

Chapter 6: Itô's formula and applications (in progress)

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1 Itô's formula

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

Theorem 1 (Itô's formula). (i) Let X be a continuous semimartingale, and $F : \mathbb{R} \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$. Then, almost surely, for every $t \geq 0$,

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2} \int_0^t F''(X_s) d\langle X \rangle_s.$$

(ii) Let $X = (X^1, \dots, X^d)$, where each X^i is a continuous semimartingale, and $F : \mathbb{R}^d \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$. Then, almost surely, for every $t \geq 0$,

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \partial_i F(X_s) dX_s^i + \frac{1}{2} \sum_{1 \leq i, j \leq d} \int_0^t \partial_{ij} F(X_s) d\langle X^i, X^j \rangle_s.$$

In particular, $F(X)$ is a continuous semimartingale.

The integrals are well defined: $F'(X)$ and $\partial_i F(X)$ are continuous adapted, hence locally bounded, and $F''(X)$, $\partial_{ij} F(X)$ are continuous, hence integrable against the finite variation processes $\langle X \rangle$ and $\langle X^i, X^j \rangle$.

Proof. (i) Fix $t > 0$, and let $\Delta_n = (0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t)$ be subdivisions of $[0, t]$ such that $\Delta_n \subset \Delta_{n+1}$ and $|\Delta_n| \rightarrow 0$. Writing a telescopic sum, and using Taylor's formula at order 2,

$$\begin{aligned} F(X_t) &= F(X_0) + \sum_{i=1}^{p_n} \left(F(X_{t_i^n}) - F(X_{t_{i-1}^n}) \right) \\ &= F(X_0) + \sum_{i=1}^{p_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) + \frac{1}{2} \sum_{i=1}^{p_n} f_{n,i} (X_{t_i^n} - X_{t_{i-1}^n})^2, \end{aligned}$$

where $f_{n,i}$ is a random variable which satisfies

$$\inf_{x \in I_{n,i}} F''(x) \leq f_{n,i} \leq \sup_{x \in I_{n,i}} F''(x), \quad I_{n,i} = [[X_{t_{i-1}^n}, X_{t_i^n}]],$$

with the notation $[[a, b]] = [a \wedge b, a \vee b]$. One can take for instance

$$f_{n,i} = 2 \int_0^1 (1 - \theta) F''(X_{t_{i-1}^n} + \theta(X_{t_i^n} - X_{t_{i-1}^n})) d\theta$$

(Taylor's formula with integral remainder), which is a random variable.

By the convergence of Riemann sums (Chapter 5), applied to the continuous adapted process $F'(X)$,

$$\sum_{i=1}^{p_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t F'(X_s) dX_s.$$

Hence the result follows from

$$\sum_{i=1}^{p_n} f_{n,i} (X_{t_i^n} - X_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t F''(X_s) d\langle X \rangle_s, \quad (1)$$

which gives the formula a.s. for every fixed t , and from the continuity of both sides in t , which gives it a.s. for every $t \geq 0$.

Proof of (1). Let $m < n$, so that $\Delta_m \subset \Delta_n$. Each interval $[t_{i-1}^n, t_i^n]$ is contained in exactly one interval $[t_{j-1}^m, t_j^m]$, and we write $\sum_{i:j}$ for the sum over the indices i such that $t_{j-1}^m \leq t_{i-1}^n < t_i^n \leq t_j^m$. Then

$$\sum_{i=1}^{p_n} f_{n,i} (X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_0^t F''(X_s) d\langle X \rangle_s = I_{n,m} + II_{n,m} + III_m,$$

with

$$\begin{aligned}
I_{n,m} &= \sum_{j=1}^{p_m} \sum_{i:j} \left(f_{n,i} - F''(X_{t_{j-1}^m}) \right) (X_{t_i^n} - X_{t_{i-1}^n})^2, \\
II_{n,m} &= \sum_{j=1}^{p_m} F''(X_{t_{j-1}^m}) \left[\sum_{i:j} (X_{t_i^n} - X_{t_{i-1}^n})^2 - \left(\langle X \rangle_{t_j^m} - \langle X \rangle_{t_{j-1}^m} \right) \right], \\
III_m &= \sum_{j=1}^{p_m} F''(X_{t_{j-1}^m}) \left(\langle X \rangle_{t_j^m} - \langle X \rangle_{t_{j-1}^m} \right) - \int_0^t F''(X_s) d\langle X \rangle_s.
\end{aligned}$$

• For m fixed, $II_{n,m} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$, by the convergence of the discrete quadratic variation of a semimartingale on each of the (finitely many) intervals $[t_{j-1}^m, t_j^m]$ (Chapter 4).

