Finance Stochast. 5, 305–325 (2001)

Finance and
Stochastics

Arbitrage and investment opportunities

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Abstract. We consider a model in which any investment opportunity is described in terms of cash flows. We don't assume that there is a numéraire, enabling investors to transfer wealth through time; the time horizon is not supposed to be finite and the investment opportunities are not specifically related to the buying and selling of securities on a financial market. In this quite general framework, we show that the assumption of no-arbitrage is essentially equivalent to the existence of a "discount process" under which the "net present value" of any available investment is nonpositive. Since most market imperfections, such as short sale constraints, convex cone constraints, proportional transaction costs, no borrowing or different borrowing and lending rates, etc., can fit in our model for a specific set of investments, we then obtain a characterization of the no-arbitrage condition in these imperfect models, from which it is easy to derive pricing formulae for contingent claims.

Key words: arbitrage, investment opportunities, numéraire, market frictions, Yan's Theorem

JEL Classification: C60, G19

Mathematics Subject Classification (1991): 46E30, 60G40, 60G44

1 Introduction

In a perfect financial model, the Fundamental Theorem of Asset Pricing (see Harrison and Kreps 1979 or Harrison and Pliska 1981) asserts that the assumption of no-arbitrage (which amounts to saying that there is no plan yielding some profit

Manuscript received: December 1998; final version received: June 2000

without a countervailing threat of loss) is essentially equivalent to the existence of an equivalent martingale measure. The problem of fair pricing of financial assets is then reduced to taking their expected value with respect to equivalent martingale measures.

We want to find in this paper the analog of the Fundamental Theorem of Asset Pricing for general models of investment opportunities. In a deterministic setting, such opportunities are described in discrete time by Cantor and Lippman (1983) and Dermody and Rockafellar (1991, 1995) and by Carassus and Jouini (1998) in continuous time. More precisely, we adopt a model where all investment opportunities are described by their cash flows; for instance, in such a model, the investment opportunity which consists in buying, in a perfect financial model, at date s_1 one unit of a risky asset, whose price process is given by $(S_t)_{t\in R_+}$ and selling at date s_2 with $s_1 \leq s_2$ the unit bought, is described by a process $(\Phi_t)_{t\in R_+}$ which is null outside $\{s_1, s_2\}$ and which satisfies $\Phi_{s_1} = -S_{s_1}$ and $\Phi_{s_2} = S_{s_2}$.
Our investment opportunities are assumed to be quite general: they are not specifically related to a market model, like in the just mentioned example. The time horizon is not supposed to be finite. The framework is the one of continuous time. We don't assume that there exists a numéraire, enabling investors to transfer wealth from one date to another and even if such possibilities exist, we do not assume that the lending rate is equal to the borrowing rate or that we have possibilities to borrow.

We find in this general model that the assumption of no-free lunch is essentially equivalent to the existence of a normalization process such that the "net present value" of any available investment opportunity is nonpositive. We emphasize that neither interest rate nor net present value are part of our model. As in Dermody and Rockafellar (1991, 1995), there is no externally given term structure, which would be needed if one were to apply the classical criterion of net present value; these notions arise however as a consequence of the assumption of no-arbitrage.
We then use this general result for specific financial market models: perfect financial model, for which we obtain the well-known characterization of the assumption of no-arbitrage given by the Fundamental Theorem of Asset Pricing in finite time, and a slightly different version in infinite time; but mainly financial models with frictions like imperfections on the numéraire, proportional transaction costs, short sale constraints, convex cone constraints, etc., for which we generalize existing results.

Initial results on the Fundamental Theorem of Asset Pricing in the perfect case were achieved by Harrison and Kreps (1979), Harrison and Pliska (1981), Kreps (1981) and Duffie and Huang (1986). Various generalizations are now available in the literature: in Dalang et al. (1989), the problem is solved in the case of finite discrete time, by only using the assumption of no-arbitrage. For discrete infinite or continuous time, the notion of free lunch is needed; Schachermayer (1994) mainly deals with the case of discrete infinite time. Continuous time models are studied by (among others) Ansel and Stricker (1990), Delbaen (1992), Delbaen and Schachermayer (1994, 1998), Frittelli and Lakner (1994), Stricker (1990). In all these models, securities markets are assumed to be frictionless.

In the context of imperfect financial markets, Jouini and Kallal (1995a) characterize the assumption of no-free lunch in a model with proportional transaction costs and give fair pricing intervals for contingent claims in such a model. Cvitanic and Karatzas (1996) study the problem of hedging contingent claims, in continuous time, for a diffusion model (with one bond and one risky asset) with proportional transaction costs, and give a dual formula for the so-called superreplication price of a contingent claim (i.e. the minimum initial wealth needed to hedge the contingent claim or in other words, to obtain, through the investment opportunities available on the market, at least the contingent claim). Delbaen et al. (1998) and Kabanov (1998) generalize this result to the multivariate case, in discrete as well as in continuous time, and with a semimartingale price process.

As for other imperfections, Jouini and Kallal (1995b) study the case of short sale constraints or shortselling costs with possibly different borrowing and lending rates. For convex constraints (and also with possible higher interest rates for borrowing), the dual formulation for the superreplication price is obtained in a diffusion framework in Cvitanic and Karatzas (1993). In a more general framework, the result is obtained in Föllmer and Kramkov (1997). Pham and Touzi (1996) consider the Fundamental Theorem of Asset Pricing with cone constraints, in a discrete and finite time setting and by only using the assumption of no-arbitrage. Brannath (1997) studies the same problem in the more general setting of convex constraints.

This paper generalizes Carassus and Jouini (1997), that considers discrete models, i.e. with finite time horizon, discrete time and finite number of states of the world at each time.

We generalize existing results in the following ways: first, we don’t assume that there exists a numéraire available to investors and allowing them to transfer wealth through time; this enables to consider any type of friction on the numéraire like no borrowing, different borrowing and lending rates, bonds with default risk, etc., which have been barely studied, or simply to take into account the fact that all investors are not equal with regard to borrowing and lending, namely some investors may enjoy special borrowing facilities while others may not; second, we are lead to introduce a new notion of no-free lunch, which is similar to the “usual one” (with deterministic times) in finite time but does not exclude a free lunch at infinity and is therefore maybe more economically meaningful; last, we characterize the assumption of no-arbitrage (or more precisely of no-free lunch) for general investments, which enables to consider investment opportunities that are not necessarily related to a market model and, more interestingly, to generalize the results obtained for imperfect markets and to obtain them all in a unified way.

The paper is organized as follows: in Sect. 2, we obtain, under a “reasonable” assumption, the characterization of the absence of free lunch in a general model with flows. Since we are not allowed to transfer wealth from one date to another, we cannot consider net gains anymore; in all papers dealing with the

Fundamental Theorem of Asset Pricing (with simple integrands), the assumption of no-arbitrage or no-free lunch essentially amounts to saying that the set $\overline{Lin \left\{\theta_s \cdot \left(S_t - S_s\right)\right\}}$, where θ describes all feasible strategies and S denotes the discounted underlying price process, contains no nonnegative nonnull random variable. Implicit in such an approach is the fact that there is an externally given term structure, enabling investors to borrow and lend money at the same rate. In the context with flows, we must use separation techniques in more complex spaces to obtain the Fundamental Theorem of Asset Pricing. We apply this characterization of the no-free lunch assumption to different market models with frictions in Sects. 3 and 4.

All the proofs are in the Appendix.

2 The fundamental theorem of asset pricing in a model with flows

As we have seen in the Introduction, since we are not necessarily allowed to transfer wealth through time, we must consider more general spaces than the classical L^p spaces. We start by introducing these spaces. Then we describe our general model with flows. Finally, we obtain, through an analog of Yan's (1980) result, the characterization of the no-free lunch assumption in such a model.

2.1 The space $L^1_P\left(\Omega, \mathscr{M}_b\right)$

For details about most notions introduced in this section, see Marle (1974) or Diestel and Uhl (1977).

We denote by $\mathscr{C}_0$ the set of continuous functions from R to R which converge to 0 at infinity; endowed with the uniform convergence topology, $\mathscr{C}_0$ is a Banach space. We denote by $\mathscr{M}_b$ the space of bounded Radon measures, i.e. the space of continuous linear functionals on $\mathscr{C}_0$; the space $\mathscr{M}_b$, endowed with the usual dual norm $\|\cdot\|_{\mathscr{M}_b}$ defined by $\|\mu\|_{\mathscr{M}_b} \equiv \sup\left\{|\mu(f)|\,; f \in \mathscr{C}_0; \|f\| \leq 1\right\}$ for all μ in $\mathscr{M}_b$, is a Banach space.

