^{p_n} H_{t_{i-1}^n} (M_{t_i^n} - M_{t_{i-1}^n}) = \int_0^t H_s^n dM_s,$$

$$\text{where} \quad H_s^n = \sum_{i=1}^{p_n} H_{t_{i-1}^n} \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s).$$

This is the definition on simple processes, except that the $H_{t_{i-1}^n}$ need not be bounded: let us check it.

Proof (2/4): identification of the sum

H^n is locally bounded, hence in $L^2_{\text{loc}}(M)$. Let

$$Y_r := \sum_{i=1}^{p_n} H_{t_{i-1}^n} (M_{t_i^n \wedge r} - M_{t_{i-1}^n \wedge r}).$$

- ▶ Y is a local martingale (localize each term, by $|M| \leq k$ and $|H_{t_{i-1}^n}| \leq k$).
- ▶ For every local martingale N (bracket as a limit in probability):

$$\begin{aligned} \langle Y, N \rangle_t &= \sum_{i=1}^{p_n} H_{t_{i-1}^n} \left(\langle M, N \rangle_{t_i^n \wedge t} - \langle M, N \rangle_{t_{i-1}^n \wedge t} \right) \\ &= \int_0^t H_s^n d\langle M, N \rangle_s. \end{aligned}$$

By uniqueness in the characterization of $H^n \cdot M$: $Y = H^n \cdot M$ on $[0, t]$.

Proof (3/4): localization

For $m \geq 1$, let

$$\tau_m = \inf\{s \geq 0, |H_s| + \langle M \rangle_s \geq m\} \xrightarrow{m \rightarrow \infty} \infty \quad \text{a.s.}$$

On $[0, \tau_m]$, $|H_s^n| \leq m$ and $|H_s| \leq m$, so

$$\int_0^{t \wedge \tau_m} (H_s^n - H_s)^2 d\langle M \rangle_s \leq 4m^2 \langle M \rangle_{t \wedge \tau_m} \leq 4m^3.$$

$M^{\tau_m} \in \mathcal{M}^2$, and by the isometry of the L^2 theory,

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^{t \wedge \tau_m} (H_s^n - H_s) dM_s \right)^2 \right] &= \mathbb{E} \left[\int_0^{t \wedge \tau_m} (H_s^n - H_s)^2 d\langle M \rangle_s \right] \\ &\xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

by dominated convergence, since $H_s^n \rightarrow H_s$ for every $s \in (0, t]$ (H continuous).

Proof (4/4): conclusion

Hence, for each fixed m ,

$$\int_0^{t \wedge \tau_m} H_s^n dM_s \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^{t \wedge \tau_m} H_s dM_s.$$

Usual argument: for $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left(\left|\int_0^t H_s^n dM_s - \int_0^t H_s dM_s\right| > \varepsilon\right) \\ \leq \mathbb{P}(\tau_m < t) + \mathbb{P}\left(\left|\int_0^{t \wedge \tau_m} (H_s^n - H_s) dM_s\right| > \varepsilon\right). \end{aligned}$$

Let $n \rightarrow \infty$, and then $m \rightarrow \infty$.

□

Integration by parts

A particular case of Itô's formula.

Proposition 6 (Integration by parts)

Let X, Y be two continuous semimartingales. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

In particular,

$$X_t^2 = X_0^2 + 2 \int_0^t X_s dX_s + \langle X \rangle_t.$$

- Compared with the classical formula: the extra term $\langle X, Y \rangle_t$.

Proof

Let $\Delta_n = (0 = t_0^n < \dots < t_{p_n}^n = t)$ with $|\Delta_n| \rightarrow 0$. By Proposition 5 and the convergence of the brackets:

$$\begin{aligned}\sum_i X_{t_{i-1}^n} (Y_{t_i^n} - Y_{t_{i-1}^n}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t X_s dY_s, \\ \sum_i Y_{t_{i-1}^n} (X_{t_i^n} - X_{t_{i-1}^n}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t Y_s dX_s, \\ \sum_i (X_{t_i^n} - X_{t_{i-1}^n}) (Y_{t_i^n} - Y_{t_{i-1}^n}) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle X, Y \rangle_t.\end{aligned}$$

Summing: telescopic sum, since

$$\begin{aligned}x_{i-1}(y_i - y_{i-1}) + y_{i-1}(x_i - x_{i-1}) \\ + (x_i - x_{i-1})(y_i - y_{i-1}) = x_i y_i - x_{i-1} y_{i-1}.\end{aligned}$$

So the formula holds a.s. for each t , hence (continuity) up to indistinguishability. □