•• We have $|I_{n,m}| \leq Z_m C_n$, where

$$C_n = \sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle X \rangle_t,$$

and

$$Z_m = \sup \left\{ |F''(X_u) - F''(X_v)|, u, v \in [0, t], |u - v| \leq |\Delta_m| \right\}.$$

Indeed, by continuity of X , $I_{n,i} \subset \{X_u, u \in [t_{i-1}^n, t_i^n]\}$, so that $f_{n,i}$ lies between two values $F''(X_u), F''(X_{u'})$ with $u, u' \in [t_{i-1}^n, t_i^n] \subset [t_{j-1}^m, t_j^m]$. Since $s \mapsto F''(X_s)$ is continuous, hence uniformly continuous on $[0, t]$, $Z_m \rightarrow 0$ a.s. as $m \rightarrow \infty$.

••• $III_m \rightarrow 0$ a.s. as $m \rightarrow \infty$, as a Riemann sum of a Stieltjes integral with continuous integrand and integrator (Chapter 5).

••• Conclusion. Let $\varepsilon > 0$ and $\eta > 0$. For every m ,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \mathbb{P} \left(\left| \sum_{i=1}^{p_n} f_{n,i} (X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_0^t F''(X_s) d\langle X \rangle_s \right| \geq \varepsilon \right) \\
\leq \limsup_{n \rightarrow \infty} \mathbb{P} \left(|I_{n,m}| \geq \frac{\varepsilon}{3} \right) + \limsup_{n \rightarrow \infty} \mathbb{P} \left(|II_{n,m}| \geq \frac{\varepsilon}{3} \right) + \mathbb{P} \left(|III_m| \geq \frac{\varepsilon}{3} \right),
\end{aligned}$$

where the second term is 0, and

$$\mathbb{P} \left(|I_{n,m}| \geq \frac{\varepsilon}{3} \right) \leq \mathbb{P}(Z_m \geq \eta) + \mathbb{P} \left(C_n \geq \frac{\varepsilon}{3\eta} \right),$$

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(C_n \geq \frac{\varepsilon}{3\eta} \right) \leq \mathbb{P} \left(\langle X \rangle_t \geq \frac{\varepsilon}{6\eta} \right).$$

Letting $m \rightarrow \infty$, and then $\eta \rightarrow 0$, we obtain (1).

(ii) Likewise, Taylor's formula yields

$$\begin{aligned} F(X_{t_i^n}) - F(X_{t_{i-1}^n}) &= \sum_{k=1}^d \partial_k F(X_{t_{i-1}^n})(X_{t_i^n}^k - X_{t_{i-1}^n}^k) \\ &\quad + \frac{1}{2} \sum_{1 \leq k, l \leq d} f_{n,i}^{k,l}(X_{t_i^n}^k - X_{t_{i-1}^n}^k)(X_{t_i^n}^l - X_{t_{i-1}^n}^l), \end{aligned}$$

with

$$\inf_{x \in I_{n,i}} \partial_{kl} F(x) \leq f_{n,i}^{k,l} \leq \sup_{x \in I_{n,i}} \partial_{kl} F(x), \quad I_{n,i} = [[X_{t_{i-1}^n}^1, X_{t_i^n}^1]] \times \cdots \times [[X_{t_{i-1}^n}^d, X_{t_i^n}^d]]$$

(take again $f_{n,i}^{k,l} = 2 \int_0^1 (1 - \theta) \partial_{kl} F(X_{t_{i-1}^n} + \theta(X_{t_i^n} - X_{t_{i-1}^n})) d\theta$, the segment being contained in the box $I_{n,i}$). The proof then goes exactly in the same way as for the one-dimensional case, using

$$\sum_{i=1}^{p_n} f_{n,i}^{k,l}(X_{t_i^n}^k - X_{t_{i-1}^n}^k)(X_{t_i^n}^l - X_{t_{i-1}^n}^l) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t \partial_{kl} F(X_s) d\langle X^k, X^l \rangle_s,$$

which is proved as (1), with $\langle X^k, X^l \rangle$ instead of $\langle X \rangle$, and $C_n = \sum_i |X_{t_i^n}^k - X_{t_{i-1}^n}^k| |X_{t_i^n}^l - X_{t_{i-1}^n}^l|$, which is bounded in probability since $C_n \leq \frac{1}{2} \sum_i ((X_{t_i^n}^k - X_{t_{i-1}^n}^k)^2 + (X_{t_i^n}^l - X_{t_{i-1}^n}^l)^2)$. $\square$