Fix a probability space (Ω, F, P). Let $\left(X, \|\cdot\|_X\right)$ be a Banach space. The set $St\,(\Omega, X)$ denotes the set of X-valued simple random variables, i.e. the set of random variables f of the form $f = \sum_{i=1}^m a_i 1_{A_i}$ for some m in N^*, A_i in F and a_i in X. The set $M_P\,(\Omega, X)$ denotes the set of P-measurable random variables, i.e. the set of random variables f such that there exists a sequence $(f_n)_{n\in N}$ in $St\,(\Omega, X)$ for which $f = \lim_n f_n$ a.s. P. Then $L^1_P\,(\Omega, X)$ denotes the set of P-measurable random variables f such that $\|f\|_X$ belongs to $L^1\,(\Omega, R)$, i.e. such that $\|f\|_{L^1_P(\Omega,X)} \equiv E^P\left[\|f\|_X\right] < \infty$.
Denote by $\left(X', \|\cdot\|_{X'}\right)$ the dual space of X, endowed with the dual norm. We know that $\left(X', \|\cdot\|_{X'}\right)$ is itself a Banach space. Let $\mathscr{L}^\infty_*\left(\Omega, P, X'\right)$ denote the set of random variables $g : \Omega \to X'$, for which

$$\langle g, e\rangle : \omega \to \langle g\,(\omega)\,, e\rangle \text{ is measurable for all } e \text{ in } X \text{ and}$$

$$\|g\|_{\mathscr{L}_*^\infty(\Omega,P,X')} \equiv \inf\{M \geq 0, \|g\|_{X'} \leq M \quad P \text{ a.s.}\} < \infty.$$

Then the set $L_*^\infty(\Omega, P, X')$ is defined to be the set of equivalence classes $g^\bullet$ of elements of $\mathscr{L}_*^\infty(\Omega, P, X')$, where $f \sim g \Leftrightarrow \langle f, e\rangle = \langle g, e\rangle$ P a.s. for all e in X. We consider the norm $\|\cdot\|_{L_*^\infty(\Omega,P,X')}$ given by $\|g\|_{L_*^\infty(\Omega,P,X')} = \inf_{g\in g^\bullet} \|g\|_{\mathscr{L}_*^\infty(\Omega,P,X')}$ for all $g \in L_*^\infty(\Omega, P, X')$.

It is shown in Schwartz (1974) that the dual space of $L_P^1(\Omega, X)$ is the space $L_*^\infty(\Omega, P, X')$; there is an isomorphism between $L_*^\infty(\Omega, P, X')$ and $\left[L_P^1(\Omega, X)\right]'$ that associates with any random variable g in $L_*^\infty(\Omega, P, X')$ the continuous linear functional Ψ on $L_P^1(\Omega, X)$ given by

$$\Psi : f \to \Psi(f) = E\left[\langle f, g\rangle_{X,X'}\right] \quad \text{for all } f \text{ in } L_P^1(\Omega, X),$$

where $\langle f, g\rangle_{X,X'} : \omega \mapsto \langle f(\omega), g(\omega)\rangle_{X,X'}$. Notice then that $L_*^\infty(\Omega, P, X')$ is a Banach space.

Let Γ denote the set of random variables γ from Ω to $\mathscr{M}_b$ of the form[1] $\gamma = \sum_{i=1}^m \gamma_{t_i}\delta_{t_i}$ i.e. $\gamma : \omega \mapsto \gamma_{t_1}(\omega)\delta_{t_1} + \ldots + \gamma_{t_m}(\omega)\delta_{t_m}$ with γ_{t_i} in $L^1(\Omega, R)$ for all $t_i \in R$. We should write $\gamma = \sum_{i=1}^{m_\gamma} \gamma_{t_i^\gamma}\delta_{t_i^\gamma}$ but we will omit the index γ for the simplicity of the notations.
We have the following immediate

Lemma 2.1 *The set Γ is included in $L_P^1(\Omega, \mathscr{M}_b)$ and for all γ in Γ, $\gamma = \sum_{i=1}^m \gamma_{t_i}\delta_{t_i}$, $\|\gamma\|_{L_P^1(\Omega,\mathscr{M}_b)} = \sum_{i=1}^m \|\gamma_{t_i}\|_{L^1(\Omega,R)}$.*

Let Γ_+ (resp. Γ_-)$\equiv \left\{\gamma \in \Gamma; \gamma = \sum_{i=1}^m \gamma_{t_i}\delta_{t_i}; \gamma_{t_i} \geq 0 \text{ (resp. } \leq 0) \ P \text{ a.s.}\right\}$.

2.2 The model

We consider a model in which agents face investment opportunities described by their cash flows. A probability space (Ω, F, P) is specified and fixed. The set Ω represents all possible states of the world. An information structure, which describes how information is revealed to investors, is given by a filtration $(F_t)_{t\in R_+}$ satisfying the “usual conditions” and such that F_0 is trivial. We model investment opportunities which are available to investors in the following way.

Definition 2.2 *An investment is an $(F_t)_{t\in R_+}$-adapted process $\Phi = (\Phi_t)_{t\in R_+}$, null outside a finite number of dates, i.e. there exists $\left(t_1^\Phi, \ldots, t_N^\Phi\right)$ such that $\Phi_t = 0$ for all $t \notin \left(t_i^\Phi\right)_{i=1}^N$, and such that Φ_t is in $L^1(\Omega, F_t, P)$ for all t in R_+.*

We consider a convex cone J of available investment opportunities: this amounts to saying that an investor has a right to subscribe to (a finite number of) different investment plans and that he can decide at the starting date of any investment opportunity which amount of this particular investment he wants to

[1] where, as usual, $\delta_t(f) \equiv f(t)$ for all $f \in \mathscr{C}_0$.

buy. We shall see in Sects. 3 and 4, with specific examples, that we are lead to consider convex cones in order to take into account the fact that investors are not necessarily able to sell an investment plan (see for instance the case of short sale constraints or transaction costs).

We introduce the following assumption.

Assumption A: there exists a sequence $d = (d_n)_{n \in N}$ in R_+ such that for all $t^* \in R_+$, for all B_{t^*} in F_{t^*} of positive probability, there exists Φ in J of the form $\Phi_{t^*} = 0$ outside B_{t^*}, $\Phi_t = 0$ for all $t < t^*$, $\Phi_t \geq 0$ for all $t > t^*$, and there exists $n \in N$, $P\left[\Phi_{d_n} > 0\right] > 0$.

In words, if a convex cone J of available investments satisfies *Assumption A*, this means that there exists a sequence of trading dates such that, for all date and for all event at that date, there exists an investment plan in our admissible set of investment opportunities that starts at that date and in that event, that can take any value at that date and in that event but that is nonnegative after that date and nonnull at one date belonging to the above mentioned sequence of dates. Roughly, *Assumption A* corresponds to the possibility of transferring "some wealth" from any date and event to some particular date. This assumption is not too restrictive: it is satisfied if we can buy at every date and event a bond with a given maturity even if this bond is defaultable and even if there is no secondary market for that bond (i.e. we have to wait until the maturity in order to recover any money with a positive probability, which may be different from 1); this includes market models with frictions on the numéraire like no borrowing, different borrowing and lending rates, bonds with default risk, different borrowing facilities among investors.

We don't specify the elements of J so far. We consider any investment $\Phi = (\Phi_t)_{t \in R_+}$ as a random variable from Ω to the set of discrete bounded Radon measures of the form $\Phi = \sum_{t \in R_+} \Phi_t \delta_t$, where $\Phi_t = 0$ for $t \notin \left(t_i^{\Phi}\right)_{i=1}^{N}$, i.e. as an element of Γ.

We now come to the notion of no-arbitrage.

Definition 2.3 *There is no arbitrage opportunity for J if and only if $J \cap \Gamma_+ = \{0\}$.*

Let us check that this definition corresponds to the usual notion of no-arbitrage, i.e. an impossibility to have access to an investment that yields a positive gain in some circumstances without a countervailing threat of loss in other circumstances. In our framework, an arbitrage opportunity would consist in a nonnegative nonnull available investment in J. And $\Phi = (\Phi_t)_{t \in R_+}$ is a nonnegative nonnull investment process if and only if $\Phi : \omega \mapsto \sum_{i=1}^{m} \Phi_{t_i}(\omega) \delta_{t_i}$, which is in $L_P^1\left(\Omega, \mathscr{M}_b\right)$, belongs to Γ_+ and is not null.

It is easy to see that the notion of no-arbitrage introduced in Definition 2.3 can be written in the form $(J - \Gamma_+) \cap \Gamma_+ = \{0\}$. A free lunch denoting the possibility of getting arbitrarily close to an arbitrage opportunity, we introduce the following

Definition 2.4 *There is no free lunch for J if and only if $\overline{(J - \Gamma_+)} \cap \Gamma_+ = \{0\}$, where the bar denotes the closure in $L_P^1\left(\Omega, \mathscr{M}_b\right)$.*

Notice that if there is a numéraire, the assumption $\overline{(J - \Gamma_+)} \cap \Gamma_+ = \{0\}$, where the bar denotes the closure in $L^1_P\left(\Omega, \mathscr{M}_b\right)$, is less restrictive than the "usual" assumption of no-free lunch (with deterministic times). The difference between the two notions lies in the fact that the nonnegative nonnull random variable that we can "almost reach" with admissible investments "is at a finite date" for the new notion whereas it can "be at infinity" for the usual notion. The notion of absence of free lunch adopted in this paper does not exclude a free lunch "at infinity" and can therefore be considered as more economically meaningful.

