Max-min optimization problem for variable annuities pricing

Christophette BLANCHET-SCALLIET † Etienne CHEVALIER ‡
Idris Kharroubi§ Thomas LIM¶

Abstract

We study the valuation of variable annuities for an insurer. We concentrate on two types of these contracts, namely guaranteed minimum death benefits and guaranteed minimum living benefits that allow the insured to withdraw money from the associated account. Here, the price of variable annuities corresponds to a fee, fixed at the beginning of the contract, that is continuously taken from the associated account. We use a utility indifference approach to determine the indifference fee rate. We focus on the worst case for the insurer, assuming that the insured makes the withdrawals that minimize the expected utility of the insurer. To compute this indifference fee rate, we link the utility maximization in the worst case for the insurer to a sequence of maximization and minimization problems that can be computed recursively. This allows to provide an optimal investment strategy for the insurer when the insured follows the worst withdrawal strategy and to compute the indifference fee. We finally explain how to approximate these quantities via the previous results and give numerical illustrations of parameter sensitivity.

Keywords: Variable annuities, insurance, indifference pricing, utility maximization, backward stochastic differential equation.


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†Université de Lyon, CNRS UMR 5208, Ecole Centrale de Lyon, Institut Camille Jordan, France, christophette.blanchet@ec-lyon.fr
‡Laboratoire de Mathématiques et Modélisation d’Evry (LaMME), Université d’Evry Val d’Essonne, UMR CNRS 8071, etienne.chevalier@univ-evry.fr.
§CEREMADE, Université Paris Dauphine, France, kharroubi@ceremade.dauphine.fr.
¶ENSIIE, Laboratoire de Mathématiques et Modélisation d’Evry (LaMME), UMR CNRS 8071, thomas.lim@ensiie.fr.
1 Introduction

Variable annuities are insurance contracts that were introduced in the 1970s in the United States (see Sloane [23]). These insurance products provide, during a given period, deferred annuities that are fund linked. More precisely, the policyholder gives at the beginning an initial amount of money to the insurer who invests this amount in a reference portfolio. In return, the insurer provides annuities that depend on the performance of this reference portfolio.

In the 1990s, insurers started adding some guarantees to these policies. They offer contracts with annuities that are at least greater than some guaranteed value. Nowadays, the most spread are Guaranteed Minimum Death Benefits (GMDB in short) and Guaranteed Minimum Living Benefits (GMLB in short) contracts. For a GMDB contract, the dependents of the insured obtain a guaranteed benefit if she dies before the maturity of the contract. On the contrary, the holder of a GMLB contract obtains a guaranteed benefit if she is still alive at the maturity of the contract. There are various ways to fix this guaranteed benefit and we refer to [4] for more details.

These newly added guarantees aroused investors’ interest in these products and made the variable annuities contracts highly demanded on financial markets. Therefore, their pricing and hedging have attracted a lot of interest in a growing literature. In their pioneering work, Boyle and Schwartz [9] used non-arbitrage models to extend the Black & Scholes framework to insurance issues. Then, Milevsky and Posner [18] applied risk neutral option pricing theory to value GMDB variable annuities. The case of withdrawal options is studied by Chu and Kwok in [11] and by Siu in [22], and a general framework to define variable annuities is presented by Bauer et al. in [4]. Milevsky and Salisbury studied in [20] the links between American put options and dynamic optimal withdrawal policies. In [5], Belanger et al. described the valuation of GMDB as an impulse control problem. They derive an HJB equation in a Markov framework and solve it numerically.

An important risk faced by the seller of a variable annuities contract concerns the characteristics of the buyer. The insurer has to take into account the behavior of the insured, i.e. her withdrawals, and her exit time from the contract, i.e. her death time. In this paper, we study a valuation of GMDB and GMLB contracts that takes into account this uncertainty on the insured. Concerning the behavior of the insured, we consider the worst case for the insured’s withdrawal strategy from the insurer point of view. Concerning the death time, we allow the death time intensity to be uncertain and to depend, for instance, on fundamental medical breakthroughs or natural disasters. This kind of unpredictable event could impact even large portfolios of policies and therefore this part of mortality risk is not diversifiable (see, for instance, [19]). Moreover, insured’s withdraw strategies may modify the mortality risk profile of the product (see [2] and [3]). Finally, in the case of indifference exponential utility pricing, it has been shown in [8] that diversification may not be consistent. Hence, we model the death time as a random time enlarging the initial filtration related to the market information which is a classical approach in credit risk. As such contracts are generally priced for a class of insured, we suppose that this random time corresponds to the death time of a representative agent in a specific class of clients that
satisfy several conditions (age, job, wealth,...). We shall assume that such a class is small enough to be unable to affect the market. From a probabilistic point of view, this justifies that the well known assumption (H) holds true, i.e. any martingale for the initial filtration remains a martingale for the initial filtration enlarged by the exit time.

Due to these risks coming from the insured, the market is incomplete and we have to choose a pricing definition. Following the approach of Chevalier et al. [10], we therefore consider the indifference pricing valuation with an exponential criterion, which is commonly considered to be pertinent for the study of insurance products (see [6]) since the insurance companies are risk-averse (see [13, 14, 24]). We suppose that the price of a variable annuities contract is defined as a continuous time fee rate $p$ that is taken from the related fund. The indifference pricing procedure consists in finding the fee rate $p$ for which $V^0 = V(p)$ where

- $V^0$ is the maximal expected utility that the insurer can get when it invests the related fund on the market,
- $V(p)$ is the maximal expected utility that the insurer can get when it invests the related fund on the market and sells a variable annuities contract at the price $p$.

The computation of $V^0$ is a classical problem that has been solved in [15] if the coefficients of the model do not depend on the death time and by [16] if these ones depend on the death time. However the computation of $V(p)$ is a challenging problem since it involves the behavior of the buyer of the variable annuities contract. Especially, we have to take into account the possible withdrawals that can be done by the insured. In [10], the insured withdrawal strategy is assumed to be a given random process. We use a more robust pricing approach by considering the worst case for the insurer, i.e. when the insured chooses the withdrawal strategy that minimizes the expected utility of the insurer. This leads us to study an optimization problem of max-min-type. Such a problem is in general difficult to solve due to the dependence of the maximizing strategy on the minimizing one. Here we take advantage of the multiplicative structure of the utility function to break this dependence. This allows us to transform the initial max-min problem into separated maximization and minimization problems on different time intervals. This decomposition allows then to give an optimal allocation strategy for the insurer and the worst withdrawals by the insured from the perspective of the insurer.

The rest of the paper is organized as follows. In Section 2, we present the probability space, and the financial market associated to our problem. In Section 3, we describe the variable annuities products and their indifference pricing valuation. We then show in Section 4 that the utility maximization in the worst case for the insurer can be reduced to a sequential utility maximization problem, and we provide the optimal strategy and worst withdrawals for the insurer. In Section 5 we explain how to solve numerically this indifference pricing and provide numerical examples. We conclude our study in Section 6 with giving an extension of our results to an optimization problem on several insureds. Finally, we postpone to the Appendix the proof of our main result (Theorem 4.1) and some additional results that are needed to complete this proof.
2 The model

2.1 Probability space

Let \((\Omega, \mathcal{G}, \mathbb{P})\) be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion \(B\) and we denote by \(\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}\) the right-continuous and complete filtration generated by \(B\). We consider on this space a random time \(\tau\), which represents the death time of a representative agent. The random time \(\tau\) is not assumed to be an \(\mathbb{F}\)-stopping time. We therefore use the standard approach of filtration enlargement by considering the smallest right-continuous extension \(\mathbb{G}\) of \(\mathbb{F}\) that turns \(\tau\) into a \(\mathbb{G}\)-stopping time. More precisely \(\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}\) is defined by

\[
\mathcal{G}_t := \bigcap_{\varepsilon > 0} \tilde{\mathcal{G}}_{t+\varepsilon},
\]

for any \(t \geq 0\), where \(\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(1_{\tau \leq u}, u \in [0, s])\), for any \(s \geq 0\).

We denote by \(\mathcal{P}(\mathbb{F})\) (resp. \(\mathcal{P}(\mathbb{G})\)) the \(\sigma\)-algebra of \(\mathbb{F}\) (resp. \(\mathbb{G}\))-predictable subsets of \(\Omega \times \mathbb{R}_+\), i.e. the \(\sigma\)-algebra generated by the left-continuous \(\mathbb{F}\) (resp. \(\mathbb{G}\))-adapted processes.

We impose the following assumption, which is classical in the filtration enlargement theory and is called H-hypothesis.

Assumption 2.1. The process \(B\) remains a \(\mathbb{G}\)-Brownian motion.

The interpretation of the H-hypothesis is an asymmetric dependence structure between \(B\) and \(\tau\). From a financial point of view, it means that the exit time \(\tau\) may depend on the financial market randomness represented by \(B\). On the contrary, the financial market does not depend on \(\tau\).

In the sequel \(N\) denotes the process \(1_{\tau \leq .}\).

Assumption 2.2. The process \(N\) admits a \(\mathbb{G}\)-compensator of the form \(\int_0^{\wedge \tau} \lambda_t dt\), i.e. \(N - \int_0^{\wedge \tau} \lambda_t dt\) is a \(\mathbb{G}\)-martingale, where \(\lambda\) is a positive bounded \(\mathcal{P}(\mathbb{F})\)-measurable process.

We denote by \(M\) the \(\mathbb{G}\)-martingale defined by

\[
M_t := N_t - \int_0^{t^{\wedge \tau}} \lambda_s ds
\]

for any \(t \geq 0\).

2.2 Financial market

We consider that the insurer can invest in a financial market which is composed of two assets. The first one is a riskless bond \(\hat{S}^0\) satisfying the following stochastic differential equation

\[
d\hat{S}^0_t = r_t \hat{S}^0_t dt, \quad t \geq 0, \quad \hat{S}^0_0 = 1,
\]

(2.1)
where \( r \) is a \( \mathcal{P}(\mathbb{F}) \)-measurable process representing the riskless interest rate. The second asset \( \hat{S} \) is a reference portfolio of risky assets underlying the variable annuities policy. \( \hat{S} \) is assumed to be solution of the linear stochastic differential equation
\[
d\hat{S}_t = \hat{S}_t(\mu_t dt + \sigma_t dB_t) \quad t \geq 0, \quad \hat{S}_0 = s \geq 0,
\]
where \( \mu \) and \( \sigma \) are \( \mathcal{P}(\mathbb{F}) \)-measurable processes.

To ensure the existence and uniqueness of the processes \( \hat{S}_0 \) and \( \hat{S} \), we make the following assumptions.

**Assumption 2.3.** The processes \( \mu, \sigma \) and \( r \) are bounded, and \( \sigma \) is lower bounded by a positive constant \( \sigma_0 \).

We shall denote by \( S \) the discounted value of \( \hat{S} \) and by \( \theta \) the risk premium of \( \hat{S} \)
\[
S_t := e^{-\int_0^t r_s ds} \hat{S}_t \quad \text{and} \quad \theta_t := \frac{\mu_t - r_t}{\sigma_t},
\]
for all \( t \in [0, T] \).

A \( \mathbb{G} \)-predictable process \( \pi = (\pi_t)_{0 \leq t \leq T} \) is called a trading strategy if \( \int \pi_t dS_t \) is well defined (for example if \( \int_0^T |\pi_t| \sigma_t|^2 dt < \infty \) \( \mathbb{P} \)-a.s.). The process \( \pi \) describes the discounted amount of money invested in the portfolio of risky assets. Assuming that the investment strategy is self-financing and denoting by \( X^{\pi}_t \) the discounted value of the insurer’s portfolio with initial capital 0 and following the strategy \( \pi \), we have
\[
X^{\pi}_t = \int_0^t \pi_s(\mu_s - r_s) ds + \int_0^t \pi_s \sigma_s dB_s \quad t \geq 0.
\]
We also denote by \( X^{\hat{S},\pi}_t \) the discounted wealth at time \( t \) when the initial capital at time \( s \) is 0 and the investment strategy is \( \pi \).

We consider an insurance company (or an insurer) with preferences given by the utility function \( U \) defined by
\[
U(y) := -e^{-\gamma y}, \quad y \in \mathbb{R},
\]
where \( \gamma \) is a positive constant.

**Remark 2.1.** Both theory and practice have shown that it is appropriate to use this utility function, see for example [6] when the company is risk-averse, which is the case for an insurance company (see for example [13, 14, 24]). It is relevant since optimal controls do not depend on the initial wealth of the insurer. Moreover an appealing feature of decision making using this utility function is that decisions are based on comparisons between moment generating functions, which capture all the characteristics of the random outcomes being compared, so that comparisons are based on a wide range of features.

In the following definition, we give insurer’s admissible strategies on a given random interval.
Definition 2.1. Let \( \nu_1 \) and \( \nu_2 \) be two \( \mathbb{G} \)-stopping times such that \( 0 \leq \nu_1 \leq \nu_2 \leq T \). The set \( \mathcal{A}[\nu_1, \nu_2] \) of admissible strategies on the stochastic interval \( [\nu_1, \nu_2] \) consists of all \( \mathbb{G} \)-predictable processes \( \pi = (\pi_t)_{0 \leq t \leq T} \) which satisfy
\[
\mathbb{E} \left[ \int_{\nu_1}^{\nu_2} |\pi_t|^2 \, dt \right] < +\infty ,
\]
and
\[
\{ \exp(-\gamma X_\pi^\nu), \nu \text{ is a stopping time such that } \nu_1 \leq \nu \leq \nu_2 \}
\]
is a uniformly integrable family.

3 Utility indifference pricing of variable annuities

3.1 Variable annuities

We consider a variable annuities product with a maturity \( T > 0 \). It is a deferred fund-linked annuity contract that we describe in the following lines.

Initial investment. The insured invests an initial capital, denoted by \( A_0 \), in the fund related to this product (also called insured account) at time \( t = 0 \).