Remark 2. (i) With $d = 2$ and $F(x, y) = xy$, we recover the integration by parts formula of Chapter 5:

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

(ii) Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set, $F : \mathcal{O} \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$, and assume that, almost surely, $X_t = (X_t^1, \dots, X_t^d) \in \mathcal{O}$ for every $t \geq 0$. Then the same result holds.

Proof of (ii). For $n \geq 1$, let $\mathcal{O}_n = \{x \in \mathbb{R}^d, d(x, \mathcal{O}^c) \geq \frac{1}{n}\}$, and let $F_n : \mathbb{R}^d \rightarrow \mathbb{R}$ be of class $\mathcal{C}^2$ such that $F_n = F$ on a neighbourhood of $\mathcal{O}_n$ (for instance $F_n = \chi_n F$, where χ_n is a smooth function equal to 1 on $\{d(\cdot, \mathcal{O}^c) > \frac{1}{2n}\}$ and to 0 on $\{d(\cdot, \mathcal{O}^c) < \frac{1}{4n}\}$, obtained by mollification of an indicator function). Itô's formula holds with F_n . Let

$$\tau_n = \inf\{t \geq 0, X_t \notin \mathcal{O}_n\}.$$

On $\{\tau_n > 0\}$, $X_t \in \mathcal{O}_n$ for $t \in [0, \tau_n]$ ($\mathcal{O}_n$ is closed and X is continuous), so that $F_n(X_t) = F(X_t)$, $\partial_i F_n(X_t) = \partial_i F(X_t)$ and $\partial_{ij} F_n(X_t) = \partial_{ij} F(X_t)$ for $t \in [0, \tau_n]$. Applying Itô's formula with F_n at time $t \wedge \tau_n$ (and using the behaviour of stochastic integrals with respect to stopping, Chapter 5), we obtain Itô's formula with F for $t \in [0, \tau_n]$. Finally, $\tau_n \rightarrow \infty$ a.s., since, for every $t \geq 0$,

$$\inf_{s \in [0, t]} d(X_s, \mathcal{O}^c) > 0 \quad \text{a.s.}$$

(X being continuous, $s \mapsto d(X_s, \mathcal{O}^c)$ is continuous, and positive since $X_s \in \mathcal{O}$ open, hence bounded below by a positive constant on the compact set $[0, t]$). $\square$

Example 3. (i) Let B be a one-dimensional $(\mathcal{F}_t)$ -Brownian motion, with $B_0 \in \mathcal{F}_0$, and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$. Applying Itô's formula to $X_t = (t, B_t)$, since $\langle \text{Id}, B \rangle_t = 0$ and $\langle \text{Id} \rangle_t = 0$, where $\text{Id}(t) = t$ has finite variation, we obtain

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(s, B_s) dB_s + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right)(s, B_s) ds.$$

Hence, if $\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} = 0$, then $F(t, B_t)$ is a local martingale. For instance

$$F(t, x) = x^2 - t, \quad F(t, x) = x^3 - 3tx, \quad \text{Hermite polynomials} \dots$$

(ii) Let $B = (B^1, \dots, B^d)$ be a d -dimensional $(\mathcal{F}_t)$ -Brownian motion, with $B_0 \in \mathcal{F}_0$. Then $\langle B^i, B^j \rangle = 0$ if $i \neq j$. Indeed, for $s \leq t$, since $B_t - B_s$ is independent of $\mathcal{F}_s$, with covariance matrix $(t - s)I_d$,

$$\begin{aligned} \mathbb{E}[B_t^i B_t^j - B_s^i B_s^j \mid \mathcal{F}_s] &= \mathbb{E}[(B_t^i - B_s^i)(B_t^j - B_s^j)] \\ &\quad + B_s^i \mathbb{E}[B_t^j - B_s^j] + B_s^j \mathbb{E}[B_t^i - B_s^i] = 0, \end{aligned}$$

so that $B^i B^j$ is a martingale. (More generally, two independent local martingales are orthogonal, see Chapter 4.) Hence, for $F : \mathbb{R}^d \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$,

$$\begin{aligned} F(B_t) &= F(B_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(B_s) dB_s^i + \frac{1}{2} \int_0^t \Delta F(B_s) ds \\ &= F(B_0) + \int_0^t \nabla F(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta F(B_s) ds. \end{aligned}$$

