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A REMARK ON CLARKE'S NORMAL CONE AND THE MARGINAL COST PRICING RULE*

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This paper constructs a closed set Y in $\boldsymbol{R}^l$ such that for all y in the boundary of Y, Clarke's normal cone to Y at y is equal to $\boldsymbol{R}^l_+$. If Y is the production set of a firm, then the marginal cost pricing rule imposes no restriction. The existence of Y is shown to be equivalent to the existence of a Lipschitzian function f from $\boldsymbol{R}^{l-1}$ to $\boldsymbol{R}$ such that the generalized gradient of f is everywhere equal to the convex hull of 0 and the simplex of $\boldsymbol{R}^{l-1}$.

1. Introduction

A marginal (cost) pricing equilibrium is a state which consists of a price vector, a list of consumption vectors, a list of production plans, which satisfy the conditions of a competitive equilibrium except for the behavior rule of the firms which are instructed to fulfil the 'first-order necessary conditions' for profit maximization (also called the marginal rule of the firm).

Following Cornet (1982), in the case where the production sets are neither assumed to be convex nor to have a smooth boundary, the marginal rule is formalized by saying that each firm j, with production set Y_j, sets the price vector p in $N_{Y_j}(y)$ the normal cone to Y_j at y, in the sense of Clarke (1975).

In this paper, we show that in certain cases Clarke's normal cone may be too large and the marginal rule, as formalized above, may impose no restriction on the price vector which is set by the firm. More precisely, we shall construct a closed (production) set $Y \subset \boldsymbol{R}^l$ such that $Y - \boldsymbol{R}^l_+ \subset Y$ (free disposal) and, for every y in the boundary ∂Y of Y, Clarke's normal cone to Y at y is equal to $\boldsymbol{R}^l_+$. We shall prove a slightly more general result, which we now state.

Theorem 1. *Let $v_1, \ldots, v_k$ be a family of independent vectors of $\boldsymbol{R}^l$, let $C = \{\sum_{i=1}^k \lambda_i v_i \mid \lambda_i \geqq 0, i=1,\ldots,k\}$ and let $C^0 = \{p \in \boldsymbol{R}^l \mid p \cdot v_i \leqq 0,$ for $i=1,\ldots,k\}$ be the negative polar cone of C. Then there exists a non-empty closed set $Y \subset \boldsymbol{R}^l$ such that $Y + C^0 \subset Y$ and, for every $y \in \partial Y$, one has $N_Y(y) = C$.*

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We shall deduce Theorem 1 from the following result, also of interest for itself, which generalizes a previous result of Rockafellar (1981) in dimension one.

Theorem 1 bis. *Let K be a d-dimensional polytope of $\boldsymbol{R}^n$ with exactly $(d+1)$ vertices or equivalently let $K=\mathrm{co}\{u_0,u_1,\ldots,u_d\}$ where $u_0,u_1,\ldots,u_d$ $(d\geqq 1)$ is a family of vectors in $\boldsymbol{R}^n$ such that $u_1-u_0,\ldots,u_d-u_0$ are independent. Then there exists a Lipschitzian function F_K such that, for all $z\in\boldsymbol{R}^n$, $K=\partial F_K(x)$, the generalized gradient of F_K at x in the sense of Clarke.*

The paper is organized as follows. In the next section we shall show that the Theorems 1 and 1 bis are equivalent and we shall also provide a constructive proof of Theorem 1 bis in dimension one ($n=1$) different from the one in Rockafellar. In section 3, we shall give the proof of Theorem 1 bis. At the end of this introduction we recall the definitions of the tangent (normal) cone and the generalized gradient in the sense of Clarke (1975) [see also Clarke (1983)]. Let Y be a closed subset of $\boldsymbol{R}^l$ then, for every $y\in Y$, the tangent cone $T_Y(y)$ in the sense of Clarke consists of all vectors $v\in\boldsymbol{R}^l$ such that, for all sequences $\{t^k\}\subset(0,+\infty)$ and $\{y^k\}\subset Y$ converging, respectively to 0 and y, there exists a sequence $\{v^k\}\subset\boldsymbol{R}^l$ converging to v, with $y^k+t^kv^k\in Y$, for all k. Clarke's normal cone is then defined by polarity as follows:

$$N_Y(y)=T_Y(y)^0=\{p\in\boldsymbol{R}^l\,|\,p\cdot v\leqq 0,\quad \text{for al } v\in T_Y(y)\}.$$

