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INCOMPLETE MARKETS AND SHORT-SALES CONSTRAINTS: AN EQUILIBRIUM APPROACH

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We consider a general discrete-time dynamic financial market with three assets: a riskless bond, a security and a derivative. The market is incomplete (*a priori*) and at equilibrium. We assume also that the agents of the economy have short-sales constraints on the stock and that the payoff at the expiry of the derivative asset is a monotone function of the underlying security price. The derivative price process is not identified *ex ante*. This leads the agents to act as if there were no market for this asset at the intermediary dates. Using some nice properties of the pricing probabilities, which are admissible at the equilibrium, we prove that it suffices to consider the subset of the risk-neutral probabilities that overestimate the low values of the security and underestimate its high values with respect to the true probability. This approach greatly reduces the interval of admissible prices for the derivative asset with respect to no-arbitrage, as showed numerically.

Keywords: Incomplete markets, information modelling, equilibrium, option pricing, short-sales constraints, trees.

1. Introduction

Pricing of contingent claims has its roots in the pioneering works of Black and Scholes [5] and Harrison and Kreps [11]. Their results are based on the key idea that the prices of the existing assets induce a unique arbitrage-free price for any new redundant asset. When it is not redundant, the price must however lie in an arbitrage-free interval. A large part of the literature has dealt with the reduction of this interval in order to restrict the bid-ask spread and to obtain an unambiguous price. An important insight is due to Bensaïd, Lesne, Pages and Scheinkman [2] who used the super-replication cost approach introduced by Kreps [16]. They gave more accurate upper and lower bounds for the price of an asset than the ones obtained by the no-arbitrage condition, by simply requiring the price functional to be non-decreasing with respect to the payoff. Unfortunately, even with those

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restrictions, the bid-ask spread remains, in general, too large as showed in Soner, Shreve and Cvitani [25]. An alternative approach has been suggested by Föllmer and Schweizer [10], where the agents are assumed to minimize the local quadratic risk. This approach leads to a unique price for a given derivative security but the assumption is very restrictive. In a transaction costs framework and in order to obtain bounds without choosing a specific utility function, Constantinides [6] considers the class of power functions, with restrictions on the relative risk aversion coefficient. He obtains bounds on the option prices which depend on the restrictions and are valid only for the considered class of utility functions.

In our paper, we propose a new approach by exploiting partial conditions issued from the equilibrium analysis. We obtain bounds valid for all the Von Neumann–Morgenstern utility functions, with no restriction on the risk-aversion. Pricing by equilibrium has been popularized in its mean-variance version due to Sharpe [24], Lintner [17] and Markowitz [18]. However, the CAPM results are not empirically satisfied and one hardly obtains tractable predictions in a general equilibrium framework, which is taking as exogenous not only the total supply of each asset but also the attitudes towards risk of the agents, which are not easy to observe in practice.

Our approach is related to the Payoff Distribution Pricing Model (PDPM) introduced by Dybvig [8] and [9], who considers the implications of the individual optimality conditions on the asset price, for agents with VNM preferences. This model has been extended in a friction market case (including incomplete markets) by Jouini and Kallal [14], who show that this approach does not reduce the interval of the admissible prices. The main difference with the present work is that we use explicitly the clearing market conditions. More precisely, the key condition is that the derivative asset is a purely financial asset. Roughly speaking, it means that its total supply on the financial market is always zero. We assume also that the derivative asset has an increasing or decreasing payoff at the maturity and that the return of the underlying security is independent of its price at each date of the model (homogeneity condition).

In fact, our analysis extends to the case where there are many assets, if we know that there exists some agent who uses the market portfolio as a benchmark. More precisely, the strategy of such an agent is such that his wealth is increasing with the market portfolio. This assumption is quite realistic, if we consider the market index as a proxy for the market portfolio.

In this framework, our main result is to establish restrictions on the family of probabilities used to price the derivative asset with respect to the classical case. These restrictions are similar to those introduced by Perrakis and Ryan [20] and Perrakis [19]. In the first reference, the restriction is discussed for a single-period model only. In the second reference, it is taken as a primitive assumption in a dynamic model, in order to obtain bounds on European and American options. In those papers, the authors propose an ordering principle on the probabilities used to price the derivative asset. In this paper, our main contribution is to prove a

similar ordering principle on the state-price deflators, in a general discrete-time model. Our restrictions are derived from the equilibrium theory, and more precisely from the market clearing conditions. They use explicitly the true probability distribution. In fact, in a model where the future states of the world are equiprobable, our main result permits us to justify Perrakis' assumption. In a general case, we propose an alternative to that assumption. In Bizid, Jouini and Koehl [3] and [4], similar ordering properties have been proved in a complete and an incomplete market framework. However, in this paper, we find more restrictive constraints on the pricing probabilities than for the general incomplete market framework, using restrictions on the payoff functional of the option and the short-sales constraints.

We illustrate also here some problems of the convergence of option prices in an incomplete market framework on practical examples, by applying the preceding methodology. In general, the convergence of discrete-time models to continuous-time ones is difficult to prove. He [12] established some convergence conditions and results on portfolio policies in a general situation. However, even in a complete framework, we can find counter-examples where the pricing of options is not satisfying (see Prigent [21] and [22] or Hubalek and Schachermayer [13]). Here, we consider a non-degenerate incomplete discrete-time model. In this setting, convergence results have been proved for the minimal martingale measure introduced by Föllmer and Schweizer (see Prigent [23]), but not for the whole set of admissible pricing probabilities. Our results allow us to reduce the pricing interval and give, in the case of a competitive market, some intuitions on the bid-ask spread of derivative assets.

The paper is organized as follows: in the next section, we present the discrete-time dynamic model. There are three assets: a riskless one, a security and a derivative product. Short-sales are prohibited on the security. The agents have general VNM utility functions. We assume also that, at each date, the expected return of the underlying asset does not depend on its price. In Sec. 3, we focus on the individual problem, we show that the information set leads to a very specific behaviour of the agent and we establish, in this framework, the first-order conditions, which give the classical bounds for the price of the derivative asset. In Sec. 4, we state our main result in an equilibrium framework: we only need to consider a subset of the previously determined probabilities; more precisely, we only take into account the probabilities, which overestimate the low values of the underlying asset and underestimate the high values with respect to the true probability. In Sec. 5, we investigate the case where the risk premium of the underlying asset price is close to zero (i.e. the expected return of the security is close to the risk-free rate). We prove that the ask-price (respectively bid-price) of a derivative globally increasing (respectively decreasing) with respect to the underlying price converges to its expectation under the true probability. In Sec. 6, we give numerical examples for a European Call and for a European Call-spread. The great reduction of the bid-ask spread justifies the interest of our approach.

2. The Model

There is one perishable consumption good produced by a single firm. At each date, the supply d_t of this good is the production of the firm. This production is distributed as dividends to shareholders who own the firm. The firm is completely financed by equity and has one share outstanding. This share is perfectly divisible and tradable at any date t for a post-dividend price p_t in terms of consumption good. After the date T', the firm becomes obsolete and is valued at zero. A quantity θ of this claim insures to the owner a quantity θd_t of the perishable good at date t. Throughout the paper, the quantity θ is constrained to be non-negative, i.e. short-sales are prohibited for this asset.

