

Chapter 6: Itô's formula and applications

Outline

1. Itô's formula
2. Exponential (semi)martingales
3. Lévy's characterization of Brownian motion
4. The Dubins–Schwarz theorem
5. Burkholder–Davis–Gundy inequality
6. Brownian martingales
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1. Itô's formula

Outline

- ▶ Itô's formula, one-dimensional and multidimensional
- ▶ Proof
- ▶ Extension to an open set
- ▶ Examples: time-dependent functions, d -dimensional Brownian motion

Itô's formula

We work on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.

Theorem 1 (Itô's formula, $d = 1$)

Let X be a continuous semimartingale, and $F : \mathbb{R} \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$. Then, almost surely, for every $t \geq 0$,

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2} \int_0^t F''(X_s) d\langle X \rangle_s.$$

- ▶ $F'(X)$ is continuous adapted, hence locally bounded.
- ▶ $F(X)$ is a continuous semimartingale.

Itô's formula: multidimensional version

Theorem 2 (Itô's formula, $d \geq 1$)

Let $X = (X^1, \dots, X^d)$, where each X^i is a continuous semimartingale, and $F : \mathbb{R}^d \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$. Then, almost surely, for every $t \geq 0$,

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{i=1}^d \int_0^t \partial_i F(X_s) dX_s^i \\ &+ \frac{1}{2} \sum_{1 \leq i, j \leq d} \int_0^t \partial_{ij} F(X_s) d\langle X^i, X^j \rangle_s. \end{aligned}$$

Proof (1/5): Taylor's formula

Fix $t > 0$, and $\Delta_n = (0 = t_0^n < \cdots < t_{p_n}^n = t)$ with $\Delta_n \subset \Delta_{n+1}$, $|\Delta_n| \rightarrow 0$.

Telescopic sum and Taylor's formula at order 2:

$$\begin{aligned} F(X_t) = F(X_0) + \sum_{i=1}^{p_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) \\ + \frac{1}{2} \sum_{i=1}^{p_n} f_{n,i}(X_{t_i^n} - X_{t_{i-1}^n})^2, \end{aligned}$$

where $f_{n,i}$ is a random variable such that

$$\inf_{x \in I_{n,i}} F''(x) \leq f_{n,i} \leq \sup_{x \in I_{n,i}} F''(x), \quad I_{n,i} = \llbracket X_{t_{i-1}^n}, X_{t_i^n} \rrbracket,$$

with $\llbracket a, b \rrbracket = [a \wedge b, a \vee b]$. For instance

$$f_{n,i} = 2 \int_0^1 (1 - \theta) F''(X_{t_{i-1}^n} + \theta(X_{t_i^n} - X_{t_{i-1}^n})) d\theta.$$

Proof (2/5): reduction

By the convergence of Riemann sums (Chapter 5):

$$\sum_{i=1}^{p_n} F'(X_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t F'(X_s) dX_s.$$

So the result follows from

$$(*) \quad \sum_{i=1}^{p_n} f_{n,i}(X_{t_i^n} - X_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t F''(X_s) d\langle X \rangle_s,$$

which gives the formula a.s. for each fixed t , and from the continuity of both sides in t .

Proof (3/5): decomposition

Let $m < n$, so $\Delta_m \subset \Delta_n$. Write $\sum_{i:j}$ for the sum over i such that $t_{j-1}^m \leq t_{i-1}^n < t_i^n \leq t_j^m$. Then

$$\sum_{i=1}^{p_n} f_{n,i} (X_{t_i^n} - X_{t_{i-1}^n})^2 - \int_0^t F''(X_s) d\langle X \rangle_s = I_{n,m} + II_{n,m} + III_m,$$

$$I_{n,m} = \sum_{j=1}^{p_m} \sum_{i:j} \left(f_{n,i} - F''(X_{t_{j-1}^m}) \right) (X_{t_i^n} - X_{t_{i-1}^n})^2,$$

$$II_{n,m} = \sum_{j=1}^{p_m} F''(X_{t_{j-1}^m}) \left[\sum_{i:j} (X_{t_i^n} - X_{t_{i-1}^n})^2 - \left(\langle X \rangle_{t_j^m} - \langle X \rangle_{t_{j-1}^m} \right) \right],$$

$$III_m = \sum_{j=1}^{p_m} F''(X_{t_{j-1}^m}) \left(\langle X \rangle_{t_j^m} - \langle X \rangle_{t_{j-1}^m} \right) - \int_0^t F''(X_s) d\langle X \rangle_s.$$

Proof (4/5): the three terms

- m fixed: $I_{n,m} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$ (quadratic variation on each $[t_{j-1}^m, t_j^m]$).