Let $f:\boldsymbol{R}^l\to\boldsymbol{R}$ be a Lipschitz function, i.e., for for some $k>0$ and for all x_1,x_2 in $\boldsymbol{R}^l$, one has $|f(x_2)-f(x_1)|\leqq k\|x_2-x_1\|$. Then, from Rademacher's theorem f is differentiable almost everywhere on V, i.e., at every element of $V\backslash\Omega_f$ where Ω_f is a subset of V of Lebesgue measure zero. Then the generalized gradient of f at x, denoted by $\partial f(x)$ is defined by

$$\partial f(x)=\mathrm{co}\left\{\lim_i \nabla f(x_i)\,|\,x_i\to x,\, x_i\notin\Omega_f\right\},$$

where $\nabla f(x_i)$ denotes the gradient vector of f at x_i and $\mathrm{co}\,C$ denotes the convex hull of a subset C of $\boldsymbol{R}^l$.

2. Production sets and associated functions

We first construct a closed (production) set $Y_2\subset\boldsymbol{R}^2$ such that $Y_2-\boldsymbol{R}^2_+\subset Y_2$ and, for every $y\in\partial Y_2$, $N_{Y_2}(y)=\boldsymbol{R}^2_+$. We let $Y_2=\{(x,y)\in\boldsymbol{R}^2\,|\,y\leqq f(x-E(x))-E(x)\}$, where $E(x)$ is the integer part of the real number x and the function $f:[0,1]\to\boldsymbol{R}$ is defined as the limit of a sequence of continuous

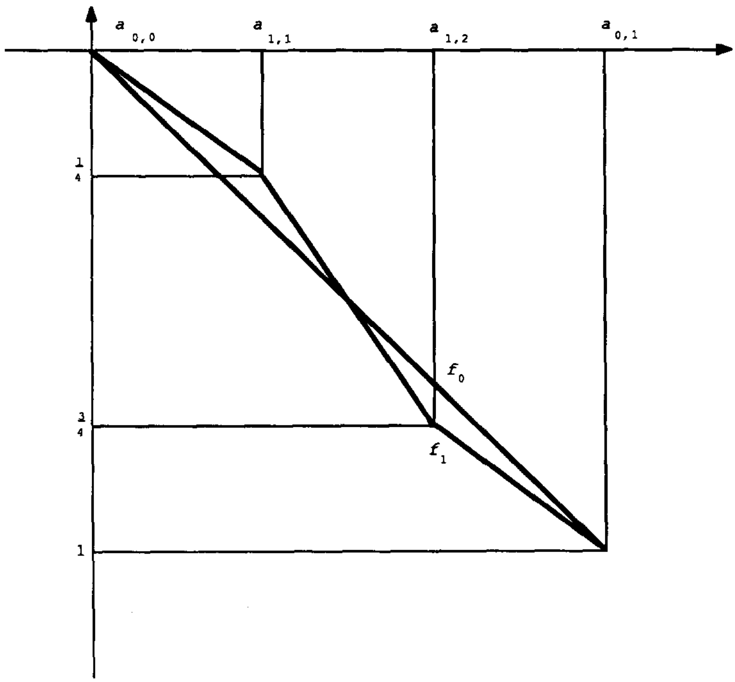

Fig. 1

functions $f_n:[0, 1] \to \boldsymbol{R}$. Before defining the sequence f_n we first need to introduce some definitions. For every integer n, we let

$$\Delta_n = \{a_{n,p} = p \cdot 3^{-n} \mid p \in \mathrm{N}, 0 < p < 3^n\} = \{x \in]0, 1[\mid 3^n \cdot x \in \boldsymbol{Z}\},$$

$$\Delta'_n = \tfrac{1}{2}(a_{n,p} + a_{n,p+1}) \mid p \in \mathrm{N}, 0 \leqq p < 3^n\} = \{x \in]0, 1[\mid 3^n \cdot x - \tfrac{1}{2} \in \boldsymbol{Z}\},$$

$$\Delta = \bigcup_{n=0}^{\infty} \Delta_n, \quad \Delta' = \bigcup_{n=0}^{\infty} \Delta'_n \qquad (\text{one has } \mathrm{cl}(\Delta) = \mathrm{cl}(\Delta') = [0, 1]),$$

where $\mathrm{cl}(\Delta)$ denotes the closure of Δ.