2.3 Characterization of the no-arbitrage assumption in a model with flows

By adapting the proof of Yan (1980), we obtain, under *Assumption A*, the following Fundamental Theorem of Asset Pricing in our general framework with flows.

Theorem 2.5 *Let J denote a convex cone of available investment opportunities satisfying Assumption A. There is no free lunch for J if and only if there exists a process $g = (g_t)_{t \in R_+}$ such that*

1. *for all $t \in R_+$, $g_t \in L^\infty(\Omega, F, P)$ and $M \equiv \sup_{t \in R_+} \|g_t\|_{L^\infty} < \infty$*
2. *for all $t \in R_+$, $g_t > 0\, P$ a.s.*
3. *for all $\Phi = (\Phi_t)_{t \in R_+} \in J$, $E\left[\sum_{t \in R_+} g_t \Phi_t\right] \leq 0$*

Moreover, the process g can be taken $(F_t)_{t \in R_+}$-adapted.

In the case where the set of available investment opportunities is related to a countable set of dates, which is the case in finite (resp. infinite) discrete time where the set J consists of $(F_t)_{t \in \mathbf{T}}$-adapted processes $\Phi = (\Phi_t)_{t \in \mathbf{T}}$ for $\mathbf{T} = \{1, ..., d\}$ (resp. $\mathbf{T} = N$), then *Assumption A* is not needed to obtain Theorem 2.5.

Corollary 2.6 *Let J denote a convex cone of investments. In finite or infinite discrete time, there is no free lunch for J if and only if there exists a P-essentially bounded process $g = (g_t)_{t \in \mathbf{T}}$ such that for all $\Phi = (\Phi_t)_{t \in \mathbf{T}} \in J$, $E\left[\sum_{t \in \mathbf{T}} g_t \Phi_t\right] \leq 0$.*

So, in continuous time, starting from the assumption of no free-lunch in a general model with flows, without any assumption on the existence of a numéraire, but under *Assumption A* (for which we have given an interpretation and that will reveal to be suited to our market frictions), we have proved the existence of a "discount process" such that, using this process as a deflator, all available investments have non-positive present value; this means that there exists a term structure such that the market consisting of the primitive investment opportunities and of the additional borrowing and lending facilities is still "arbitrage-free". Besides, the existence of such a discount process prevents from any arbitrage opportunity. In other words, there is no free lunch for a convex cone of available investments satisfying *Assumption A* if and only if a given convex set of "admissible" discount processes is non-void. We recall that *Assumption A* is not needed if we deal with discrete, possibly infinite, time.

We have seen that the "discount process" can be taken adapted to our filtration. We have obtained so far no other regularity conditions (such as right-continuity or existence of left limits); we shall see in the following section that under some regularity conditions on the available investments, we obtain regularity properties on the "discount process" itself. Besides, the taking into account of frictions on the numéraire will enable to better understand the nature of these admissible discount processes.

Since most market models with frictions can fit in the model with flows for a specific convex cone of available investments, our model provides a unified framework for the study of the characterization of the absence of free lunch in such imperfect market models. Notice however that economies with fixed transaction costs do not fall in the framework with flows, since the set of all available investments is not a cone.

3 Application to models with frictions on the numéraire

In this section, we still consider a general framework with flows, but we introduce a numéraire, possibly submitted to constraints. This enables us to give a better interpretation of the "admissible" discount processes found in the previous section. Moreover, we obtain a characterization of the no-free lunch assumption in general models with flows with possible frictions on the numéraire. Since market imperfections like convex cone constraints, proportional transaction costs, etc. can fit in our model with flows, our approach will enable us in the next section to characterize the absence of free lunch in these imperfect models, when there are, in addition, imperfections on the numéraire.

We introduce a few notations. Let $\mathcal{G}$ denote the set of all (equivalence classes) of adapted processes $g = (g_t)_{t\in R_+}$, such that for all $t \in R_+$, $0 < g_t \leq M^g$ a.s. P for some scalar M^g. For a convex cone K in Γ, let $\mathcal{G}_K$ denote the convex set of processes $g \in \mathcal{G}$ such that $E\left[\sum_{t\in R_+} g_t \Phi_t\right] \leq 0$ for all $\Phi = \sum_{t\in R_+} \Phi_t \delta_t$ in K.

3.1 With a "perfect" numéraire

We assume that there is a perfect numéraire, i.e. that there are possibilities to transfer wealth through time, without friction. Let $S^0 = \left(S_t^0\right)_{t\in R_+}$ denote a positive, adapted (numéraire) process such that for all $(t_1, t_2) \in R_+^2$, for all $\theta \in L^\infty\left(\Omega, F_{t_1\wedge t_2}, P\right)$, the process denoted by $\Phi^{(0;\theta,t_1,t_2)}$ and given by

$$\Phi_t^{(0;\theta,t_1,t_2)} = \theta\left[-S_{t_1}^0 1_{t=t_1} + S_{t_2}^0 1_{t=t_2}\right] \qquad \text{for all } t \in R_+$$

belongs to Γ. Notice that this is equivalent to $S_t^0 \in L^1\left(\Omega, F_t, P\right)$ for all $t \in R_+$. Let J^{num} denote the convex cone generated by a given convex cone J in Γ and all the investments $\Phi^{(0;\theta,t_1,t_2)}$. The set J^{num} corresponds to all available investments in a model with flows, where agents can invest in a perfect numéraire. Notice that J^{num} satisfies *Assumption A* so that, according to Theorem 2.5, there is no free lunch in a model with a perfect numéraire if and only if the set $\mathscr{G}_{J^{\text{num}}}$ is non-void. We obtain the following characterization of the set of admissible discount processes.

Corollary 3.1 *The set $\mathscr{G}_{J^{\text{num}}}$ consists of all processes $g \in \mathscr{G}_J$ such that gS^0 is a P-martingale for $(F_t)_{t\in R_+}$.*

This means that if we introduce a term structure given by a numéraire S^0, then in the absence of free lunch, the process g must be equal to $\frac{1}{S^0}$ multiplied by a positive martingale (which is the stochastic analog of a (positive) constant function). Up to a martingale, the process g gives us the possible term structures, i.e. which would be compatible with the assumption of no-free lunch. Note that the process gS^0 admits a right-continuous left-limited (RCLL) modification, so that if $1/S^0$ is RCLL, then g itself admits a RCLL modification.

In finite time, according to Corollary 3.1, the absence of free lunch for J^{num} implies that there exists an equivalent probability measure under which the net present value (using S^0 as a deflator) of any available investment is nonpositive, i.e. there exists a probability measure $Q \sim P$, such that $E^Q\left[\sum_{t\in R_+}\frac{\Phi_t}{S_t^0}\right] \leq 0$ for all $\Phi \in J$. In infinite time, the situation is somewhat different.

Lemma 3.2 1. *If S^0 is uniformly integrable, then there is no free lunch for J^{num} if and only if there exists a probability measure $\bar{P}$ on (Ω, F), absolutely continuous with respect to P and such that $\left(\frac{1}{S_t^0}E\left[\frac{d\bar{P}}{dP} \mid F_t\right]\right)_{t\in R_+} \in \mathscr{G}$ and $E^{\bar{P}}\left[\sum_{t\in R_+}\frac{\Phi_t}{S_t^0}\right] \leq 0$ for all investment Φ in J.*

2. *If Ω is the canonical space of all continuous functionals on R_+ taking values in $R \cup \{\infty\}$, then there is no free lunch for J^{num} if and only if there exists a probability measure $\bar{P}$ on (Ω, F), such that $\left(\frac{1}{S_t^0}\frac{d\bar{P}|_{F_t}}{dP|_{F_t}}\right)_{t\in R_+} \in \mathscr{G}$ and $E^{\bar{P}}\left[\sum_{t\in R_+}\frac{\Phi_t}{S_t^0}\right] \leq 0$ for all investment Φ in J.*

For 1., notice that unlike in the "classical case" with a perfect numéraire constantly equal to 1 (see e.g. Stricker (1990)), in the infinite time framework, we only find an absolutely continuous probability measure, whose restricted density to all F_t is positive, instead of an equivalent probability measure. This is due to the fact that, as mentioned in Sect. 2, our assumption of no-free lunch is less restrictive than the "usual" one, since it does not exclude free lunches at infinity. For 2., instead of considering a specific set Ω, we could also impose conditions on the filtration $(F_t)_{t\in R_+}$. Our problem is related to the existence of the exit measure (or Föllmer measure) of the martingale process gS^0. See Föllmer (1972), Meyer (1970) or Azéma and Jeulin (1976) for more details on this topic.