- $|I_{n,m}| \leq Z_m C_n$, where

$$C_n = \sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \langle X \rangle_t,$$

$$Z_m = \sup \left\{ |F''(X_u) - F''(X_v)|, u, v \in [0, t], |u - v| \leq |\Delta_m| \right\} \xrightarrow[m \rightarrow \infty]{\text{a.s.}} 0$$

($f_{n,i}$ lies between two values of $F''(X_u)$, $u \in [t_{i-1}^n, t_i^n]$, and $F''(X)$ is uniformly continuous on $[0, t]$).

- $III_m \rightarrow 0$ a.s.: Riemann sum of a Stieltjes integral, continuous integrand and integrator.

Proof (5/5): conclusion

For $\varepsilon > 0$, $\eta > 0$, and every m , the probability of the event $\{|\dots| \geq \varepsilon\}$ in (*) satisfies

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}(|\dots| \geq \varepsilon) &\leq \limsup_{n \rightarrow \infty} \mathbb{P}(|I_{n,m}| \geq \tfrac{\varepsilon}{3}) \\ &\quad + \underbrace{\limsup_{n \rightarrow \infty} \mathbb{P}(|II_{n,m}| \geq \tfrac{\varepsilon}{3}) + \mathbb{P}(|III_m| \geq \tfrac{\varepsilon}{3})}_{=0}, \end{aligned}$$

For $\eta > 0$,

$$\mathbb{P}(|I_{n,m}| \geq \tfrac{\varepsilon}{3}) \leq \mathbb{P}(Z_m \geq \eta) + \mathbb{P}(C_n \geq \tfrac{\varepsilon}{3\eta}).$$

and for small enough η

$$\limsup_{n \rightarrow \infty} \mathbb{P}(C_n \geq \tfrac{\varepsilon}{3\eta}) \leq \mathbb{P}(\langle X \rangle_t \geq \tfrac{\varepsilon}{6\eta}).$$

Let $m \rightarrow \infty$, then $\eta \rightarrow 0$.

□

Proof of the multidimensional version

Taylor's formula:

$$\begin{aligned} F(X_{t_i^n}) - F(X_{t_{i-1}^n}) &= \sum_{k=1}^d \partial_k F(X_{t_{i-1}^n}) (X_{t_i^n}^k - X_{t_{i-1}^n}^k) \\ &\quad + \frac{1}{2} \sum_{1 \leq k, l \leq d} f_{n,i}^{k,l} (X_{t_i^n}^k - X_{t_{i-1}^n}^k) (X_{t_i^n}^l - X_{t_{i-1}^n}^l), \end{aligned}$$

with $f_{n,i}^{k,l}$ between the inf and sup of $\partial_{kl}F$ on the box

$$I_{n,i} = \prod_k \llbracket X_{t_{i-1}^n}^k, X_{t_i^n}^k \rrbracket.$$

Same proof as for $d = 1$, using

$$\sum_{i=1}^{p_n} f_{n,i}^{k,l} (X_{t_i^n}^k - X_{t_{i-1}^n}^k) (X_{t_i^n}^l - X_{t_{i-1}^n}^l) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t \partial_{kl} F(X_s) d\langle X^k, X^l \rangle_s,$$

with $\langle X^k, X^l \rangle$ instead of $\langle X \rangle$.

□

Remarks

Remark 3

1. *With $d = 2$ and $F(x, y) = xy$, we recover the integration by parts formula:*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t.$$

2. *Let $\mathcal{O} \subset \mathbb{R}^d$ be open, $F : \mathcal{O} \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$, and assume that, a.s., $X_t \in \mathcal{O}$ for every $t \geq 0$. Then Ito's formula holds in the same setting.*

Proof of 2.

$\mathcal{O}_n = \{x \in \mathbb{R}^d, d(x, \mathcal{O}^c) \geq \frac{1}{n}\}$, and $F_n : \mathbb{R}^d \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$ with $F_n = F$ on a neighbourhood of $\mathcal{O}_n$.

