Journal of Mathematical Economics 34 (2000) 547–558

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

Price functionals with bid–ask spreads: an axiomatic approach

Elyès Jouini*,1

CEREMADE, Université Paris IX Dauphine, Place De Lattre-de-Tossigny, 75775 Paris Cedex 16, France

Received 7 July 1997; received in revised form 14 April 1999; accepted 17 June 1999

Abstract

In Jouini and Kallal [Jouini, E., Kallal, H., 1995. Martinagles and arbitrage in securities markets with transaction costs. Journal of Economic Theory 66 (1) 178-197], the authors characterized the absence of arbitrage opportunities for contingent claims with cash delivery in the presence of bid–ask spreads. Other authors obtained similar results for a more general definition of the contingent claims but assuming some specific price processes and transaction costs rather than bid–ask spreads in general (see for instance, Cvitanic and Karatzas [Cvitanic, J., Karatzas, I., 1996. Hedging and portfolio optimization under transaction costs: a martinangle approach. Mathematical Finance 6, 133-166]). The main difference consists of the fact that the bid–ask ratio is constant in this last reference. This assumption does not permit to encompass situations where the prices are determined by the buying and selling limit orders or by a (resp. competitive) specialist (resp. market-makers). We derive in this paper some implications from the no-arbitrage assumption on the price functionals that generalizes all the previous results in a very general setting. Indeed, under some minimal assumptions on the price functional, we prove that the prices of the contingent claims are necessarily in some minimal interval. This result opens the way to many empirical analyses.

Keywords: Price functionals; Bid–ask spreads; No-arbitrage assumption; Market-makers; Contingent claims

1. Introduction

There an important literature on the contingent claims pricing problem under transaction costs on the primitive assets. For instance, Leland (1985) studied the replication price for a contingent claim in a discrete time setting. In this paper, when the horizon is kept fixed and

* Tel.: +33-1-4405-4675.

E-mail address: jouini@dauphine.fr (E. Jouini).

[1] This paper was written when the author was affiliated with CREST-ENSAE and visiting the Stern School of Business, New York University, USA.

PII: S0304-4068(99)00023-3

the number N of time periods goes to infinity, the price of the primitive asset is assumed to converge to a diffusion process. If we further assume that the transaction costs go to zero as the square root of N, Leland (1985) claims then that the replication price for a call option converges to the Black and Scholes price of this option in a model without transaction costs but with a correctly modified volatility for the primitive asset. For a correct proof of Leland's result see Kabanov (1997). In Boyle and Vorst (1992), the authors do not assume that the transaction costs go to zero and characterize the replication cost as an integral of the future prices relatively to a signed measure which is not, in general, a probability measure as in the frictionless model.

Bensaid et al. (1992) in the same year revolutionized the transaction costs literature considering dominating strategies instead of replicating ones. Indeed, the authors note that the replication cost is not necessarily, as in the transaction costless framework, the minimum cost necessary to obtain at least the same payoffs as those of the considered contingent claim. They propose then, in a discrete time setting, an algorithm in order to compute the so-called domination price: the minimum cost necessary to obtain at least the same payoffs as those of the considered contingent claim. Furthermore, they characterize the situation where the replication price is equal to the domination price and where the replication strategy is in some sense optimal.

In the same year and after the seminal work of Bensaid et al. (1992), Jouini and Kallal characterized, in a paper published in 1995, this domination price in a general setting. They prove that this price is equal for a given contingent claim to the supremum of the future payoffs expected value. This supremum is taken over all the equivalent martingale measures associated to one of the processes lying between the bid and the ask price processes. Furthermore, they characterize the absence of arbitrage opportunities in the model by the existence of a process lying between the bid and the ask price processes and of an equivalent probability measure for which the considered process is a martingale.

More recently, Shirakawa and Konno (1995) in a stationary binomial framework, Kusuoka (1995) in a discrete time and finite number of states of the world framework and Cvitanic and Karatzas (1996) in a diffusion setting, obtained results similar to some of Jouini and Kallal (1995a) [2] in a different setting. Indeed, in Jouini and Kallal (1995a), the authors only consider contingent claims with cash delivery. Note that this restriction is innocuous in the transaction costless framework but this is not at all the case in our framework.

Nevertheless, it is important to remark that in all these papers, the authors assume the existence of some price process S satisfying some classical conditions implying the absence of arbitrage opportunities in a frictionless framework (diffusion, binomial process...). The bid and the ask price processes are obtained multiplying S by $(1+\lambda)$ and $(1-\mu)$. In this setting, the transaction costs are proportional to the price S and the bid and ask price processes have the same behaviour. The Jouini and Kallal (1995a) paper is the only one with two independent price processes: a bid price process and an ask price process. The bid–ask spread can be interpreted as transaction costs but can be explained by the buying and selling limit orders on the markets. These prices are the prices for which a buyer or a seller is sure to find an immediate counterpart. From this point of view the bid–ask spread includes

[2] For instance, Cvitanic and Karatzas do not characterize the absence of arbitrage opportunities but only the domination price. Indeed, the choice of a diffusion framework implies the absence of arbitrage opportunities.

the possible transaction costs but is not reduced to these costs. With this interpretation we cannot assume that the relative bid–ask spread is constant. Indeed, Hamon and Jacquillat (1992) established in an empirical study that the relative bid–ask spread can typically be multiplied by three on the same year and by two during the same day. Furthermore, it appears that the relative bid–ask spread is positively correlated to the volatility of the security with a coefficient near to 0.5. These results are not compatible with the previous references.