Withdrawals. Let \( T := (t_i)_{0 \leq i \leq n} \) be the set of policy anniversary dates, with \( t_0 = 0 \) and \( t_n = T \). By convention we set \( t_{n+1} = +\infty \).

At any date \( t_i \), for \( i \in \{1, \ldots, n-1\} \), the insured, if she is still alive, is allowed to withdraw an amount of money. This should be lower than a bounded non-negative \( \mathbb{G}_{t_i} \)-measurable random variable \( \hat{G}_i \), which may depend on previous withdrawals, on previous account values, and on some guarantees determined in the policy.

We define \( \mathcal{W} \) as a finite subset of \([0, 1]\) which contains 0 and 1 and introduce the set of admissible withdrawal policies
\[
\hat{\mathcal{E}} = \left\{ (\alpha_i \hat{G}_i)_{1 \leq i \leq n-1} : \alpha_i \text{ is a } \mathbb{G}_{t_i} \text{-measurable random variable such that } \alpha_i \in \mathcal{W} \text{ for all } i \in \{1, \ldots, n-1\} \right\} .
\]

For \( \hat{\xi} \in \hat{\mathcal{E}} \) and \( i \in \{1, \ldots, n-1\} \), \( \hat{\xi}_i \) is the withdrawal made by the insured at time \( t_i \) and we introduce the family \( (\xi_i)_{1 \leq i \leq n-1} \) such that \( \xi_i := e^{-\int_0^{t_i} r_s \, ds} \hat{\xi}_i \) is the discounted withdrawal made at time \( t_i \). We define by \( \mathcal{E} \) the admissible discounted withdrawal policies with \( \xi \in \mathcal{E} \) if and only if the vector \( \hat{\xi} \in \hat{\mathcal{E}} \). For any \( k \in \{0, \ldots, n-2\} \) and \( i \in \{1, \ldots, n-k-1\} \), we also define the set \( \mathcal{E}_k^i \) by
\[
\mathcal{E}_k^i = \left\{ \xi \in \mathcal{E} \text{ s.t. } \xi_j = 0 \text{ for all } j \in \{k+1, \ldots, k+i\} \right\} .
\]
\( \mathcal{E}_k^i \) is the set of admissible withdrawal policies such that all withdrawals are made between times \( t_{k+1} \) and \( t_{k+i} \).
Dynamics of the related fund. We denote by $A_t^p$ the discounted value at time $t$ of the fund related to the variable annuities contract sold at fee rate $p$. If the insured follows the withdrawal policy $\hat{\xi} \in \hat{\mathcal{E}}$, we have

$$
\begin{align*}
\frac{dA_t^p}{dt} &= A_t^p[(\mu_t - r_t - p)dt + \sigma_t dB_t], \quad \text{for } t \not\in T, \\
A_{t_i}^p &= (A_{t_i}^p - f_i) \lor 0, \quad \text{for } 1 \leq i \leq n - 1,
\end{align*}
$$

(3.3)

where $f_i$ is a $\mathcal{G}_{t_i}$-measurable random variable greater than $\xi_i$ for any $i \in \{1, \ldots, n - 1\}$ and depending on previous withdrawals, on previous account values and on some guarantees determined in the policy. The simplest case would be to have $f_i = \xi_i$ but variable annuities contracts may be more complex. For instance, for a given withdrawal $\hat{\xi}_i$, the insurer may withdraw a larger amount of money from the insured account.

We now focus on the dependencies between $f_i$ and $\hat{\mathcal{G}}_i$, and begin with introducing two sets of functions defined on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$.

Let $\mathcal{I}$ (resp. $\mathcal{J}$) be the set of bounded, non-negative functions $\phi$ (resp. $\psi$) defined on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$ such that for any $i \in \{1, \ldots, n - 1\}$ and $(t, x, e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, the function $y \mapsto \phi(t, x, e)$ is non-increasing (resp. $y \mapsto \psi(t, x, e)$ is non-decreasing) and for any $j \in \{1, \ldots, n + 1\}$, the function $y \mapsto \phi(t, x, e)$ is non-decreasing (resp. $y \mapsto \psi(t, x, e)$ is non-increasing).

$\hat{\mathcal{G}}_i$ is the maximum amount that can be withdrawn at time $t_i$, hence it decreases with respect to previous withdrawals and increases with previous values of the fund related to the variable annuities contract. Hence, we assume that there exists $\hat{g} \in \mathcal{I}$ such that, for any $i \in \{1, \ldots, n - 1\}$, we have

$$
\hat{g} = \hat{g}(t_i, \hat{A}_{t_0}^p, \ldots, \hat{A}_{t_{i-1}}^p, \hat{A}_{t_i}^p, 0, \ldots, 0, \hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, 0, \ldots, 0),
$$

where $\hat{A}_t^p = e^{\int_0^t r_s ds} A_s^p$ for all $t \in [0, T]$.

In the same vein, the random variables $(f_i)_{1 \leq i \leq n-1}$ corresponds to penalties for early withdrawals. It seems reasonable to assume that they increase with previous withdrawals and, for marketing considerations, decrease with previous values of the fund. Therefore, we assume that there exists $\hat{f} \in \mathcal{J}$ such that, for any $i \in \{1, \ldots, n - 1\}$,

$$
f_i := f(t_i, \hat{A}_{t_0}^p, \ldots, \hat{A}_{t_{i-1}}^p, \hat{A}_{t_i}^p, 0, \ldots, 0, \hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, 0, \ldots, 0) = e^{\int_0^{t_i} r_s ds} \hat{f}(t_i, \hat{A}_{t_0}^p, \ldots, \hat{A}_{t_{i-1}}^p, \hat{A}_{t_i}^p, 0, \ldots, 0, \hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, 0, \ldots, 0).
$$

We give concrete examples of functions $\hat{g}$ and $f$ in the next subsection.

Pay off contract. The last quantity to define is the pay-off of the variable annuity. The pay-off is obviously increasing with respect to the values of the account and decreasing with respect to the withdrawals. Let $\hat{F}^L$ and $\hat{F}^D$ belong to $\mathcal{I}$, the pay off is paid at time $T \land \tau$ to the insured or her dependents, and is equal to the following random variable

$$
\hat{F}(p, \hat{\xi}) := \hat{F}^L(T, \hat{a}^p, \hat{\xi}) 1_{\{T < \tau\}} + \hat{F}^D(\tau, \hat{a}^p, \hat{\xi}) 1_{\{\tau \leq T\}},
$$

(3.4)
where \( \tilde{a}^p := (\tilde{A}^p_{t,\tau})_{0 \leq i \leq n} \). \( \hat{F}^L \) is the pay-off if the policyholder is alive at time \( T \) and \( \hat{F}^D \) is the pay-off if the policyholder is dead at time \( \tau \). Notice that \( \hat{F}(p, \xi) \) is \( G_{T \wedge \tau} \)-measurable.

In the following, we denote by \( F(p, \xi) \) the discounted pay-off defined by

\[
F(p, \xi) = e^{-\int_0^{T \wedge \tau} \kappa \, ds} \hat{F}(p, \xi).
\]

### 3.2 GMDB and GMLB contracts

Usual examples of variable annuity riders are GMDB and GMLB. In that case, we have to precisely define guarantees. We introduce \( \hat{G}^D \), \( \hat{G}^L \) and \( \hat{G}^W \) belonging to \( I \) such that, for any \( Q \in \{D, L\} \), on \([0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{-1} \), we have

\[
\hat{F}^Q(t, x, e) = x_{n+1} \vee \hat{G}^Q(t, x, e),
\]

and, on \([0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{-1} \),

\[
\hat{g}(t, x, e) = \sum_{i=0}^{n} \left[ x_{i+1} \vee \hat{G}^W(t, x_0, x_1, 0, \ldots, 0, e_1, \ldots, e_{i-1}, 0, \ldots, 0) \right] \mathbb{I}_{\{t \leq t_i < t_{i+1}\}}.
\]

In that case, the penalty function \( f \) is often given by

\[
f(t_i, x, e) = \begin{cases} e_i & \text{if } e_i \leq G_i, \\ G_i + \kappa (e_i - G_i) & \text{if } e_i > G_i, \end{cases}
\]

where \( \kappa > 1 \) and \( G_i := G^W(t_i, x_0, x_1, 0, \ldots, 0, e_1, \ldots, e_{i-1}, 0, \ldots, 0) \). The insurer takes a fee if the insured withdraws more than the guarantee \( G_i \); this fee is equal to \((\kappa - 1)(e_i - G_i)\).

The usual guarantee functions used to define GMDB and GMLB are listed below (see [4] for more details).

- **Constant guarantee.** For \( i \in \{0, \ldots, n\} \) and \( t_i \leq t < t_{i+1} \), we set

  \[
  \hat{G}^Q(t, x, e) = x_1 - \sum_{k=1}^{i} \hat{f}(t_k, x, e) \quad \text{on } [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{-1}.
  \]

  Hence, following the withdrawal strategy \( \xi \in \mathcal{E} \), the insured will get

  \[
  F(p, \xi) = A^p_{T \wedge \tau} \vee \left( e^{-\int_0^{T \wedge \tau} \kappa \, ds} \sum_{i=0}^{n} \left( A_0 - \sum_{k=1}^{i} \hat{f}(t_k, x, e) \right) \mathbb{I}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} \right).
  \]

- **Roll-up guarantee.** For \( \eta > 0 \), \( i \in \{0, \ldots, n\} \) and \( t_i \leq t < t_{i+1} \), we set

  \[
  \hat{G}^Q(t, x, e) = x_1 (1 + \eta)^i - \sum_{k=1}^{i} \hat{f}(t_k, x, e) (1 + \eta)^{i-k} \quad \text{on } [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{-1},
  \]

  and then if the insured follows the withdrawal strategy \( \xi \in \mathcal{E} \), she will get

  \[
  F(p, \xi) = A^p_{T \wedge \tau} \vee \left( e^{-\int_0^{T \wedge \tau} \kappa \, ds} \sum_{i=0}^{n} \left( A_0 (1 + \eta)^i - \sum_{k=1}^{i} \hat{f}(t_k, x, e) (1 + \eta)^{i-k} \right) \mathbb{I}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} \right).
  \]

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Ratchet guarantee. The guarantee depends on the path of $A$ in the following way

$$
\hat{G}^Q(t, x, e) = \sum_{i=0}^{n} \max \left( x_1 - \sum_{k=1}^{i} \hat{f}(k, x, e), \ldots, x_i - \sum_{k=1}^{i} \hat{f}(t_i, x, e), x_{i+1} \right) 1_{\{t_i \leq T \land \tau < t_{i+1}\}},
$$

for any $(t, x, e) \in [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$. The insured will get

$$
F(p, \hat{\xi}) = A_p^T \lor \left( e^{-\int_0^{T \lor \tau} r_s \, ds} \sum_{i=0}^{n} \max \left( \hat{a}_0^p - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}_p, \hat{\xi}), \ldots, \hat{a}_p^i \right) 1_{\{t_i \leq T \lor \tau < t_{i+1}\}} \right).
$$

Remark 3.2. We notice that such pay-offs are not bounded. Unfortunately, we need to suppose them to be bounded in our approach (see Remark B.1). From an economic point of view, the boundedness of the pay-offs can be justified by saying that the insurer can provide at most an amount $m$ which corresponds to its cash account. Therefore, the real pay-off that the insurer can provide is not $F(p, \hat{\xi})$ but $F(p, \hat{\xi}) \land m$.

3.3 Utility maximization and indifference pricing

Since the financial market is incomplete we propose to use an indifference pricing approach to determine the fee rate. We look for, if it exists, a fee rate $p^*$ such that

- the insurer is better of selling the policy if the fee rate is greater than $p^*$,
- it is worse of not selling the contract if the fee rate is smaller than $p^*$.

The optimal fee rate $p^*$ is then the smallest $p$ such that

$$
\sup_{\pi \in \mathcal{A}[0, T]} \mathbb{E} \left[ U(X^\pi_T) \right] \leq \sup_{\pi \in \mathcal{A}[0, T]} \inf_{\xi \in \mathcal{E}} \mathbb{E} \left[ U \left( A_0 + X^\pi_T - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - F(p, \hat{\xi}) \right) \right].
$$

A solution of inequality (3.5) will be called an indifference fee rate. Notice that, since the utility function is an exponential function, indifference fee rates will not depend on the initial wealth invested by the insurer but only on the initial deposit $A_0$ made by the insured. For this reason, we do not consider the initial wealth of the insurer and we assume w.l.o.g. that her initial endowment is zero.

In order to find the indifference fees, we shall compute the following quantities

$$
V^0 := \sup_{\pi \in \mathcal{A}[0, T]} \mathbb{E} \left[ U \left( X^\pi_T \right) \right],
$$

and

$$
V(p) := \sup_{\pi \in \mathcal{A}[0, T]} \inf_{\xi \in \mathcal{E}} \mathbb{E} \left[ U \left( A_0 + X^\pi_T - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - F(p, \hat{\xi}) \right) \right] = -e^{-\gamma A_0} \omega(p), \quad p \in \mathbb{R},
$$

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where \( w \) is defined for any \( p \in \mathbb{R} \) by
\[
  w(p) := \inf_{\pi \in \mathcal{A}[0,T]} \sup_{\xi \in E} \mathbb{E}\left[u\left(X_T^\pi - \sum_{i=1}^{n-1} \xi_i1_{t_i \leq \tau} - F(p, \hat{\xi})\right)\right],
\]
with \( u(y) := e^{-\gamma y} \) for all \( y \in \mathbb{R} \).

The quantity \( V^0 \) corresponds to the maximal expected utility at time \( T \) when the insurance company has not sold the variable annuities policy. We can characterize this value function \( V^0 \) and the optimal strategy \( \pi^* \) by mean of BSDEs as done by Hu et al. in [15]. To this end we define the following spaces.