- ▶ Itô's formula holds with F_n .
- ▶ $\tau_n = \inf\{t \geq 0, X_t \notin \mathcal{O}_n\}$: for $t \in [0, \tau_n]$, $X_t \in \mathcal{O}_n$, so F_n and F and their derivatives coincide at X_t .
- ▶ Hence Itô's formula with F holds on $[0, \tau_n]$.
- ▶ $\tau_n \rightarrow \infty$ a.s., since for every $t \geq 0$,

$$\inf_{s \in [0, t]} d(X_s, \mathcal{O}^c) > 0 \quad \text{a.s.}$$

$(s \mapsto d(X_s, \mathcal{O}^c))$ is continuous and positive on the compact $[0, t]$. □

Example: time-dependent functions

Let B be a one-dimensional $(\mathcal{F}_t)$ -Brownian motion, $B_0 \in \mathcal{F}_0$, and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$. Itô's formula for $X_t = (t, B_t)$:

$$F(t, B_t) = F(0, B_0) + \int_0^t \frac{\partial F}{\partial x}(s, B_s) dB_s + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \right)(s, B_s) ds$$

(since $\langle \text{Id}, B \rangle = 0$ and $\langle \text{Id} \rangle = 0$).

Hence, if

$$\boxed{\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} = 0}$$

then $F(t, B_t)$ is a **local martingale**.

For instance: $x^2 - t$, $x^3 - 3tx$, Hermite polynomials...

Example: d -dimensional Brownian motion

Let $B = (B^1, \dots, B^d)$ be a d -dimensional $(\mathcal{F}_t)$ -Brownian motion.

$\langle B^i, B^j \rangle = 0$ for $i \neq j$: for $s \leq t$, $B^i B^j$ is a martingale: indeed, write

$$B_t^i B_t^j = (B_s^i + (B_t^i - B_s^i))(B_s^j + (B_t^j - B_s^j))$$

so that

$$\mathbb{E}[B_t^i B_t^j \mid \mathcal{F}_s] = B_s^i B_s^j + 0 + 0 + \mathbb{E}[(B_t^i - B_s^i)(B_t^j - B_s^j) \mid \mathcal{F}_s] = B_s^i B_s^j.$$

Hence, for $F : \mathbb{R}^d \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$,

$$\begin{aligned} F(B_t) &= F(B_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(B_s) dB_s^i + \frac{1}{2} \int_0^t \Delta F(B_s) ds \\ &= F(B_0) + \int_0^t \nabla F(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta F(B_s) ds. \end{aligned}$$

Exercise: exit from an annulus

Exercise

Let B be a d -dimensional Brownian motion, $d \geq 2$, with $B_0 = x \neq 0$, and, for $a > 0$,

$$T_a = \inf\{t \geq 0, |B_t| = a\}.$$

For $0 < r < |x| < R$, compute $\mathbb{P}(T_r < T_R)$.

Hint: apply Itô's formula on $\mathcal{O} = \mathbb{R}^d \setminus \{0\}$ to a radial function F with $\Delta F = 0$ on $\mathcal{O}$.

2. Exponential (semi)martingales

Outline

- ▶ The equation $Z_t = e^{X_0} + \int_0^t Z_s dX_s$
- ▶ Existence and uniqueness, explicit formula
- ▶ Complex parameters

Exponential semimartingale

Theorem 4

Let X be a continuous semimartingale. There exists a unique continuous semimartingale Z such that

$$Z_t = e^{X_0} + \int_0^t Z_s dX_s, \quad t \geq 0.$$

Notation $Z = \mathcal{E}(X)$, with the explicit representation

$$\mathcal{E}(X)_t = \exp \left(X_t - \frac{1}{2} \langle X \rangle_t \right).$$

If X is a local martingale, so is $\mathcal{E}(X)$.

Example: $X_t = \theta B_t$, $\mathcal{E}(X)_t = \exp \left(\theta B_t - \frac{\theta^2}{2} t \right)$.