In addition to the equity claim described above, a European option maturing at date $T \leq T'$ is also available. There is no assumption on the fact that this option will be tradable in the future. The option is exercised at date T, and its payoff is a function of the underlying asset price at the maturity $h(p_T + d_T)$. The option being a purely financial asset, its total supply is always zero; of course short-sales are then possible for this asset, at least for some agent in the economy. Agent n invests at date 0 a quantity α^n (positive or negative) of the derivative asset, at a price q_0. As there is no assumption on the clearing procedure for the option at intermediary dates, each agent will follow a "worst case" strategy in order to "hedge" his position in the option until the final date (at least in his maximization program at date 0). Note that, at each date t, we can associate with the option a "utility value" denoted by q_t, which depends on the price dynamics of the underlying asset. Indeed, as time passes, agents are less "uncertain" of the price of the underlying asset at maturity and can only make more restrictive assumptions on the expected value of the option. Note that this "value" depends clearly on the risk-aversion of the agent. Moreover after expiry, the value of the derivative is set to zero.

Formally, we consider a model with a finite number of many states and dates, where all random processes share a common filtered probability space in which P is the (true) probability and $E_t[\cdot]$ denotes the expectation conditional on what is known at t. We denote by Σ_t the set of all date t-nodes and, for $\sigma_t \in \Sigma_t$, $f(\sigma_t)$ is the set of the immediate successors of the date t-node σ_t. If σ' is an immediate successor of σ, $P_\sigma(\sigma')$ is the transition probability of P between σ and σ'.

We also assume that, at each date, the expected stock return is independent of its price. It means that the probability law followed by the process $\rho_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$ is known at date t and does not depend on the node $\sigma_t \in \Sigma_t$, and then on p_t. Therefore, we have in particular:

$$\forall\,(\sigma_1, \sigma_2) \in (\Sigma_t)^2, \frac{E_t[p_{t+1} + d_{t+1}|\sigma_1]}{p_t(\sigma_1)} = \frac{E_t[p_{t+1} + d_{t+1}|\sigma_2]}{p_t(\sigma_2)}\,. \tag{2.1}$$

There are N consumers. The nth consumer has a classical Von Neumann-Morgenstern utility function $U^n(\cdot)$ which associates with any consumption process

$(C_t)_{0\leq t\leq T'}$ the level:

$$U^n(C) = E\left[\sum_{t=0}^{T'} u_t^n(C_t)\right], \tag{2.2}$$

where u_t^n maps $\mathbb{R}^{+*}$ in $\mathbb{R}$. For all t, $t = 0, \ldots, T'$, we require the following classical properties on u_t^n:

Assumption 2.1. *For all n, u_t^n is continuously differentiable, increasing and strictly concave. Moreover, we impose the following Inada condition*:

$$u_t^n(x) \xrightarrow[x\to 0^+]{} -\infty\,.$$

Furthermore, we assume that the consumers face a positive interest rate r_t between date $t-1$ and date t; r_t is assumed to be known at date $t-1$ and results from the equilibrium on the consumption transfers between agents and between dates. We assume also that r_t does not depend on p_{t-1}.

At each date, the following happen in order: first, the firm produces and distributes the dividends among the shareholders; second, consumption, new portfolios and new prices occur, where prices come from the equilibrium conditions and, as usual, are considered as given for the agents in their utility maximization program. A precise definition of the equilibrium will be given in Sec. 4.

3. Information Modelling and Choice of Strategy

In this section, we focus on the problem of a given consumer, so we drop the superscript n. We will show how the information set of an agent will impact his strategy selection. For all t between 0 and T', we will denote by:

- $\theta_t \geq 0$ the quantity of the equity claim owned by the consumer at date t;
- α_t the quantity (positive or negative) of the derivative asset owned by the agent at date t;
- β_t the quantity of the consumption good (including the interests) transferred between date $t-1$ and date t through the interest rate market.

The initial wealth comes only from the initial share θ_0 — i.e. $\alpha_0 = \beta_0 = 0$ and there are no endowments after date 0.

The consumption strategy of the agent at each date t is such that his consumption C_t depends on date t information, whereas θ_t, α_t and β_t depend on date $t-1$ information. We take, by convention, $\beta_{T'+1} = \theta_{T'+1} = \alpha_{T'+1} = r_{T'+1} = 0$. Note also that for any date $t > T$, we can set $\alpha_t = 0$.

For given price processes $(p_t)_t$, $(d_t)_t$, $(r_t)_t$ and $(q_t)_t$, the agent clearly tries to maximize his utility level. The maximization program is defined by the optimal utility level $Y(p, d, r, q)$:

$$Y(p, d, r, q) = \sup_{S\in\mathcal{A}(p,d,r,q)} U(C(S))\,, \tag{3.1}$$

where $\mathcal{A}(p,d,r,q)$ denotes the convex set of the admissible strategies, i.e. the strategies $\{(C_t)_{t=0,\dots,T'};(\theta_t)_{t=1,\dots,T'};(\beta_t)_{t=1,\dots,T'};(\alpha_t)_{t=1,\dots,T'}\}$ such that at any date t, $t=1,\dots,T'$, we have

$$C_t+\theta_{t+1}p_t+\beta_{t+1}(1+r_{t+1})^{-1}+\alpha_{t+1}q_t=\theta_t(p_t+d_t)+\beta_t+\alpha_t q_t \tag{3.2}$$

with $C_t\geq 0$ and $\theta_t\geq 0$. Note that, *a priori*, Y could be infinite.

We assume that, at date 0, the agent knows:

- the distribution of $(p_t)_t$, $(d_t)_t$, $(r_t)_t$ under the true probability (in practice, he estimates it with an econometric study);
- the payoff function of the derivative asset (i.e. the functional h).

However, he has no other information at date 0 on the price process followed by the derivative asset at future dates. Therefore, we assume that he adopts a "worst case" strategy when investing at date 0 in the derivative asset. Note that, however, the agent's behavior may be quite different at future dates, if he gets some information on the transfers in the option.