- \( S^\infty_\mathbb{F} \) (resp. \( S^\infty_\mathbb{G} \)) is the set of càdlàg \( \mathbb{F} \) (resp. \( \mathbb{G} \))-adapted essentially bounded processes.
- \( L^2_\mathbb{F} \) (resp. \( L^2_\mathbb{G} \)) is the set of \( \mathbb{P}(\mathbb{F}) \) (resp. \( \mathbb{P}(\mathbb{G}) \))-measurable processes \( z \) such that \( \mathbb{E}\int_0^T |z_s|^2 ds < \infty \).
- \( L^2(\lambda) \) is the set of \( \mathbb{P}(\mathbb{G}) \)-measurable processes \( u \) such that \( \mathbb{E}\int_0^{T \wedge \tau} \lambda_s |u_s|^2 ds < \infty \).

We then have the following result which is a consequence of Theorem 7 in [15].

**Proposition 3.1.** The value function \( V^0 := \sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}\left[U\left(X_T^\pi\right)\right] \) is given by
\[
  V^0 = -\exp(\gamma y_0),
\]
where \((y, z)\) is the solution in \( S^\infty_\mathbb{F} \times L^2_\mathbb{F} \) to the BSDE
\[
\begin{align*}
  dy_t &= \left(\frac{\theta^2}{2\gamma} + \theta_t z_t\right)dt + z_t dB_t, \\
  y_T &= 0.
\end{align*}
\]

Moreover, the optimal strategy associated to this problem is defined by
\[
  \pi^*_t := \frac{\theta_t}{\gamma \sigma_t} + \frac{z_t}{\sigma_t}, \quad \forall t \in [0, T].
\]

In the usual indifference pricing setting, the indifference price \( p \) of a contingent claim \( \xi \) is solution of the following equation
\[
  \exp(-\gamma p) \sup_{\pi} \mathbb{E}\left[-\exp(-\gamma (X_T^\pi - \xi))\right] = \sup_{\pi} \mathbb{E}\left[-\exp(-\gamma X_T^\pi)\right].
\]

Therefore, we can isolate \( p \) and we get a semi-explicit formula for the indifference price. A difficulty with our approach is that fees are continuously payed by the insured and that the fee rate \( p \) appears in the pay-off \( F(p, \hat{\xi}) \). Therefore, one cannot use algebraic properties of the utility function to get semi-explicit formula for indifference fees. Nevertheless, we can prove some monotonicity results on the value function \( V \) which will be used to prove that the indifference fee rate exists or not, and to compute it.

**Proposition 3.2.** The value function \( V \) is non-decreasing on \( \mathbb{R} \).
Proof. Let \( p_1 \) and \( p_2 \) be two real numbers such as \( p_1 < p_2 \) and \( \mathcal{E}^{p_1} \) (resp. \( \mathcal{E}^{p_2} \)) the set of admissible discounted withdrawal policies for fee \( p_1 \) (resp. \( p_2 \)). It is obvious that \( \mathcal{E}^{p_2} \subset \mathcal{E}^{p_1} \). Moreover, the function \( p \mapsto \mathbb{E} \left[u \left(X_T^p - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - F(p, \hat{\xi})\right)\right] \) is non-increasing with respect to \( p \) because of the monotonicity properties of the functions \( \hat{g}, f \) and \( F^Q \) for any \( Q \in \{L, D\} \). The result follows from the definition of \( V \).

This monotonicity property of the function \( V \) allows to conclude the existence of indifference fees.

- If \( V(-\infty) < V^0 < V(+\infty) \), then there exists \( p^* \) such if \( p < p^* \), the insurance company has no interest to sell the contract, and if \( p \geq p^* \) then the company has interest to sell the contract.

- If \( V(-\infty) > V^0 \), the insurance should always sell the contract.

- If \( V(+\infty) < V^0 \), the insurance should never sell the contract.

The asymptotic behavior of \( V \) is then studied in the following subsection for usual guarantees.

3.4 Indifference fee for usual guarantees

In this part, we consider usual guarantees and study conditions for the existence of indifference fees.

**Proposition 3.3** (Ratchet guarantee). Let \( m > A_0 \). We assume that

\[
F(p, \hat{\xi}) = m \wedge \left[A^{p}_{T \wedge \tau} \vee \left(e^{-\int_{0}^{T \wedge \tau} r_s \, ds} \sum_{i=0}^{n} \max \left(\hat{a}_0 - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}_p, \hat{\xi}), \ldots, \hat{a}_p\right) 1_{\{t_i \leq T \wedge \tau < t_{i+1}\}}\right]\right]
\]

for \((p, \xi) \in \mathbb{R} \times \mathcal{E}\). Then, there exists \( p^* \in \mathbb{R} \cup \{-\infty\} \) such that \( V(p) \geq V^0 \) for all \( p \geq p^* \) and \( V(p) < V^0 \) for all \( p < p^* \).

**Proof.** From Proposition 3.2, we just have to show that \( \lim_{p \to +\infty} V(p) \geq V^0 \).

It will then follow from the monotonicity of \( V \) that there exists \( p^* \in \mathbb{R} \cup \{-\infty\} \) such that \( V(p) \geq V^0 \) for \( p \geq p^* \) and \( V(p) < V^0 \) for \( p < p^* \). First, notice that we may deduce from Assumption 2.3 that there exists a positive constant \( C \) such that, for any \( t \in [0, T] \), \( \mathbb{E}[A^p_t] \leq Ce^{-pt} \). Therefore, as \( A^p_T \geq 0 \), we get

\[
\lim_{p \to +\infty} A^p_t = 0 \quad \text{a.s. for any } t \in (0, T].
\]

Now, set \( i \in \{1, \ldots, n-1\} \), as \( f_i \geq \xi_i \) we obtain that

\[
\lim_{p \to +\infty} \hat{G}_i \leq \lim_{p \to +\infty} \max \left( A_0 - \sum_{k=1}^{i-1} \hat{\xi}_k, \hat{A}^{p}_{t_1} - \sum_{k=2}^{i-1} \hat{\xi}_k, \ldots, \hat{A}^{p}_{t_i}\right)
\]

\[
= A_0 - \sum_{k=1}^{i-1} \hat{\xi}_k.
\]
In the same way, we get

\[
\lim_{p \to +\infty} F(p, \hat{\xi}) \leq \lim_{p \to +\infty} m \wedge \left[ A^p_{T \wedge \tau} \vee \left( e^{-\int_0^{T \wedge \tau} r_s \, ds} \sum_{i=0}^{n} \max(\hat{a}_0^i - \sum_{k=1}^{i} \hat{\xi}_k, \ldots, \hat{a}_i^p) \mathbf{1}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} \right) \right] \\
\leq e^{-\int_0^{T \wedge \tau} r_s \, ds} \sum_{i=0}^{n} (A_0 - \sum_{k=1}^{i} \hat{\xi}_k^i) \mathbf{1}_{\{t_i \leq T \wedge \tau < t_{i+1}\}}.
\]

(3.10)

We now study the limit of \( V \) at \(+\infty\). For all \( \pi \in \mathcal{A}[0, T] \), there exists \( \xi^{\pi,*} \in \mathcal{E} \) such that

\[
V(p) \geq \inf_{\xi \in \mathcal{E}} \mathbb{E} \left[ -\exp \left( -\gamma \left( X_T^\pi + A_0 - \sum_{i=1}^{n-1} \hat{\xi}_i^\pi \mathbf{1}_{t_i \leq \tau} - F(p, \hat{\xi}) \right) \right) \right] \\
= \mathbb{E} \left[ -\exp \left( -\gamma \left( X_T^\pi + A_0 - \sum_{i=1}^{n-1} \hat{\xi}_i^\pi \mathbf{1}_{t_i \leq \tau} - F(p, \hat{\xi}^{\pi,*}) \right) \right) \right].
\]

The last equality follows from the fact that \( \mathcal{E} \) is a finite set and \( \hat{\xi}_i \geq \xi_i \). We deduce from the monotone convergence theorem and the inequality (3.10), that

\[
\lim_{p \to +\infty} V(p) \geq \mathbb{E} \left[ -\exp \left( -\gamma \left( X_T^\pi + A_0 - \sum_{i=1}^{n} \hat{\xi}_i^\pi \mathbf{1}_{t_i \leq \tau} - \lim_{p \to +\infty} F(p, \hat{\xi}^{\pi,*}) \right) \right) \right] \\
= \mathbb{E} \left[ -\exp \left( -\gamma \left( X_T^\pi + \sum_{i=0}^{n} (A_0 - \sum_{k=1}^{i} \hat{\xi}_k^\pi \mathbf{1}_{t_i \leq T \wedge \tau < t_{i+1}}) \left( 1 - e^{-\int_0^{T \wedge \tau} r_s \, ds} \right) \right) \right] \\
\geq \mathbb{E} \left[ -\exp(-\gamma X_T^\pi) \right].
\]

We recall that, from Proposition 3.1, there exists \( \pi^* \in \mathcal{A}[0, T] \), such that \( V^0 = \mathbb{E}[-\exp(-\gamma X_T^\pi^*)] \). Therefore, we obtain that \( \lim_{p \to +\infty} V(p) \geq V^0 \) which is the expected result.

\[\Box\]

**Proposition 3.4** (Roll-up guarantee). Let \( m > A_0 \) and \( \eta \geq 0 \). Assume that

\[
F(p, \hat{\xi}) = m \wedge \left[ A^p_{T \wedge \tau} \vee \left( e^{-\int_0^{T \wedge \tau} r_s \, ds} \sum_{i=0}^{n} \left( A_0(1 + \eta)^i - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}^p, \hat{\xi})(1 + \eta)^{i-k} \right) \mathbf{1}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} \right) \right],
\]

for all \( (p, \xi) \in \mathbb{R} \times \mathcal{E} \). There exists \( \eta_* \geq 0 \) such that for all \( \eta \in [0, \eta_*] \), there exists \( p^* \in \mathbb{R} \cup \{-\infty\} \) such that \( V(p) \geq V^0 \) for all \( p \geq p^* \) and \( V(p) < V^0 \) for all \( p < p^* \).

**Proof.** Let \( \eta \geq 0 \). Following the proof of Proposition 3.3, we can prove that, for \( i \in \{1, \ldots, n - 1\} \), we have

\[
\lim_{p \to +\infty} \hat{G}_i \leq \lim_{p \to +\infty} \hat{A}_i^p \vee \left( A_0(1 + \eta)^i - \sum_{k=1}^{i} \hat{\xi}_k(1 + \eta)^{i-k} \right) \\
= A_0(1 + \eta)^i - \sum_{k=1}^{i} \hat{\xi}_k(1 + \eta)^{i-k}.
\]
In the same way, we get
\[
\lim_{p \to +\infty} F(p, \hat{\xi}) \leq \lim_{p \to +\infty} m \wedge \left[ A_{T \wedge \tau}^p \vee \left( e^{-\int_0^{T \wedge \tau} r_s \, ds} \sum_{i=0}^n \left( A_0 (1 + \eta)^i - \sum_{k=1}^i \hat{\xi}_k (1 + \eta)^{i-k} \right) \mathbb{1}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} \right) \right] \\
\leq e^{-\int_0^{T \wedge \tau} r_s \, ds} \sum_{i=0}^n \left( A_0 (1 + \eta)^i - \sum_{k=1}^i \hat{\xi}_k (1 + \eta)^{i-k} \right) \mathbb{1}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} .
\] (3.11)

From Proposition 3.1, there exists \( \pi^* \in \mathcal{A}[0, T] \), such that
\[
V^0 = \mathbb{E}[-\exp(-\gamma X_T^{\pi^*})].
\]
From the fact that \( \mathcal{E} \) is a finite set, we deduce that
\[
V(p) \geq \inf_{\xi \in \mathcal{E}} \mathbb{E}\left[ -\exp\left( -\gamma \left( X_T^{\pi^*} + A_0 - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - F(p, \hat{\xi}) \right) \right) \right] \\
= \mathbb{E}\left[ -\exp\left( -\gamma \left( X_T^{\pi^*} + A_0 - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - F(p, \hat{\xi}^*) \right) \right) \right].
\]
It follows from the monotone convergence theorem and the inequality (3.11), that
\[
\lim_{p \to +\infty} V(p) \geq \mathbb{E}\left[ -\exp\left( -\gamma \left( X_T^{\pi^*} + A_0 - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \leq \tau} - \lim_{p \to +\infty} F(p, \hat{\xi}^*) \right) \right) \right] \\
\geq \Phi(\eta),
\]
where we have set
\[
\Phi(\eta) = \sum_{i=0}^n \mathbb{E}\left[ -\exp\left( -\gamma \left( X_T^{\pi^*} + \Phi_i(\eta) \right) \right) \mathbb{1}_{\{t_i \leq T \wedge \tau < t_{i+1}\}} \right],
\]
with, for \( i \in \{1, \ldots, n\},
\[
\Phi_i(\eta) := A_0 \left( 1 - e^{-\int_0^{T \wedge \tau} r_s \, ds} (1 + \eta)^i \right) - \sum_{k=1}^i \hat{\xi}_k^* \left( e^{\int_0^{t_k} r_s \, ds} - e^{-\int_0^{T \wedge \tau} r_s \, ds} (1 + \eta)^{i-k} \right).
\]
Obviously, \( \Phi \) is continuous and non-increasing on \( \mathbb{R}_+ \). Moreover, we have
\[
\Phi(0) \geq \mathbb{E}[ -\exp(-\gamma X_T^{\pi^*}) ] = V^0 \quad \text{and} \quad \lim_{\eta \to +\infty} \Phi(\eta) = -\infty.
\]
From the mean value theorem, we may define \( \eta_* \geq 0 \) as
\[
\eta_* := \sup\{ \eta \geq 0 : \Phi(\eta) = V^0 \}.
\]
We conclude the proof by noticing that for \( 0 \leq \eta \leq \eta_* \), we have
\[
\lim_{p \to +\infty} V(p) \geq V^0.
\]
4 Min-Max optimization problem

In this section we study the optimization problem (3.8). We determine the value function

\[ w \]

using a sequential utility maximization.