Exercise. Let B be a d -dimensional Brownian motion, $d \geq 2$, with $B_0 = x \neq 0$, and, for $a > 0$,

$$T_a = \inf\{t \geq 0, |B_t| = a\},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. For $0 < r < |x| < R$, compute $\mathbb{P}(T_r < T_R)$. (Hint: apply the previous example, with the open set $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$, to a radial function F such that $\Delta F = 0$ on $\mathcal{O}$.)

2 Exponential (semi)martingales

Theorem 4. *Let X be a continuous semimartingale. There exists a unique continuous semimartingale Z such that*

$$Z_t = e^{X_0} + \int_0^t Z_s dX_s, \quad t \geq 0. \quad (2)$$

We use the notation $Z = \mathcal{E}(X)$, and we have the explicit representation

$$\mathcal{E}(X)_t = \exp \left(X_t - \frac{1}{2} \langle X \rangle_t \right). \quad (3)$$

If X is a local martingale, so is $\mathcal{E}(X)$, by (2).

For instance, with $X_t = \theta B_t$, where B is a Brownian motion with $B_0 = 0$ and $\theta \in \mathbb{R}$, $\mathcal{E}(X)_t = \exp \left(\theta B_t - \frac{\theta^2}{2} t \right)$.

Proof. Existence. Start from (3), and apply Itô's formula with $F(x) = e^x$ to the semimartingale $X - \frac{1}{2} \langle X \rangle$, whose bracket is $\langle X \rangle$ (the bracket does not see the finite variation part):

$$\begin{aligned} e^{X_t - \frac{1}{2} \langle X \rangle_t} &= e^{X_0} + \int_0^t e^{X_s - \frac{1}{2} \langle X \rangle_s} d \left(X_s - \frac{1}{2} \langle X \rangle_s \right) + \frac{1}{2} \int_0^t e^{X_s - \frac{1}{2} \langle X \rangle_s} d \langle X \rangle_s \\ &= e^{X_0} + \int_0^t e^{X_s - \frac{1}{2} \langle X \rangle_s} dX_s. \end{aligned}$$

Uniqueness. With $G(x) = e^{-x}$ and $Y_t = \exp \left(-X_t + \frac{1}{2} \langle X \rangle_t \right)$, Itô's formula gives

$$Y_t = e^{-X_0} - \int_0^t Y_s dX_s + \int_0^t Y_s d \langle X \rangle_s.$$

Let $\tilde{Z}$ be a solution of (2). By the integration by parts formula,

$$\begin{aligned} Y_t \tilde{Z}_t &= Y_0 \tilde{Z}_0 + \int_0^t Y_s d\tilde{Z}_s + \int_0^t \tilde{Z}_s dY_s + \langle \tilde{Z}, Y \rangle_t \\ &= 1 + \int_0^t Y_s \tilde{Z}_s dX_s + \int_0^t \tilde{Z}_s (-Y_s dX_s + Y_s d \langle X \rangle_s) - \int_0^t Y_s \tilde{Z}_s d \langle X \rangle_s = 1, \end{aligned}$$

since $\langle \tilde{Z}, Y \rangle_t = - \int_0^t \tilde{Z}_s Y_s d \langle X \rangle_s$. Hence $Y_t \tilde{Z}_t = 1$, i.e. $\tilde{Z}_t = \frac{1}{Y_t} = \mathcal{E}(X)_t$. $\square$

As before, we have the generalization to complex parameters.