Proof: existence

Itô's formula with $F(x) = e^x$, applied to the semimartingale $X - \frac{1}{2}\langle X \rangle$, whose bracket is $\langle X \rangle$:

$$\begin{aligned} e^{X_t - \frac{1}{2}\langle X \rangle_t} &= e^{X_0} + \int_0^t e^{X_s - \frac{1}{2}\langle X \rangle_s} d\left(X_s - \frac{1}{2}\langle X \rangle_s\right) \\ &\quad + \frac{1}{2} \int_0^t e^{X_s - \frac{1}{2}\langle X \rangle_s} d\langle X \rangle_s \\ &= e^{X_0} + \int_0^t e^{X_s - \frac{1}{2}\langle X \rangle_s} dX_s. \end{aligned}$$

Proof: uniqueness

With $Y_t = \exp \left(-X_t + \frac{1}{2} \langle X \rangle_t \right)$, Itô's formula gives

$$Y_t = e^{-X_0} - \int_0^t Y_s dX_s + \int_0^t Y_s d\langle X \rangle_s.$$

Let $\tilde{Z}$ be a solution. Integration by parts:

$$\begin{aligned} Y_t \tilde{Z}_t &= 1 + \int_0^t Y_s d\tilde{Z}_s + \int_0^t \tilde{Z}_s dY_s + \langle \tilde{Z}, Y \rangle_t \\ &= 1 + \int_0^t Y_s \tilde{Z}_s dX_s + \int_0^t \tilde{Z}_s \left(-Y_s dX_s + Y_s d\langle X \rangle_s \right) \\ &\quad - \int_0^t Y_s \tilde{Z}_s d\langle X \rangle_s = 1. \end{aligned}$$

Hence $\tilde{Z}_t = \frac{1}{Y_t} = \mathcal{E}(X)_t$.

□

Complex parameters

Corollary 5

Let M be a real-valued local martingale and $\lambda \in \mathbb{C}$. Then

$$\mathcal{E}(\lambda M)_t = \exp\left(\lambda M_t - \frac{\lambda^2}{2} \langle M \rangle_t\right)$$

is a $\mathbb{C}$ -valued local martingale (real and imaginary parts are local martingales), and

$$\mathcal{E}(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \mathcal{E}(\lambda M)_s dM_s.$$

Proof.

Itô's formula for the real and imaginary parts of

$F(x, y) = e^{\lambda x - \frac{\lambda^2}{2} y}$, applied to $(M_t, \langle M \rangle_t)$. □

3. Lévy's characterization of Brownian motion

Outline

- ▶ A local martingale with bracket t is a Brownian motion
- ▶ Proof via the complex exponential martingale
- ▶ Example: $\int_0^t \operatorname{sgn}(B_s) dB_s$

Lévy's characterization

If $X = (X^1, \dots, X^d)$ is an $(\mathcal{F}_t)$ -Brownian motion in $\mathbb{R}^d$, then $\langle X^i, X^j \rangle_t = \delta_{ij} t$. The converse is true:

Theorem 6 (Lévy)

Let $M^1, \dots, M^d$ be continuous local martingales, with $M_0^i = 0$, such that $\langle M^i, M^j \rangle_t = t \delta_{ij}$. Then $M = (M^1, \dots, M^d)$ is an $(\mathcal{F}_t)_{t \geq 0}$ -Brownian motion.

In particular, a local martingale M with $M_0 = 0$ and $\langle M \rangle_t = t$ is a Brownian motion.

Proof (1/2)

Let $\xi \in \mathbb{R}^d$. $N_t := \xi \cdot M_t$ is a local martingale with

$$\langle N \rangle_t = \sum_{j,k} \xi_j \xi_k \langle M^j, M^k \rangle_t = |\xi|^2 t.$$

By Corollary 5 with $\lambda = i$,

$$\mathcal{E}(iN)_t = \exp \left(i\xi \cdot M_t + \frac{|\xi|^2}{2} t \right)$$

is a $\mathbb{C}$ -valued local martingale, and

$$\sup_{0 \leq s \leq t} |\mathcal{E}(iN)_s| \leq \exp \left(\frac{|\xi|^2}{2} t \right) \in L^1(\mathbb{P}).$$

Hence $(\mathbb{E}[\sup_{s \leq t} |M_s|] < \infty \Rightarrow \text{true martingale})$ $\mathcal{E}(iN)$ is a **true martingale**.