In the sequel we use the following notations. For \( x \in \mathbb{R}^n \) and \( 1 \leq k \leq n \) we denote by \( x^{(k)} \) the vector of \( \mathbb{R}^k \) defined by

\[ x^{(k)} := (x_1, \ldots, x_k) . \]

For \( y \in \mathbb{R}^k \) we denote by \( \hat{y} \) the vector

\[ \hat{y} := (y_1 e^{\int_0^{t_1} r_s ds}, \ldots, y_k e^{\int_{t_k}^{t} r_s ds}) . \]

4.1 Sequential utility maximization

The problem (3.8) is not classical for the following two reasons.

- The terminal wealth is \( \mathcal{G}_T \)-measurable and the pay-off is \( \mathcal{G}_{T \wedge \tau} \)-measurable.
- There are a maximization w.r.t. the withdrawals \( \xi \) and a minimization w.r.t. the
  investment strategies \( \pi \).

We first modify the problem to get a \( \mathcal{G}_{T \wedge \tau} \)-measurable wealth and a \( \mathcal{G}_{T \wedge \tau} \)-measurable pay-off. The following Proposition is based on the fact that after the insured’s death time \( \tau \), we consider that the contract will run up to its maturity \( T \). Between \( T \wedge \tau \) and \( T \), the insurer has to maximize its utility without facing risks coming from the insured withdrawals or death. Therefore, we begin with solving this problem and it will lead us to consider a

problem with maturity \( T \wedge \tau \) and a modified pay off, called \( H \) in the following Proposition.

For a stopping time \( \nu \) and an investment strategy \( \pi \in \mathcal{A}[0, T] \), we denote by \( (X_{t}^{\nu, \pi})_{t \geq 0} \) the process defined by

\[ X_{t}^{\nu, \pi} = \int_{\nu}^{t \wedge \nu} \pi_s (\mu_s - r_s) ds + \int_{\nu}^{t \wedge \nu} d\pi_s \sigma_s dB_s , \quad t \in [0, T] . \]  

(4.12)

\( X_{t}^{\nu, \pi} \) corresponds to the wealth at time \( t \) when we follow the strategy \( \pi \) by starting at time \( \nu \) with the wealth 0.

Proposition 4.5 (Initialization). For any \( p \in \mathbb{R} \), we have

\[ w(p) = \inf_{\pi \in \mathcal{A}[0, T \wedge \tau]} \sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ u(X_{T \wedge \tau}^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - H(p, \hat{\xi})) \right] , \]

with

\[ H(p, \hat{\xi}) := F(p, \hat{\xi}) + \frac{1}{\gamma} \log \left( \mathbb{E} \left[ u(X_{T \wedge \tau}^{\pi, \hat{\xi}}) \mathbb{G}_{T \wedge \tau} \right] \right) , \]

where \( X_{T}^{T \wedge \tau, \pi} \) is given by (4.12).
Proof. We first prove the following inequality

\[
 w(p) \geq \inf_{\pi \in \mathcal{A}[0,T]} \sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - H(p,\hat{\xi}) \right) \right]. \tag{4.13}
\]

For any \( \pi \in \mathcal{A}[0,T] \) and \( \xi \in \mathcal{E} \), it follows from the fact that \( u(x+y) = u(x)u(y) \) for any \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \) that

\[
\mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - F(p,\hat{\xi}) \right) \right] = \mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - F(p,\hat{\xi}) \right) \right] \mathbb{E} \left[ u \left( X_T^{\pi} | G_{T \wedge \tau} \right) \right] \geq \mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - H(p,\hat{\xi}) \right) \right].
\]

Therefore, we can see that the inequality (4.13) holds.

We now prove the following inequality

\[
 w(p) \leq \inf_{\pi \in \mathcal{A}[0,T]} \sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - H(p,\hat{\xi}) \right) \right].
\]

From Lemma A.1 we know that there exists \( \pi^{*,\tau} \in \mathcal{A}[T \wedge \tau, T] \) such that

\[
 \text{ess inf}_{\pi \in \mathcal{A}[T \wedge \tau, T]} \mathbb{E} \left[ u \left( X_T^{\pi} | G_{T \wedge \tau} \right) \right] = \mathbb{E} \left[ u \left( X_T^{\pi^{*,\tau}} | G_{T \wedge \tau} \right) \right].
\]

Then, we consider the subset \( \mathcal{A}^*[0,T] \) of \( \mathcal{A}[0,T] \) defined by \( \mathcal{A}^*[0,T] := \{ \pi_0, 1_{t \leq T \wedge \tau} + \pi^{*,\tau} 1_{t > T \wedge \tau}, \pi \in \mathcal{A}[0,T \wedge \tau] \}. \) Since \( \mathcal{A}^*[0,T] \subset \mathcal{A}[0,T] \), we get

\[
 w(p) \leq \inf_{\pi \in \mathcal{A}^*[0,T]} \sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - F(p,\hat{\xi}) \right) \right] \leq \inf_{\pi \in \mathcal{A}[0,T \wedge \tau]} \sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ u \left( X_T^{\pi} - \sum_{i=1}^{n-1} \xi_i 1_{t_i \leq \tau} - H(p,\hat{\xi}) \right) \right].
\]

Hence, we get the equality. \( \square \)

**Remark 4.3.** If \( F \) is a bounded random variable then \( H \) is also bounded since, from Lemma A.1, we have

\[
 \text{ess inf}_{\pi \in \mathcal{A}[T \wedge \tau, T]} \mathbb{E} \left[ u \left( X_T^{\pi} | G_{T \wedge \tau} \right) \right] = \exp \left( \gamma Y_{T \wedge \tau}^{(n)} \right),
\]

and we know that \( Y^{(n)} \) is bounded.

We now decompose the initial problem in \( n \) subproblems.

**Theorem 4.1.** The value function \( w \) is given by

\[
 w(p) = \inf_{\pi \in \mathcal{A}[0,T]} \mathbb{E} \left[ u \left( X_{T_1}^{\pi} \right) v(1) \right],
\]

where
Moreover there exists \( y \) We prove by backward induction on \( \xi \in \mathcal{E} \) by

\[
\begin{align*}
v(n,\xi^{(n-1)}) &:= e^{\gamma H(p,\xi^{(n-1)})}, \\
v(i,\xi^{(i-1)}) &:= \text{ess sup}_{\xi \in \mathcal{E}^{i+1}_{i-1}} \text{ess inf}_{\pi \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau]} J(i, \pi, \xi^{(i-1)}, \zeta),
\end{align*}
\]

with for any \( i \in \{1, \ldots, n-1\} \), \( \pi \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau] \) and \( \zeta \in \mathcal{E}^{i+1}_{i-1} \)

\[
J(i, \pi, \xi^{(i-1)}, \zeta) := \mathbb{E}\left[u\left(X_{t_{i+1} \wedge \tau}^{\pi} - \zeta \mathbb{1}_{t_i < \tau}\right) v(i+1, (\xi^{(i-1)}, \zeta)) \middle| \mathcal{G}_{t_i \wedge \tau}\right],
\]

- \( v(1) := \text{ess sup}_{\zeta \in \mathcal{E}^1_0} \text{ess inf}_{\pi \in \mathcal{A}[t_i \wedge \tau, t_2 \wedge \tau]} \mathbb{E}\left[u\left(X_{t_2 \wedge \tau}^{\pi} - \zeta \mathbb{1}_{t_i < \tau}\right) v(2, \zeta) \middle| \mathcal{G}_{t_1 \wedge \tau}\right]. \)

The proof of this theorem is postponed to Subsection C of the Appendix.

In the sequel, by abuse of notation, we write \( v(1, \xi^0) \) for \( v(1) \).

### 4.2 Optimal investment and worst withdrawals for the insurer

In the following result, we provide the withdrawal \( \xi_t^* \) and the investment strategy \( \pi_t^* \) that attain the value functions \( v(i, \cdot) \) for any \( i \in \{1, \ldots, n\} \). From Theorem 4.1, they correspond to the optimal investment strategy and the worst withdrawal for the insurer.

**Proposition 4.6.** For any \( i \in \{1, \ldots, n-1\} \), there exists a strategy \( \pi_t^* \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau] \), a withdrawal \( \xi_t^* \in \mathcal{E}^1_{i+1} \), and a map \( y^{(i)} \) from \( \mathcal{V}^{i-1} \) to \( L^\infty(\Omega, \mathcal{G}_{t_i \wedge \tau}, \mathbb{P}) \) such that

\[
v(i, \xi^{(i-1)}) = \mathbb{E}\left[u\left(X_{t_{i+1} \wedge \tau}^{\pi_t^*} - \xi_t^* \mathbb{1}_{t_i < \tau}\right) v(i+1, (\xi^{(i-1)}, \xi_t^*)) \middle| \mathcal{G}_{t_i \wedge \tau}\right] = \exp\left(\gamma y^{(i)}(\xi^{(i-1)})\right).
\]

Moreover there exists \( y^{(0)} \) such that the value function \( v \) of the initial problem (3.8) is given by

\[
w(p) = \exp(\gamma y^{(0)}).
\]

**Proof.** We prove by backward induction on \( i \in \{1, \ldots, n-1\} \) that

- the map \( H^i \) defined on \( \mathcal{V}^{i-1} \) by

\[
H^i(x_1, \ldots, x_{i-1}) = v\left(i, \left(x_1 e^{\int_0^{t_1} r_s ds}, \ldots, x_{i-1} e^{\int_0^{t_{i-1}} r_s ds}\right)\right), \quad x^{(i-1)} \in \mathcal{V}^{i-1},
\]

is valued in \( L^\infty(\Omega, \mathcal{G}_{t_i} \wedge \tau, \mathbb{P}) \),

- there exists a strategy \( \pi_t^* \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau] \), a withdrawal \( \xi_t^* \in \mathcal{E}^1_{i+1} \), and a map \( y^{(i)} \) from \( \mathcal{V}^{i-1} \) to \( L^\infty(\Omega, \mathcal{G}_{t_i \wedge \tau}, \mathbb{P}) \) such that

\[
v(i, \xi^{(i-1)}) = \mathbb{E}\left[u\left(X_{t_{i+1} \wedge \tau}^{\pi_t^*} - \xi_t^* \mathbb{1}_{t_i < \tau}\right) v(i+1, (\xi^{(i-1)}, \xi_t^*)) \middle| \mathcal{G}_{t_i \wedge \tau}\right] = \exp\left(\gamma y^{(i)}(\xi^{(i-1)})\right).
\]
Fix $i = n - 1$. Since $H$ is valued in $L^\infty(\Omega, \mathcal{G}_{t_n} \land \tau, \mathbb{P})$, we can apply Lemma B.3 and we get a strategy $\pi^{*,n-1}(\xi^{(n-2)}, \zeta) \in \mathcal{A}[t_{n-1} \land \tau, t_n \land \tau]$ and a map $y^{(n-1)}$ from $\hat{W}^{n-1}$ to $L^\infty(\Omega, \mathcal{G}_{t_1 \land \tau}, \mathbb{P})$ such that

$$
v(n-1, \xi^{(n-2)}) = \text{ess sup}_{\zeta \in \mathcal{E}^1_{n-2}} \mathbb{E} \left[ u \left( X^{t_{n-1} \land \tau, \pi^{*,n-1}(\xi^{(n-2)}, \zeta)}_{t_{n-1} \land \tau} - \zeta \mathbb{1}_{t_{n-1} < \tau} \right) v(n, (\xi^{(n-2)}, \zeta)) \bigg| \mathcal{G}_{t_{n-1} \land \tau} \right] = \text{ess sup}_{\zeta \in \mathcal{E}^1_{n-2}} \exp \left( \gamma y^{(n-1)}(\xi^{(n-2)}, \zeta) \right).
$$

We can then apply Lemma B.4 and we get a withdrawal $\xi^{*}_{n-1} \in \mathcal{E}^1_{n-2}$ such that

$$
v(n-1, \xi^{(n-2)}) = \exp \left( \gamma y^{(n-1)}(\xi^{(n-2)}, \xi^{*}_{n-1}) \right).
$$

Then $\xi^{*}_{n-1} \in \mathcal{E}^1_{n-2}$, $\pi^{*,n-1}(\xi^{(n-2)}, \xi^{*}_{n-1}) \in \mathcal{A}[\tau \land t_{n-1}, \tau \land t_n]$ and the map $y^{(n-1),*}$ defined by

$$
y^{(n-1),*}(.) = y^{(n-1)}(., \xi^{*}_{n-1} \int_0^{t_{n-1}} r_s ds)
$$

satisfy the conditions. Moreover, since $y^{(n-1),*}$ is uniformly bounded, we get the same property for $H^{n-1}$.