The "worst case" utility level is defined as the infimum over the utility levels associated with any price process for the option:

$$Y^*(p,d,r)=\inf_q Y(p,d,r,q)=\inf_q \sup_{S\in\mathcal{A}(p,d,r,q)} U(C(S))\,. \tag{3.3}$$

A "worst case" strategy is a strategy which guarantees at least the utility level $Y^*(p,d,r)$ for any choice of the price process q. Note that if Y^* is finite, such a strategy exists. In the next section, we will give an existence condition for Y^*. We will first prove the following result:

Theorem 3.1. *The "worst case" strategy involves actually not investing in the derivative asset at intermediary dates.*

Proof. It is clear that, at each date, any optimal strategy associated with a price process q for the derivative asset is better for the agent than any strategy with no trading in the derivative, since it is less constrained (note that no trade in the derivative asset is compatible with any price process q). Then, more formally, if we denote by $\mathcal{A}^\emptyset(p,d,r)$ the set of admissible strategies with no trading in the derivative, we have:

$$\forall\, S^\emptyset\in\mathcal{A}^\emptyset(p,d,r)\,,\quad U(C(S^\emptyset))\leq\inf_q\sup_{S\in\mathcal{A}(p,d,r,q)} U(C(S))\,. \tag{3.4}$$

Conversely, let us assume that the agent's maximization problem is:

$$Y^\emptyset(p,d,r)=\sup_{S^\emptyset\in\mathcal{A}^\emptyset(p,d,r)} U(C(S^\emptyset))\,. \tag{3.5}$$

Using Assumption 2.1 (strict concavity, differentiability and Inada condition at 0), a classical result gives that if $Y^\emptyset(p,d,r)$ is finite, the Riesz representation $(\zeta_t)_t$ of

∇U at the optimum permits to define, by induction the process $q^{\emptyset}$:

$$\forall\, t < T\,, \quad q_t^{\emptyset} = \frac{1}{\zeta_t}\, E_t[\zeta_{t+1} \cdot q_{t+1}^{\emptyset}]\,, \quad \text{with } q_T^{\emptyset} = h(p_T + d_T)\,. \tag{3.6}$$

Let us assume now that the price of the derivative asset follows the process $(q_t^{\emptyset})_t$. By definition, the optimal strategy $S^{\emptyset,*}$ is in $\mathcal{A}(p, d, r, q^{\emptyset})$. Since $S^{\emptyset,*}$ satisfies the first-order conditions for the allocations in the risky and the riskless assets, it is easy to check that $S^{\emptyset,*}$ also satisfies the first-order condition with respect to the derivative asset and then, is an optimal strategy in $\mathcal{A}(p, d, r, q^{\emptyset})$. Then:

$$\sup_{S \in \mathcal{A}(p,d,r,q^{\emptyset})} U(C(S)) = U(C(S^{\emptyset,*})) = \sup_{S^{\emptyset} \in \mathcal{A}^{\emptyset}(p,d,r)} U(C(S^{\emptyset}))\,. \tag{3.7}$$

We have then the converse inequality, which ends the proof of the Theorem. □

In the following, we assume that there is no trade in the derivative market at the intermediary dates. More precisely, the agent acts at date 0 as if there is no trading possibility for the derivative asset afterwards. Any admissible strategy $S^{\emptyset}$ in $\mathcal{A}^{\emptyset}$ is defined by $\{(C_t)_{t=0,\ldots,T'}; (\theta_t)_{t=1,\ldots,T'}; (\beta_t)_{t=1,\ldots,T'}; \alpha\}$, where $\alpha = \alpha_1$ depends on date 0 information.

The agent's budget constraint at any date t, $t = 1, \ldots, T'$, becomes then:

$$\begin{aligned} C_t + \theta_{t+1} p_t + \beta_{t+1}(1 + r_{t+1})^{-1} \\ = \theta_t(p_t + d_t) + \beta_t + \alpha h(p_t + d_t) \cdot \mathbb{I}_{t=T} = W_t\,, \end{aligned} \tag{3.8}$$

where W_t is interpreted as the wealth "available" at date t before consumption and $\mathbb{I}_{t=T}$ values 1 if $t = T$ and 0 elsewhere. Moreover, the total wealth at date 0 satisfies:

$$\theta_0(p_0 + d_0) = C_0 + \theta_1 p_0 + \beta_1(1 + r_1)^{-1} + \alpha q_0 = W_0\,. \tag{3.9}$$

Any strategy in $\mathcal{A}^{\emptyset}$ must satisfy the budget constraints (3.8 and 3.9), the consumption constraint $C_t \geq 0$ for all t between 0 and T', and the short sales constraint $\theta_t \geq 0$ for all t between 1 and T'.

Since the quantity α in the option appears only at date 0 in the optimization program, we can separate the flows induced by the option and by the wealth process for any date $t > 0$. Moreover, as the option payoff depends only on the underlying asset price at date T, the state variables at any date are W and $p + d$ and the auxiliary programs associated with the maximization problem (3.5) are defined by backward induction:

$$V_{T'}(W; p + d) = u_{T'}(W)\,, \tag{3.10}$$

and, for all t, $t = 1, \ldots, T' - 1$,

$$\begin{aligned} V_t(W; p + d) = \max_{\substack{C_t \geq 0 \\ \theta_{t+1} \geq 0}} & u_t(C_t) + E_t[V_{t+1}((W - C_t)(1 + r_{t+1}) \\ & + \theta_{t+1}(p_{t+1} + d_{t+1} - p_t(1 + r_{t+1})); p_{t+1} + d_{t+1})] \end{aligned}$$

$$= \max_{\substack{C_t \geq 0 \\ \theta_{t+1} \geq 0}} u_t(C_t) + E_t[V_{t+1}((W - C_t)(1 + r_{t+1})$$

$$+ \theta_{t+1} p_t(\rho_{t+1} - (1 + r_{t+1})); \rho_{t+1} p_t)]$$

$$= \max_{\substack{C_t \geq 0 \\ \theta_{t+1} \geq 0}} u_t(C_t) + E_t[V_{t+1}((W - C_t + \theta_{t+1}\delta_{t+1} p_t)$$

$$\cdot (1 + r_{t+1}); \rho_{t+1} p_t)] \,, \tag{3.11}$$

where we denote $(\delta_t)_{t=1,\ldots,T'}$ the process

$$\delta_t = \frac{p_t + d_t}{p_{t-1}}(1 + r_t)^{-1} - 1 = \rho_t(1 + r_t)^{-1} - 1 \,. \tag{3.12}$$

By hypothesis, the model is such that δ_t does not depend on p_{t-1}.
Moreover, at date $t = 0$, the program is the following:

$$V_0(W; p_0 + d_0) = \max_{\substack{C_t \geq 0 \\ \theta_{t+1} \geq 0; \alpha}} u_0(C_0) + E[V_1((W - C_0 + \theta_1 \delta_1 p_0 - \alpha q_0)$$

$$\cdot (1 + r_1); \rho_1 p_0)] \,. \tag{3.13}$$

For all t between 0 and T', V_t maps $\mathbb{R}^+ \times \mathbb{R}^+$ in $\mathbb{R}$. Note that the wealth must be non-negative, but thanks to the Inada condition, this constraint is not binding in the maximization problem.

In other words, $V_t(W; p + d)$ is the best that the agent can do at date t with the wealth W if the underlying asset price is $p + d$. It is easy to check that, for all t, $t = 0, \ldots, T'$, the function $V_t(.\,; p_t + d_t)$ is increasing and concave. Moreover, $V_t(.\,; p_t + d_t)$ is differentiable for $W = W_t^*$ (Proposition 3.1, see proof in Appendix).