Proof (2/2)

So, for $s < t$, $\mathbb{E}[\mathcal{E}(iN)_t \mid \mathcal{F}_s] = \mathcal{E}(iN)_s$, i.e.

$$\mathbb{E}\left[e^{i\xi \cdot (M_t - M_s)} \mid \mathcal{F}_s\right] = e^{-\frac{|\xi|^2}{2}(t-s)}, \quad \forall \xi \in \mathbb{R}^d.$$

For $A \in \mathcal{F}_s$ with $\mathbb{P}(A) > 0$: the characteristic function of $M_t - M_s$ under $\mathbb{P}(\cdot \mid A)$ is $e^{-\frac{|\xi|^2}{2}(t-s)}$. Hence

$$M_t - M_s \sim \mathcal{N}(0, (t-s) \mathbf{I}_d), \quad \text{independent of } \mathcal{F}_s,$$

and M is continuous and adapted, with $M_0 = 0$. It is a BM □

Example

Let B be an $(\mathcal{F}_t)$ -Brownian motion with $B_0 = 0$, and

$$\beta_t = \int_0^t \operatorname{sgn}(B_s) dB_s, \quad \operatorname{sgn}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x\leq 0\}}.$$

- ▶ $\operatorname{sgn}(B_s)$ is bounded and progressive: β is a continuous martingale.
- ▶ $\langle \beta \rangle_t = \int_0^t (\operatorname{sgn}(B_s))^2 ds = t$.

Hence β is an $(\mathcal{F}_t)$ -Brownian motion.

4. The Dubins–Schwarz theorem

Outline

- ▶ A continuous local martingale is a time-changed Brownian motion
- ▶ Intervals of constancy of M and $\langle M \rangle$
- ▶ Proof of the theorem

The Dubins–Schwarz theorem

Theorem 7 (Dubins–Schwarz)

Let M be a continuous local martingale such that $M_0 = 0$ and $\langle M \rangle_\infty = \infty$ a.s. Then there exists a Brownian motion β such that

$$M_t = \beta_{\langle M \rangle_t}, \quad t \geq 0.$$

Remark 8

1. β is not an $(\mathcal{F}_t)$ -Brownian motion in general, but a $(\mathcal{G}_r)$ -Brownian motion, where

$$\mathcal{G}_r = \mathcal{F}_{\tau_r}, \quad \tau_r = \inf\{t \geq 0, \langle M \rangle_t > r\}.$$

2. The assumption $\langle M \rangle_\infty = \infty$ can be relaxed, up to enlarging the probability space.
3. The proof is simpler if $t \mapsto \langle M \rangle_t$ is strictly increasing.

Intervals of constancy

Lemma 9

Let M be a local martingale with $M_0 = 0$. Then, almost surely, M and $\langle M \rangle$ have the same intervals of constancy:

$$\forall 0 \leq a < b : \quad M_t = M_a \quad \forall t \in [a, b] \quad \Longleftrightarrow \quad \langle M \rangle_b = \langle M \rangle_a.$$

Proof ($\Rightarrow$).

By continuity, it suffices to consider fixed (rational) $a < b$.

$\{M_t = M_a \quad \forall t \in [a, b]\} \subset \{\langle M \rangle_b = \langle M \rangle_a\}$: approximation of the bracket by the discrete quadratic variation (along an a.s. converging subsequence). □

Proof of the lemma ($\Leftarrow$)

Let $\tau = \inf\{t > a, \langle M \rangle_t \neq \langle M \rangle_a\}$, and

$$N_t := \int_0^t \mathbf{1}_{\{a < s \leq \tau\}} dM_s = M_{t \wedge \tau} - M_{t \wedge a}.$$

N is a local martingale with

$$\langle N \rangle_t = \int_0^t \mathbf{1}_{\{a < s \leq \tau\}} d\langle M \rangle_s = \langle M \rangle_{\tau \wedge t} - \langle M \rangle_{a \wedge t} = 0.$$

Hence $N = 0$, i.e. $M_{t \wedge \tau} = M_a$ for every $t \geq a$.

On $\{\langle M \rangle_b = \langle M \rangle_a\}$, $\tau \geq b$, hence $M_t = M_a$ for $t \in [a, b]$. □

Proof of the theorem (1/4): time change

For $r \geq 0$, let

$$\tau_r = \inf\{t \geq 0, \langle M \rangle_t > r\} < \infty \quad \text{a.s.}$$

- ▶ $r \mapsto \tau_r$ is non-decreasing and càdlàg, and for $r > 0$,

$$\tau_{r-} = \lim_{s \uparrow r} \tau_s = \inf\{t \geq 0, \langle M \rangle_t \geq r\}.$$