Corollary 5. Let M be a (real-valued) local martingale and $\lambda \in \mathbb{C}$. Then

$$\mathcal{E}(\lambda M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2} \langle M \rangle_t\right)$$

is a $\mathbb{C}$ -valued local martingale (i.e. its real and imaginary parts are both local martingales), and

$$\mathcal{E}(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \mathcal{E}(\lambda M)_s dM_s.$$

Proof. Apply Itô's formula to the real and imaginary parts of $F(x, y) = e^{\lambda x - \frac{\lambda^2}{2} y}$ (a $\mathcal{C}^2$ function of $(x, y) \in \mathbb{R}^2$) and to $(M_t, \langle M \rangle_t)$; the computation is the same as in the real case. $\square$

3 Lévy's characterization of Brownian motion

If $X = (X^1, \dots, X^d)$ is an $(\mathcal{F}_t)$ -Brownian motion with values in $\mathbb{R}^d$, then $\langle X^i, X^j \rangle_t = \delta_{ij} t$. The converse is true:

Theorem 6 (Lévy). Let $M^1, \dots, M^d$ be continuous local martingales, with $M_0^i = 0$, such that $\langle M^i, M^j \rangle_t = t \delta_{ij}$. Then $M = (M^1, \dots, M^d)$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion.

In particular, if M is a local martingale such that $M_0 = 0$ and $\langle M \rangle_t = t$, then it is a Brownian motion.

Proof. Let $\xi \in \mathbb{R}^d$. Then $N_t := \xi \cdot M_t$ is a local martingale with bracket

$$\langle N \rangle_t = \sum_{1 \leq j, k \leq d} \xi_j \xi_k \langle M^j, M^k \rangle_t = \sum_{j=1}^d \xi_j^2 \langle M^j \rangle_t = |\xi|^2 t.$$

Moreover, by Corollary 5 (with $\lambda = i$),

$$\mathcal{E}(iN)_t = \exp\left(i\xi \cdot M_t + \frac{|\xi|^2}{2} t\right)$$

is a local martingale, with values in $\mathbb{C}$, and

$$\forall t \geq 0 : \quad \sup_{0 \leq s \leq t} |\mathcal{E}(iN)_s| \leq \exp\left(\frac{|\xi|^2}{2} t\right) \in L^1(\mathbb{P}).$$

Hence (general result: if M is a local martingale such that $\mathbb{E}[\sup_{0 \leq s \leq t} |M_s|] < \infty$ for every t , then M is a true martingale), $(\mathcal{E}(iN)_t)_{t \geq 0}$ is actually a true martingale, and

$$\forall s < t : \quad \mathbb{E}[\mathcal{E}(iN)_t \mid \mathcal{F}_s] = \mathcal{E}(iN)_s,$$

i.e.

$$\mathbb{E}[e^{i\xi \cdot (M_t - M_s)} \mid \mathcal{F}_s] = e^{-\frac{|\xi|^2}{2}(t-s)}.$$

This holds for every $\xi \in \mathbb{R}^d$. Hence, for $A \in \mathcal{F}_s$ with $\mathbb{P}(A) > 0$, the characteristic function of $M_t - M_s$ under $\mathbb{P}(\cdot \mid A)$ is $\xi \mapsto e^{-\frac{|\xi|^2}{2}(t-s)}$, and we have everything:

$$M_t - M_s \sim \mathcal{N}(0, (t-s)\mathbf{I}_d), \quad \text{independent of } \mathcal{F}_s,$$

and M is continuous and adapted, with $M_0 = 0$. □

Example 7. Let B be an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion with $B_0 = 0$, and

$$\beta_t = \int_0^t \operatorname{sgn}(B_s) dB_s, \quad \operatorname{sgn}(x) = \mathbf{1}_{\{x > 0\}} - \mathbf{1}_{\{x \leq 0\}}.$$

The process $\operatorname{sgn}(B_s)$ is bounded and progressive, so β_t is well defined as a continuous (true) martingale, and

$$\langle \beta \rangle_t = \int_0^t (\operatorname{sgn}(B_s))^2 ds = t.$$

Hence β is an $(\mathcal{F}_t)$ -Brownian motion.

4 Dubins–Schwarz representation of continuous martingales

Theorem 8 (Dubins–Schwarz). Let M be a continuous local martingale such that $M_0 = 0$ and $\langle M \rangle_\infty = \infty$ a.s. Then there exists a Brownian motion β such that

$$M_t = \beta_{\langle M \rangle_t}, \quad t \geq 0.$$

Remark 9. (i) $(\beta_r)_{r \geq 0}$ is not an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion in general, but a $(\mathcal{G}_r)_{r \geq 0}$ -Brownian motion, where

$$\mathcal{G}_r = \mathcal{F}_{\tau_r}, \quad \tau_r = \inf\{t \geq 0, \langle M \rangle_t > r\}.$$

(ii) The assumption $\langle M \rangle_\infty = \infty$ a.s. can be relaxed, up to enlarging the probability space (we omit the proof).