We now turn to cases where the numéraire is subject to some constraints.

3.2 With lending and borrowing opportunities

Let $S^0 = \left(S_t^0\right)_{t \in R_+}$ and $S^1 = \left(S_t^1\right)_{t \in R_+}$ denote two positive and adapted processes with $S_0^0 = S_0^1 = 1$, such that for all $(t_1, t_2) \in R_+^2$ with $t_1 \leq t_2$, for all $\theta \in L_+^\infty\left(\Omega, F_{t_1}, P\right)$, the processes denoted by $\Phi^{(0;\theta,t_1,t_2)}$ and $\Phi^{(1;\theta,t_1,t_2)}$ and given by

$$\begin{aligned} \Phi_t^{(0;\theta,t_1,t_2)} &= \theta\left[-S_{t_1}^0 1_{t=t_1} + S_{t_2}^0 1_{t=t_2}\right] \qquad \text{for all } t \in R_+ \\ \Phi_t^{(1;\theta,t_1,t_2)} &= \theta\left[S_{t_1}^1 1_{t=t_1} - S_{t_2}^1 1_{t=t_2}\right] \qquad \text{for all } t \in R_+ \end{aligned}$$

belong to Γ. Let J^{lb} denote the convex cone generated by a given convex cone J in Γ and all the investments $\Phi^{(0;\theta,t_1,t_2)}$ and $\Phi^{(1;\theta,t_1,t_2)}$. The set J^{lb} corresponds to all available investments in a model with flows where agents have lending and borrowing opportunities, but not in the same conditions. Notice that J^{lb} satisfies *Assumption A* so that, according to Theorem 2.5, there is no free lunch in a model with such constraints on the lending and borrowing opportunities if and only if the set $\mathcal{G}_{J^{\text{lb}}}$ is non-void. We obtain the following characterization of the set $\mathcal{G}_{J^{\text{lb}}}$.

Corollary 3.3 *The set $\mathcal{G}_{J^{\text{lb}}}$ consists of all processes $g \in \mathcal{G}_J$ such that gS^0 is a P-supermartingale and gS^1 is a P-submartingale for $(F_t)_{t \in R_+}$.*

We now introduce additional conditions on the processes S^0 and S^1.

(**C1**) The processes S^0 and S^1 are right-continuous and for all (t, t_1, t_2) with $t \leq t_1 \leq t_2$, for all $A \in F_{t_1}$, the process $\Phi^{\left(k;\frac{1_A}{S_t^k},t_1,t_2\right)} \in J^{\text{lb}}$ for $k = 0, 1$.

(**C2**) For all $s \leq t$, $S_t^0/S_s^0 \leq S_t^1/S_s^1$ a.s. P.

(**C3**) The processes S^0 and S^1 can be written in the form $S_\cdot^0 = \exp\int_0^\cdot r_s^0 ds$ and $S_\cdot^1 = \exp\int_0^\cdot r_s^1 ds$ for some processes r^0 and r^1 in[2] $L^\infty\left(R_+ \times \Omega, \mathcal{B}(R_+) \otimes F, \lambda \otimes P\right)$.

Condition (C1) is essentially a right-continuity condition; the additional integrability condition is satisfied if for all $t \in R_+$, $1/S_t^0$ as well as $1/S_t^1$ are bounded. Condition (C2) essentially means that the "borrowing rate" is greater than or equal to the "lending rate". Condition (C3) means that the processes S^0 and S^1 are associated to interest rates, with a possible spread. Notice that contrarily to Jouini and Kallal (1995b), we do not suppose $r^0 \leq r^1$, since we get it as a consequence of the absence of free lunch; besides we shall see that if there is no-free lunch for J^{lb}, (**C3**) implies (**C1**) and (**C2**).

We now obtain more specific characterizations of the set $\mathcal{G}_{J^{\text{lb}}}$ under the different conditions.

Lemma 3.4 *1. Under (**C1**), any $g \in \mathcal{G}_{J^{\text{lb}}}$ admits a right-continuous modification.*

*2. Under (**C1**) and (**C2**), the set $\mathcal{G}_{J^{\text{lb}}}$ consists of all processes $g \in \mathcal{G}_J$ for which for all $s \in R_+$, there exists a process $\left(Z_t^s\right)_{t \geq s}$ satisfying $S_t^0/S_s^0 \leq Z_t^s \leq S_t^1/S_s^1$ for all $t \geq s$ such that $\left(g_t Z_t^s\right)_{t \geq s}$ is a (right-continuous) martingale for $(F_t)_{t \geq s}$.*

[2] We let λ denote the Lebesgue measure on $\mathcal{B}(R_+)$.

*3.a. Under (**C3**), if there is no free lunch for J^{lb}, then we can take $r^0 \le r^1$ and (**C1**) and (**C2**) are satisfied.*

*3.b. Under (**C3**), the set $\mathcal{G}_{J^{\mathrm{lb}}}$ consists of all processes $g \in \mathcal{G}_J$ for which there exists a process $Z \equiv \exp \int_0^{\cdot} r_s ds$ for some bounded measurable process r satisfying $r^0 \le r \le r^1$ and such that gZ is a (right-continuous) martingale for $(F_t)_{t \in R_+}$.*

Part 1 says that the right-continuity of the borrowing and lending processes induce the right-continuity of the admissible discount processes. Part 2 says that up to a martingale, the returns of the admissible discount processes lie between the returns of the lending and borrowing processes; notice that if $1/S^0$ is left limited, then any admissible discount process admits a left limited modification. Part 3 says that if the lending and borrowing processes are associated to interest rates, then so are the admissible discount processes.

3.3 With lending opportunities only

With the notations adopted in the previous subsection, let J^{b} denote the convex cone generated by a given convex cone J in Γ and all the investments $\Phi^{(0;\theta,t_1,t_2)}$. The set J^{b} corresponds to all available investments in a market model where investors have lending opportunities, but not necessarily borrowing opportunities. Notice that J^{b} satisfies *Assumption A* so that there is no free lunch in a model with no borrowing if and only if the set $\mathcal{G}_{J^{\mathrm{b}}}$ is non-void.

Corollary 3.5 *The set $\mathcal{G}_{J^{\mathrm{b}}}$ consists of all processes $g \in \mathcal{G}_J$ such that gS^0 is a supermartingale for $(F_t)_{t \in R_+}$.*

We can now turn to frictions not only involving the numéraire.

4 Application to other market models with frictions

We shall consider market models with frictions on the numéraire and “imperfections” such as models with dividends, short sale constraints (or more generally convex cone constraints), proportional transaction costs.

4.1 Models with dividends

We consider a model of financial market consisting of N financial assets, possibly paying dividends to their holders. We denote by $\left(S^k\right)_{1 \le k \le N}$ the adapted price process of the securities and by $\left(D^k\right)_{1 \le k \le N}$ the associated (possibly null) adapted dividends process. We assume that the dividends are discrete, i.e. that there is no dividend paid outside a countable set of dates. The random variable D_t^k corresponds to the dividends paid by the security k at time t. We treat S_t^k by convention as the post-dividends market value of the security k at time t. In other words, if the dividend process jumps at time t, the market value S_t^k reflects the jump as already having been paid out, or S_t^k is ex-dividend.

For $1 \leq k \leq N$, for all $(t_1, t_2) \in R_+^2$ with $t_1 \leq t_2$, and for all $A \in F_{t_1}$, we assume that the process $\Phi^{(k;A,t_1,t_2)}$ given by

$$\Phi_t^{(k;A,t_1,t_2)} = 1_A \left[-S_{t_1}^k 1_{t=t_1} + D_t^k 1_{t_1 < t \leq t_2} + S_{t_2}^k 1_{t=t_2} \right] \qquad \text{for all } t \in R_+$$

belongs to Γ and we let J_{Div} denote the linear space generated by all these investments. We suppose that *Assumption A* is satisfied. Using Theorem 2.5, we get that

Corollary 4.1 *There is no free lunch in the model with dividends (or equivalently for J_{Div}) if and only if there exists a process $g = (g_t)_{t \in R_+} \in \mathscr{G}$ such that for all $t_1 \leq t_2$*

$$E \left[g_{t_2} \left(S_{t_2} + \sum_{t_1 < t \leq t_2} g_t D_t \right) \mid F_{t_1} \right] = g_{t_1} S_{t_1}.$$

Note that in the perfect case with no dividend, a finite time horizon, and a "perfect" and bounded numéraire, we obtain the "classical" Fundamental Theorem of Asset Pricing, which asserts that there is no free lunch if and only if there exists an equivalent probability measure with bounded density, which makes the discounted price process of the traded securities a martingale.