Let

$$S^{\emptyset,*} = \{(C_t^*)_{t=0,\ldots,T'}; (\theta_t^*)_{t=1,\ldots,T'}; (\beta_t^*)_{t=1,\ldots,T'}; \alpha^*\}$$

be an admissible strategy achieving $Y^{\emptyset}$. Since the function $u_t(\cdot)$ is increasing, an immediate application of the dynamic programming principle shows that (C_t^*, θ_{t+1}^*) achieves the maximum of the auxiliary program which defines $V_t(.; p_t + d_t)$, for the wealth:

$$W = W_t^* = \theta_t^*(p_t + d_t) + \beta_t^* + \alpha^* h(p_t + d_t) \cdot \mathbb{I}_{t=T} \,. \tag{3.14}$$

The budget constraint gives, for $t > 0$:

$$\frac{\beta_{t+1}^*}{(1 + r_{t+1})} = W_t^* - C_t^* - \theta_{t+1}^* p_t \tag{3.15}$$

and for $t = 0$:

$$\frac{\beta_1^*}{(1 + r_1)} = W_0^* - C_0^* - \theta_1^* p_0 - \alpha^* q_0 \,. \tag{3.16}$$

For t between 0 and $T - 1$, the properties on the function V_t ensure that (C_t^*, θ_{t+1}^*) must satisfy the first order conditions with respect to θ and C.

From now on, $V_t^{1,0}(;)$ denotes the partial derivative of V_t with respect to its first variable.[a]

Since $V_{t+1}^{1,0}(.\,;p_{t+1}+d_{t+1})$ is always positive, the first-order condition in θ gives:

$$E_t[(p_{t+1}+d_{t+1}-(1+r_{t+1})p_t)V_{t+1}^{1,0}(W_{t+1}^*;p_{t+1}+d_{t+1})] \leq 0\,, \tag{3.17}$$

when the constraint $\theta_{t+1} \geq 0$ is binding (resp.

$$E_t[(p_{t+1}+d_{t+1}-(1+r_{t+1})p_t)V_{t+1}^{1,0}(W_{t+1}^*;p_{t+1}+d_{t+1})] = 0\,, \tag{3.18}$$

when it is not).

The previous equations can be rewritten:
for all $t \geq 0$

$$p_t \geq \frac{1}{(1+r_{t+1})}E_t^\circ[p_{t+1}+d_{t+1}] \left(\text{resp. } p_t = \frac{1}{(1+r_{t+1})}\,E_t^\circ[p_{t+1}+d_{t+1}]\right)\,, \tag{3.19}$$

where $E_t^\circ[\cdot]$ stands for the conditional expectation under a new probability P° defined by the following transition probabilities between the node σ_t and its successors:

$$P_{\sigma_t}^\circ(\cdot) = \frac{V_{t+1}^{1,0}(W_{t+1}^*(\cdot);p_{t+1}+d_{t+1})}{E_t[V_{t+1}^{1,0}(W_{t+1}^*;p_{t+1}+d_{t+1})|\sigma_t]}P_{\sigma_t}(\cdot)\,. \tag{3.20}$$

It is clear that P° is equivalent to P. Moreover, the first-order condition with respect to α at the optimum gives that

$$\begin{aligned} E\left[h(p_T+d_T)\prod_{k=0}^{T-1}((1+r_{k+1})^{-1}V_{t+1}^{1,0}(W_{k+1}^*;\rho_{k+1}p_k))\right] \\ = q_0 E[V_1^{1,0}(W_1^*;\rho_1 p_0)]\,, \end{aligned} \tag{3.21}$$

and recursively, we obtain:

$$q_0 = E^\circ\left[\prod_{k=0}^{T-1}(1+r_{k+1})^{-1}h(p_T+d_T)\right]\,. \tag{3.22}$$

Let us denote by:

$$(\bar{p}_t)_{t=0,\ldots,T'} = \left(\frac{p_t+\sum_{i=0}^t d_i\prod_{j=i+1}^t(1+r_j)}{\prod_{i=1}^t(1+r_i)}\right)_{t=0,\ldots,T'}\,, \tag{3.23}$$

[a]In the next, $V_t^{i+1,j}(;)$ (resp. $V_t^{i,j+1}(;)$) denotes the partial derivative of $V_t^{i,j}$ with respect to its first (resp. second) variable.

Note that these partial derivatives do not always exist everywhere. However, using the concavity of the function V, these derivatives exist a.e. and the arguments in the following remain valid in this situation.

where $\bar{p}_t$ is in fact the total gain obtained by an agent which holds the underlying security between 0 and t, evaluated at date 0; and let us denote by:

$$(\bar{q}_t)_{t=1,\dots,T} = \left(\frac{E_t^\circ[\prod_{k=t}^{T-1}(1+r_{k+1})^{-1}h(p_T+d_T)]}{\prod_{k=0}^{t-1}(1+r_{k+1})} \right)_{t=1,\dots,T}, \tag{3.24}$$

where $\bar{q}_t$ is actually the expected value of the option at date t, evaluated at date 0. Since we do not know if the short-sales constraint is binding or not for the considered agent, Eqs. (3.19) and (3.22) give that $(\bar{p}_t)_{t=0,\dots,T'}$ is a supermartingale under the probability P° and that $(\bar{q}_t)_{t=0,\dots,T}$ with $\bar{q}_0 = q_0$ is a martingale under P°.

It is clear that we have just characterized a state-price deflator which is compatible with the information known at date 0 and with a no-trade (or equivalently a "worst case") optimal strategy (if, of course, such a strategy exists).

Recall that, in an incomplete market framework, a classical arbitrage pricing procedure (Jouini and Kallal [15]) would be to determine the set $\mathcal{P}$ of probabilities under which the discounted process $(\bar{p}_t)_{t=0,\dots,T'}$ is a supermartingale, and then, to compute the interval

$$[\inf_{Q\in\mathcal{P}} q_0(Q); \sup_{Q\in\mathcal{P}} q_0(Q)], \tag{3.25}$$

where we use the following notation:

$$q_0(Q) = E^Q\left[h(p_T+d_T)\prod_{k=0}^{T-1}(1+r_{k+1})^{-1}\right], \tag{3.26}$$

in order to estimate the price of the option. When short-sales are not prohibited, the interval is $[\inf_{Q^*\in\mathcal{M}} q_0(Q^*); \sup_{Q^*\in\mathcal{M}} q_0(Q^*)]$, where $\mathcal{M}\subset\mathcal{P}$ is the set of $(\bar{p}_t)_{t=0,\dots,T'}$ martingale measures.

We show in the next section that we can consider a more constrained set of probabilities in an equilibrium framework.

4. Equilibrium Restrictions

We specified in the preceding section the behaviour of an individual. More precisely, we showed that the "worst case" strategy is actually the optimal strategy with no trading in the derivative asset at intermediate dates ; and we characterized, in this situation, an admissible state-price deflator. We assume now that the market model is at the equilibrium and that all the agents follow a worst case strategy:

Assumption and Definition 4.1. *Given the utility functions and the initial endowments of the agents, we assume that there exists an equilibrium, i.e. a vector*:

$$(S^1,\dots,S^N,(p_t+d_t)_{t=0,\dots,T'};(q_t)_{t=0,\dots,T})_{\Delta t}, \tag{4.1}$$

such that S^n is the strategy of agent n and solves

$$\sup_{S\in\mathcal{A}^{\emptyset}(p,d,r)} U^n(C(S)) \qquad \text{(individual optimality)}$$

and

$$\sum_{i=1}^{N} S^i = ((d_t)_t, (1)_t, (0)_t, 0)\,. \qquad \text{(market clearing condition)}$$

Note that, by definition, when there exists an equilibrium, the supremum exists (i.e. $\sup_{S\in\mathcal{A}^{\emptyset}} U^n(C(S))$ is finite).