- ▶ For every $r > 0$, by continuity of $\langle M \rangle$:

$$\langle M \rangle_{\tau_{r-}} = r = \langle M \rangle_{\tau_r}.$$

Proof of the theorem (2/4): definition of β

$\mathcal{G}_r := \mathcal{F}_{\tau_r}$ is a filtration (satisfying the usual conditions). Let

$$\beta_r := M_{\tau_r} \quad ((\mathcal{G}_r)\text{-adapted}).$$

- $M_t = \beta_{\langle M \rangle_t}$, i.e. $M_t = M_{\tau_{\langle M \rangle_t}}$: M is constant on $[t, \tau_{\langle M \rangle_t}]$ by Lemma 9, since

$$\langle M \rangle_t = \langle M \rangle_{\tau_{\langle M \rangle_t}} \quad (\text{as } \langle M \rangle_{\tau_r} = r \text{ with } r = \langle M \rangle_t).$$

- $\beta_0 = M_{\tau_0} = 0$, since $0 = M_0 = M_{\tau_{\langle M \rangle_0}} = M_{\tau_0}$.

Proof of the theorem (3/4): continuity, martingale

- β is continuous: càdlàg (M continuous, $r \mapsto \tau_r$ càdlàg), and

$$\beta_{r-} = M_{\tau_{r-}} = M_{\tau_r} = \beta_r$$

by Lemma 9, since $\langle M \rangle_{\tau_{r-}} = \langle M \rangle_{\tau_r}$.

- M^{τ_r} is a local martingale with $\langle M^{\tau_r} \rangle_{\infty} = \langle M \rangle_{\tau_r} = r$ bounded, so

$$M^{\tau_r} \quad \text{and} \quad (M^{\tau_r})^2 - \langle M^{\tau_r} \rangle$$

are uniformly integrable martingales.

Proof of the theorem (4/4): conclusion

Optional stopping, for $s \leq r$ ($\tau_s \leq \tau_r$):

$$\mathbb{E}[\beta_r \mid \mathcal{G}_s] = \mathbb{E}[M_\infty^{\tau_r} \mid \mathcal{F}_{\tau_s}] = M_{\tau_s} = \beta_s,$$

$$\mathbb{E}[\beta_r^2 - r \mid \mathcal{G}_s] = \mathbb{E}[(M_\infty^{\tau_r})^2 - \langle M^{\tau_r} \rangle_\infty \mid \mathcal{F}_{\tau_s}] = \beta_s^2 - s.$$

So β is a continuous $(\mathcal{G}_r)$ -martingale with $\langle \beta \rangle_r = r$.

By Lévy's characterization: β is a $(\mathcal{G}_r)$ -Brownian motion. □

5. Burkholder–Davis–Gundy inequality

Outline

- ▶ Statement: $\mathbb{E}[(M^*)^p]$ is comparable to $\mathbb{E}[\langle M \rangle^{p/2}]$
- ▶ Step 1: upper bound, $p \geq 2$
- ▶ Step 2: lower bound, $p \geq 4$
- ▶ A domination lemma, and the remaining values of p

BDG inequality

$$M_t^* = \sup_{s \in [0, t]} |M_s|, \quad M_\infty^* = \sup_{s \geq 0} |M_s|.$$

Theorem 10 (BDG)

For every $p > 0$, there exist $0 < c_p < C_p$ such that, for every continuous local martingale M with $M_0 = 0$,

$$c_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right] \leq \mathbb{E} \left[(M_\infty^*)^p \right] \leq C_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right].$$

- ▶ If τ is a stopping time, $M \mapsto M^\tau$ gives the same with ∞ replaced by τ .
- ▶ $p = 2$: Itô's isometry and Doob's inequality.

Proof: localization

Replacing M by M^{τ_n} ,

$$\tau_n = \inf\{t \geq 0, |M_t| + \langle M \rangle_t \geq n\},$$

we may assume M and $\langle M \rangle$ bounded; the general case follows by monotone convergence.

Likewise, we work on $[0, t]$ and let $t \rightarrow \infty$.