Jouini and Kallal (1995a) proved that the absence of arbitrage opportunities is equivalent to the existence of a frictionless arbitrage free process (i.e., a process which could be transformed into a martingale under a well-chosen probability measure) lying between the bid and the ask price processes. Consequently, all the models with constant proportional transaction costs applied to some frictionless arbitrage-free price process S, are obviously arbitrage-free. The converse is false and if a model with constant proportional transaction costs applied to some price process S is arbitrage-free then S is not necessarily a frictionless arbitrage-free process.

In a recent paper, Koehl et al. (1996) consider, in a discrete time framework, such a model with proportional transaction costs but without any specific assumption on S. Nevertheless, they assume that the absence of arbitrage opportunities assumption is satisfied even with a little bit smaller bid–ask spread. But if this bid–ask spread is the result of all the buying and selling limit orders in a market with competitive market-makers (as on the MONEP, Paris) and not by a monopolistic specialist (as on the NYSE), then it seems natural to assume that the bid–ask spread is in some sense minimal. The only reason for which the bid–ask spread is not smaller appears then as the existence of arbitrage opportunities for little bit smaller bid–ask spreads. The condition imposed by Koehl et al. (1996) is then not so innocuous.

In the present paper, we consider a model according to Jouini and Kallal (1995a) for the description of the primitive assets. We prove in this setting, that the valuation formula obtained by Jouini and Kallal (1995a) for derivative assets with cash delivery extends for general derivative assets. This extension is important because on the markets, the traded contracts can impose cash delivery but also asset delivery or can let the choice to the derivative's holder and the domination price in these three situations is not the same at all as shown on some examples by Bensaid et al. (1992). This result is obtained as a corollary of the result of Jouini and Kallal (1995a) and generalizes the result of Cvitanic and Karatzas (1996). Indeed, our result is obtained under an absence of arbitrage opportunities assumption (obviously satisfied in Cvitanic and Karatzas (1996)) weaker than the classical analogous assumption in the continuous time models since we only consider simple strategies rather than general continuous time strategies. Our set of strategies is then smaller and the absence of strategies leading to an arbitrage a weaker assumption.

Our approach is an axiomatic one and constitutes a methodological innovation. Indeed, we shall first introduce the minimal assumptions for a price functional in order to be admissible. Then, we will prove that such an admissible price functional necessarily lies between the supremum and the infimum of the previously defined expected values. Furthermore, these maximum and minimum appear as admissible bid–ask prices. Our approach is now used in some posterior papers like in Koehl and Pham (2000).

From an economic point of view our result has many different interpretations.

First, our result can be seen as a necessary relation satisfied at the equilibrium (and then under the absence of arbitrage opportunities condition) by the primitive assets prices and the derivative assets prices. Our result is then particularly useful for econometricians who typically restrict their attention to a small number of traded securities (either because of data availability or for tractability reasons) and work out the implications of the data they have collected on them. Assuming the absence of arbitrage opportunities, a set of state price densities compatible with the data (in our framework, a set of martingale measures) can be derived. From there, it is possible to compute, for instance, the bounds on the mean and variance of the state prices, as in Hansen and Jagannathan (1991) (in a frictionless setup) and provide common diagnostic for a whole class of models. How to take into account the transaction costs in such an analysis is up to now a discussed question and we can refer to Rubinstein (1994) and Jackwerth and Rubinstein (1996) for a discussion of this point.

Second, if we consider a model in which we introduce regularly new standardized assets (for instance, 3-month calls at the money each trimester), we can assume that the introduction of these new assets is completely anticipated by the market and then that the introduction of these new assets will not modify significantly the trend or the volatility of the primitive assets price processes. The no-arbitrage condition implies then that the price of the new asset has to be between our bounds.

Third, if we keep in mind that an important part of the transactions on the derivative assets are over the counter, it seems reasonable in that case to think that the introduction of a new asset discussed between only two individuals and designed by one of them in order to satisfy particular needs of the other one will not modify the fundamentals of the economy. The unique rule for the seller is then to fix a price below the buyer's manufacturing cost.

Fourth, assuming that we are at the equilibrium before the introduction of the new assets, Jouini and Kallal (1996) proved that our bounds define the tightest bid–ask interval for the new asset for which a new equilibrium can be found without any modification of the other asset prices.