(iii) The proof is simpler if $t \mapsto \langle M \rangle_t$ is a.s. strictly increasing.

Lemma 10. *Let M be a local martingale with $M_0 = 0$. Then, almost surely, $t \mapsto M_t$ and $t \mapsto \langle M \rangle_t$ have the same intervals of constancy:*

$$\forall 0 \leq a < b : \quad M_t = M_a \quad \forall t \in [a, b] \quad \Longleftrightarrow \quad \langle M \rangle_b = \langle M \rangle_a.$$

Proof. By continuity of M and $\langle M \rangle$, only the result for fixed a, b (say rational) needs to be proved: a.s., both properties for all $0 \leq a < b$ follow from the same properties for all rational $a < b$, by approximating $[a, b]$ from inside by intervals with rational end points.

First, $\{M_t = M_a \quad \forall t \in [a, b]\} \subset \{\langle M \rangle_b = \langle M \rangle_a\}$ up to a null set, obviously (by approximation of the bracket by the discrete quadratic variation, along a subsequence converging a.s.).

Conversely, let

$$\tau = \inf\{t > a, \langle M \rangle_t \neq \langle M \rangle_a\} (\geq a).$$

The process

$$N_t := \int_0^t \mathbf{1}_{\{a < s \leq \tau\}} dM_s = M_{t \wedge \tau} - M_{t \wedge a}$$

is a local martingale, and, by definition of τ and continuity of $\langle M \rangle$,

$$\langle N \rangle_t = \int_0^t \mathbf{1}_{\{a < s \leq \tau\}} d\langle M \rangle_s = \langle M \rangle_{\tau \wedge t} - \langle M \rangle_{a \wedge t} = 0.$$

Hence $N_t = 0$ a.s. for every $t \geq 0$, i.e. $M_{t \wedge \tau} = M_a$ for every $t \geq a$. On the event $\{\langle M \rangle_b = \langle M \rangle_a\}$, we have $\tau \geq b$, hence $M_t = M_a$ for every $t \in [a, b]$. $\square$

Proof of Theorem 8. Step 1. For $r \geq 0$, define

$$\tau_r = \inf\{t \geq 0, \langle M \rangle_t > r\}.$$

Since $\langle M \rangle_\infty = \infty$ a.s., $\tau_r < \infty$ a.s. for every r (we discard the null set where $\langle M \rangle_\infty < \infty$). The function $r \mapsto \tau_r$ is non-decreasing and càdlàg, and, if $r > 0$,

$$\tau_{r-} = \lim_{s \uparrow r} \tau_s = \inf\{t \geq 0, \langle M \rangle_t \geq r\}.$$

For every $r > 0$, $\langle M \rangle_{\tau_{r-}} = r = \langle M \rangle_{\tau_r}$, by continuity of $t \mapsto \langle M \rangle_t$.

Step 2. For $r \geq 0$, let $\mathcal{G}_r := \mathcal{F}_{\tau_r}$; $(\mathcal{G}_r)_{r \geq 0}$ is a filtration (τ_r is a stopping time, the filtration $(\mathcal{F}_t)$ being right-continuous). It satisfies the usual conditions, by right-continuity of $r \mapsto \tau_r$ and of $(\mathcal{F}_t)$. Let

$$\beta_r := M_{\tau_r}, \quad r \geq 0;$$

β is $(\mathcal{G}_r)_{r \geq 0}$ -adapted. We have

$$M_t = \beta_{\langle M \rangle_t}, \quad \text{i.e.} \quad M_t = M_{\tau_{\langle M \rangle_t}}.$$

To see this, it suffices to check that M is constant on $[t, \tau_{\langle M \rangle_t}]$ (note that $t \leq \tau_{\langle M \rangle_t}$), which is a consequence of Lemma 10, since $\langle M \rangle_t = \langle M \rangle_{\tau_{\langle M \rangle_t}}$ (as $\langle M \rangle_{\tau_r} = r$ for every r , with the choice $r = \langle M \rangle_t$).