We now suppose that the result holds for some $i \in \{2, \ldots, n - 1\}$ and we prove it for $i - 1$. By definition we have

$$
v(i-1, \xi^{(i-2)}) = \text{ess sup}_{\zeta \in \mathcal{E}^1_{i-2}} \text{ess inf}_{\pi \in \mathcal{A}[t_{i-1} \land \tau, t_i \land \tau]} \mathbb{E} \left[ u \left( X^{t_{i-1} \land \tau, \pi}_{t_{i-1} \land \tau} - \zeta \mathbb{1}_{t_{i-1} < \tau} \right) v(i, (\xi^{(i-2)}, \zeta)) \bigg| \mathcal{G}_{t_{i-1} \land \tau} \right] = \text{ess sup}_{\zeta \in \mathcal{E}^1_{i-2}} \text{ess inf}_{\pi \in \mathcal{A}[t_{i-1} \land \tau, t_i \land \tau]} \mathbb{E} \left[ u \left( X^{t_{i-1} \land \tau, \pi}_{t_{i-1} \land \tau} - \zeta \mathbb{1}_{t_{i-1} < \tau} \right) H^i(\xi^{(i-2)}, \zeta, \int_0^{t_{i-1}} r_s ds) \bigg| \mathcal{G}_{t_{i-1} \land \tau} \right].
$$

By the induction hypothesis $H^i$ is valued in $L^\infty(\Omega, \mathcal{G}_{t_i \land \tau}, \mathbb{P})$. We can therefore apply Lemma B.3 and we get a strategy $\pi^{*,i-1}(\xi^{(i-2)}, \zeta) \in \mathcal{A}[t_{i-1} \land \tau, t_i \land \tau]$ and a map $y^{(i-1)}$ from $\hat{W}^{i-1}$ to $L^\infty(\Omega, \mathcal{G}_{t_1 \land \tau}, \mathbb{P})$ such that

$$
v(i-1, \xi^{(i-2)}) = \text{ess sup}_{\zeta \in \mathcal{E}^1_{i-2}} \mathbb{E} \left[ u \left( X^{t_{i-1} \land \tau, \pi^{*,i-1}(\xi^{(i-2)}, \zeta)}_{t_{i-1} \land \tau} - \zeta \mathbb{1}_{t_{i-1} < \tau} \right) v(i, (\xi^{(i-2)}, \zeta)) \bigg| \mathcal{G}_{t_{i-1} \land \tau} \right] = \text{ess sup}_{\zeta \in \mathcal{E}^1_{i-2}} \exp \left( \gamma y^{(i-1)}(\xi^{(i-2)}, \zeta) \right).
$$

We can then apply Lemma B.4 and we get a withdrawal $\xi^{*}_{i-1} \in \mathcal{E}^1_{i-2}$ such that

$$
v(i-1, \xi^{(i-2)}) = \mathbb{E} \left[ u \left( X^{t_{i-1} \land \tau, \pi^{*,i-1}(\xi^{(i-2)}, \zeta)}_{t_{i-1} \land \tau} - \xi^{*}_{i-1} \mathbb{1}_{t_{i-1} < \tau} \right) v(i, (\xi^{(i-2)}, \xi^{*}_{i-1})) \bigg| \mathcal{G}_{t_{i-1} \land \tau} \right].
$$

Then $\xi^{*}_{i-1} \in \mathcal{E}^1_{i-2}$, $\pi^{*,i-1}(\xi^{(i-2)}, \xi^{*}_{i-1}) \in \mathcal{A}[\tau \land t_{i-1}, \tau \land t_i]$ and the map $y^{(i-1),*}$ defined by

$$
y^{(i-1),*}(.) = y^{(i-1)}(., \xi^{*}_{i-1} \int_0^{t_{i-1}} r_s ds)
$$

satisfy the conditions. Moreover, since $y^{(i-1),*}$ is uniformly bounded, we get the same property for $H^{i-1}$.
To finish the proof it is sufficient to prove that there exists \( y^{(0)} \) such that \( w(p) = \exp(\gamma y^{(0)}) \).

We know that

\[
  w(p) = \inf_{\pi \in \mathcal{A}[0,t_1 \wedge \tau]} \mathbb{E} \left[ \exp(\gamma (X^{\pi}_{t_1 \wedge \tau} + y^{(1),*}) \right]
\]

By Lemma B.3, we know that there exists \( \pi^* \in \mathcal{A}[0,t_1 \wedge \tau] \) and \( y^{(0)} \) such that

\[
  \inf_{\pi \in \mathcal{A}[0,t_1 \wedge \tau]} \mathbb{E} \left[ \exp(\gamma (X^{\pi}_{t_1 \wedge \tau} + y^{(1),*}) \right] = \mathbb{E} \left[ \exp(\gamma (X^{\pi^*}_{t_1 \wedge \tau} + y^{(1),*}) \right) = \exp(\gamma y^{(0)})
\]

\[\square\]

5 Numerical resolution

5.1 Approximation procedure

**Max-min problem.** We first propose a scheme to solve the problem (3.7) by using Theorem 4.1. We describe the procedure in an inductive way. We present the step 0, which corresponds to the initialization given by Proposition 4.5. Then, step \( i \) corresponds to the computation of the function \( v(n-i,.) \) and the optimal strategy \( \pi^{*,n-i} \) and the worst withdrawal \( \xi^{*,n-i} \) once the previous steps have been done.

**Step 0:** We solve the following problem

\[
  \text{ess inf} \quad \mathbb{E} \left[ u(X^{T \wedge \tau,\pi}_T) \big| \mathcal{G}_{T \wedge \tau} \right].
\]

From Lemma A.1 we know that

\[
  \text{ess inf} \quad \mathbb{E} \left[ u(X^{T \wedge \tau,\pi}_T) \big| \mathcal{G}_{T \wedge \tau} \right] = \exp(\gamma Y^{(n)}_{T \wedge \tau}),
\]

where \( Y^{(n)} \) is the solution to the linear BSDE

\[
  \begin{cases}
    dY^{(n)}_t = \left[ \frac{\theta_t^2}{\gamma} + \theta_t Z^{(n)}_t \right] dt + Z^{(n)}_t dB_t, \\
    Y^{(n)}_T = 0.
  \end{cases}
\]

Therefore, we have

\[
  Y^{(n)}_{T \wedge \tau} = \mathbb{E}_Q \left[ - \int_{T \wedge \tau}^{\tau} \frac{\theta_t^2}{\gamma} dt \big| \mathcal{G}_{T \wedge \tau} \right],
\]

with \( dQ/dP|_{\mathcal{G}_t} = \mathcal{E}(- \int_0^t \theta_s dB_s)_t \).

**Step 1:** We solve the following problem

\[
  v(n-1, \xi^{(n-2)}) = \text{ess sup} \quad \text{ess inf} \quad \mathbb{E} \left[ u \left( X^{t_{n-1} \wedge \tau,\pi}_{t_{n-1} \wedge \tau} - \xi \mathbb{1}_{t_{n-1} \wedge \tau} - F(p, (\xi^{(n-2)}, \hat{\xi})) - Y^{(n)}_{T \wedge \tau}) \big| \mathcal{G}_{t_{n-1} \wedge \tau} \right],
\]
For that we first solve the infimum problem, and we know from Lemma B.3 that there exists a r.v. \( y^{(n-1)}(\xi^{(n-2)}, \zeta) \) such that

\[
\underset{\pi \in \mathcal{A}[t_{n-1} \wedge \tau, t_n \wedge \tau]}{\text{ess inf}} \mathbb{E} \left[ u \left( X^{t_{n-1} \wedge \tau, \pi}_{t_{n-1} \wedge \tau} - \zeta 1_{t_{n-1} \wedge \tau} - F(p, (\hat{\xi}^{(n-2)}, \hat{\zeta})) - Y^{(n)}_{T \wedge \tau} \right) \mid \mathcal{G}_{t_{n-1} \wedge \tau} \right] = \exp \left( \gamma y^{(n-1)}(\xi^{(n-2)}, \zeta) \right).
\]

The r.v. \( y^{(n-1)}(\xi^{(n-2)}, \zeta) \) is defined by \( \zeta 1_{t_{n-1} \wedge \tau} + Y^{(n-1)}_{t_{n-1}}(\xi^{(n-2)}, \zeta) \), where \( Y^{(n-1)}(\xi^{(n-2)}, \zeta) \) solves the following BSDE

\[
\begin{cases}
-dY^{(n-1)}_t = \left( \lambda_t \frac{\gamma y_t^{(n-1)}}{\gamma} - \theta_t Z_t^{(n-1)} - \frac{|\theta_t|^2}{2\gamma} \right) dt - Z_t^{(n-1)} dB_t - U_t^{(n-1)} dN_t, \\
Y^{(n-1)}_{T \wedge \tau} = F(p_t, (\xi^{(n-2)}, \zeta)).
\end{cases}
\]

We also get the optimal investment strategy \( \pi^{*,n-1} \) by the formula

\[
\pi^{*,n-1} = \frac{1}{\sigma} \left[ \frac{\theta}{\gamma} + Z^{(n-1)} \right].
\]

We can now find \( \xi^{*-1}_{n-1} \in \mathcal{E}_{n-2}^1 \) such that

\[
\underset{\zeta \in \mathcal{E}_{n-1}^1 \pi \in \mathcal{A}[t_{n-1} \wedge \tau, t_{n-1} \wedge \tau]}{\text{ess sup}} y^{(n-1)}(\xi^{(n-2)}, \zeta) = y^{(n-1)}(\xi^{(n-2)}, \xi^{*-1}_{n-1}).
\]

\underline{Step i:} We now compute

\[
v(n-i, \xi^{(n-i-1)}) = \underset{\zeta \in \mathcal{E}_{n-1}^1 \pi \in \mathcal{A}[t_{n-i} \wedge \tau, t_{n-i+1} \wedge \tau]}{\text{ess sup}} \text{ess inf} \mathbb{E} \left[ u \left( X^{t_{n-i} \wedge \tau, \pi}_{t_{n-i} \wedge \tau} - \zeta 1_{t_{n-i} \wedge \tau} - y^{(n-i+1)}(\xi^{(n-i-1)}, \zeta, \xi^{*-1}_{n-i+1}) \right) \mid \mathcal{G}_{t_{n-i} \wedge \tau} \right].
\]

As previously, the infimum problem is solved using Lemma B.3. We have just adapted the terminal condition in the BSDE. Let \( Y^{(n-i)}(\xi^{(n-i-1)}, \zeta) \) solving the following BSDE

\[
\begin{cases}
-dY^{(n-i)}_t = \left( \lambda_t \frac{\gamma y_t^{(n-i)}}{\gamma} - \theta_t Z_t^{(n-i)} - \frac{|\theta_t|^2}{2\gamma} \right) dt - Z_t^{(n-i)} dB_t - U_t^{(n-i)} dN_t, \\
Y^{(n-i)}_{t_{n-i} \wedge \tau} = y^{(n-i+1)}(\xi^{(n-i-1)}, \zeta, \xi^{*-1}_{n-i+1}).
\end{cases}
\]

Then the r.v. \( y^{(n-i)}(\xi^{(n-i-1)}, \zeta) \) defined by \( \zeta 1_{t_{n-i} \wedge \tau} + Y^{(n-i)}_{t_{n-i}}(\xi^{(n-i-1)}, \zeta) \) satisfies

\[
\underset{\pi \in \mathcal{A}[t_{n-i} \wedge \tau, t_{n-i+1} \wedge \tau]}{\text{ess inf}} \mathbb{E} \left[ u \left( X^{t_{n-i} \wedge \tau, \pi}_{t_{n-i} \wedge \tau} - \zeta 1_{t_{n-i} \wedge \tau} - y^{(n-i+1)}(\xi^{(n-i-1)}, \zeta, \xi^{*-1}_{n-i+1}) \right) \mid \mathcal{G}_{t_{n-i} \wedge \tau} \right] = \exp \left( \gamma y^{(n-i)}(\xi^{(n-i-1)}, \zeta) \right).
\]

We get the optimal investment strategy for the insurer by the formula

\[
\pi^{*,n-i} = \frac{1}{\sigma} \left[ \frac{\theta}{\gamma} + Z^{(n-i)} \right].
\]

Finally we get the worst withdrawal \( \xi^{*-1}_{n-i} \in \mathcal{E}_{n-i-1}^1 \) such that

\[
\underset{\zeta \in \mathcal{E}_{n-i-1}^1}{\text{ess sup}} y^{(n-i)}(\xi^{(n-i-1)}, \zeta) = y^{(n-i)}(\xi^{(n-i-1)}, \xi^{*-1}_{n-i}).
\]
Step $n$: We finish by solving the optimisation problem
\[
\text{ess inf}_{\pi \in \mathcal{A}[t_0,t_1 \wedge \tau]} \mathbb{E}\left[u\left(X_{t_1 \wedge \tau}^{t_0,\pi} - y^{(1)}(\zeta_1^*)\right)\right].
\]

From Lemma B.3, there exists a r.v. $y^{(0)}$ such that
\[
\text{ess inf}_{\pi \in \mathcal{A}[t_0,t_1 \wedge \tau]} \mathbb{E}\left[u\left(X_{t_1 \wedge \tau}^{t_0,\pi} - y^{(1)}(\zeta_1^*)\right)\right] = \exp(\gamma y^{(0)})
\]
where $y^{(0)}$ is the value at time 0 of the solution to the BSDE
\[
\begin{cases}
-dY_t^{0} = \left(\lambda_t e^{\gamma U^{0}_t} - \theta_t Z_t^{0} - \frac{\theta_t^2}{2\gamma} \right) dt - Z_t^{0} dB_t - U_t^{0} dN_t, \\
Y_{t_1 \wedge \tau}^{(0)} = y^{(1)}(\zeta_1^*). 
\end{cases}
\]

We get the optimal investment strategy for the insurer by the formula
\[
\pi^{*,0} = \frac{1}{\frac{\theta}{\gamma} + Z^{(0)}}.
\]

The value function associated to the optimization problem (3.7) is given by
\[-\exp(\gamma (y^{(0)} - A_0)),
\]
and the worst withdrawals for the insurer are given by $(\zeta_1^*, \ldots, \zeta_{n-1}^*)$.

We can compute $(Y^{(n-i)}, Z^{(n-i)}, U^{(n-i)})$ for any $1 \leq i \leq n$ by using discretization methods for BSDEs (see for example [10]).

**Indifference price.** We now know how to calculate $V^0$ and $V(p)$ for any $p \in \mathbb{R}$. The next step is to calculate the indifference fee rate $p^*$ which is defined by
\[
p^* = \inf\{p \in \mathbb{R} : V^0 \leq V(p)\}.
\]

From the previous results, we can rewrite this problem as follows
\[
p^* = \inf\{p \in \mathbb{R} : -\exp(\gamma y_0) \leq -\exp(\gamma (y^{(0)}(p) - A_0))\},
\]
where $y^{(0)}(p)$ is defined by the step $n$.