2. The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X = L^2\ (\Omega, \mathcal{F}, P)$ the space of square integrable random variables on $(\Omega, \mathcal{F}, P)$, that we assume to be separable. In fact, X is the space of classes of random variables that coincide almost everywhere. If $B \in \mathcal{F}$, we denote by 1_B the element of X equal to 1 on B and to 0 elsewhere. Let $\tilde{\Omega}$ be the space equal to $\Omega \times \{0, \ldots, K\}$ endowed with $(\tilde{\mathcal{F}}, \tilde{P})$ the natural probability structure defined by $(\mathcal{F}, P)$. Let $\tilde{X}$ be the set defined by $\tilde{X} =, L^2(\tilde{\Omega}\tilde{\mathcal{F}}, \tilde{P})$. The set $\tilde{X}$ can be identified with X^{K+1}. Let $\tilde{X}_+$ be the set of random variables $x \in \tilde{X}$ such that $\tilde{P}(x \geq 0) = 1$ and $\tilde{P}(x > 0)$. A linear functional on ψ is $\tilde{X}$ said to be positive if $\psi(x) \geq 0$ for all $x \in \tilde{X}$ such that $\tilde{P}(x \geq 0) = 1$ and $\psi(x) > 0$ for all $x \in \tilde{X}_+$.

We consider a multiperiod economy where agents can trade a finite number of securities at all dates $t \in \mathcal{T}$, with $\mathcal{T} \subset [0, T]$. Although we impose a finite horizon there is no other restriction on market timing: our framework includes discrete as well as continuous time models. Without loss of generality we shall assume that agents can trade at the initial and the final date, i.e. $\{0, T\} \subset \mathcal{T}$. Each security k, with $k = 0, \ldots, K$, can be bought for its ask price $Z_k(t)$ and can be sold for its bid price $Z'_x(t)$ at any time $t \in T$. A right-continuous[3] filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ models the information structure of our economy, where the σ-algebra $\mathcal{F}_t$ represents the information available to agents at date t. We also make the following assumption.

Assumption (P). (i) Z_k and Z'_k are right-continuous and adapted to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$, for all $k = 0, \ldots, K$. (ii) $E((Z_k)^2(t)) < \infty$ and $E((Z'_k)^2(t)) < \infty$ for all $t \in \mathcal{T}$ and $k = 0, \ldots, K$. (iii) $Z_k \geq Z'_k > 0$ for all t and for almost all ω. (iv) Z_0 and Z'_0 are constant equal to 1.

Assumption (*i*) says that the bid and the ask prices of traded securities belong to the information set available to agents. For convenience, we shall also assume that $\mathcal{F}_0$ is the trivial σ-algebra, and that $\mathcal{F}_T = \mathcal{F}$. Assumption (*ii*) is technical. Assumption (*iii*) means that all the prices remain positive and that the buying price is greater than or equal to the selling price. This last condition is obviously satisfied under the no-arbitrage condition and can then be dropped without any loss. The last assumption means that there is no transaction costs on the cash. It is easy to relax the equality to one dividing all the prices by Z_0.

A contingent claim is then defined by the contingent traded securities quantities delivered at the final date.

Definition 1. A contingent claim C is defined by $(C_0, \ldots, C_K) \in X^{K+1}$ the contingent portfolio guaranteed by C.

This definition of a contingent claim permits us to consider, for instance, call options with asset delivery. In this last case $C_i = 1_{S_i \geq K}$ and $C_0 = -K1_{S_i \geq K}$ if the primitive asset is the *i*-th one. Furthermore, it is easy to see that one contingent unit of a given asset has not the same effect on the agent's portfolio than the "equivalent" amount (in fact, we cannot define the equivalent amount at all since the buying and the selling prices differ). There is many other situations, where the derivative asset cannot be described by a contingent amount but by a contingent portfolio like in the national loan where we have to deliver at the due date the less costly bond in some given basket. Furthermore, since the considered filtration is, in general, the filtration generated by the price processes, all contingent claims can be expressed as random functions of assets and our definition permits then to encompass the most general situations.

A price functional in this setting is a function p defined on the contingent claims space X^{K+1} and which takes its values in $R \cup \{\infty\}$ where $p(C)$ represents the price at which the contingent claim C can be bought. The following conditions characterize the admissible price functionals.

Axiom 1. *The price functional p is a sublinear form (i.e., for all pair (C,C′) of elements of X^{K+1} and for all non-negative real number λ we have $p(C + C') \leq p(C) + p(C')$ and $p(\lambda C) = p(C)$).*

[3] I.e., for all $t \in [0, T]$, $\mathcal{F}_t$ is the intersection of the -algebras $\mathcal{F}_s$, where $s > t$. This assumption, as well as the right-continuity of the bid–ask price processes, are not necessary if there are no transaction costs at the final date (i.e., if $Z(T) = Z'(T)$ a.e.).