We will establish our main result for discrete-time financial markets which are close enough (in some sense) to continuous-time ones. In order to formalize this idea, let us introduce a family of financial models $M_{\Delta t}((\Omega, F, P), p, q)$ parametrized by Δt and defined by:

- an information set (i.e. a probability space (Ω, F, P)).
- a primitive asset p and a derivative one q (with a zero total net supply).
- a horizon date T (we assume now that Δt is the constant time step of the model).
- N agents.

In the following, we assume smoothness properties of the objective function with respect to the time period Δt. More precisely, when $\Delta t \to 0$, for each agent n, the value function $(V)^n_{\Delta t}$ (resp. $(V^{1,1})^n_{\Delta t}$, resp. $(V^{2,0})^n_{\Delta t}$) (prolonged on the time interval $[0, T]$) has a uniformly continuous limit $(\bar{V})^n$ (resp. uniformly continuous limit $(\bar{V}^{1,1})^n$, resp. uniformly continuous limit $(\bar{V}^{2,0})^n$) and:

$$\frac{Var_t[\delta_{t+\Delta t}]}{\Delta t} \xrightarrow[\Delta t\to 0]{} v_t^2 > 0 \quad (v_t \text{ finite}) \tag{4.2}$$

and

$$\frac{E_t[\delta_{t+\Delta t}]}{\Delta t} \xrightarrow[\Delta t\to 0]{} \mu_t < +\infty\,. \tag{4.3}$$

Note that the two last hypothesis make the limit continuous-time model non-degenerate, even if it can be incomplete. Indeed, the volatility term can be stochastic in very specific cases (e.g. (v_t) can be a "toss up" process). However, processes like jumps on the price of the underlying asset are not included in the scope of the paper.

Our objective is to prove that we can consider a more constrained set of pricing probabilities using an equilibrium argument. The result is not only valid at the limit, but it holds actually for Δt which are sufficiently small. We recall that the financial market models are supposed to be at equilibrium (at least for Δt sufficiently small).

We now define the comonotonicity between two functions.

Definition 4.1. Let t between 0 and $T - \Delta t$ and $\sigma_t \in \Sigma_t$. Let X and Y be two functions defined on $f(\sigma_t)$ (the set of the immediate successors of σ_t). We say that

X and Y are locally comonotonic in σ_t and we note $X \underset{\sigma_t}{\sim} Y$ if

$$\forall\,(\sigma_{t+\Delta t}, \sigma'_{t+\Delta t}) \;\in f(\sigma_t)^2, (X(\sigma_{t+\Delta t})(X(\sigma'_{t+\Delta t}))(Y(\sigma_{t+\Delta t}) - Y(\sigma'_{t+\Delta t})) \geq 0\,. \tag{4.4}$$

When $X \underset{\sigma_t}{\sim} -Y$, we will say that X and Y are locally anti-comonotonic.

From now on, we consider the standard situation where the payoff of the financial derivative is a monotonous function of the underlying price at the final date. In fact, the results proved in the section can be extended to more general payoffs ; however, for technical reasons, these situations won't be specified.

With this assumption, the following result holds:

Theorem 4.1. *For Δt that are small enough, there exist two probability measures P^* and Q^* with positive weights such that*:

(i) *The discounted process $(\bar{p}_t)_t$ is a supermartingale (resp. a martingale) under the probability P^* (resp. Q^*).*

(ii) $$\forall\,\sigma_t \in \Sigma_t\,, \quad \frac{P^*_{\sigma_t}}{P_{\sigma_t}} \underset{\sigma_t}{\sim} -\bar{p}_{t+\Delta t}\,. \tag{4.5}$$

(iii) *The discounted process $(\bar{q}_t)_t$ is a martingale under P^* and Q^*.*

Proof. See Appendix. □

Remark 4.1. With many assets, the same result can be obtained assuming directly that, at each node, there exists an agent who uses the market portfolio as a benchmark, and then, is such that his wealth and the market portfolio value $(p_{t+\Delta t} + d_{t+\Delta t})$ are comonotonic. However, we shall only consider a derivative asset, which payoff is a monotonous function of the wealth.

Corollary 4.1. *Let $\mathcal{P}^*$ (resp. $\mathcal{M}$) be the set of the probabilities P^* (resp. Q^*) such that*:

(i) *The discounted process $(\bar{p}_t)_t$ is a supermartingale (resp. martingale) under the probability P^* (resp. Q^*)*

(ii) $$\forall\,\sigma_t \in \Sigma_t, \;\; \frac{P_{\sigma_t^*}}{P_{\sigma_t}} \underset{\sigma_t}{\sim} -\bar{p}_{t+\Delta t}\,. \tag{4.6}$$

Then:

$$q_0 \in [\inf_{P^*\in\mathcal{P}^*} q_0(P^*);\; \sup_{P^*\in\mathcal{P}^*} q_0(P^*)] \cap [\inf_{Q^*\in\mathcal{M}} q_0(Q^*);\; \sup_{Q^*\in\mathcal{M}} q_0(Q^*)]\,. \tag{4.7}$$

Proof. Immediate. □

As in the previous remark, if there are many assets and if some agent uses the market portfolio as a benchmark, the result holds as well.

For standard option payoffs, we can obtain even more interesting results on the boundaries of the pricing interval. We have to state first the following Lemma:

Lemma 4.1. *For Δt that are small enough, the process $(\bar{p}_t)_t$ is submartingale with respect to the true probability P.*

Proof. See Appendix. □

Hence, the risk premium of the underlying asset is now supposed to be positive at each node σ_t of the discrete-time model.

The following result now holds:

Theorem 4.2. *Assume that the payoff of the derivative asset is a non-decreasing (resp. non-increasing) function of the underlying price. Then, if the risk premium of the underlying asset is always positive, we have:*

(i) *the supremum $\sup_{P^* \in \mathcal{P}^*} q_0(P^*)$ (resp. infimum $\inf_{P^* \in \mathcal{P}^*} q_0(P^*)$) is reached for a probability in $\mathcal{M}$, i.e. a martingale measure locally anti-comonotonic with respect to the stock price process and*

(ii) $\inf_{P^* \in \mathcal{P}^*} q_0(P^*) \leq \inf_{Q^* \in \mathcal{M}} q_0(Q^*)$ *(resp.* $\sup_{P^* \in \mathcal{P}^*} q_0(P^*) \geq \sup_{Q^* \in \mathcal{M}} q_0(Q^*)$*).*

Consequently,

$$q_0 \in [\inf_{Q^* \in \mathcal{M}} q_0(Q^*); \sup_{P^* \in \mathcal{P}^* \cap \mathcal{M}} q_0(P^*)]$$
$$(resp. [\inf_{P^* \in \mathcal{P}^* \cap \mathcal{M}} q_0(P^*); \sup_{Q^* \in \mathcal{M}} q_0(Q^*)]), \tag{4.8}$$

where $\mathcal{P}^*$ (resp. $\mathcal{P}^* \cap \mathcal{M}$) is the set of supermartingale (resp. martingale) measures locally anti-comonotonic with the stock price process.