The sequence of continuous functions $f_n:[0, 1] \to \boldsymbol{R}$ is then defined by induction as follows (see fig. 1). We let $f_0(x) = -x$ and we let f_{n+1} be defined by

(a) f_{n+1} is affine on $[a_{n+1,p}, a_{n+1,p+1}]$, for all $p < 3^{n+1}$,
(b) $f_{n+1}(a_{n+1,p}) = f_n(a_{n,q}) + \frac{1}{4}\varphi(p-3q)(f_n(a_{n,q+1}) - f_n(a_{n,q}))$, where $\varphi(x) = \frac{1}{2}x(x+1)$ for all x, and $q = E(p/3)$.

The properties of Y_2 are then a direct consequence of the following steps, the proof of which is left to the reader.

Step 1. f_n is decreasing, and for all $p<3^n$, for all $x\in]a_{n,p}, a_{n,p+1}[$, f_n is differentiable at x.

Step 2. Let $x\in\Delta_{n_0}\cup\Delta'_{n_0}$, for all $n>n_0$, $f_n(x)=f_{n_0}(x)$.

Step 3. The sequence $\{f_n\}$ converges uniformly to f and f is continuous and decreasing.

Step 4. If $x-E(x)\in\Delta_n$ then $(0,1)\in N_{Y_2}(x,f(x))$, and if $x-E(x)\in\Delta'_n$, $(1,0)\in N_{Y_2}(x,f(x))$.

We now define the function $\lambda_1\colon \boldsymbol{R}\to\boldsymbol{R}$ by

$$\lambda_1(x)=\inf\{\lambda\in\boldsymbol{R}\,|\,(x,\,-x)-\lambda(1,1)\in Y_2\},$$

for $x\in\boldsymbol{R}$, and we claim that λ_1 is Lipschitz and that

$$\partial\lambda_1(x)=[-1,\,+1]\quad\text{for}\quad\text{every } x\in\boldsymbol{R}.$$

This assertion is in fact a consequence of more general considerations which allow us to show that Theorems 1 and 1 bis are, in fact, equivalent. For this, we shall associate to every set $Y\subset\boldsymbol{R}^l$ a function $f\colon\boldsymbol{R}^{l-1}\to\boldsymbol{R}$, and conversely. We first recall the following result of Bonnisseau and Cornet (1985, 1988).

Lemma 1. Let Y be a non-empty closed subset of $\boldsymbol{R}^l$ such that $Y\neq\boldsymbol{R}^l$ and $Y+Q\subset Y$, where Q is a closed convex cone of $\boldsymbol{R}^l$ with vertex 0 and with a non-empty interior. We let $e\in\operatorname{int}Q$ such that $\|e\|=1$ and

$$\lambda(x)=\inf\{\lambda\in\boldsymbol{R}\,|\,x+\lambda e\in Y\}\quad\textit{and}\quad\Lambda(x)=x+\lambda(x)e\quad\textit{for}\quad x\in e^{\perp}.$$

(*a*) *The mapping $\Lambda\colon e^{\perp}\to\partial Y$ is a homeomorphism with inverse the restriction of* $\operatorname{proj}_{e^{\perp}}$ *to ∂Y.*
(*b*) *The function λ is Lipschitzian and $\partial\lambda(x)=\{p\in e^{\perp}\,|\,p-e\in N_Y(x+\lambda(x)e\}$.*

We note that Lemma 1 implies the above property of λ_1 after a change of variables. We now show that it also allows us to deduce Theorem 1 bis from Theorem 1. Indeed, let $u_0, u_1,\dots,u_d$ be vectors in $\boldsymbol{R}^n$ such that $u_1-u_0,\dots,u_d-u_0$ are independent. Let $l=n+1$, let $Y\subset\boldsymbol{R}^l$ be the closed set associated with the vectors $v_i=(u_i,1)\in\boldsymbol{R}^l$, for $i=0,\dots,d$, let $Q=C^0$ where $C=\{\sum_{i=0}^{d}\lambda_i v_i\,|\,\lambda_i\geqq 0,\ i=0,\dots,d\}$ and let $e=(0,\,-1)$. Then one sees that the Lipschitz function $\lambda\colon e^{\perp}\to\boldsymbol{R}$ associated with Y by Lemma 1 satisfies, for every $x\in e^{\perp}$, $\partial\lambda(x)=\{\sum_{i=0}^{d}\lambda_i(u_i,0)\,|\,\lambda_i\geqq 0, i=0,\dots,d$ and $\sum_{i=0}^{d}\lambda_i=1\}$ (which, up to a change of variables, is the conclusion of Theorem 1 bis).