4.2 Convex cone constraints and application to short sale constraints

We consider now a model of financial market where the quantities of the N different risky assets held by investors are constrained to lie in a convex cone C (in R^N). Notice that this situation includes the one with no constraint ($C = R^N$). We denote by $\left(S^k\right)_{1 \leq k \leq N}$ the adapted price process of the risky assets. For $1 \leq k \leq N$, for all $(t_1, t_2) \in R_+^2$, with $t_1 \leq t_2$ and for all $A \in F_{t_1}$, we assume that the process $\Phi^{(k;A,t_1,t_2)}$ given by

$$\Phi_t^{(k;A,t_1,t_2)} = 1_A \left[-S_{t_1}^k 1_{t=t_1} + S_{t_2}^k 1_{t=t_2} \right] \qquad \text{for all } t \in R_+$$

belongs to Γ. Let $\Phi^{(A,t_1,t_2)} = \left(\Phi^{(k;A,t_1,t_2)}\right)_{1 \leq k \leq N}$. We denote by J_{Vex} the convex cone generated by

$$\left\{ x \cdot \Phi^{(A,t_1,t_2)}; x \in C, (t_1, t_2) \in R_+^2, A \in F_{t_1} \right\}.$$

The set J_{Vex} corresponds to all available investments in a model where portfolios are constrained to lie in a convex cone.
Let $C^0 = \left\{ y \in R^N; y \cdot x \leq 0, \text{ for all } x \in C \right\}$. In the unconstrained case, we have $C^0 = \{0\}$. We get the following characterization of the set $\mathscr{G}_{J_{\mathrm{Vex}}}$.

Corollary 4.2 *The set $\mathscr{G}_{J_{\mathrm{Vex}}}$ consists of all processes $g \in \mathscr{G}$ such that for all $t_1 \leq t_2$, $E\left[g_{t_2} S_{t_2} - g_{t_1} S_{t_1} \mid F_{t_1}\right]$ takes values in C^0.*

We can apply this result to market models with short sale constraints. We consider a model of financial markets where two sorts of securities can be traded. Short selling the first type of securities is not allowed, i.e. they can only be held in nonnegative amounts, whereas the second type of securities can only be held in nonpositive amounts. The model includes situations where holding negative amounts of a security is possible but costly as well as situations where some (or all) securities are not subject to any constraint, since we may include a security twice in the model, in the first and in the second set of securities. This model has been studied in Jouini and Kallal (1995b), in finite time, for price processes which are in $L^2(\Omega, F, P)$ and right-continuous.

This situation falls in our framework with convex cone constraints for a convex cone C_s of the form

$$C_s = \left\{x = (x_1, ..., x_N) \in R^N; x_k \geq 0 \text{ for } k \in K_1 \text{ and } x_k \leq 0 \text{ for } k \in K_2\right\}$$

for two disjoint subsets K_1 and K_2 of $\{1, ..., N\}$. Then

$$C_s^0 = \left\{y = (y_1, ..., y_N) \in R^N; y_k \leq 0 \text{ for } k \in K_1 \text{ and } y_k \geq 0 \text{ for } k \in K_2\right\}.$$

We easily obtain the following characterization of the set $\mathscr{G}_{J_S}$, where J_S is given by J_{Vex} for $C = C_s$. The set J_S corresponds to all available investments in a model with short sale constraints.

Corollary 4.3 *The set $\mathscr{G}_{J_S}$ of admissible discount processes in the case with short sale constraints consists of all processes $g \in \mathscr{G}$, such that for any security k that cannot be sold short, gS^k is a supermartingale, and for any security k that can only be sold short, gS^k is a submartingale.*

4.3 Proportional transaction costs

We consider a model of financial market where the securities are subject to bid-ask spreads: at each date, there is not a unique price for a security but an ask price, at which investors can buy the security and a bid price, at which they can sell the security. Notice that this model includes situations where there is a unique price process S and where the transaction cost remains constant overtime, i.e. situations where at each time $t \in R_+$, investors must pay $S_t(1 + c)$ for some positive constant c to buy the security and receive $S_t(1 - c)$ when selling it.

Such a model with transaction costs has been studied in Jouini and Kallal (1995a) using simple trading strategies like in Harrison and Kreps (1979), assuming that the time horizon is finite (equal to T), that there exists a numéraire process (identically equal to one) and that for all $t \in R_+$, the random variables corresponding to the bid and ask prices at date t are in $L^2(\Omega, F_t, P)$. We intend here to obtain the result of Jouini and Kallal (1995a) by using our general approach with flows, and to extend it mainly to the case with frictions on the numéraire and to infinite time.

More precisely, we consider $(N-1)$ securities and for $2 \leq k \leq N$, we let $\left(S_t^k\right)_{t\in R_+}$ and $\left(S_t'^k\right)_{t\in R_+}$ denote respectively the adapted ask and bid price process. We assume that the processes S and S' are right-continuous and of class $\mathscr{D}_f$, i.e. that the families $\{S_\tau\}_{\tau\in\mathscr{S}^f}$ and $\{S'_\tau\}_{\tau\in\mathscr{S}^f}$ are uniformly integrable, where $\mathscr{S}^f$ denotes the collection of stopping times of $(F_t)_{t\in R_+}$, taking only a finite number of values in $[0,+\infty[$. Let $\mathscr{S}_t^f$ denote the collection of stopping times τ in $\mathscr{S}^f$ such that $\tau \geq t$ a.s. P.

For each k, for any stopping times τ_1 and τ_2 in $\mathscr{S}^f$, we consider the process $\Phi^{(k;\tau_1,\tau_2)}$ given by

$$\Phi_t^{(k;\tau_1,\tau_2)} = -S_{\tau_1}^k 1_{t=\tau_1} + S_{\tau_2}'^k 1_{t=\tau_2} \qquad \text{for all } t \in R_+.$$

It is easy to see that for each k, for any stopping times τ_1 and τ_2 in $\mathscr{S}^f$, the process $\Phi^{(k;\tau_1,\tau_2)}$ is an investment as defined in Definition 2.2 and we define the set J_{Costs} as the convex cone generated by all these investments.

With the notations adopted in Subsect. 3.2, we make the additional assumption that there exist two processes S^0 and S^1 satisfying Condition (C1) and we denote by $J_{\text{Costs}}^{\text{lb}}$ the convex cone generated by J_{Costs} and all the investments $\Phi^{(0;\theta,t_1,t_2)}$ and $\Phi^{(1;\theta,t_1,t_2)}$. We denote by $\mathscr{G}_{\text{lb}}$ the set of processes $g \in \mathscr{G}$ such that gS^0 is a P-supermartingale and gS^1 is a P-submartingale for $(F_t)_{t\in R_+}$. According to Corollary 3.3, $\mathscr{G}_{\text{lb}}$ is the set of discount processes compatible with the absence of free lunch for the convex cone generated by all investments related to the borrowing and lending processes.

Notice that the no-arbitrage assumption implies that the bid price process lies below the ask price process. We then have the following

Lemma 4.4 *There is no free lunch for $J_{\text{Costs}}^{\text{lb}}$ if and only if there exist an adapted process $g = (g_t)_{t\in R_+} \in \mathscr{G}_{\text{lb}}$ and some adapted price process $\tilde{S}$ lying between the bid and ask price processes such that $g\tilde{S}$ is a (right-continuous) P-martingale for $(F_t)_{t\in R_+}$.*

If we now suppose that there is a perfect numéraire $S^0 \equiv 1$, we get the result obtained by Jouini and Kallal (1995a) in a finite time setting, i.e. there is no free lunch with proportional transaction costs if and only if there exists an equivalent probability measure, with bounded density (with density in $L^2(\Omega, F_T, P)$ in the setting of Jouini and Kallal (1995a)) that transforms some process between the bid and the ask price processes into a martingale. Lemma 4.4 enables to consider market models with proportional transaction costs and frictions on the numéraire, in an infinite time setting.

5 Conclusion

We have characterized the assumption of no-arbitrage in financial markets, where any investment opportunity is described by the cash flows that it generates; the absence of free lunch is equivalent to the existence of a normalization process such that the “net present value” of any available investment is nonpositive, i.e.

there exists a process such that the “net present value” of the normalized cash flows is nonpositive. We have then applied this very general result to specific financial market models and mainly financial models with frictions like imperfections on the numéraire, which had been barely studied, but also proportional transaction costs, short sale constraints, convex cone constraints, for which we have generalized, in a unified way, existing results.

