
Sheet 2. Brownian motion. Markovian properties

Denote by $B = (B_t)_{t \geq 0}$ the Brownian motion started at the origin and by $\mathcal{F}_t = \sigma(B_s, s \leq t)$ the associated filtration. For all stopping time τ , we define $\mathcal{F}_\tau = \sigma(B_s, s \leq \tau)$. For all $a \in \mathbb{R}$, τ_a denotes the first hitting time of a , that is $\tau_a = \inf\{t \geq 0 : B_t = a\}$.

Exercise 1 Prove that almost surely, $\sup_{0 \leq t \leq s} B_t > 0$ for all $s > 0$ and $\sup_{t \geq 0} B_t = \infty$.

Exercise 2 1. Show that for all $a \geq 0$, $\tau_a = a^2 \tau_1$ in distribution.

2. Let $0 \leq a \leq b < \infty$. Justify that $\tau_b - \tau_a$ has the same distribution as τ_{b-a} and is independent of $\mathcal{F}_{\tau_a}$.

Exercise 3 Show that almost surely,

$$\liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty, \quad \limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty.$$

Deduce that for all $s > 0$, B is almost surely not right differentiable at s .

Exercise 4 Let $(t_n)_{n \geq 1}$ be a sequence of positive real numbers decreasing towards 0. Prove that a.s., $B_{t_n} > 0$ for infinitely many n , and $B_{t_n} < 0$ for infinitely many n .

Exercise 5 Let $\tau = \inf\{t \geq 0 : B_t = 1\}$ and $\eta = \inf\{t \geq \tau : B_t = 0\}$.

1. Is η a stopping time?
2. Determine the distribution of η .

Exercise 6 (Arcsine law) Let $\tau = \inf\{t \geq 1 : B_t = 0\}$ and $\eta = \sup\{0 \leq t \leq 1 : B_t = 0\}$.

1. Is τ a stopping time? Is η a stopping time?
2. Determine the law of τ and deduce that $P(\eta \leq t) = \frac{2}{\pi} \arcsin(t^{1/2})$.

Exercise 7 Let $S_t = \sup_{s \in [0, t]} B_s$. Prove that $S_2 - S_1$ is distributed as $\max\{|N| - |N'|, 0\}$, where N and N' are independent Gaussian $\mathcal{N}(0, 1)$ random variables.

Exercise 8 Prove that as $t \rightarrow \infty$, $(\int_0^t e^{B_s} ds)^{1/t^{1/2}} \rightarrow e^{|N|}$ in distribution, where N is a Gaussian $\mathcal{N}(0, 1)$ random variable.

Exercise 9 Prove that $\int_0^\infty e^{-|B_t|^2} dt = \infty$ a.s.

Exercise 10 (Zeros of Brownian motion) Let $H := \{t \in [0, 1] : B_t = 0\}$. Using the strong Markov property, show that H is a.s. a compact set without any isolated point and of zero Lebesgue measure.

Exercise 11 (Local maxima) Show that the local maxima of Brownian motion are distinct: for any given $p < q < r < s$,

$$\sup_{p \leq t \leq q} B_t \neq \sup_{r \leq t \leq s} B_t \quad a.s.$$

Exercise 12 (Law of the iterated logarithms) Let $S_t = \sup_{s \in [0, t]} B_s$ and $h(t) = \sqrt{2t \log \log t}$.

1. Let $\varepsilon > 0$. Prove that $\sum_n P(S_{t_{n+1}} \geq (1 + \varepsilon)h(t_n))$ converges, where $t_n = (1 + \varepsilon)^n$. Deduce that $\limsup_{t \rightarrow \infty} \frac{S_t}{h(t)} \leq 1$ a.s.
2. Show that

$$\limsup_{t \rightarrow \infty} \frac{\sup_{s \in [0, t]} |B_s|}{h(t)} \leq 1 \quad a.s.$$

3. Let $\theta > 1$, and let $s_n = \theta^n$. Prove that for all $\alpha \in (0, \sqrt{1 - \theta^{-1}})$, the sum $\sum_n P(B_{s_n} - B_{s_{n-1}} > \alpha h(s_n))$ diverges. Deduce that $\limsup_{t \rightarrow \infty} \frac{B_t}{h(t)} \geq \alpha - \frac{2}{\sqrt{\theta}}$ a.s.
4. Justify that
- $$\limsup_{t \rightarrow \infty} \frac{B_t}{h(t)} = 1 \quad a.s.$$
5. Let $X_1(t) = |B_t|$, $X_2(t) = S_t$ and $X_3(t) = \sup_{s \in [0, t]} |B_s|$. What can be said about $\limsup_{t \rightarrow \infty} \frac{X_i(t)}{h(t)}$ for $i = 1, 2, 3$.
6. What can be said about $\liminf_{t \rightarrow \infty} \frac{B_t}{h(t)}$ and $\limsup_{t \rightarrow 0} \frac{B_t}{\sqrt{2t \log \log(1/t)}}$.