Conversely we shall now deduce Theorem 1 bis. Let $v_1,\dots,v_k$ $(k\geqq 1)$ be a family of independent vectors of $\boldsymbol{R}^l$ and let $C=\{\sum_{i=1}^{k}\lambda_i v_i\,|\,\lambda_i\geqq 0, i=1,\dots,k\}$.

Then there clearly exists $e \in \boldsymbol{R}^l$, such that $\|e\|=1$ and $e \cdot v_i<0$, for $i=1,\ldots,k$ (which implies that $e \in C^0$). We let $u_i=\operatorname{proj}_{e^\perp}[v_i/-e \cdot v_i]$, $i=1,\ldots,k$ and, from Theorem 1 bis there exists a Lipschitz function $f: e^\perp \to \boldsymbol{R}$ such that, for every $x \in e^\perp$, $\partial f(x)=\operatorname{co}\{u_1,\ldots,u_k\}$. We now define the set $Y \subset \boldsymbol{R}^l$ by $Y=\{x+te \mid x \in e^\perp, t \geqq f(x)\}$, i.e., the epigraph of f in $e^\perp \times \boldsymbol{R}e$ identified with $\boldsymbol{R}^l$. Consequently by Clarke (1983, Propositions 2.9.6 and 2.9.7) one then deduces that, for $y=x+f(x)e$,

$$N_Y(y)=\bigcup_{\lambda \geqq 0} \lambda[\partial f(x)-\{e\}]=\left\{\sum_{i=1}^{k} \lambda_i(v_i/-e \cdot v_i) \,\middle|\, \lambda_i \geqq 0, i=1,\ldots,k\right\}=C.$$

3. Proof of Theorem 1 bis

We first prove Theorem 1 bis in a particular case, i.e., we assume that $d=n$, $u_0=0$, and $u_i=\sum_{h=1}^{i} e_h^n$ $(i=1,\ldots,n)$ where $e_1^n,\ldots,e_n^n$ denotes the canonical basis of $\boldsymbol{R}^n$ (e_i^n has all its coordinates equal to zero but the ith one which is equal to one). We shall then deduce the general case in a second step.

Step 1. We assume that $d=n, u_0^n=0$ and $u_i^n=\sum_{h=1}^{i} e_h^n$ $(i=1,\ldots,n)$. Then

$$T_n=\operatorname{co}\{u_0^n,\ldots,u_n^n\}=\{x=(x_1,\ldots,x_n) \in \boldsymbol{R}^n \mid 0 \leqq x_n \leqq \cdots \leqq x_1 \leqq 1\}.$$

We define, for $n \geqq 1$, the function λ_n as follows. We choose an arbitrary Lipschitzian function $\lambda_1: \boldsymbol{R} \to \boldsymbol{R}$ such that, for all $x \in \boldsymbol{R}$, $\partial \lambda_1(x)=[0,1]=T_1$. This is posible from section 2 or Rockafellar (1981). Then the sequence of functions λ_n is uniquely defined by

$$\lambda_{n+1}(x_1,\ldots,x_{n+1})=\lambda_n(x_1,\ldots,x_{n-1},x_n+\lambda_1(x_{n+1})).$$

Each function λ_n is Lipschitzian (by induction) since λ_1 is Lipschitzian and the two following claims will prove that, for every $n \geqq 1$, and every $x \in \boldsymbol{R}^n$, $\partial \lambda_n(x)=T_n$.

Claim 1. For every $n \geqq 1$, for every $x \in \boldsymbol{R}^n, T_n \subset \partial \lambda_n(x)$.