Step 1: upper bound, $p \geq 2$ (1/2)

$\varphi(x) = |x|^p$ is $\mathcal{C}^2$. Itô's formula:

$$|M_t|^p = \int_0^t p |M_s|^{p-1} \operatorname{sgn}(M_s) dM_s + \frac{1}{2} \int_0^t p(p-1) |M_s|^{p-2} d\langle M \rangle_s.$$

The stochastic integral is a true martingale (bounded integrand and bracket). Hence

$$\begin{aligned} \mathbb{E}[|M_t|^p] &= \frac{p(p-1)}{2} \mathbb{E}\left[\int_0^t |M_s|^{p-2} d\langle M \rangle_s\right] \\ &\leq \frac{p(p-1)}{2} \mathbb{E}\left[(M_t^*)^{p-2} \langle M \rangle_t\right] \\ &\leq \frac{p(p-1)}{2} \mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}} \mathbb{E}\left[\langle M \rangle_t^{p/2}\right]^{\frac{2}{p}} \end{aligned}$$

(Hölder with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$).

Step 1: upper bound, $p \geq 2$ (2/2)

Doob's inequality:

$$\begin{aligned}\mathbb{E}\left[(M_t^*)^p\right] &\leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_t|^p] \\ &\leq \left(\frac{p}{p-1}\right)^p \frac{p(p-1)}{2} \mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}} \mathbb{E}\left[\langle M \rangle_t^{p/2}\right]^{\frac{2}{p}}.\end{aligned}$$

Divide by $\mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}}$; note $1 - \frac{p-2}{p} = \frac{2}{p}$. Letting $t \rightarrow \infty$:

$$\begin{aligned}\mathbb{E}\left[(M_\infty^*)^p\right] &\leq C'_p \mathbb{E}\left[\langle M \rangle_\infty^{p/2}\right], \\ C'_p &= \left(\frac{p}{p-1}\right)^{p^2/2} \left(\frac{p(p-1)}{2}\right)^{p/2}.\end{aligned}$$

Step 2: lower bound, $p \geq 4$ (1/2)

Integration by parts:

$$\langle M \rangle_t = M_t^2 + L_t, \quad L_t = -2 \int_0^t M_s dM_s.$$

Convexity ($\frac{p}{2} \geq 1$): $|a + b|^{p/2} \leq 2^{\frac{p}{2}-1} (|a|^{p/2} + |b|^{p/2})$. Hence

$$\begin{aligned} \mathbb{E} \left[\langle M \rangle_t^{p/2} \right] &\leq 2^{\frac{p}{2}-1} \left(\mathbb{E} [|M_t|^p] + \mathbb{E} [|L_t|^{p/2}] \right) \\ &\leq 2^{\frac{p}{2}-1} \left(\mathbb{E} \left[(M_t^*)^p \right] + C'_{p/2} \mathbb{E} \left[\langle L \rangle_t^{p/4} \right] \right), \end{aligned}$$

by Step 1 applied to L with exponent $\frac{p}{2} \geq 2$.

Moreover

$$\langle L \rangle_t = 4 \int_0^t M_s^2 d\langle M \rangle_s \leq 4 (M_t^*)^2 \langle M \rangle_t.$$

Step 2: lower bound, $p \geq 4$ (2/2)

With $A_p = 2^{p-1}C'_{p/2}$:

$$2^{\frac{p}{2}-1}C'_{p/2} \mathbb{E} \left[\langle L \rangle_t^{p/4} \right] \leq A_p \mathbb{E} \left[(M_t^*)^{p/2} \langle M \rangle_t^{p/4} \right].$$

Using $A_p xy \leq \frac{1}{2}x^2 + \frac{A_p^2}{2}y^2$, with $x = \langle M \rangle_t^{p/4}$, $y = (M_t^*)^{p/2}$, we get the key inequality

$$\mathbb{E} \left[\langle M \rangle_t^{p/2} \right] \leq \frac{1}{2} \mathbb{E} \left[\langle M \rangle_t^{p/2} \right] + \left(2^{\frac{p}{2}-1} + \frac{A_p^2}{2} \right) \mathbb{E} \left[(M_t^*)^p \right].$$

Since $\mathbb{E}[\langle M \rangle_t^{p/2}] < \infty$, letting $t \rightarrow \infty$:

$$c'_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right] \leq \mathbb{E} \left[(M_\infty^*)^p \right], \quad c'_p = \frac{1}{2^{p/2} + A_p^2}.$$

So far

$$\underbrace{c'_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right] \leq \mathbb{E} \left[(M_\infty^*)^p \right]}_{p \geq 4}$$

$$\underbrace{\mathbb{E} \left[(M_\infty^*)^p \right] \leq C'_p \mathbb{E} \left[\langle M \rangle_\infty^{p/2} \right]}_{p \geq 2}$$

It remains to treat small values of p : a domination lemma.