Therefore $p^*$ is defined by
\[
p^* = \inf\{p \in \mathbb{R} : y^{(0)}(p) \leq y_0 + A_0\}.
\]

We may approximate $p^*$ by bisection or dichotomy method.
5.2 Simulations

In this section we present numerical illustrations of parameter sensitivity for indifference fee rates. We compute solutions for both optimization problems: $V^0$, the utility maximization problem without variable annuities, and $V^{(0)}(p)$, the utility maximization problem with variable annuities. We use the method described in [16] to decompose BSDEs with a jump in a recursive system of two Brownian BSDEs. Brownian BSDEs involved are then simulated thanks to the discretization scheme studied in [7]. For the computation of the conditional expectations, we use a parametric regression method with polynomial basis. We find $p^*$ such that

$$p^* = \inf\{ p \in \mathbb{R} , V_0 \leq V(p) \} .$$

For simplicity, we assume that the insured can withdraw only every ten years. We shall give the following numerical values to parameters

$$\gamma = 1.3, \quad T = 30, \quad A_0 = 1, \quad r_0 = 0.01, \quad \mu_0 = 0.04, \quad \sigma = 0.1 .$$

We describe the dependence with respect to the market parameters: the initial drift, the volatility, the initial interest rate and the initial value of the account $A_0$.

Notice that indifference fee rates decrease with respect to the drift up to a certain level. The bigger is the drift the less usefull are the guarantees, then the fees payed to get these guarantees decrease. When $\mu$ is very big with respect to $r$ and $\sigma$ then an investor has rather to invest only in the risky assets. Selling guarantees compels the insurer to hedge against interest rates variations. That is why the fees increase when $\mu$ is greater than 0, 7 in our case.
Once again, we can get a financial interpretation of the monotonicity of the fees with respect to market volatility. The bigger is the volatility the more useful are the guarantees, then the fees payed to get these guarantees have to increase.
Dependence with respect to the interest rate is more complex. We notice that, in that case, indifference fee rates increase when the absolute value of the difference between drift and interest rate increases. On the one hand, if \( r \) is smaller than \( \mu \), when it increases the discounting on future payments make them to become worth less and the price of guarantees decreases. On the other hand, when \( r \) is greater than the drift, an exponential utility maximizer should not expose its portfolio to the market volatility. Hence, she should receive a bigger compensation to do so. If the insurer sell its product, she has to hedge her portfolio against volatility and to have a non-zero position on the risky assets. She will sell her product at a bigger price if interest rate is greater than \( \mu \) and increases.

![Figure 4: Indifference fee rate w.r.t. \( A_0 \)](image)

Since the utility function is an exponential one, indifference fee rates will not depend on the initial wealth invested by the insurer but strongly on the initial deposit \( A_0 \) made by the insured (see inequality (3.5)). As fees are proportional, the more the insured invests, the more the insurer will get from the contract. Therefore, indifference fee rates will decrease when the initial deposit \( A_0 \) increases.

### 6 Multi-contracts

In this paper, we have considered the valuation of a single variable annuities contract. However, an insurance company usually sells several contracts and it will be interesting to consider the indifference pricing of a given number of variable annuities contracts. Our approach can then be generalized to this case by considering several stopping times \( \tau_1, \ldots, \tau_N \) instead of the single stopping time \( \tau \) where \( N \) is the number of contracts sold by the insur-
ance company. For any \(1 \leq i \leq N\), \(\tau_i\) corresponds to the death time of the policyholder \(i\).

In this case we enlarge the filtration \(F\) by the death times \((\tau_i)_{1 \leq i \leq N}\). For that we consider the filtration \(G\) given by \(G_t := \tilde{G}_{t+}\), where \(\tilde{G}_t := F_t \vee D_1^t \vee \cdots \vee D_N^t\) for all \(t \geq 0\), and \(D_i^t := \sigma(\mathbb{1}_{\tau_i \leq s}, s \leq t)\) for any \(1 \leq i \leq N\).

The indifference fee \(p^*\) is the smallest \(p\) such that

\[
\sup_{\pi \in A[0,T]} \mathbb{E}[U(X_T^\pi)] \leq \sup_{\pi \in A[0,T]} \inf_{\xi \in \mathcal{E}} \mathbb{E}\left[U\left(\sum_{i=1}^N A_0(i) + X_T^\pi - \sum_{j=1}^{n-1} \sum_{i=1}^N \xi_{i,j} \mathbb{1}_{t_j \leq \tau_i} - \sum_{i=1}^N F(i,p,\hat{\xi}_i)\right)\right],
\]

where

- \(\mathcal{E}\) is the set of matrices \(N \times (n-1)\) where the line \(i\), for \(1 \leq i \leq N\), of the matrix is the discounted withdrawal strategy of the insured \(i\), and the column \(j\), for \(1 \leq j \leq n-1\), of the matrix is the discounted withdrawals of the \(N\) insureds at time \(t_j\),

- for any \(k \in \{0, \ldots, n-2\}\) and \(l \in \{1, \ldots, n-k-1\}\), we also define the set \(\mathcal{E}_k^l\) by

\[
\mathcal{E}_k^l = \left\{ \xi \in \mathcal{E} \text{ s.t. } \xi_{i,j} = 0 \text{ for all } j \notin \{k+1, \ldots, k+i\} \text{ and } i \in \{1, \ldots, N\} \right\},
\]

- \(A_0(i)\) is the initial value of the account of the insured \(i\),

- \(\xi_{i,j}\) is the discounted withdrawal of the insured \(i\) at time \(t_j\),

- \(\hat{\xi}_i\) is the vector of withdrawals of the insured \(i\),

- \(F(i,p,\hat{\xi}_i)\) is the pay-off for the insured \(i\) at time \(T \wedge \tau_i\) if this one has followed the strategy \(\hat{\xi}_i\).

To solve the second control problem we can use a similar result as Theorem 4.1. For this purpose, we introduce the notation \(\tau_{(N)}\) which corresponds to the last death time.

**Theorem 6.2.** The value function \(w\) is given by

\[
w(p) = \inf_{\pi \in A[0,T_{\wedge \tau_{(N)}}]} \mathbb{E}[u(X_{T_{\wedge \tau_{(N)}}}) v(1)],
\]

where

- \(v(i,\xi^{(i-1)})\) is defined recursively for any \(i \in \{2, \ldots, n\}\) and \(\xi \in \mathcal{E}\) by

\[
\begin{aligned}
&v(n,\xi^{(n-1)}) := \exp\left(\gamma \sum_{j=1}^N F(j,p,\hat{\xi}_j)\right) \inf_{\pi \in A[T_{\wedge \tau_{(N)}}]} \mathbb{E}\left[u\left(X_{T_{\wedge \tau_{(N)}}}^\pi\right)|G_{T_{\wedge \tau_{(N)}}}\right], \\
v(i,\xi^{(i-1)}) := \sup_{\xi \in \mathcal{E}_{i+1}^i} \inf_{\pi \in A[t_i \wedge \tau_{(N)}]} \mathbb{E}\left[u\left(X_{t_{i+1} \wedge \tau_{(N)}}^\pi\right)|G_{t_i \wedge \tau_{(N)}}\right],
\end{aligned}
\]

with for any \(i \in \{1, \ldots, n-1\}\), \(\pi \in A[t_i \wedge \tau_{(N)}, t_{i+1} \wedge \tau_{(N)}]\) and \(\xi \in \mathcal{E}_{i+1}^i\)

\[
J(i,\pi,\xi^{(i-1)},\zeta) := \mathbb{E}\left[u\left(X_{t_{i+1} \wedge \tau_{(N)}}^\pi - \sum_{j=1}^{t_{i+1} \wedge \tau_{(N)}} \xi_{j}\mathbb{1}_{t_j < \tau_i}\right) v(i+1, \xi^{(i-1)}, \zeta)|G_{t_i}\right],
\]

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v(1) := \text{ess sup}_{\zeta \in \mathcal{E}} \text{ess inf}_{\pi \in \mathcal{A}(t_1 \wedge \tau(N), t_2 \wedge \tau(N))} \mathbb{E} \left[ u(X_{t_1 \wedge \tau(N)}^{t_1 \wedge \tau(N), \pi} - \sum_{j=1}^{N} \zeta_j 1_{t_1 < \tau_j}) v(2, \zeta) \middle| \mathcal{G}_{t_1 \wedge \tau(N)} \right].

Proof. The proof of this theorem is based on the results of Lemma B.3 and Lemma B.4, which hold in the case with \( N \) variable annuities. However, in Lemma B.3 we now have to study BSDEs of the following form, instead of the BSDE (B.4)

\[ Y_t = F - \int_t^{k+1 \wedge \tau(N)} \theta(s) \xi(s) ds - \int_t^{k+1 \wedge \tau(N)} \zeta(s) dB_s - \int_t^{k+1 \wedge \tau(N)} \nu(s) dN_s, \]

where \( N_s := \sum_{i=1}^{N} 1_{t_i \leq s} \). We refer to [16] for the existence and uniqueness results. For Lemma B.4 we use a vector of size \( N \) for \( \zeta \).

Since the results of Lemma B.3 and Lemma B.4 hold, we can use the arguments of the proof in Annexe C to prove Theorem 6.2.

\[ \square \]

\section{A Appendix: Preliminary results}

\subsection{A.1 Preliminary results}

\textbf{Lemma A.1.} There exists a strategy \( \pi^{*,n} \in \mathcal{A}[T \wedge \tau, T] \) such that

\[ \inf_{\pi \in \mathcal{A}[T \wedge \tau, T]} \mathbb{E} \left[ \exp(-\gamma X_T^{T \wedge \tau, \pi}) \middle| \mathcal{G}_{T \wedge \tau} \right] = \mathbb{E} \left[ \exp(-\gamma X_T^{T \wedge \tau, \pi^{*,n}}) \middle| \mathcal{G}_{T \wedge \tau} \right]. \]

Moreover, there exists a process \( Y^{(n)} \) such that

\[ \inf_{\pi \in \mathcal{A}[T \wedge \tau, T]} \mathbb{E} \left[ \exp(-\gamma X_T^{T \wedge \tau, \pi}) \middle| \mathcal{G}_{T \wedge \tau} \right] = \exp(\gamma Y^{(n)}_{T \wedge \tau}), \]

where \( (Y^{(n)}, Z^{(n)}) \) is solution of the BSDE

\[ \begin{cases} dY^{(n)}_t = \left[ \frac{|\theta_t|^2}{\gamma} + \theta_t Z^{(n)}_t \right] dt + Z^{(n)}_t dB_t, \\
Y^{(n)}_T = 0. \end{cases} \]

The optimal strategy \( \pi^{*,n} \) is given by

\[ \pi^{*,n}_t = \frac{1}{\sigma_t \gamma + Z^{(n)}_t}. \]

Proof. The proof of this lemma is similar to the proof of Theorem 7 in [15] therefore we only give a sketch of the proof. We look for a process \( Y^{(n)} \) such that the family of processes \( \{J^{(n)}(\pi), \pi \in \mathcal{A}[T \wedge \tau, T]\} \) defined for any \( \pi \in \mathcal{A}[T \wedge \tau, T] \) by

\[ J^{(n)}_t(\pi) := \exp \left( -\gamma (X_T^{T \wedge \tau, \pi} - Y^{(n)}_t) \right), \quad \forall t \in [T \wedge \tau, T], \]

satisfies the following conditions

(i) \( J^{(n)}_t(\pi) \) is a random variable \( \mathcal{G}_{T \wedge \tau} \)-measurable whose value does not depend on the value of \( \pi \) on \( [T \wedge \tau, T] \).

(ii) \( J^{(n)}_T(\pi) = \exp(-\gamma X_T^{T \wedge \tau, \pi}). \)
(iii) $J^{(n)}(\pi)$ is a submartingale for any $\pi \in \mathcal{A}[T \wedge \tau, T]$ on the time interval $[T \wedge \tau, T]$.

(iv) There exists a strategy $\pi^{*,n} \in \mathcal{A}[T \wedge \tau, T]$ such that $J^{(n)}(\pi^{*,n})$ is a martingale on the time interval $[T \wedge \tau, T]$.

Assume that the process $Y^{(n)}$ solves the following BSDE
\begin{equation}
\begin{cases}
- dY^{(n)}_t = f(t, Y^{(n)}_t, Z^{(n)}_t) dt - Z^{(n)}_t dB_t , \\
Y^{(n)}_T = 0.
\end{cases} 
\tag{A.1}
\end{equation}

Recall that we must choose a function $f$ for which the family $\{J^{(n)}(\pi), \mathcal{A}[T \wedge \tau, T]\}$ satisfies the previous conditions. From usual results, we obtain that the function $f$ is defined by
$$f(t, y, z) = -|\theta_t|^2 \gamma - \theta_t z,$$
and the candidate to be $\pi^{*,n}$ is given by
$$\pi^{*,n}_t = \frac{1}{\sigma_t} \left[ \frac{\theta_t}{\gamma} + Z^{(n)}_t \right].$$

Since the generator $f$ is Lipschitz, we know that there exists a unique solution $(Y^{(n)}, Z^{(n)})$ to the BSDE (A.1). To finish the proof we must prove that the family $\{J^{(n)}(\pi), \mathcal{A}[T \wedge \tau, T]\}$ satisfies the previous conditions, but it is the same proof as the one given in [15].