Indeed, let $n \geqq 1$, it suffices to show that, for every $x \in \boldsymbol{R}^n$, and every $i \in \{0,1,\ldots,n\}$ the vector u_i^n belongs to $\partial \lambda_n(x)$. From the definition of the generalized gradient $\partial \lambda_n(x)$ it clearly suffices to show that, for $\varepsilon>0$ small enough, and every $i \in \{0,1,\ldots,n\}$ the following set:

$$H_{i,n}^{\varepsilon}=\{x \in \boldsymbol{R}^n \mid \nabla \lambda_n(x) \text{ exists and } \nabla \lambda_n(x) \in B(u_i^n,\varepsilon)\}$$

is dense in $\boldsymbol{R}^n$, i.e., $\boldsymbol{R}^n=\operatorname{cl} H_{i,n}^{\varepsilon}$, the closure of $H_{i,n}^{\varepsilon}$.

We shall prove this assertion on n. If $n=1$, we prove that, for every $\varepsilon>0$, the set $H^{\varepsilon}_{0,1}$ is dense in $\boldsymbol{R}$ (and the proof is similar for $H^{\varepsilon}_{1,1}$). Suppose on the contrary that, for some $\varepsilon>0$ and some $x_0\in\boldsymbol{R}$, there is a neighborhood V of x_0 such that, whenever $x\in V$ and $\lambda_1'(x)$ exists one has $|\lambda_1'(x)|\geqq\varepsilon$, hence $\lambda_1'(x)\geqq\varepsilon$ since λ_1 is clearly non-decreasing. Then, from the definition of $\partial\lambda_1(x_0)$, one has $\partial\lambda_1(x_0)\subset[\varepsilon,+\infty)$ which contradicts the fact that $\partial\lambda_1(x_0)=[0,1]$.

Let us now suppose that the assertion is true for dimension n and we shall now show that the set $H^{\varepsilon}_{i,n+1}$ is dense in $\boldsymbol{R}^{n+1}$. Indeed, we notice that $\lambda_{n+1}=\lambda_n\circ\Pi_n\circ\theta_{n+1}$ where $\Pi_n:\boldsymbol{R}^{n+1}\to\boldsymbol{R}^n$ and $\theta_{n+1}:\boldsymbol{R}^{n+1}\to\boldsymbol{R}^{n+1}$ are defined, for $x=(x_1,\ldots,x_{n+1})$ by

$$\Pi_n(x)=(x_1,\ldots,x_n)\quad\text{and}\quad \theta_{n+1}(x)=(x_1,\ldots,x_{n-1},x_n+\lambda_1(x_{n+1}),x_{n+1}),$$

and we point out that θ_{n+1} is a homeomorphism. We let $\varepsilon'=\varepsilon/\sqrt{2}$ and we define $i_n\in\{0,1,\ldots,n\}$ and $i_1\in\{0,1\}$ as follows. If $i\leqq n$, we let $i_n=i$ and $i_1=0$ and if $i=n+1$, we let $i_n=n$ and $i_1=1$. From the induction assumption the set $H^{\varepsilon'}_{i_n,n}\times H^{\varepsilon'}_{i_1,n}$ is dense in $\boldsymbol{R}^{n+1}$, hence $\theta^{-1}_{n+1}(H^{\varepsilon'}_{i_n,n}\times H^{\varepsilon'}_{i_1,1})$ is also dense in $\boldsymbol{R}^{n+1}$. The proof of Claim 1 will then be complete if we show that

$$\theta^{-1}_{n+1}(H^{\varepsilon'}_{i_n,n}\times H^{\varepsilon'}_{i_1,1})\subset H^{\varepsilon}_{i,n+1}.$$

We now show that the above inclusion holds. Let $x\in\theta^{-1}_{n+1}(H^{\varepsilon'}_{i_n,n}\times H^{\varepsilon'}_{i_1,1})$ then $\hat{x}=\Pi_n(i_{n+1}(x))=(x_1,\ldots,x_{n-1},x_n+\lambda_1(x_{n+1}))\in H^{\varepsilon'}_{i_n,n}$ and $x_{n+1}\in H^{\varepsilon'}_{i_1,1}$. Hence λ_n is differentiable at $\hat{x}$, λ_1 is differentiable at x_{n+1}, $\nabla\lambda_n(\hat{x})\in B(u^n_{i_n},\varepsilon')$ and $\lambda_1'(x_{n+1})\in B(i_1,\varepsilon')$. Consequently, $\lambda_{n+1}=\lambda_n\circ\Pi_n\circ\theta_{n+1}$ is differentiable at x and one easily sees that $\nabla\lambda_{n+1}(x)=A\nabla\lambda_n(\hat{x})$ where A is the following matrix:

$$\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 1\\ 0 & 0 & \cdots & \lambda_1'(x_{n+1})\end{pmatrix}$$

Then $\nabla\lambda_{n+1}(x)\in B(u^{n+1}_i,\varepsilon)$. This ends the proof of Claim 1.