This means that it is less expensive to buy the sum $C + C'$ of two contingent claims than to buy the claims C and C' separately and add up the prices. It is easy to see why if we think in terms of hedging costs: the sum of two strategies that hedge the claims C and C' hedges the claim $C + C'$ but some orders to buy and sell the same security at the same date might cancel out. Some of the transaction costs might be saved this way. But even if the price differs from the hedging cost our assumption seems to be satisfied in the real world and it is well-known, for instance, that the theoretical call–put parity (obtained under a linearity assumption on the price functionals) is not satisfied in general. In particular our condition implies that the buying price $p(C)$ is greater than or equal to the selling price $-p(-C)$. The multiplicative condition seems to be less natural but is assumed in all the classical financial market models. Furthermore, the multiplicative effect is not clear since the price is influenced by two diametrically opposed effects: increasing returns to scale (possibility to obtain better prices from the broker for large quantities) and exhaustion of the best bid and ask offers which implies a greater bid–ask spread and decreasing returns to scale. Without further informations on the relative size of these effects, the assumption $p(\lambda C) = \lambda p(C)$ seems to be acceptable. This assumption is compatible with the sublinearity one and seems to be satisfied in the real world for reasonable values of λ.

Axiom 2. *The price functional p is lower semi-continuous (i.e., if a sequence (C_n) converges to C in X^{K+1} and if $p(C_n)$ converges to λ then $p(C) \leq \lambda$).*

This assumption is not only a technical one but is also a natural one. Indeed if some payoffs arbitrarily close to the payoff C can be obtained at a price lower to some given price, it seems to be obvious that no one will accept to pay more than this given price to obtain C. The lower semi-continuity of p is then a classical consequence of this property.

Axiom 3. *The pricing functional p induces no-arbitrage (i.e., if $C \in \bar{X}_+$ then $p(C) > 0$).*

This assumption is a classical one and we can remark that this formulation is the weaker one. Indeed, our assumption concerns the absence of arbitrage opportunities under the price p in a static setting and not the absence of free lunches in a dynamic setting. Furthermore, in general, free lunches are defined as limits and here we have no such complicated construction.

In order to introduce our last condition on p we have to describe more precisely the strategy space of the agents. In fact and even in a model with a continuous resolution of uncertainty, it seems to be more realistic to assume that the agents do not trade at each date but only on a finite set of dates as in Harrison and Kreps (1979). This set is chosen by the agents and depends on the strategy choice.

The set of admissible strategies in our framework is then smaller than the admissible strategies set in Harrison and Pliska (1979), Dybvig and Huang (1988) in a frictionless setting and in Cvitanic and Karatzas (1996) with transaction costs. The absence of arbitrage opportunities *assumption* which imposes the non-existence of strategies leading to positive payoffs at a non-positive cost is then *weaker* in our framework.

Definition 2. A simple strategy is a pair (θ, θ') of $K + 1$ processes such that, (*i*) (θ, θ') is adapted to $\{\mathcal{F}_t\}_{t \in \mathcal{T}}$, (*ii*) θ_k and θ'_k are non-negative and non-decreasing processes for $k = 0, \ldots, K$, (*iii*) $E((\theta_k Z_k)^2(t)) < \infty$, $E((\theta_k Z'_k)^2(t)) < \infty$, $E((\theta_k Z'_k)^2(t)) < \infty$, and $E((\theta'_k Z_k)^2(t)) < \infty$, for all $t \in \mathcal{T}$ and $k = 0, \ldots, K$, (*iv*) there exists an integer *N* and a set of dates $\{t_0, \ldots, t_N\} \subset \mathcal{T}$, with $0 = t_0 \leq \ldots \leq t_N = T$, such that $(\theta(t, \omega), \theta'(t, \omega))$ is constant, for every ω, over the interval $[t_{n-1}, t_n)$, for $n = 1, \ldots, N$.

Since the bid and the ask prices possibly differ, we separate strategies in a long cumulative component θ and in a short cumulative component θ', i.e., $\theta_k(t)$ is the total quantity of the *k*-th security bought up to time *t* and $\theta'_k(t)$ is the total quantity sold up to time *t*. Hence, $\theta_k(t) - \theta'_k(t)$ is the amount of the *k*-th security owned at time *t*. Assumption (*i*) says that consumers can trade only on current and past information. Assumption (*ii*) translates the fact that θ and θ' are cumulative long and short positions. Assumption (*iii*) is technical. Assumption (*iv*) says that any given strategy must have a finite (but arbitrarily large) number of trading dates decided in advance (at date 0). We assume here that trading dates are decided in advance. It is possible, however, to let trading dates be stopping times and impose only their number to be decided in advance. Note that when we are concerned by the characterization of the absence of arbitrage opportunities, the assumption (*iv*) restricts the set of strategies and for a same conclusion, a result obtained under (*iv*) is stronger than a result obtained without (*iv*).

Agents are assumed not to have external sources of financing, and since they consume only at dates 0 and *T* they must sell (or short) some securities in order to purchase others at intermediary dates. Hence, we define self-financing strategies as the admissible (in the sense of the budget constraints) strategies.