**Lemma A.2.** Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables valued in $\hat{\mathcal{W}}$. Then there exists a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ of $(X_n)_{n \in \mathbb{N}}$ and a random variable $X_\infty$ such that
$$X_{n_k} \rightarrow X_\infty \quad \text{as} \quad k \rightarrow +\infty \quad \mathbb{P} - \text{a.s.}$$

**Proof.** We first notice that since $\hat{\mathcal{W}}$ is finite, the set $\hat{\mathcal{W}}^\mathbb{N}$ is countable. For a sequence $w = (w_n)_{n \in \mathbb{N}} \in \hat{\mathcal{W}}^\mathbb{N}$, we define the subset $\Omega_w$ of $\Omega$ by
$$\Omega_w = \left\{ \omega \in \Omega : (X_n(\omega))_{n \in \mathbb{N}} = w \right\}. \tag{A.2}$$

Then $\Omega_w \in \mathcal{G}$ for all $w \in \hat{\mathcal{W}}^\mathbb{N}$, $\Omega_w \cap \Omega_w' = \emptyset$ for $w, w' \in \hat{\mathcal{W}}^\mathbb{N}$ such that $w \neq w'$ and
$$\Omega = \bigcup_{w \in \hat{\mathcal{W}}^\mathbb{N}} \Omega_w. \tag{A.3}$$

Since $\hat{\mathcal{W}}^\mathbb{N}$ is countable, we can enumerate its elements as
$$\hat{\mathcal{W}}^\mathbb{N} = \left\{ w^1, w^2, w^3, \ldots \right\} = \left\{ w^k, \ k \in \mathbb{N} \right\}.$$

Since $w^1$ is valued in $\hat{\mathcal{W}}$ which is finite (and hence compact), there exists an increasing function $\varphi_1 : \mathbb{N} \rightarrow \mathbb{N}$ such that $(w^1_{\varphi_1(n)})_{n \in \mathbb{N}}$ converges to some $w^1_\infty \in \hat{\mathcal{W}}$.

Consider now the sequence $(w^2_{\varphi_1(n)})_{n \in \mathbb{N}}$. It is also valued in the compact set $\mathcal{W}$. We can therefore find an increasing function $\varphi_2 : \mathbb{N} \rightarrow \mathbb{N}$ such that $(w^2_{\varphi_1 \circ \varphi_2(n)})_{n \in \mathbb{N}}$ converges to some $w^2_\infty \in \hat{\mathcal{W}}$. 

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We proceed in the same way for the following integers to get a sequence \((\varphi^n)_{n \in \mathbb{N}}\) of non-decreasing functions from \(\mathbb{N}\) to \(\mathbb{N}\) such that \((w^k_{n\varphi_1 \cdots \varphi_k(n)})_{n \in \mathbb{N}}\) converges to some \(w^k_{\infty}\) for all \(k \in \mathbb{N}\).

Next, we define the function \(\varphi : \mathbb{N} \to \mathbb{N}\) by
\[
\varphi(n) = \varphi_1 \circ \cdots \circ \varphi_n(n)
\]
for all \(n \in \mathbb{N}\). From the construction of the functions \((\varphi_n)_{n \in \mathbb{N}}\) and the definition of \(\varphi\) we obtain that \((w^k_{\varphi(n)})_{n \in \mathbb{N}}\) converges to \(w^k_{\infty}\) for all \(k \in \mathbb{N}\).

Using (A.2) and (A.3) we get that the sequence \((X_{\varphi(n)})_{n \in \mathbb{N}}\) converges \(\mathbb{P}\)-a.s. to the random variable \(X_{\infty}\) defined by
\[
X_{\infty}(\omega) = w^k_{\infty}
\]
for all \(\omega \in \Omega_{w^k}\) and all \(k \in \mathbb{N}\). \(\square\)

## B Optimal strategies for the sub-period problems

In this part, we show that there exists optimal strategies for each sub-period problem. The two following lemmas give the optimal strategies in \(\pi\) (resp. in \(\xi\)). These two results will be used to prove Theorem 4.1.

**Lemma B.3.** Fix \(k \in \{0, \ldots, n-1\}\). Let \(H_{k+1}\) be an application from \(\mathcal{W}^k\) to \(L^\infty(\Omega, \mathcal{G}_{[t_{k+1} \wedge \tau], \mathbb{P}})\). Then for any \(\xi \in \mathcal{E}\) there exists a strategy \(\pi^{*,k}(\xi(k))\) and a random variable \(y^{(k)}(\xi(k)) \in L^\infty(\Omega, \mathcal{G}_{t_{k+1} \wedge \tau}, \mathbb{P})\) such that
\[
\mathbb{E}\left[ \exp\left( -\gamma (X_{t_{k+1} \wedge \tau}) - H_{k+1}(\hat{\xi}(k)) \right) \mid \mathcal{G}_{t_{k+1} \wedge \tau} \right] = \mathbb{E}\left[ \exp\left( -\gamma (X_{t_{k+1} \wedge \tau}) - y^{(k)}(\xi(k)) \right) \mid \mathcal{G}_{t_{k+1} \wedge \tau} \right] = \exp\left( \gamma y^{(k)}(\xi(k)) \right).
\]

**Proof.** As for the proof of Lemma A.1 we look for a process \(Y^{(k)}\) such that the family of processes \((R^{(k)}(\pi))_{\pi \in \mathcal{A}[t_{k} \wedge \tau, t_{k+1} \wedge \tau]}\), where \(R^{(k)}(\pi)\) is defined by
\[
R^{(k)}_t(\pi) = \exp\left( -\gamma (X_{t}^{t_{k} \wedge \tau}) - Y_t^{(k)} \right),
\]
satisfies the following conditions

(i) \(R^{(k)}_{t_{k} \wedge \tau}(\pi)\) is a random variable \(\mathcal{G}_{t_{k} \wedge \tau}\)-measurable whose value does not depend on the value of \(\pi\) on \([t_{k} \wedge \tau, t_{k+1} \wedge \tau]\).

(ii) \(R^{(k)}_{t_{k+1} \wedge \tau}(\pi) = \exp(-\gamma (X_{t_{k+1}}^{t_{k} \wedge \tau}) - H_{k+1}(\hat{\xi}(k)))\).

(iii) \(R^{(k)}(\pi)\) is a submartingale for any \(\pi \in \mathcal{A}[t_{k} \wedge \tau, t_{k+1} \wedge \tau]\) on the time interval \([t_{k} \wedge \tau, t_{k+1} \wedge \tau]\).

(iv) There exists a strategy \(\pi^{*,k}(\xi(k)) \in \mathcal{A}[t_{k} \wedge \tau, t_{k+1} \wedge \tau]\) such that \(R^{(k)}(\pi^{*,k}(\xi(k)))\) is a martingale on the time interval \([t_{k} \wedge \tau, t_{k+1} \wedge \tau]\).
Assume that the process $Y^{(k)}$ solves the following BSDE

$$
\begin{align*}
-dY_t^{(k)} &= f(t, Y_t^{(k)}, Z_t^{(k)}, U_t^{(k)})dt - Z_t^{(k)}dB_t - U_t^{(k)}dN_t, \\
Y_{t_{k+1} \wedge \tau} &= H_{k+1}(\hat{\xi}^{(k)}).
\end{align*}
$$

(B.4)

After some calculus we get that the candidate $\pi^{*,k}(\xi^{(k)})$ is given by

$$
\pi_t^{*,k}(\xi^{(k)}) = \frac{Z_t^{(k)}}{\sigma_t} + \frac{\theta_t}{\gamma} \sigma_t,
$$

and

$$
f(t, y, z, u) = \lambda_t \left( e^{\gamma u} - 1 \right) - \theta_t z - \frac{|\theta_t|^2}{2\gamma}.
$$

To prove that the BSDE (B.4) admits a solution we use the result given in [10], and the end of the proof is similar to the verification theorem given in this companion paper. Therefore if we choose $g^{(k)}(\xi^{(k)}) = Y_{t_k}^{(k)}$ we get the result. \(\square\)

**Remark B.1.** The hypothesis that $H_{k+1} \in L^\infty(\Omega, \mathcal{G}_{t_{k+1} \wedge \tau}, \mathbb{P})$ is crucial to obtain a solution to the BSDE (B.4). That is why we assume that the pay-off of the contract is bounded, otherwise $H_n$ will not be bounded.

**Lemma B.4.** Fix $k \in \{0, \ldots, n-1\}$. Let $H_{k+1}$ be an application from $\mathcal{W}^k$ to $L^\infty(\Omega, \mathcal{G}_{t_{k+1} \wedge \tau}, \mathbb{P})$. Then for any $\xi \in \mathcal{E}$ there exists $\zeta^* \in \mathcal{E}_{k-1}^1$ such that

$$
\begin{align*}
\text{ess sup}_{\zeta \in \mathcal{E}_{k-1}^1} \mathbb{E}_t \left[ u(X_{t_k + 1}^{\wedge \tau}, \xi - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta})) \big| \mathcal{G}_{t_k \wedge \tau} \right] \\
&= \mathbb{E}_t \left[ u(X_{t_k + 1}^{\wedge \tau}, \zeta^* - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta})) \big| \mathcal{G}_{t_k \wedge \tau} \right],
\end{align*}
$$

for all $\pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]$.

**Proof.** From a classical result on the essential supremum of a family of random variables (see e.g. [21]), there exists a sequence $(\zeta^\ell)_{\ell \in \mathbb{N}}$ valued in $\mathcal{E}_{k-1}^1$ such that

$$
u\left( -\zeta^\ell \mathbb{1}_{t_k \wedge \tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^\ell) \right) \rightarrow \text{ess sup}_{\zeta \in \mathcal{E}_{k-1}^1} u\left( -\zeta \mathbb{1}_{t_k \wedge \tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}) \right)
$$

\(\text{P-a.s. as } \ell \text{ goes to infinity. We now apply Lemma A.2 to the sequence } (\zeta^\ell e^{\int_0^t r_s ds})_{\ell \in \mathbb{N}} \text{ and we obtain that, up to a subsequence, } (\zeta^\ell)_{\ell \in \mathbb{N}} \text{ converges } \text{P-a.s. to some random variable } \zeta^*.

From (B.5) we get

$$
u\left( -\zeta^* \mathbb{1}_{t_k \wedge \tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}^*) \right) = \text{ess sup}_{\zeta \in \mathcal{E}_{k-1}^1} u\left( -\zeta \mathbb{1}_{t_k \wedge \tau} - H_{k+1}(\hat{\xi}^{(k-1)}, \hat{\zeta}) \right).
$$

(B.6)
We then notice from the multiplicative structure of the function \( u \) that for any \( \pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau] \) and any \( \zeta \in \mathcal{E}_{k-1}^1 \) we have

\[
\mathbb{E}\left[u(X_{t_{k+1} \wedge \tau}^{t_k \wedge \tau, \pi} - \zeta \mathbb{1}_{t_k < \tau} - H_{k+1}(\hat{\zeta}^{(k-1)}, \hat{\zeta}^*)\right| \mathcal{G}_{t_k \wedge \tau}] \leq \mathbb{E}\left[u(X_{t_{k+1} \wedge \tau}^{t_k \wedge \tau, \pi} - \zeta \mathbb{1}_{t_k < \tau} - H_{k+1}(\hat{\zeta}^{(k-1)}, \hat{\zeta}^*)\right| \mathcal{G}_{t_k \wedge \tau}]
\]

Therefore we get

\[
\mathbb{E}\left[u(X_{t_{k+1} \wedge \tau}^{t_k \wedge \tau, \pi} - \zeta \mathbb{1}_{t_k < \tau} - H_{k+1}(\hat{\zeta}^{(k-1)}, \hat{\zeta}^*)\right| \mathcal{G}_{t_k \wedge \tau}] = \text{ess sup}_{\zeta \in \mathcal{E}_{k-1}^1} \mathbb{E}\left[u(X_{t_{k+1} \wedge \tau}^{t_k \wedge \tau, \pi} - \zeta \mathbb{1}_{t_k < \tau} - H_{k+1}(\hat{\zeta}^{(k-1)}, \hat{\zeta}^*)\right| \mathcal{G}_{t_k \wedge \tau}],
\]

for all \( \pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau] \). \( \square \)

\section*{C Proof of Theorem 4.1}

We shall now prove the result in two steps. For each step, we use method of induction (forward in the first, backward in the second).

**First step.** We first show that the following inequality holds

\[
w(p) \geq \inf_{\pi \in \mathcal{A}[0,t_1 \wedge \tau]} \mathbb{E}\left[u(X_{t_1 \wedge \tau}^\pi) v(1)\right]. \quad (C.7)
\]

We prove this inequality by induction on the number \( k \) of anniversary dates. More precisely, we show that for any \( k \in \{1, \ldots, n\} \) and any map \( H_k \) from \( \mathcal{W}^{k-1} \) to \( L^\infty(\Omega, \mathcal{G}_{t_1 \wedge \tau}, \mathbb{P}) \), we have

\[
v_{H_k} := \inf_{\pi \in \mathcal{A}[0,t_1 \wedge \tau]} \sup_{\xi \in \mathcal{E}_{k-1}^0} \mathbb{E}\left[u(X_{t_k \wedge \tau}^{t_1 \wedge \tau} - \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H_k(\hat{\xi}^{(k-1)}))\right] \geq \inf_{\pi \in \mathcal{A}[0,t_1 \wedge \tau]} \mathbb{E}\left[u(X_{t_k \wedge \tau}^{t_1 \wedge \tau}) \hat{v}_{H_k}(1)\right], \quad (C.8)
\]

where \( \hat{v}_{H_k}(i, \xi^{(i-1)}) \) is defined recursively for any \( i \in \{1, \ldots, k\} \) and \( \xi \in \mathcal{E}_{k-1}^0 \) by

\[
\begin{align*}
\hat{v}_{H_k}(k, \xi^{(k-1)}) &:= e^{\gamma H_k(\xi^{(k-1)})}, \\
\hat{v}_{H_k}(i, \xi^{(i-1)}) &:= \text{ess sup}_{\zeta \in \mathcal{E}_{i-1}^1} \text{ess inf}_{\pi \in \mathcal{A}[i \wedge \tau, t_{i+1} \wedge \tau]} \hat{J}(i, \pi, \xi^{(i-1)}, \zeta),
\end{align*}
\]

with

\[
\hat{J}(i, \pi, \xi^{(i-1)}, \zeta) := \mathbb{E}\left[u(X_{t_{i+1} \wedge \tau}^{t_i \wedge \tau} - \zeta \mathbb{1}_{t_i < \tau}) \hat{v}_{H_k}(i+1, (\xi^{(i-1)}, \zeta))\right| \mathcal{G}_{t_i \wedge \tau}]
\]
for any \( i \in \{1, \ldots, k-1 \} \), \( \pi, \zeta \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau] \) and \( \xi \in \mathcal{E}_{t_{i-1}}^1 \).