Step 3. $\beta_0 = M_{\tau_0} = 0$, since, as in Step 2, $M_0 = M_{\tau_{\langle M \rangle_0}}$, which is rewritten as $0 = M_{\tau_0}$.

Moreover, $\beta_r = M_{\tau_r}$ is continuous: it is càdlàg (since M is continuous and $r \mapsto \tau_r$ is càdlàg), and, for every $r > 0$, $\beta_{r-} = \beta_r$, i.e. $M_{\tau_{r-}} = M_{\tau_r}$, by Lemma 10, since $\langle M \rangle_{\tau_{r-}} = \langle M \rangle_{\tau_r}$.

Step 4. For every $r \geq 0$, M^{τ_r} is an $(\mathcal{F}_t)_{t \geq 0}$ -local martingale with

$$\langle M^{\tau_r} \rangle_\infty = \langle M \rangle_{\tau_r} = r$$

bounded; hence $(M_t^{\tau_r})_{t \geq 0}$ and $((M_t^{\tau_r})^2 - \langle M^{\tau_r} \rangle_t)_{t \geq 0}$ are uniformly integrable $(\mathcal{F}_t)_{t \geq 0}$ -martingales (Chapter 4). By the optional stopping theorem, for $s \leq r$ (so that $\tau_s \leq \tau_r$),

$$\mathbb{E}[\beta_r \mid \mathcal{G}_s] = \mathbb{E}[M_\infty^{\tau_r} \mid \mathcal{F}_{\tau_s}] = M_{\tau_s}^{\tau_r} = M_{\tau_s} = \beta_s,$$

and

$$\mathbb{E}[\beta_r^2 - r \mid \mathcal{G}_s] = \mathbb{E}[(M_\infty^{\tau_r})^2 - \langle M^{\tau_r} \rangle_\infty \mid \mathcal{F}_{\tau_s}] = (M_{\tau_s}^{\tau_r})^2 - \langle M \rangle_{\tau_s} = \beta_s^2 - s.$$

Step 5. Finally, β is a continuous $(\mathcal{G}_r)_{r \geq 0}$ -martingale such that $\langle \beta \rangle_r = r$; by Lévy's characterization (Theorem 6), it is a $(\mathcal{G}_r)_{r \geq 0}$ -Brownian motion. $\square$

5 Burkholder–Davis–Gundy inequality

For a continuous local martingale M , we let

$$M_t^* = \sup_{s \in [0, t]} |M_s|, \quad M_\infty^* = \sup_{s \geq 0} |M_s|.$$

Theorem 11 (BDG). *For every $p > 0$, there exist constants $0 < c_p < C_p$ such that, for every continuous local martingale M with $M_0 = 0$, we have*

$$c_p \mathbb{E}[\langle M \rangle_\infty^{p/2}] \leq \mathbb{E}[(M_\infty^*)^p] \leq C_p \mathbb{E}[\langle M \rangle_\infty^{p/2}].$$

If τ is a stopping time, then $M \mapsto M^\tau$ gives

$$c_p \mathbb{E}[\langle M \rangle_\tau^{p/2}] \leq \mathbb{E}[(M_\tau^*)^p] \leq C_p \mathbb{E}[\langle M \rangle_\tau^{p/2}].$$

The case $p = 2$ is Itô's isometry and Doob's inequality.

Proof. By localization, i.e. replacing M by M^{τ_n} , where

$$\tau_n = \inf\{t \geq 0, |M_t| + \langle M \rangle_t \geq n\},$$

we can argue with M and $\langle M \rangle$ bounded, and obtain the general case by monotone convergence. Likewise, we work on $[0, t]$, and let $t \rightarrow \infty$ by monotone convergence.

Step 1: upper bound for $p \geq 2$. The function $\varphi(x) := |x|^p$ is of class $\mathcal{C}^2$. By Itô's formula,

$$|M_t|^p = \int_0^t p |M_s|^{p-1} \operatorname{sgn}(M_s) dM_s + \frac{1}{2} \int_0^t p(p-1) |M_s|^{p-2} d\langle M \rangle_s.$$