Claim 2. For every $n\geqq 1$, for every $x\in\boldsymbol{R}^n$, $\partial\lambda_n(x)\subset T_n$.

We shall prove the claim by induction on n. If $n=1$ the claim is clearly true from our choice of the function λ_1. Let us now suppose that the claim is true up to dimension n and, let $x\in\boldsymbol{R}^{n+1}$, we now show that $\partial\lambda_{n+1}(x)\subset T_{n+1}$. We recall that $\lambda_{n+1}=\lambda_n\circ h$ where $h=\Pi_n\circ\theta_{n+1}$, i.e., $h(x)=$

$(x_1, \ldots, x_{n-1}, x_n + \lambda_1(x_{n+1})$ and we let $h_i(x)$ be the ith coordinate of $h(x)$. Then, from the Chain Rule for Lipschitzian mappings [Clarke (1983, Theorem 2.3.9)] one gets

$$\partial\lambda_{n+1}(x) \subset \operatorname{clco}\left\{\sum_{i=1}^{n} \alpha_i \xi_i \,\middle|\, \xi_i \in \partial h_i(x), \alpha = (\alpha_i) \in \partial\lambda_n(h(x))\right\}.$$

But, from the induction assumption, $\partial\lambda_n(h(x)) = T_n$ and clearly $\partial h_i(x) = e_i^{n+1}$, for $i \in \{1, \ldots, n-1\}$ and $\partial h_n(x) \subset \{e_n^{n+1} + \beta e_{n+1}^{n+1} \mid \beta \in \partial\lambda_1(x_{n+1}) = [0,1]\}$. Consequently,

$$\partial\lambda_{n+1}(x) \subset \operatorname{clco}\{(\alpha_1, \ldots, \alpha_n, \beta\alpha_n) \mid \beta \in [0,1] \quad \text{and}$$

$$0 \leqq \alpha_n \cdots \leqq \alpha_1 \leqq 1\} = T_{n+1}.$$

Step 2. We nhow give the proof of Theorem 1 in the general case and let $u_0, \ldots, u_d$ be vectors in $\boldsymbol{R}^n$ such that $u_1 - u_0, \ldots, u_d - u_0$ are independent. We first assume that $u_0 = 0$. Then there exists a unique one-to-one linear mapping $L: \boldsymbol{R}^d \to \boldsymbol{R}^n$ such that $L(u_i^d) = u_i$, for $i \in \{0, \ldots, d\}$ where u_i^d is defined as in Step 1. We define the function $f: \boldsymbol{R}^n \to \boldsymbol{R}$ by $f(x) = \lambda_n(x))$, where L^* denotes the adjoint mapping of L. We notice that L^* is onto, since L is one-to-one. Consequently, from the Chain Rule for Lipschitzian mappings [Clarke (1983, Theorem 2.3.10)] and from Step 1, one gets, for every x,

$$\partial f(x) = L^{**}(\partial\lambda_n(L^*(x))) = L(T_n) = \operatorname{co}\{u_0, \ldots, u_d\}.$$

Let us now suppose that u_o is arbitrary. From above, there exists $f: \boldsymbol{R}^n \to \boldsymbol{R}$ such that, for every $x \in \boldsymbol{R}^n$, $\partial f(x) = \operatorname{co}\{0, u_1 - u_0, \ldots, u_d - u_0\}$ and we let $g: \boldsymbol{R}^n \to \boldsymbol{R}$ be defined by $g(x) = f(x) + x \cdot u_0$. Then from Clarke (1983, Corollary 1 of 2.3.3), for every $x \in \boldsymbol{R}^n$, $\partial g(x) = \partial f(x) + \{u_0\} = \operatorname{co}\{u_0, u_1, \ldots, u_d\}$. This ends the proof of Theorem 1 bis.

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