Proof. See Appendix. □

5. The Case of a Weak Risk Premium

Before computing some practical examples, we consider, in this section, the situation of a not very Risk-Averse World. It is likely (but not necessarily) the case in the true world, for very liquid markets, where competition leads to very low margins on stock prices. Note that, however, the incompleteness of the market can explain the larger bid-ask spread for derivative securities. This section helps mainly by giving an intuition on the following computational examples, but it can be skipped in a first reading.

Let us define $\mathcal{P}^*(P)$ as the set of probabilities P^* such that the discounted process $(\bar{p}_t)_t$ is a supermartingale under the probability P^* and $\forall\, \sigma_t \in \Sigma_t, \frac{P^*_{\sigma_t}}{P_{\sigma_t}} \underset{\sigma_t}{\sim}$

$-\bar{p}_{t+\Delta t}$. The following lemma states that the derivative pricing interval obtained for a true probability P is continuous with respect to P.

Lemma 5.1. *Consider a sequence P^n of probabilities converging to P. Then*:

$$\begin{cases} \inf\limits_{P^{n,*}\in\mathcal{P}^*(P^n)} E^{P^{n,*}}[\bar{q}_T] \xrightarrow[n\to+\infty]{} \inf\limits_{P^*\in\mathcal{P}^*(P)} E^{P^*}[\bar{q}_T]\,, \\ \sup\limits_{P^{n,*}\in\mathcal{P}^*(P^n)} E^{P^{n,*}}[\bar{q}_T] \xrightarrow[n\to+\infty]{} \sup\limits_{P^*\in\mathcal{P}^*(P)} E^{P^*}[\bar{q}_T]\,. \end{cases} \tag{5.1}$$

Proof. Remember first that, for any P, the probability such that the transition probability is always equal to one for the lowest value of the underlying asset, is in $\mathcal{P}^*(P)$, which is then non-empty.

Now consider the general problem:

$$\max_{g(x,a)\geq 0} h(x) = H(a)\,, \tag{5.2}$$

where h is continuous, a is a parameter and g is continuous with respect to a and linear in x. Let a^* be fixed and assume that the set $\{g(x,a^*) > 0\}$ is non-empty. We prove now that H is continuous in a^*. Let x^* be such that $H(a^*) = h(x^*)$ (it exists by a compactness argument) and x' in $\{g(x,a^*) > 0\}$. Note that the set $\{g(x,a^*) \geq 0\}$ is convex and non-empty. Since g is linear in x, the convex combination $x_\lambda = \lambda x' + (1-\lambda)x^*$ is in $\{g(x,a^*) > 0\}$ for λ in $(0;1]$. Then for a given ε and λ sufficiently small, we have $h(x_\lambda) > h(x^*) - \varepsilon$. Furthermore, for a sufficiently close to a^*, we have $g(x_\lambda, a) > 0$ and consequently $H(a) > H(a^*) - \varepsilon$.

Assume now that there exists a sequence a_n converging to a^* such that $H(a_n) > H(a^*) + \varepsilon$ and let us denote by x_n the sequence satisfying $h(x_n) = H(a_n)$. By a compactness argument, it is easy to find a y in $\{g(x,a^*) \geq 0\}$ such that $h(y) \geq H(a^*) + \varepsilon$, which contradicts the definition of H. This ends the proof of the continuity of H at a^*.

In order to prove this lemma, it only remains to show that the set defined as $\mathcal{P}^*(P)$, but with strict inequality constraints, is non-empty.

Let us construct an element of this set by its transition probabilities. Since we are considering the equilibrium problem, there is no arbitrage in our model, which implies that, following Jouini and Kallal [15], there exists at least one equivalent-supermartingale measure. Therefore, for a given node σ_t in Σ_t, with successors $(\sigma^i_{t+\Delta t})_{i=1,\ldots,I}$ such that $\bar{p}(\sigma^1_{t+\Delta t}) > \cdots > \bar{p}(\sigma^I_{t+\Delta t})$, we have $\bar{p}(\sigma^I_{t+\Delta t}) < \bar{p}(\sigma_t)$. If we consider the transition probabilities with weights $\{\varepsilon, 2\varepsilon, \ldots, (I-1)\varepsilon, 1-\frac{I(I-1)}{2}\varepsilon\}$, where ε is sufficiently small, it is easy to check that it leads to an element of $\mathcal{P}^*(P)$ with strict inequality constraints. □

We consider now the case where the true probability P is in $\mathcal{M}$ (i.e. P is martingale).

Lemma 5.2. *If P is in $\mathcal{M}$,*

$$\mathcal{P}^* \cap \mathcal{M} = \{P\}\,. \tag{5.3}$$

Remark 5.1. If the assumptions of Theorem 4.2 hold, it results that the quantity $\sup_{P^*\in\mathcal{P}^*} E^{P^*}[\bar{q}_T]$ (resp. $\inf_{P^*\in\mathcal{P}^*} E^{P^*}[\bar{q}_T]$) is equal to $E[\bar{q}_T]$.

Proof. We will work directly with the discounted price process (in order to eliminate the risk-free rate r and the dividends). The proof will be obtained by induction. Let us consider a given node σ_t in Σ_t. For the sake of simplicity, we will denote by $u_1 > \cdots > u_I$ the returns of the process $\bar{p}$ between σ_t and its successors. Moreover, $\pi_1, \ldots, \pi_I$ (resp. $\hat{\pi}_1, \ldots, \hat{\pi}_I$) are the transition probabilities between σ_t and its successors, for the true probability P (resp. for a probability $\hat{P} \in \mathcal{P}^* \cap \mathcal{M}$). We have

$$\sum_{i=1}^{I} \hat{\pi}_i = 1 \cdot \frac{E_t^{\hat{P}}[\bar{p}_{t+\Delta t}]}{\bar{p}_t} = \sum_{i=1}^{I} \hat{\pi}_i u_i = 1\,. \tag{5.4}$$

Therefore

$$\begin{aligned} \hat{\pi}_{I-1} &= \frac{(1-u_I) - \sum_{i=1}^{I-2} \hat{\pi}_i (u_i - u_I)}{(u_{I-1} - u_I)}\,. \\ \hat{\pi}_I &= \frac{(u_{I-1} - 1) + \sum_{i=1}^{I-2} \hat{\pi}_i (u_i - u_{I-1})}{(u_{I-1} - u_I)}\,. \end{aligned} \tag{5.5}$$

Since $\hat{P} \in \mathcal{P}^*$, we have:

$$0 \le \frac{\hat{\pi}_1}{\pi_1} \le \cdots \le \frac{\hat{\pi}_i}{\pi_i} \le \cdots \le \frac{\hat{\pi}_I}{\pi_I}\,. \tag{5.6}$$

Note that we necessarily have $\frac{\hat{\pi}_1}{\pi_1} \le 1 \le \frac{\hat{\pi}_I}{\pi_I}$, and if $\frac{\hat{\pi}_1}{\pi_1} = 1$, then $\frac{\hat{\pi}_i}{\pi_i} = 1$ for all i.