Appendix

Proof of Lemma 2.1 1) We prove it for $\gamma = \gamma_t \delta_t$ in Γ. If γ_t is a simple real random variable, i.e. if it only takes a finite number of real values $(a_i)_{i=1}^{I}$, then we can write γ in the form $\gamma = \sum_{i=1}^{I} 1_{A_i} (a_i \delta_t)$ for a partition $(A_i)_{1 \leq i \leq I}$ of Ω, and $\gamma \in St\left(\Omega, \mathscr{M}_b\right)$. For a general γ_t, γ_t can be written as the P a.s. limit of a sequence $\left(\gamma_t^n\right)_{n \in N}$ of simple real random variables; then $\gamma = \lim_n \left(\gamma_t^n \delta_t\right)$ a.s. P, and $\gamma \in M_P\left(\Omega, \mathscr{M}_b\right)$. Now, since $\|\gamma(\omega)\|_{\mathscr{M}_b} = \sup_{\|f\| \leq 1, f \in \mathscr{C}_0} |\gamma_t(\omega) f(t)|$, we get $E\left[\|\gamma\|_{\mathscr{M}_b}\right] = E\left[|\gamma_t(\omega)|\right] < \infty$.

2) For all γ in Γ, $\gamma = \sum_{i=1}^{m} \gamma_{t_i} \delta_{t_i}$, we have $\|\gamma(\omega)\|_{\mathscr{M}_b} = \sum_{i=1}^{m} |\gamma_{t_i}(\omega)|$ because for all $(t_1, \ldots t_m)$ in $(R_+)^m$, we can find f in $\mathscr{C}_0$, satisfying $\|f\| \leq 1$ and $f(t_i) = 1$ (resp. $f(t_i) = -1$) if $\gamma_{t_i}(\omega) \geq 0$ (resp. $\gamma_{t_i}(\omega) \leq 0$). □

Proof of Theorem 2.5 Let $C \equiv (J - \Gamma_+)$. Assume first that there exists a process g satisfying the conditions of the Theorem. Let $x \in \Gamma$, $x = \lim_{L_P^1(\Omega, \mathscr{M}_b)} x^n$, $x^n \in C$. Since $x^n \in C \subseteq \Gamma$, $E\left[\left|\sum_{t \in R_+} g_t x_t^n - \sum_{t \in R_+} g_t x_t\right|\right] \leq M \|x - x_n\|_{L_P^1(\Omega, \mathscr{M}_b)}$ (Lemma 2.1), thus $E\left[\sum_{t \in R_+} g_t x_t^n\right] \to_n E\left[\sum_{t \in R_+} g_t x_t\right]$. Since for all n, $E\left[\sum_{t \in R_+} g_t x_t^n\right] \leq 0$, we get $E\left[\sum_{t \in R_+} g_t x_t\right] \leq 0$ and $x \notin \Gamma_+ \setminus \{0\}$.

Suppose now that $\overline{C} \cap \Gamma_+ = \{0\}$. Let $\mu \in \Gamma_+ \setminus \{0\}$. We apply the Hahn-Banach separation Theorem in the normed vector space $L_P^1\left(\Omega, \mathscr{M}_b\right)$ to find $\left(\alpha_\mu, \beta_\mu\right) \in R^2$ and $\mathfrak{g} \neq 0$ in $\mathscr{L}_*^\infty\left(\Omega, \left(\mathscr{M}_b\right)'\right)$, i.e. satisfying

1. $\langle \mathfrak{g}, \nu \rangle_{(\mathscr{M}_b)', \mathscr{M}_b} : \omega \to \langle \mathfrak{g}(\omega), \nu \rangle_{(\mathscr{M}_b)', \mathscr{M}_b}$ is measurable for all ν in $\mathscr{M}_b$
2. $\|\mathfrak{g}\|_{\mathscr{L}_*^\infty(\Omega, (\mathscr{M}_b)')} \equiv \inf\left\{M \geq 0, \|\mathfrak{g}\|_{(\mathscr{M}_b)'} \leq M \quad P \text{ a.s.}\right\} < \infty,$

for which, for all c in C, $E\left[\langle \mathfrak{g}, c \rangle_{(\mathscr{M}_b)', \mathscr{M}_b}\right] \leq \alpha_\mu < \beta_\mu \leq E\left[\langle \mathfrak{g}, \mu \rangle_{(\mathscr{M}_b)', \mathscr{M}_b}\right]$, where $\langle \mathfrak{g}, c \rangle_{(\mathscr{M}_b)', \mathscr{M}_b} : \omega \mapsto \langle \mathfrak{g}(\omega), c(\omega) \rangle_{(\mathscr{M}_b)', \mathscr{M}_b}$. Since C is a convex cone, we can take $\alpha_\mu = 0$.

Let now for all $t \in R_+$, $g_t \equiv \langle \mathfrak{g}, \delta_t \rangle_{(\mathscr{M}_b)', \mathscr{M}_b}$ and $g \equiv (g_t)_{t \in R_+}$. Then, using 1., g_t is measurable. Moreover, using 2., since for all $t \in R_+$, $\delta_t \in \mathscr{M}_b$ and satisfies $\|\delta_t\|_{\mathscr{M}_b} \leq 1$, we get the existence of M in R_+^* such that for all $t \in R_+$, $g_t \leq M \quad P$ a.s.

We claim that for all $t \in R_+$, g_t is almost surely nonnegative. Suppose that there exists $t \in R_+$ such that $V_t \equiv \left\{\langle \mathfrak{g}, \delta_t \rangle_{(\mathscr{M}_b)', \mathscr{M}_b} < 0\right\}$ has positive P-probability. The random variable ϕ_t in $L_P^1\left(\Omega, \mathscr{M}_b\right)$ given by ϕ_t :

$\omega \rightarrow -1_{V_t}(\omega)\,\delta_t$ belongs to Γ_-, so according to the separation Theorem $E\left[\langle \mathfrak{g}, \phi_t\rangle_{(\mathscr{M}_b)',\mathscr{M}_b}\right] \le 0$. But $E\left[\langle \mathfrak{g}, \phi_t\rangle_{(\mathscr{M}_b)',\mathscr{M}_b}\right] = E\left[-1_{V_t}\langle \mathfrak{g}, \delta_t\rangle_{(\mathscr{M}_b)',\mathscr{M}_b}\right] > 0$, a contradiction.

We claim that for all $\Phi \equiv \sum_{t\in R_+} \Phi_t \delta_t$ in J, $E\left[\sum_{t\in R_+} g_t \Phi_t\right] \le 0$. This is immediate using 2. and the equality $\left\langle \mathfrak{g}(\omega), \sum_{t\in R_+} \Phi_t \delta_t\right\rangle_{(\mathscr{M}_b)',\mathscr{M}_b} = \sum_{t\in R_+} \Phi_t(\omega)$ $\langle \mathfrak{g}(\omega), \delta_t\rangle_{(\mathscr{M}_b)',\mathscr{M}_b}$.

Denote by G the set of all equivalence classes of processes $g = (g_t)_{t\in R_+}$ such that for all $t \in R_+$, $0 \le g_t \le M^g$ a.s. P and $E\left[\sum_{t\in R_+} g_t \Phi_t\right] \le 0$ for all $\Phi \in J$. We claim that for all $s \in R_+$, there exists a process g^s in G satisfying $g^s_s > 0$ almost surely. Fix s in R_+ and let $\mathscr{S}$ be the family of equivalence classes of subsets of Ω formed by the supports of the g_s for all g in G. We see that the family $\mathscr{S}$ is not reduced to the empty set by considering $\mu = \delta_s$ in the separation Theorem. The family $\mathscr{S}$ is closed under countable unions: indeed, consider a sequence $(g^n)_{n\in N}$ in G, and let $(a_n)_{n\in N}$ denote a sequence of positive scalars such that $\sum_{n\in N} a_n M^{g^n} < \infty$; the process h given by $h_t \equiv \sum_{n\in N} a_n g^n_t$ a.s. P for all $t \in R_+$, belongs to G. Hence there is g^s in G such that for $S^* = \{g^s_s > 0\}$, we have $P(S^*) = \sup\{P(S); S \in \mathscr{S}\}$. We shall now prove that $P(S^*) = 1$. If $P(S^*) < 1$, then we can apply the separation Theorem to $\mu = 1_{(\Omega\setminus S^*)}\delta_t$ which belongs to $\Gamma_+\setminus\{0\}$ and proceeding as above, we find $g'^s \in G$ such that $E\left[1_{(\Omega\setminus S^*)} g'^s_s\right] > 0$. Then, $g^s + g'^s$ would be an element of G, whose support has P-probability strictly greater than $P(S^*)$: a contradiction.

In the same way, we get that there exists g in G such that $g_{d_n} > 0$ almost surely for all $n \in N$, where $d = (d_n)_{n\in N}$ is the sequence introduced in *Assumption A*. We consider the process g such that for all $t \in R_+$, $g_t \equiv \sum_{n\in N} b_n g^{d_n}_t$, where $(b_n)_{n\in N}$ is a sequence of positive scalars such that $\sum_{n\in N} b_n M^{g^{d_n}} < \infty$.