Definition 3. A simple strategy (θ, θ') is self-financing if for $n = 1, \ldots, N$ we have

$$(\theta(t_n) - \theta(t_{n-1})) \cdot Z(t_n) \leq (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z'(t_n).$$

This means that at every trading date (after the initial date) the value of the securities that are bought is less than or equal to the value of the securities that are sold: in other words sales must finance purchases. The set of simple self-financing strategies is denoted by Θ. It turns out that the set of simple self-financing trading strategies is stable by addition and multiplication by a non-negative scalar, i.e., it is a convex cone of the space of simple strategies.

A strategy $(\theta, \theta') \in \Theta$ costs $\theta(0) \cdot Z(0) - \theta'(0) \cdot Z'(0)$ units of date 0 consumption, and provides $(\theta_k - \theta'_k)(T)$ units of security *k* at date *T*.

We have already seen that when there are transaction costs, it is not true that the cheapest way to obtain a given minimal contingent payoff at date *T* is to duplicate it by dynamic trading. This fact has been pointed out by Bensaid et al. (1992) in a discrete time and states framework. A simple example can illustrate it. Assume that a call option on a stock is to be hedged using a riskless bond and the underlying stock only. Also suppose that there are transaction costs in trading the stock at intermediate dates (between now and maturity). It is then easy to see that if transaction costs are prohibitively high it is cheaper to buy the stock and hold it until maturity (which leads to a payoff that is strictly larger than the payoff of the call) than to try to duplicate the call. In fact, the same conclusion is obtained in 1995 by Levental and Skorohod (1995), Soner et al. (1995) and Dubourg (1997), in continuous time models even with small transaction cost. Hence, we shall consider the price functional π defined for every claim $C \in X^{K+1}$ by

$$\pi(C) = \inf\{\theta(0) \cdot Z(0) - \theta'(0) \cdot Z'(0)\} : (\theta, \theta') \in \Theta \ \text{ and } \ (\theta - \theta')(T) \geq C\}.$$

In words, $\pi(C)$ represents the infimum cost necessary to get at least the final contingent portfolio C at date T. Note that a contingent claim C is not necessarily attainable or at least dominatable by a strategy belonging to θ. In this case, we have $\pi(C) = \infty$. Note that π is not defined as in Jouini and Kallal (1995a) taking limits and inf-limits but directly from the dominating strategies set.

We can now introduce our last condition.

Axiom 4. *For all $C \in X^{K+1}$, we have $p(C) \leq \pi(C)$.*

This assumption is only a monotonicity assumption. We impose that if it is possible to obtain a better payoff than C at a cost $\pi(C)$ then no one will accept to pay more than $\pi(C)$ in order to obtain C. Remark that even if C is duplicable and if its duplication cost is the minimum cost necessary to obtain at least the same payoff we do not impose $p(C) = \pi(C)$. In particular, if C is the k-th traded security we have $\pi(C) = Z_k(T)$ [4] and we do not impose $p(C) = \pi(C)$. Indeed, $\pi(C)$ and $-\pi(-C)$ can be interpreted as the best prices proposed by the market-makers. It is obvious then that transactions can occur at both prices. Furthermore, no agent will accept to pay more than $\pi(C)$ in order to receive C or to receive less than $-\pi(-C)$ if he sells C. Nevertheless, a buyer and a seller can accept any intermediary price. Consequently, the price at which a transaction will occur is not necessarily one of the bounds imposed by the market-makers but can be any price between these bounds. Furthermore, it is easy to see that if the new asset is introduced on the market at a buying price greater than $\pi(C)$ and a selling price smaller than $-\pi(-C)$ then no one will buy or sell this asset and the equilibrium prices and allocation will not be modified. Consequently, if we want to introduce this new asset and if we want to see it traded by some agent, it seems to be reasonable to look for a buying or a selling price in our interval.

We are now in a position to state our main result.

Theorem 1. *(i) There exists at least one admissible price functional p if and only if there exists at least a probability measure P^* equivalent to P (i.e., P and P^* have exactly the same zero measure sets) with $E((\mathrm{d}P^*/\mathrm{d}P)^2) < \infty$ and a process Z^* satisfying, for all t, $Z'(t) \leq Z^*(t) \leq Z(t)$, a.e., such that $E(((\mathrm{d}P^*/\mathrm{d}P)Z^*(T))^2) < \infty$ and Z^* is a martingale with respect to the filtration $\{\mathcal{F}_t\}$ and the probability measure P^*. (ii) If p satisfies conditions (A-1) to (A-4) then for all contingent claim C we have $p(C) \in [\inf E^*(C \cdot Z^*(T)), \sup E^*(C \cdot Z^*(T))] = [-p^*(-C), p^*(C)]$ where the infimum and the supremum are taken over all the expectation operators E^* associated to a probability measure P^* and all the processes*

[4] In fact, in order to have $\pi(C) = Z_k(T)$ we have to impose also that the considered security cannot be dominated by a combination of the others. This is, in particular, true under some independence conditions on the traded assets.

Z^* such that (P^*,Z^*) satisfy the conditions of (i). Furthermore, the functional p^* defined as above satisfies conditions (A-1) to (A-4) and is then an admissible price functional.