As the true probability makes $\bar{p}$ martingale, by a backward induction on i, we have:

$$\forall\, i, i = 1, \ldots, I-1\,, \quad \hat{\pi}_i \in [B_{\inf}(\hat{\pi}_0, \ldots, \hat{\pi}_{i-1}), B_{\sup}(\hat{\pi}_0, \ldots, \hat{\pi}_{i-1})]\,, \tag{5.7}$$

where

$$\begin{aligned} &B_{\inf}(\hat{\pi}_0, \ldots, \hat{\pi}_{i-1}) \\ &\quad = \max\left[\hat{\pi}_{i-1}\frac{\pi_i}{\pi_{i-1}}\,, \frac{\sum_{l=i+1}^{I} \pi_l \left[(1-u_l) - \sum_{k=0}^{i-1} \hat{\pi}_k (u_k - u_l)\right]}{\sum_{l=i+1}^{I} \pi_l (u_i - u_l)}\right], \\ &B_{\sup}(\hat{\pi}_0, \ldots, \hat{\pi}_{i-1}) = \frac{\pi_i \left[(1-u_I) - \sum_{k=0}^{i-1} \hat{\pi}_k (u_k - u_I)\right]}{\sum_{l=i}^{I} \pi_l (u_l - u_I)}\,, \end{aligned} \tag{5.8}$$

and $\hat{\pi}_0 = 0$, $\pi_0 = 1$.

We see that for $i = 1, B_{\inf}(\hat{\pi}_0) = B_{\sup}(\hat{\pi}_0) = \pi_1$, which implies

$$\forall\, i, i = 1, \ldots, I\,, \quad \hat{\pi}_i = \pi_i\,. \tag{5.9}$$

This ends the proof. □

The following proposition is an immediate application of the two previous lemmata and Theorem 4.2. If $\bar{q}$ is an increasing function of $\bar{p}$ at the maturity and if the true probability is sufficiently close to a martingale probability, then the supremum is close to the expectation of the discounted price process under the true probability. When $\bar{q}$ is increasing with $-\bar{p}$, the result holds with the infimum. It has a simple interpretation: when the derivative asset is positively correlated with the underlying one, it is cheaper than when it is negatively correlated. Indeed, in the second case, the option can insure the portfolio of an owner of the equity claim.

Proposition 5.1. *Let P be the true probability, and assume that $\bar{q}$ is increasing with $\bar{p}$. Then, for all $\alpha > 0$, there exists $\varepsilon > 0$ such that, if there exists $\hat{P} \in M$, for all σ_t, $|\hat{P}(\sigma_t) - P(\sigma_t)| < \varepsilon$, then:*

$$|\sup_{P^* \in \mathcal{P}^*(P)} E^{P^*}[\bar{q}_T] - E^P[\bar{q}_T]| < \alpha\,. \tag{5.10}$$

Remark 5.2. Suppose that the true probability P is near to being a martingale for the underlying discounted price (i.e. the risk premium of the agents is close to zero). This can happen because of competition, for instance. However, we cannot deduce that the discounted price of a derivative asset is necessarily close to the expectation of its payoff. In fact, the preceding Proposition gives only a convergence result for one bound of the pricing interval.

We provide in the following section a numerical investigation of the trinomial lattice case for European calls.

6. One Example: The Trinomial Lattice

We study here a market model using the lattice approach suggested by Cox, Ross and Rubinstein [7]. Lattice methods for valuing options arise from discrete random walk models for the underlying security. The discount rate r is assumed to be equal to zero. The process $(\bar{p}_t)_t$, introduced in Sec. 3, follows a discrete diffusion sustained by a trinomial tree. This means that each node of the tree has three immediate successors till the final date of the model. If there are only two assets (a risky security and a riskless one), the financial market is clearly incomplete. Another non-redundant asset is required at each node in order to complete the market. We consider a discrete-time model $M_{\Delta t}$ tree with $\frac{T}{\Delta t} + 1$ dates and such that its true associated probability distribution is known. We assume that Δt is small enough in order to satisfy the ordering property of Theorem 4.1.

We want to price at date zero, a European option written on the underlying security and maturing at date T. We consider an option such that its payoff is an increasing function of the underlying asset. We first study the case where, at each date, the return process of the underlying asset is independent of its value. Second, we investigate the specific case of a stationary tree. In both cases, Theorem 4.2 ensures that the set $\mathcal{M}$ of the probabilities, which make the process $(\bar{p}_t)_t$ martingale,

is not empty. As the market is incomplete, the European option cannot be evaluated exactly only with the available information on the bond and the underlying asset, but its price lies in an interval that we can compute.

6.1. *Independent returns*

In this section, we assume that, for all t, $t = 0, \ldots, T - \Delta t$, the random variable $\frac{\bar{p}_{t+\Delta t}}{\bar{p}_t}$ is independent of $\bar{p}_t$. Then, for a given date t, $t = 0, \ldots, T - \Delta t$, we can consider the true transition probabilities $\{\pi_1, \pi_2, \pi_3\}$, where π_1 (resp. π_2, resp. π_3) corresponds to the value u_1 (resp. u_2, resp. u_3) of the return process $\frac{\bar{p}_{t+\Delta t}}{\bar{p}_t}$, with $u_1 > u_2 > u_3$. We denote by $\{\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3\}$ the transition probability at date t of an element of $\mathcal{M}$.

As the underlying price process is a submartingale under the true probability, we have

$$\pi_1 u_1 + \pi_2 u_2 + \pi_3 u_3 \geq 1\,. \tag{6.1}$$

From the definition of $\mathcal{M}$, we get:

$$\begin{cases} \hat{\pi}_1 u_1 + \hat{\pi}_2 u_2 + \hat{\pi}_3 u_3 = 1 \\ \hat{\pi}_1 + \hat{\pi}_2 + \hat{\pi}_3 = 1 \end{cases} \Leftrightarrow \begin{cases} \hat{\pi}_1 = \hat{\pi}_1 \\ \hat{\pi}_2 = \dfrac{1 - u_3}{u_2 - u_3} - \dfrac{u_1 - u_3}{u_2 - u_3}\hat{\pi}_1 \\ \hat{\pi}_3 = \dfrac{u_2 - 1}{u_2 - u_3} + \dfrac{u_1 - u_2}{u_2 - u_3}\hat{\pi}_1 \end{cases} \tag{6.2}$$

Remark that, in order to have $\mathcal{M} \neq \{\emptyset\}$, the tree must satisfy: $u_1 > 1 > u_3$. Remark also that

$$\hat{\pi}_1 \in \left[\max\left(0, \frac{1 - u_2}{u_1 - u_2}\right), \frac{1 - u_3}{u_1 - u_3}\right]\,. \tag{6.3}$$