We have found so far a process g such that for all $t \in R_+$, g_t is in $L^\infty_+(\Omega, F, P; R)$ and $\|g_t\|_{L^\infty} \le M$, $g_{d_n} > 0$ almost surely for all $d_n \in d$ and $E\left[\sum_{t\in R_+} g_t \Phi_t\right] \le 0$ for all $\Phi = \sum_{t\in R_+} \Phi_t \delta_t$ in J. Then it is easy to check that the same holds for the $(F_t)_{t\in R_+}$-adapted process $\tilde{g} = (\tilde{g}_t)_{t\in R_+}$ such that $\tilde{g}_t = E\left[g_t \mid F_t\right]$ P a.s. for all t. To finish the proof, we only need to show that for all $t \in R_+$, $\tilde{g}_t > 0$ a.s. P. Assume that for some T outside the set of dates $\{d_n; n \in N\}$ we have just considered, the event $B_T \equiv \{\tilde{g}_T = 0\}$ has positive P-probability; we know that there exists $\Phi = \sum_{t\in R_+} \Phi_t \delta_t$ in C such that $\Phi_T = 0$ outside B_T, $\Phi_t = 0$ for all $t < T$, $\Phi_t \ge 0$ for all $t > T$, $\exists n \in N$, $P\left[\Phi_{d_n} > 0\right] > 0$. For this particular Φ, we would have $E\left[\sum_{t\in R_+} g_t \Phi_t\right] \ge E\left[g_{d_n} \Phi_{d_n}\right] > 0$, which is impossible and completes our proof. □

Proof of Corollary 2.6 The proof remains the same as for Theorem 2.5, replacing the sequence $(d_n)_{n\in N}$ by $\{1, \ldots, d\}$ or N. □

Proof of Corollary 3.1 The set $\mathscr{G}_{J^{\mathrm{num}}}$ consists of all processes $g \in \mathscr{G}$ such that for all Φ in J^{num}, $E\left[\sum_{t\in R_+} g_t \Phi_t\right] \le 0$. Applying this inequality to $\Phi^{(0;1_A,t_1,t_2)}$ and $\Phi^{(0;-1_A,t_1,t_2)}$ for all $t_1 \le t_2$, and $A \in F_{t_1}$, we get that for all $t_1 \le t_2$, $E\left[1_A g_{t_2} S^0_{t_2}\right] \le$

$E\left[1_A g_{t_1} S^0_{t_1}\right]$ and $E\left[1_A g_{t_2} S^0_{t_2}\right] \geq E\left[1_A g_{t_1} S^0_{t_1}\right]$, thus the process $\left(g_t S^0_t\right)_{t\in R_+}$ is a P-martingale for $(F_t)_{t\in R_+}$. The converse is immediate. □

Proof of Lemma 3.2 For 1. and 2., one implication is immediate taking $g_t \equiv \left(\frac{1}{S^0_t}\right)\frac{d\bar{P}|_{F_t}}{dP|_{F_t}}$ and applying Corollary 3.1 and Theorem 2.5. For the converse implication, we know by Corollary 3.1 that for all $g \in \mathscr{G}_{J^{\text{num}}}$, gS^0 is a martingale. The process gS^0 admits a right-continuous modification, that can be chosen so as to be a martingale with respect to $(F_t)_{t\in R_+}$ (see Dellacherie and Meyer (1980)). For 1., the process $\left(g_t S^0_t\right)_{t\in R_+}$ is uniformly integrable, so it converges P a.s. to an integrable random variable that we shall denote by $g_\infty S^0_\infty$, such that $\left\{g_t S^0_t; F_t; 0 \leq t \leq \infty\right\}$ is a martingale (see Dellacherie and Meyer (1980)). Since for all $t \in R_+$, $g_t S^0_t$ is P a.s. positive, we define an absolutely continuous probability measure $\bar{P}$ (with respect to P) on (Ω, F) by setting $\frac{d\bar{P}}{dP} \equiv \frac{g_\infty S^0_\infty}{E\left[g_\infty S^0_\infty\right]}$. Its density restricted to any F_t, which is equal to $\frac{g_t S^0_t}{E\left[g_\infty S^0_\infty\right]}$, is positive. Then for all investment Φ in J, $\frac{\Phi_t}{S^0_t}$ belongs to $L^1\left(\Omega, F_t, \bar{P}\right)$ for all $t \in R_+$ and $E^{\bar{P}}\left[\sum_{t\in R_+}\frac{\Phi_t}{S^0_t}\right] = \frac{1}{E\left[g_\infty S^0_\infty\right]}E\left[\sum_{t\in R_+} g_t\Phi_t\right] \leq 0$.

For 2., the process gS^0 is a right-continuous nonnegative martingale and therefore admits an exit measure (see Föllmer (1972), Meyer (1970)). There exists a probability measure $\bar{P}$ on (Ω, F) such that for all stopping time T, $\bar{P}\left[T < \infty\right] = E\left[g_T S^0_T 1_{T<\infty}\right]$. For all $A \in F_t$, taking $T = t1_A + \infty 1_{A^c}$ gives us $\bar{P}(A) = E\left[g_t S^0_t 1_A\right]$ and the result wanted. □

Proof of Corollary 3.3 Any $g \in \mathscr{G}_{J^{\text{lb}}}$ satisfies $E\left[\sum_{t\in R_+} g_t \Phi^{(0;1_A,t_1,t_2)}_t\right] \leq 0$ for all $t_1 \leq t_2$, for all $A \in F_{t_1}$. Then for all $A \in F_{t_1}$, $E\left[1_A g_{t_2} S^0_{t_2}\right] \leq E\left[1_A g_{t_1} S^0_{t_1}\right]$ and the process gS^0 is a supermartingale. In an analogous way, we find that the process gS^1 is a submartingale. The converse is immediate. □

Proof of Lemma 3.4 1. Using Condition $(C1)$ and proceeding like in Corollary 3.3**,** we get that for all $s \in R_+$, $X^s \equiv \left(g_t \frac{S^0_t}{S^0_s}\right)_{t\geq s}$ is an $(F_t)_{t\geq s}$-supermartingale and $Y^s \equiv \left(g_t \frac{S^1_t}{S^1_s}\right)_{t\geq s}$ is an $(F_t)_{t\geq s}$-submartingale. We know that P a.s., for all $t \in R_+$, the limits X^0_{t+} exist, thus P a.s., for all $t \in R_+$, the limits g_{t+} exist. Moreover, for all $s \geq 0$, $X^s_s \geq E\left[X^s_{s+} \mid F_s\right]$ and $Y^s_s \leq E\left[Y^s_{s+} \mid F_s\right]$(Dellacherie and Meyer (1980)). Using the right-continuity of the filtration, we get that $g_s = g_{s+}$ a.s. P, and g admits a right-continuous modification.

2. One implication is immediate noticing that for all s, $Z^s_s = 1$ and applying Corollary 3.3. We prove the converse implication for $s = 0$. We know by Corollary 3.3 and 1. that up to a modification, for $g \in \mathscr{G}_{J^{\text{lb}}}$, gS^0 (resp. gS^1) is a right-continuous super-(resp. sub-)martingale. Moreover, by Condition $(C2)$, we have $gS^0 \leq gS^1$. Then we use the approach adopted in Jouini and Kallal (1995a, Lemma 3) or Choulli and Stricker (1997), adapting it to the case of infinite time to get the existence of a right-continuous martingale process $(m_t)_{t\in R_+}$ which satisfies $\left(gS^0\right)(t) \leq m_t \leq \left(gS^1\right)(t)$ for all $t \in R_+$. Since for all $t \in R_+$, g_t is almost surely positive, we may define the process $(Z_t)_{t\in R_+}$ by $Z_t \equiv \frac{m_t}{g_t}$ for all $t \in R_+$.