The stochastic integral is a true martingale, since its integrand and $\langle M \rangle$ are bounded. Hence

$$\begin{aligned} \mathbb{E}[|M_t|^p] &= \frac{p(p-1)}{2} \mathbb{E}\left[\int_0^t |M_s|^{p-2} d\langle M \rangle_s\right] \leq \frac{p(p-1)}{2} \mathbb{E}\left[(M_t^*)^{p-2} \langle M \rangle_t\right] \\ &\leq \frac{p(p-1)}{2} \mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}} \mathbb{E}\left[\langle M \rangle_t^{p/2}\right]^{\frac{2}{p}}, \end{aligned}$$

by Hölder's inequality (with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$). By Doob's inequality,

$$\mathbb{E}\left[(M_t^*)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_t|^p] \leq \left(\frac{p}{p-1}\right)^p \frac{p(p-1)}{2} \mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}} \mathbb{E}\left[\langle M \rangle_t^{p/2}\right]^{\frac{2}{p}}$$

(using what we just proved). Divide by $\mathbb{E}[(M_t^*)^p]^{\frac{p-2}{p}}$ on both sides (which is finite; the inequality is trivial if it is 0), and observe that $1 - \frac{p-2}{p} = \frac{2}{p}$. Letting $t \rightarrow \infty$ by monotone convergence, we get

$$\mathbb{E}\left[(M_\infty^*)^p\right] \leq C'_p \mathbb{E}\left[\langle M \rangle_\infty^{p/2}\right], \quad \text{with} \quad C'_p = \left(\frac{p}{p-1}\right)^{p^2/2} \left(\frac{p(p-1)}{2}\right)^{p/2}.$$

Step 2: lower bound for $p \geq 4$. By the integration by parts formula,

$$\langle M \rangle_t = M_t^2 + L_t, \quad L_t = -2 \int_0^t M_s dM_s,$$

and, by convexity of $x \mapsto |x|^{p/2}$ (since $\frac{p}{2} \geq 1$), $|a + b|^{p/2} \leq 2^{\frac{p}{2}-1}(|a|^{p/2} + |b|^{p/2})$. Hence

$$\mathbb{E}\left[\langle M \rangle_t^{p/2}\right] \leq 2^{\frac{p}{2}-1} \left(\mathbb{E}[|M_t|^p] + \mathbb{E}[|L_t|^{p/2}]\right)$$

$$\leq 2^{\frac{p}{2}-1} \left(\mathbb{E} \left[(M_t^*)^p \right] + C'_{p/2} \mathbb{E} \left[\langle L \rangle_t^{p/4} \right] \right),$$

by Step 1 applied to the (bounded) local martingale L with the exponent $\frac{p}{2} \geq 2$.
Now

$$\langle L \rangle_t = 4 \int_0^t M_s^2 d\langle M \rangle_s \leq 4 (M_t^*)^2 \langle M \rangle_t,$$

so that

$$2^{\frac{p}{2}-1} C'_{p/2} \mathbb{E} \left[\langle L \rangle_t^{p/4} \right] \leq A_p \mathbb{E} \left[(M_t^*)^{p/2} \langle M \rangle_t^{p/4} \right], \quad A_p = 2^{p-1} C'_{p/2}.$$

Using $A_p xy \leq \frac{1}{2} x^2 + \frac{A_p^2}{2} y^2$ with $x = \langle M \rangle_t^{p/4}$ and $y = (M_t^*)^{p/2}$, and putting everything together, we obtain the key inequality

$$\mathbb{E} \left[\langle M \rangle_t^{p/2} \right] \leq \frac{1}{2} \mathbb{E} \left[\langle M \rangle_t^{p/2} \right] + \left(2^{\frac{p}{2}-1} + \frac{A_p^2}{2} \right) \mathbb{E} \left[(M_t^*)^p \right].$$

Since $\mathbb{E}[\langle M \rangle_t^{p/2}] < \infty$ ($\langle M \rangle$ being bounded), it follows that $\mathbb{E}[\langle M \rangle_t^{p/2}] \leq (2^{p/2} + A_p^2) \mathbb{E}[(M_t^*)^p]$, and, letting $t \rightarrow \infty$,

$$c'_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right] \leq \mathbb{E} \left[(M_\infty^*)^p \right], \quad c'_p = \frac{1}{2^{p/2} + A_p^2}.$$

So far, we have proved

$$\underbrace{c'_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right] \leq \mathbb{E} \left[(M_\infty^*)^p \right]}_{p \geq 4} \quad \text{and} \quad \underbrace{\mathbb{E} \left[(M_\infty^*)^p \right] \leq C'_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right]}_{p \geq 2}.$$

□