In order to prove the main result we have to introduce the function $\tilde{\pi}$ defined as follows and to establish the following lemma,

$$\tilde{\pi}(C) = \inf\{\liminf_n\{\theta_n(0)Z(0) - \theta'_n(0)Z'(0)\} : (\theta_n, \theta'_n) \in \Theta,$$
$$(\theta_n - \theta'_n)(T) \geq C_n, (C_n) \to C\}.$$

In words $\tilde{\pi}(C)$ represents the infimum cost necessary to get at least a final contingent portfolio arbitrarily close to C at date T.

Lemma 2. *The functionals π and $\tilde{\pi}$ are sublinear and $\tilde{\pi}$ is the largest l.s.c. functional that lies below π.*

Proof of the lemma. If we remark that Θ is a convex cone, it is relatively easy to prove that π is a sublinear functional and by a limit argument that $\tilde{\pi}$ is also a sublinear functional. Let M be the set defined by $\{x \in X : \tilde{\pi}(x) < \infty\}$. □

Let $\lambda \in R$ and C^n be a sequence in M converging to $C \in M$ such that $\tilde{\pi}(C^n) \leq \lambda$, for all n. Then, by a diagonal extraction process, there exist a sequence $\tilde{C}_n$ and a sequence $(\theta, \theta'_n) \in \Theta$ such that $||\tilde{C}_n - C_n|| \leq 1/n$, $(\theta_n - \theta'_n)(T) \geq \tilde{C}_n$ and $\theta_n(0)\cdot Z(0) - \theta'_n(0)\cdot Z'(0) \leq \lambda + (1/n)$. Since $\tilde{C}_n$ converges to C we must then have, by definition of $\tilde{\pi}$, $\tilde{\pi}(C) \leq \lambda$. Hence, the set $\{C \in M : \tilde{\pi}(C) \leq \lambda\}$ is closed and $\tilde{\pi}$ is l.s.c.

Let C_n be a sequence of elements of X^{K+1} converging to a claim C and let (θ_n, θ'_n) be a sequence of strategies in Θ such that $(\theta_n - \theta'_n)(T) \geq C_n$. It is clear that $\theta_n(0) \cdot Z(0) - \theta'_n(0) \cdot Z'(0) \geq \pi(C_n)$ and consequently, $\tilde{\pi}(C) \geq \inf\{\liminf_n \pi(C_n) : C_n \to C\}$. Moreover, it is clear that $\tilde{\pi}(C) \leq \pi(C)$ for all $C \in X^{K+1}$. Since $\tilde{\pi}$ is l.s.c. we must have $\tilde{\pi}(C) \geq \inf\{\liminf_n \tilde{\pi}(C_n) : C_n \to C\}$ which implies that $\tilde{\pi}(C) \leq \inf\{\liminf_n \pi(C_n) : C_n \to C\}$. Consequently, $\tilde{\pi}(C) = \inf\{\liminf_n \pi(C_n) : C_n \to C\}$. An analogous argument gives, for every l.s.c. functional $f\colon X^{K+1} \to R$ such that $f \leq \pi$, that $f(C) \leq \inf\{\liminf_n \pi(C_n)\colon C_n \to C\}$ and hence, $F \leq \tilde{\pi}$.

Proof of the theorem. First, let P^* be a probability measure equivalent to P and let Z^*, with $Z' \leq Z^* \leq Z$, be a martingale with respect to P^* and $\{\mathcal{F}_t\}$. Define the linear functional p by $p(C) = E^*(Z^*(T) \cdot C)$ for all $C \in X^{K+1}$. Since $\rho Z^*(T) = (\mathrm{d}P^*/\mathrm{d}P)Z^*(T) \in X$ we have that $p(C) = E^*(Z^*(T) \cdot C) = E(\rho Z^*(T) \cdot C)$ is continuous. Since P and P^* are equivalent, it is easy to see that p is positive. The price functional p satisfies then assumptions (A-1) to (A-3).

Let $C \in X^{K+1}$ and let $(\theta, \theta') \in \Theta$ with trading dates $0 = t_0 \leq t_1 \leq \ldots t_N = T$, such that $(\theta - \theta')(T) \geq C$. Since $Z' \leq Z^* \leq Z$ and (θ, θ') is non-decreasing and self-financing, we have, for $n = 1, \ldots, N$,

$$E^*((\theta(t_n) - \theta(t_{n-1})) \cdot Z^*(t_n) - (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z^*(t_n)|\mathcal{F}_{t_{n-1}})$$
$$\leq E^*((\theta(t_n) - \theta(t_{n-1})) \cdot Z(t_n) - (\theta'(t_n) - \theta'(t_{n-1})) \cdot Z'(t_n)|\mathcal{F}_{t_{n-1}} \leq 0.$$

Using the fact that Z^* is a martingale with respect to $\{\mathcal{F}_t\}$ and P^*, we have

$$E^*((\theta-\theta')(t_n)\cdot Z^*(t_n)|\mathcal{F}_{t_{n-1}})$$
$$\leq E^*((\theta-\theta')(t_{n-1})\cdot Z^*(t_n)|\mathcal{F}_{t_{n-1}}) \leq (\theta-\theta')(t_{n-1})\cdot Z^*(t_{n-1}).$$