3.a. Since $(C3)$ implies $(C1)$, using 1), there is a right-continuous process $g \in \mathscr{G}$ such that for all $t_1 \leq t_2$, $E\left[g_{t_2}\left(\frac{S^0_{t_2}}{S^0_{t_1}} - \frac{S^1_{t_2}}{S^1_{t_1}}\right) \mid F_{t_1}\right] \leq 0$. If there is no free lunch for J^{lb}, we have for all $(t_1, t_2) \in (R_+)^2$, $E\left[g_{t_2}\left(\frac{S^0_{t_2}}{S^0_{t_1}} - \frac{S^1_{t_2}}{S^1_{t_1}}\right) \mid F_{t_1}\right] \leq 0$. Then, letting for $i = 0, 1$, $\overline{r^i(t,\omega)} \equiv \overline{\lim}\left[\left(\frac{S^i(t+1/n,\omega)}{S^i(t,\omega)} - 1\right) n\right]$, we get

$$E\left[g_{t_1}\left(\overline{r^0_{t_1}} - \overline{r^1_{t_1}}\right) \mid F_{t_1}\right] \leq \overline{\lim} E\left[g_{t_1+1/n}\left(\frac{S^0_{t_1+1/n}}{S^0_{t_1}} - \frac{S^1_{t_1+1/n}}{S^1_{t_1}}\right) \mid F_{t_1}\right]$$

or $g_{t_1}\left(\overline{r^0_{t_1}} - \overline{r^1_{t_1}}\right) \leq 0$, so that for all $s \in R_+$, $\overline{r^0_s} \leq \overline{r^1_s}$ a.s. P. Since for $i = 0, 1$, for all ω, $r^i_s = \overline{r^i_s}$ a.s. λ, we can write $S^i_\cdot = \exp \int_0^\cdot \overline{r^i_s} ds$. We now claim that $(C2)$ is satisfied. Let $A \in \mathscr{B}(R_+) \otimes F$ be given by $A \equiv \left\{(s,\omega), r^0_s(\omega) > r^1_s(\omega)\right\}$. We get that $\lambda \otimes P(A) = 0$, so that $\lambda(A_\omega) = 0$ a.s. P (where as usual, $A_\omega \equiv \{s, (s,\omega) \in A\}$) and for all $t_1 \leq t_2$, $\int_{t_1}^{t_2} r^0_s ds \leq \int_{t_1}^{t_2} r^1_s ds$ almost surely.

3.b. Consider a finite set of dates $(t_i)_{i \in \{0,\ldots,N\}}$ in $[0,T]$ such that $0 = t_0 < \ldots < t_N = T$, where T is a given positive real number. We deduce from 2. that there exists a process $X^0 = \left(X^0_t\right)_{t \in (t_i)}$ such that $\left(g_t X^0_t\right)_{t \in (t_i)}$ is a P-martingale for $(F_t)_{t \in (t_i)}$ and for all $i \in \{0, \ldots, N-1\}$,

$$\frac{S^0_{t_{i+1}}}{S^0_{t_i}} \leq \frac{X^0_{t_{i+1}}}{X^0_{t_i}} \leq \frac{S^1_{t_{i+1}}}{S^1_{t_i}}.$$

We extend the process found this way for each time grid $(t_i)_{i \in \{0,\ldots,N\}}$ into a continuous time process, defined for $t \in [0,T]$ by letting $X^0_\cdot = \exp \int_0^\cdot y_s ds$ with $y(s) = y_i(s)$ for every $t \in [t_i, t_{i+1}[$ where $\frac{X^0_{t_{i+1}}}{X^0_{t_i}} = \exp \int_{t_i}^{t_{i+1}} y_i(s) ds$. Note that X^0 is absolutely continuous and can be written $X^0_t = 1 + \int_0^t u_s ds$ where $u_s = y_s \exp \int_0^s y_v dv$. We do this for finer and finer grids of union equal to some dense subset of $[0,T]$. Since $|u(s,\omega)| \leq \overline{r}_1$ where $\overline{r}_1$ denotes the upper bound for r_1, the sequence of slopes $u(s,\omega)$ constructed this way is bounded in $L^\infty\left([0,T] \times \Omega, \mathscr{B}[0,T] \otimes F_T, \lambda_{[0,T]} \otimes P\right)$; so it admits a convergent (for the weak-star topology $\sigma\left(L^\infty, L^1\right)$) subsequence. Let us denote by $\tilde{u}$ its limit, and consider the process $Z = 1 + \int_0^\cdot \tilde{u}_s ds$, defined for $t \in [0,T]$, which can be written in the form $Z_t = \exp \int_0^t r_s ds$, for some bounded process r.
For all $t \in [0,T]$, the "associated" subsequence $gX^0(t)$ is convergent for the weak-star topology $\sigma\left(L^\infty(\Omega, F_t, P), L^1(\Omega, F_t, P)\right)$ with limit equal to $gZ(t)$. Using the fact that for all time grid $(t_i)_{i \in \{0,\ldots,N\}}$, the associated process gX^0 is a P-martingale for $(F_t)_{t \in (t_i)}$, it is easy to see that gZ is a P-martingale for $(F_t)_{t \in [0,T]}$. We adopt the same approach between nT and $(n+1)T$ for $n \in N^*$, to find a process Z, defined for $t \in R_+$, which can be written in the form $Z_t = \exp \int_0^t r_s ds$, for some bounded process r and such that gZ is a P-martingale for $(F_t)_{t \in R_+}$. Proceeding like in the proof of 3.a., we get that $r^0 \leq r \leq r^1$. □

Proof of Corollary 3.5 Immediate. □

Proof of Corollary 4.1 Immediate. □

Proof of Corollary 4.2 If $g \in \mathscr{G}_{\mathrm{JVex}}$, then for $x \in C$, we have for all $A \in F_{t_1}$, $E\left[1_A x \cdot \left(g_{t_2}S_{t_2} - g_{t_1}S_{t_1}\right)\right] \leq 0$ thus, $x \cdot E\left[\left(g_{t_2}S_{t_2} - g_{t_1}S_{t_1}\right) \mid F_{t_1}\right] \leq 0$ and $E\left[g_{t_2}S_{t_2} - g_{t_1}S_{t_1} \mid F_{t_1}\right]$ almost surely takes values in C^0. Conversely, if $E\left[g_{t_2}S_{t_2} - g_{t_1}S_{t_1} \mid F_{t_1}\right]$ takes values in C^0, then for all $A \in F_{t_1}$ and $x \in C$, $1_A x \cdot E\left[g_{t_2}S_{t_2} - g_{t_1}S_{t_1} \mid F_{t_1}\right] \leq 0$ a.s. P, thus $E\left[\sum_{t \in R_+}\left(x \cdot \Phi^{(A,t_1,t_2)}\right)_t g_t\right] \leq 0$. □

Proof of Corollary 4.3 Immediate applying Corollary 4.2 to C of the form C_s. □

Proof of Lemma 4.4 One implication is easy noticing that for all stopping times τ_1 and τ_2 in $\mathscr{S}^f$, we have, using the fact that $\tilde{S}$ lies between the bid and ask price processes, the law of iterated expectations and the optional sampling Theorem,

$$E\left[\sum_{t \in R_+} \Phi_t^{(k;\tau_1,\tau_2)} g_t\right] = E\left[g_{\tau_2}S'^k_{\tau_2} - g_{\tau_1}S^k_{\tau_1}\right] \leq E\left[g_{\tau_2}\tilde{S}^k_{\tau_2} - g_{\tau_1}\tilde{S}^k_{\tau_1}\right] = 0.$$

The result then follows from Theorem 2.5.

For the converse implication, we know by Corollary 3.3 that there exists a process $g = (g_t)_{t \in R_+} \in \mathscr{G}_{\mathrm{lb}}$ such that for all k, for all τ_1 and τ_2 in $\mathscr{S}_t^f$,

$$E\left[-g_{\tau_1}S^k_{\tau_1} + g_{\tau_2}S'^k_{\tau_2} \mid F_t\right] \leq 0$$

or

$$E\left[g_{\tau_2}S'^k_{\tau_2} \mid F_t\right] \leq E\left[g_{\tau_1}S^k_{\tau_1} \mid F_t\right]. \tag{5.1}$$

We consider the two $(N-2)$-dimensional processes Z and Z' given by

$$\begin{aligned} Z'_t &= \operatorname*{ess\,sup}_{\tau \in \mathscr{S}_t^f} E\left[g_\tau S'_\tau \mid F_t\right] \quad \text{for all} \quad t \in R_+ \\ Z_t &= \operatorname*{ess\,inf}_{\tau \in \mathscr{S}_t^f} E\left[g_\tau S_\tau \mid F_t\right] \quad \text{for all} \quad t \in R_+ \end{aligned}$$

In words, Z'^k is the supremum of the conditional expected value of the proceeds from the strategy that consists in going short in one security k (and investing the proceeds in security 0) after time t (but not necessarily at the same time in different events). The process Z is defined symmetrically. We have for all $t \in R_+$, $S'_t g_t \leq Z'_t$ as well as $Z_t \leq S_t g_t$ because for all $t \in R_+$, the stopping time $\tau = t$ belongs to $\mathscr{S}_t^f$.

It is a standard result in optimal stopping that Z' is a P-supermartingale for $(F_t)_{t \in R_+}$ and that Z is a P-submartingale for $(F_t)_{t \in R_+}$. Using inequality (5.1), we have $Z' \leq Z$. The process g can be taken right-continuous (Lemma 3.4, 1.). As in Jouini and Kallal (1995a, Lemma 3), we get that there is a process $\tilde{Z}$ lying between $\left(S'g\right)$ and (Sg), which is a P-martingale for $(F_t)_{t \in R_+}$. Now, letting $\tilde{S} \equiv \frac{\tilde{Z}}{g}$, we get that $g\tilde{S}$ is a right-continuous P-martingale and for all $t \in R_+$, $S'_t \leq \tilde{S}_t \leq S_t$ P a.s. □

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