By abuse of notation, we denote \( \hat{v}_{H_k}(1, \xi(0)) \) for \( \hat{v}_{H_k}(1) \).

For \( k = 1 \), the inequality (C.8) is obvious.

We now assume the result holds for some \( k \in \{1, \ldots, n-1 \} \). Let \( H_{k+1} \) be a map from \( \hat{W}^k \) to \( L^\infty(\Omega, \mathcal{G}_{t_k+1} \wedge \tau, \mathbb{P}) \). We define

\[
K(k+1, \pi, \xi) := \mathbb{E}\left[ u\left( X_{t_k+1}^{\pi} - \sum_{i=1}^{k} \xi_i I_{t_i \leq \tau} - H_{k+1}(\hat{\xi}(k)) \right) \right]
\]

for any \( \pi \in \mathcal{A}[0, t_{k+1} \wedge \tau] \) and \( \xi \in \mathcal{E}_0^{k} \). From Lemma B.3 there exists \( \pi^{*,k}(\xi) \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau] \) such that

\[
\hat{v}_{H_{k+1}}(k, \xi^{(k-1)}) = \text{ess sup}_{\xi \in \mathcal{E}_{t_{k-1}}^{k}} \mathbb{E}\left[ u\left( X_{t_k+1}^{\pi} - \xi I_{t_k < \tau} - H_{k+1}(\hat{\xi}(k), \xi^{(k-1)}, \xi^{\pi^{*,k}(\xi^{(k-1)}, \xi^{*})}) \right) \right].
\]

From Lemma B.4 and the previous equality, there exists \( \xi^* \in \mathcal{E}_{t_{k-1}}^{k} \) such that

\[
K(k+1, \pi, \xi) \geq K(k+1, \pi I_{[0, t_k \wedge \tau]} + \pi^{*,k}(\xi^{(k)}) I_{[t_k \wedge \tau, t_{k+1} \wedge \tau]}, \xi),
\]

for any \( \pi \in \mathcal{A}[0, t_{k+1} \wedge \tau] \) and \( \xi \in \mathcal{E}_0^{k} \). This implies

\[
\sup_{\xi \in \mathcal{E}_0^{k}} K(k+1, \pi, \xi) \geq \sup_{\xi \in \mathcal{E}_0^{k}} K(k+1, \pi I_{[0, t_k \wedge \tau]} + \pi^{*,k}(\xi^{(k)}) I_{[t_k \wedge \tau, t_{k+1} \wedge \tau], \xi})
\]

From the definition of \( K \) we have

\[
K(k+1, \pi I_{[0, t_k \wedge \tau]} + \pi^{*,k}(\xi^{(k-1)}, \xi^{*}) I_{[t_k \wedge \tau, t_{k+1} \wedge \tau], \xi^{(k-1)}, \xi^{*}})
\]

\[
= \mathbb{E}\left[ u\left( X_{t_k+1}^{\pi} - \sum_{i=1}^{k-1} \xi_i I_{t_i \leq \tau} \right) \mathbb{E}\left[ u\left( X_{t_k+1}^{\pi, \pi^{*,k}(\xi^{(k-1)}, \xi^{*})} - \xi^* I_{t_k < \tau} - H_{k+1}(\xi^{(k-1)}, \xi^{*}) \right) \right] \mathbb{E}\left[ G_{t_k} \right] \right]
\]

\[
= \mathbb{E}\left[ u\left( X_{t_k+1}^{\pi} - \sum_{i=1}^{k} \xi_i I_{t_i \leq \tau} \right) \hat{v}_{H_{k+1}}(k, \xi^{(k-1)}) \right].
\]
Therefore we get
\[
\sup_{\xi \in \mathcal{E}_0^k} K(k + 1, \pi, \xi) \geq \sup_{\xi \in \mathcal{E}_0^{k-1}} \mathbb{E}\left[u\left(X_{t_k}^{\pi} - \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H_{k+1}(k, \xi^{(k-1)})\right)\right]. \tag{C.9}
\]
We now define the application \( H_k \) by
\[
H_k(x_1, \ldots, x_{k-1}) := \frac{1}{\gamma} \log \left( \hat{v}_{H_{k+1}}(k, (x_1 e^{-\int_0^t r_s ds}, \ldots, x_{k-1} e^{-\int_0^{k-1} r_s ds})) \right),
\]
for all \((x_1, \ldots, x_{k-1}) \in \mathcal{W}^{k-1}\). From Lemma B.3, we have
\[
H_k(x_1, \ldots, x_{k-1}) = y^{(k)}(x_1, \ldots, x_{k-1}, \xi^*) \in L^\infty(\Omega, \mathcal{G}_{t_1} \wedge \tau, \mathbb{P})
\]
We can then use the induction hypothesis and we get
\[
\inf_{\pi \in \mathcal{A}[0, t_1 \wedge \tau]} \sup_{\xi \in \mathcal{E}_0^{k-1}} \mathbb{E}\left[u(X_{t_k}^{\pi} - \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H_k(\xi^{(k-1)})\right] \geq \inf_{\pi \in \mathcal{A}[0, t_1 \wedge \tau]} \mathbb{E}\left[u(X_{t_1}^{\pi}) \hat{v}_{H_k}(1)\right]. \tag{C.10}
\]
Taking the infimum in (C.9) we obtain
\[
\inf_{\pi \in \mathcal{A}[0, t_{k+1} \wedge \tau]} \sup_{\xi \in \mathcal{E}_0^{k}} K(k + 1, \pi, \xi) \geq \inf_{\pi \in \mathcal{A}[0, t_1 \wedge \tau]} \sup_{\xi \in \mathcal{E}_0^{k-1}} \mathbb{E}\left[u(X_{t_k}^{\pi} - \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{t_i \leq \tau} - H_k(\xi^{(k-1)})\right].
\]
Using (C.10) we get
\[
\inf_{\pi \in \mathcal{A}[0, t_{k+1} \wedge \tau]} \sup_{\xi \in \mathcal{E}_0^{k}} K(k + 1, \pi, \xi) \geq \inf_{\pi \in \mathcal{A}[0, t_1 \wedge \tau]} \mathbb{E}\left[u(X_{t_1}^{\pi}) \hat{v}_{H_k}(1)\right].
\]
We conclude the proof of the inequality (C.7) by noticing that
\[
\hat{v}_{H_k}(k, \xi^{(k-1)}) = \hat{v}_{H_{k+1}}(k, \xi^{(k-1)}).
\]
Hence, we have \( \hat{v}_{H_k}(1) = \hat{v}_{H_{k+1}}(1) \) and then the inequality (C.8) holds for any \( k \in \{1, \ldots, n\} \).

**Second step.** We now prove the following inequality
\[
w(p) \leq \inf_{\pi \in \mathcal{A}[0, t_1 \wedge \tau]} \mathbb{E}\left[u(X_{t_1}^{\pi}) v(1)\right]. \tag{C.11}
\]
To this end, we show by a backward induction on \( k \in \{1, \ldots, n-1\} \) that for \( \xi^{(k)} \in \mathcal{E}_0^k \) and \( H \) a bounded \( \mathcal{G}_{T \wedge \tau} \)-measurable random variable, we have
\[
\operatorname{ess inf}_{\pi \in \mathcal{A}[t_k \wedge \tau, T \wedge \tau]} \operatorname{ess sup}_{\xi \in \mathcal{E}_0^{n-k-1}} \mathbb{E}\left[u(X_{t_k}^{\pi} - \xi_k \mathbb{1}_{t_k < \tau} - \sum_{j=1}^{n-k-1} \xi_j \mathbb{1}_{t_{k+j} < \tau} - H)\mathbb{1}_{\mathcal{G}_{t_k} \wedge \tau}\right]
\leq \operatorname{ess inf}_{\pi \in \mathcal{A}[t_k \wedge \tau, t_{k+1} \wedge \tau]} \mathbb{E}\left[u(X_{t_{k+1}}^{\pi} - \xi_k \mathbb{1}_{t_k < \tau}) v(k+1, \xi^{(k)})\mathbb{1}_{\mathcal{G}_{t_k} \wedge \tau}\right]. \tag{C.12}
\]
For $k = n - 1$, the previous inequality obviously holds from the definition of $v(n, \cdot)$.
We now assume that the inequality (C.12) holds for some $k \in \{2, \ldots, n - 1\}$ and we prove it for $k - 1$.

We first write $\pi = (\pi^1, \ldots, \pi^{n-1})$ for $\pi \in A[t_k \wedge \tau, \ldots, T \wedge \tau]$ with $\pi^k \in A[t_k \wedge \tau, t_{k+1} \wedge \tau]$.
We then define the map $L$ by

$$L(\ell, \pi^1, \ldots, \pi^{n-1}, \xi(\ell)) = \text{ess sup}_{\xi \in \xi^\ell} E \left[ u(X_{t_\ell \wedge \tau} - \xi_\ell 1_{t_\ell < \tau} - \sum_{j=1}^{n-\ell} \xi_{\ell+j} 1_{t_{\ell+j} < \tau} - H) \mid G_{t_\ell \wedge \tau} \right],$$

for $\ell \in \{1, \ldots, n - 1\}$, $\xi \in \mathcal{E}$ and $\pi \in A[0, T \wedge \tau]$.

Let $H$ be a bounded $G_{T \wedge \tau}$-measurable random variable, $\pi \in A[t_{k-1} \wedge \tau, T \wedge \tau]$ and $\xi \in \mathcal{E}$.
By definition of $L$ we know that

$$L(k - 1, \pi, \xi^{\ell+1}) \leq \text{ess sup}_{\xi \in \xi^{\ell+1}} E \left[ u(X_{t_{k-1} \wedge \tau} - \xi_{k-1} 1_{t_{k-1} < \tau}) \mid G_{t_{k-1} \wedge \tau} \right]. \quad (C.13)$$

Using Lemma B.4, we can find $\xi_k^k \in \xi_k^1$ (depending only on $\pi^{k+1}, \ldots, \pi^{n-1}$) such that

$$\text{ess sup}_{\xi_k \in \xi_k^1} L(k, \pi^k, \pi^{k+1}, \ldots, \pi^{n-1}, (\xi^{k-1}, \xi_k)) = L(k, \pi^k, \pi^{k+1}, \ldots, \pi^{n-1}, (\xi^{k-1}, \xi_k^k)),$$

for all $\pi^k \in A[t_k \wedge \tau, t_{k+1} \wedge \tau]$. We also know from Lemma B.3 that there exists $\pi^{k*} \in A[t_k \wedge \tau, t_{k+1} \wedge \tau]$ (also depending on $\pi^{k+1}, \ldots, \pi^{n-1}$) such that

$$L(k, \pi^{k*}, \pi^{k+1}, \ldots, \pi^{n-1}, (\xi^{k-1}, \xi_k^*) = \text{ess inf}_{\pi^k \in A[t_k \wedge \tau, t_{k+1} \wedge \tau]} L(k, \pi^k, \pi^{k+1}, \ldots, \pi^{n-1}, (\xi^{k-1}, \xi_k^*)). \quad (C.15)$$

For $\pi \in A[0, T \wedge \tau]$, define $\bar{\pi}$ by

$$\bar{\pi} := \pi 1_{[0, t_{k} \wedge \tau]} + \pi^{k*} 1_{[t_k \wedge \tau, t_{k+1} \wedge \tau]} + \pi^{k+1} 1_{[t_{k+1} \wedge \tau, t_{k+2} \wedge \tau]} + \cdots + \pi^{n-1} 1_{[t_{n-1} \wedge \tau, T \wedge \tau]}.$$

From (C.13), (C.14) and (C.15) we deduce

$$L(k - 1, \bar{\pi}, (\xi^{k-1})) \leq E \left[ u(X_{t_{k-1} \wedge \tau} - \xi_{k-1} 1_{t_{k-1} < \tau}) \mid G_{t_{k-1} \wedge \tau} \right] \quad \text{ess inf}_{\pi \in A[t_{k-1} \wedge \tau, T \wedge \tau]} L(k, \pi, (\xi^{k-1}, \xi_k^*), H) \mid G_{t_{k-1} \wedge \tau}. \quad (C.16)$$

By taking the infimum we get

$$\text{ess inf}_{\pi \in A[t_{k-1} \wedge \tau, T \wedge \tau]} L(k - 1, \pi, (\xi^{k-1}), H) \leq \text{ess inf}_{\pi \in A[t_{k-1} \wedge \tau, t_k \wedge \tau]} L(k - 1, \bar{\pi}, (\xi^{k-1}), H).$$
From this last inequality, (C.16) and the induction assumption we get

\[ \text{ess inf} \pi \in A(t_{k-1} \wedge \tau, \tau \wedge \tau) L(k-1, \pi, \xi^{(k-1)}, H) \leq \text{ess inf} \pi \in A(t_{k-1} \wedge \tau, t_k \wedge \tau) \mathbb{E}\left[ u(\chi_{t_{k-1}}^{\tau}, \pi, \xi_{t_{k-1}}^{\tau} \mathbb{1}_{t_{k-1} < \tau}) v(k, \xi^{(k-1)}) \left| G_{t_{k-1}} \right. \right], \]

which gives the result for \( k - 1 \) from the definition of the map \( L \).

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**References**