By iteration, $E^*((\theta-\theta')(T)\cdot Z^*(T)) \leq (\theta-\theta')(0)\cdot Z^*(0) \leq \theta(0)\cdot Z(0)-\theta'(0)\cdot Z'(0)$. □

Furthermore, by definition of p we have $p(C) = E^*(Z^*(T)\cdot C) \leq E^*((\theta-\theta')(T)\cdot Z^*(T))$. Hence, $p(C) \leq \theta(0)\cdot Z(0)-\theta'(0)\cdot Z'(0)$ and taking the infimum over the strategies $(\theta,\theta') \in \Theta$ such that $(\theta-\theta')(T) \geq C$, we obtain that $p(C) = E^*(C) \leq \pi(C)$ for all $C \in X^{K+1}$ and p satisfies also the condition (A-4).

Assume now that there exists at least one admissible price functional. Following Jouini and Kallal (1995a) (Definition 2.1), we will call a free lunch in $\tilde{X}$ a sequence of real numbers r_n that converges to some $r^* \geq 0$, a sequence (x_n) in $\tilde{X}$ that converges to some $x^* \geq$ such that $r^*+x^* \in \tilde{X}_+$, and a sequence of claims C^n such that $C^n \geq x_n$ and $r_n+\tilde{\pi}(C^n) \leq 0$ [5], for all n. We have then the following result.

Lemma 3. *If there exists at least one admissible price functional then there is no free lunch.*

Proof. Consider a free lunch as defined above. We have $x_n \in \tilde{X}$ and $r_n+\tilde{\pi}(x_n) \leq 0$. If $r^* \geq 0$ and since $\tilde{\pi}$ is l.s.c., we have then $\tilde{\pi}(x^*) \leq 0$. Recalling that $p \leq \pi$ is l.s.c. and that $\tilde{\pi}$ is the largest l.s.c. functional that lies below π, we have $p(x^*) \leq 0$ with $x^* \in \tilde{X}_+$ which constitutes an arbitrage. Then, by assumption (A-3) there is no free lunch. □

Remark. If there is no free lunches then $\tilde{\pi} = p^*$. This is a direct result ofJouini and Kallal (1995a,b) but can also be proved directly using Theorem and Lemma 1. Indeed, it is easy to see that $\tilde{\pi}$ satisfies assumptions (A-1), (A-2) and (A-4) and if there is no free lunches then (A-3) is also satisfied. We have then $\tilde{\pi}(C) \leq \sup p^*(C)$ and since $\tilde{\pi}$ is the largest l.s.c. functional that lies below π the converse inequality holds. When there are free lunches, then following Jouini and Kallal (1995a), there does not exist "martingale-measures" and p^* is not defined.

Assume now that there is no-arbitrage and consequently that there is no free lunch defined as above. Let us consider M be the subset of $\tilde{X}$ defined by $M = \{m \in \tilde{X}| : \tilde{\pi}(m) < \infty\}$ and let us denote by $\tilde{\Psi}$ the set of positive linear forms on $\tilde{X}$. Consider $\psi \in \tilde{\Psi}$ such that $\psi|_M \leq \tilde{\pi}$, as guaranteed by Jouini and Kallal (1995a, Theorem 2.1) under the no free lunch condition. Since ψ is continuous, by the Riesz representation theorem there exists a random variable $\rho \in \tilde{X}$ such that $\psi(x) = E(\rho, x)$, for all $x \in \tilde{X}$ or equivalently there exists $(\rho_0,\ldots,\rho_K)$ in X^{K+1} such that $\psi(C) = E(\rho\cdot C)$, for all $C \in X^{K+1}$. Define P^* from ψ by $P^*(B) = E(\rho_0 1_B)$ for all $B \in \mathcal{F}$. By linearity and strict positivity of ψ it is clear that P^* is a measure equivalent to P. Using the fact that $Z_0 = Z'_0 = 1\psi(1_\Omega,\ldots,0) \leq 1$ and $\psi(1_\Omega,\ldots,0) \leq -1$ which implies that $P^*(1_\Omega) = 1$ and $\mathrm{d}P^*/\mathrm{d}P = \rho_0$ is square integrable.

[5] Note that our functions π and $\tilde{\pi}$ are denoted, respectively, by $\tilde{\pi}$ and π in the mentioned reference.

It remains to show that there exists a process Z^*, with $Z'_k \leq Z^*_k \leq Z_k$, and such that Z^*_k is a martingale with respect to P^* and $\{\mathcal{F}_t\}$, for $k = 1, \ldots, K$. In fact, we will prove that the martingale relatively to P^* and $\{\mathcal{F}_t\}$ defined by $Z^*_k(t) = E^*(\rho_k/\rho_0|\mathcal{F}_t)$ lies between Z'_k and Z_k.

Let $k \in \{1, \ldots, K\}$, $t \in \mathcal{T}$ and $B \in \mathcal{F}_t$. Let C the contingent claim defined by $C_k = 1_B$, $C_0 = -Z_k(t)1_B$ and $C_h = 0$ for $h \neq 0, k$. The contingent claim C is duplicable. It suffices to buy at t, if $\omega \in B$, one unit of the security k and to pay with security 0 units. This strategy costs nothing and we have then $E^*((-Z_k(t) + \rho_k\rho_0)1_B) = E((-\rho_0 Z_k(t) + \rho_k)1_B = \psi(C) \leq \tilde{\pi}(C) \leq (C) \leq 0$. Then, we have $E^*(\rho_k/\rho_0 1_B) \leq E^*(Z_k(t)1_B)$, for all t and all $B \in \mathcal{F}_t$. This implies that $Z^*_k \leq Z_k$. By a symmetric argument we obtain $Z^*_k \geq Z'_k$. Furthermore, by construction, $(\mathrm{d}P^*/\mathrm{d}P)Z^*(T)$ is square integrable which achieves to prove the point (i) of the theorem.

In fact, we have also proved that every $\psi \in \tilde{\Psi}$ such that $\psi|_M \leq \pi$ is equal to $E^*(C{\cdot}Z^*)$ for some process Z^* between Z' and Z and some probability measure P^* such that Z^* is a martingale relatively to P^* and conversely.

Following Jouini and Kallal (1995a, Theorem 2.2), $\tilde{\pi}(C) = \sup\psi(C)$ where the supremum is taken over all the functionals $\psi \in \tilde{\Psi}$ such that $\psi|_M \leq \tilde{\pi}$. Consequently, if p is an admissible price functional, by (A-1), (A-2) and (A-4) we have that $p \leq \tilde{\pi}$ and applying this result to C and $-C$ for a given C in $\tilde{X}$, we obtain $p(C) \in [\inf E^*(C \cdot Z^*(T)),\ \sup E^*(C \cdot Z^*(T))]$ where the infimum and the supremum are taken over all the expectation operators E^* associated to a probability measure P^* and all the processes Z^* such that (P^*, Z^*) satisfy the conditions of (i).

Since p^* satisfies conditions (A-1) to (A-4), this achieves the proof of the theorem.

References

Bensaid, B., Lesne, J.P., Pages, H., Scheinkman, J., 1992. Derivative asset pricing with transaction costs. Mathematical Finance 2, 63–86.

Boyle, P., Vorst, T., 1992. Option replication in discrete time with transaction costs. Journal of Finance 47, 272–293.

Cvitanic, J., Karatzas, I., 1996. Hedging and portfolio optimization under transaction costs: a martingale approach. Mathematical Finance 6, 133–166.

Dubourg, N., 1997. Inefficiency of contingent claims domination with proportional transaction costs. Ph.D. Dissertation, University of Paris 1.

Hamon, J., Jacquillat, B., 1992. Le marché français des actions, Etudes empiriques 1977–1991. Presses Universitaires de France, Paris.

Hansen, L.P., Jagannathan, R., 1991. Implications of security market data for models of dynamic economies. Journal of Political Economy 99, 252–262.

Harrison, J., Kreps, D., 1979. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20, 381–408.

Harrison, J., Pliska, S., 1979. Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications 11, 215–260.

Jackwerth, J.C., Rubinstein, M., 1996. Recovering probability distributions from option prices. Journal of Finance 51 (5), 1611–1631.

Jouini, E., Kallal, H., 1995a. Martingales and arbitrage in securities markets with transaction costs. Journal of Economic Theory 66 (1), 178–197.

Jouini, E., Kallal, H., 1995b. Arbitrage in securities markets with short sales constraints. Mathematical Finance (5).

Jouini, E., Kallal, H., 1996. Viability and equilibrium in securities markets with frictions. CREST Working Paper 9707. Mathematical Finance, to appear.

Koehl, P.-F., Pham, H., 2000. Sublinear price functionals under portfolio constraints. Journal of Mathematical Economics 33, 339–351.

Koehl, P.-F., Pham, H., Touzi, N., 1996. Option pricing under transaction costs: a martingale approach. CREST Working Paper 9647.

Kusuoka, S., 1995. Limit theorem on option replication with transaction costs. Annals of Applied Probability 5, 198–221.

Leland, H., 1985. Option pricing and replication with transaction costs. Journal of Finance 40, 1283–1301.

Levental, S., Skorohod, A.V., 1995. On the possibility of hedging options in the presence or transaction costs, Annals of Applied Probability, forthcoming.

Rubinstein, M., 1994. Implied binomial trees. Journal of Finance 49, 771–818.

Shirakawa, H., Konno, H., 1995. Pricing of Options Under The Proportional Transaction Costs. Tokyo Institute of Technology, preprint.

Soner, H.M., Shreve, S.E., Cvitanic, J., 1995. There is no nontrivial hedging portfolio for option pricing with transaction costs. Annals of Applied Probability 5, 327–355.