Corollary 4.1 showed that we only need to consider the set $\mathcal{P}^*$ in order to compute the admissible interval of the derivative asset prices. Moreover, Theorem 4.2 proves that the supremum on the admissible European call prices is in $\mathcal{P}^* \cap \mathcal{M}$. We consider the martingale probabilities such that:

$$\frac{\hat{\pi}_1}{\pi_1} \leq \frac{\hat{\pi}_2}{\pi_2} \leq \frac{\hat{\pi}_3}{\pi_3}\,. \tag{6.4}$$

From the proof of Lemma 5.2, we must have at each node

$$\hat{\pi}_1 \in [B_{\inf}, B_{\sup}]\,, \tag{6.5}$$

with

$$\begin{aligned} B_{\inf} &= \max\left(0, \frac{1 - u_2}{u_1 - u_2}, \frac{(1 - u_3)(1 - \pi_1) - (u_2 - u_3)\pi_2}{(u_1 - u_3)(1 - \pi_1) - (u_2 - u_3)\pi_2}\right)\,. \\ B_{\sup} &= \frac{(1 - u_3)\pi_1}{(u_1 - u_3)\pi_1 + (u_2 - u_3)\pi_2}\,. \end{aligned} \tag{6.6}$$

Our methodology is now as follows. We compute the bounds, obtained in Theorem 4.2, $\inf_{Q^*\in\mathcal{M}} E^{Q^*}[\bar{q}_T]$ and $\sup_{P^*\in\mathcal{M}\cap\mathcal{P}^*} E^{P^*}[\bar{q}_T]$, which involve only martingales, and we compare them with the classical no-arbitrage bounds $\inf_{Q^*\in\mathcal{M}} E^{Q^*}[\bar{q}_T]$ and $\sup_{Q^*\in\mathcal{M}} E^{Q^*}[\bar{q}_T]$ (without short-sales constraints) and $\inf_{Q\in\mathcal{P}} E^{Q}[\bar{q}_T]$ and $\sup_{Q\in\mathcal{P}} E^{Q}[\bar{q}_T]$ (with short sales constraints).

Note that $\sup_{Q\in\mathcal{P}} E^{Q}[\bar{q}_T] = \sup_{Q^*\in\mathcal{M}} E^{Q^*}[\bar{q}_T]$ (it is easy to adapt the proof of Theorem 4.2).

6.2. *Stationary tree*

The framework and notation of this subsection are the same as those used previously, except that we assume also: $u_1 = u > u_2 = 1 > u_3 = u^{-1}$, where u does not depend on time. This condition insures that the tree is reconnecting and greatly simplifies the computation. The martingale probabilities must satisfy:

$$\begin{cases} \hat{\pi}_1 = \hat{\pi}_1 \\ \hat{\pi}_2 = 1-(1+u)\hat{\pi}_1 \\ \hat{\pi}_3 = u\hat{\pi}_1 \end{cases}, \quad \text{where } \hat{\pi}_1 \in \left[0, \frac{1}{1+u}\right]. \tag{6.7}$$

Since we consider a linear maximization program, the solution is an extreme one. It implies that we have to choose, at each node, between the two following systems:

$$\begin{cases} \hat{\pi}^{no}_{1,1} = 0 \\ \hat{\pi}^{no}_{2,1} = 1 \\ \hat{\pi}^{no}_{3,1} = 0 \end{cases} \quad \text{or} \quad \begin{cases} \hat{\pi}^{no}_{1,2} = \dfrac{1}{1+u} \\ \hat{\pi}^{no}_{2,2} = 0 \\ \hat{\pi}^{no}_{3,2} = \dfrac{u}{1+u} \end{cases} \tag{6.8}$$

Remark 6.1. It is clear that the first vector $(0,1,0)$ corresponds to a security with a zero volatility (no change in the value of the asset). On the other hand, the vector $(\frac{1}{1+u}, 0, \frac{u}{1+u})$ is associated with a binomial tree, which relative variance under the true probability is $\frac{(u-1)^2}{u}$ at each period. As $u > 1$, we can write $u = e^{\alpha\sqrt{\Delta t}}$ where α is strictly positive. Then, when $\Delta t \to 0$, Cox, Ross and Rubinstein [7] proved that this lattice converges to a Black and Scholes model, with a volatility of α. And the price of a call option tends to the Black and Scholes price associated with this volatility. In fact, we can show that the market model such that all the martingale measures are admissible, converges to the following log-normal model:

There exists a probability measure Q such that $(\bar{p}_t)_t$ is martingale under Q and follows a lognormal diffusion:

$$d\bar{p}_t = \varsigma_t \cdot \bar{p}_t \cdot dW^Q_t\,, \tag{6.9}$$

with $\forall\, t \in [0,T]$, $W^Q_t \sim N(0,t)$ and $\varsigma_t \in [0;\alpha]$ when $u = e^{\alpha\sqrt{\Delta t}}$ and $\Delta t \to 0$. This incomplete model has been studied by Avellaneda, Levy and Paras [1]. They

showed that, for a convex payoff, the price of the option lies in the interval $[BS(\alpha_{\min}), BS(\alpha_{\max})]$ (where $BS(\alpha)$ is the Black and Scholes price associated with a volatility α). However, for a non-convex pay-off, this interval can be much larger.

Note that, under the true probability P in the limit model, the discounted process $(\bar{p}_t)_t$ evolves as follows:

$$d\bar{p}_t = \mu \cdot \bar{p}_t \cdot dt + \varsigma_t \cdot \bar{p}_t \cdot dW_t^P \,. \tag{6.10}$$

The drift μ is constant because the tree structure is stationary. We get in this case: $E^P[\bar{p}_T] = \bar{p}_0 \cdot e^{\mu \cdot T}$.

We fix now numerical values for the true transition probabilities.

We assume for instance that: $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$ and that the horizon date T is one year. Then, we can compute easily by backward induction the drift μ:

Table 1. Drift vs volatility and number of steps of the model.

vol vs time steps	$T/\Delta t = 100$	$T/\Delta t = 1000$
$\alpha = \sqrt{\frac{3}{2}}.10\%$	0.5%	0.5%
$\alpha = \sqrt{\frac{3}{2}}.15\%$	1.125%	1.125%
$\alpha = \sqrt{\frac{3}{2}}.20\%$	2.0%	2.0%

We find that the drift of the underlying asset is very small; therefore, the true probability is close to a martingale. However, note that this drift does not converge to zero when $\Delta t \to 0$. Moreover, the instantaneous risk premium $\lambda_t = \frac{\mu}{\varsigma_t}$ is not known (it may change with time) and lies in the interval $[\frac{\mu}{\alpha}, +\infty[$. This uncertainty induces an uncertainty on the price of the derivative asset in the general incomplete market framework, while, with our equilibrium assumptions, we get that

$$\hat{\pi}_1^o \in \left[\frac{1}{1+2u}, \frac{1}{2+u}\right] .$$

Then the optimization program is done on the two systems:

$$\begin{cases} \hat{\pi}_{1,1}^o = \dfrac{1}{1+2u} \\ \hat{\pi}_{2,1}^o = \dfrac{u}{1+2u} \\ \hat{\pi}_{3,1}^o = \dfrac{u}{1+2u} \end{cases} \quad \text{or} \quad \begin{cases} \hat{\pi}_{1,2}^o = \dfrac{1}{2+u} \\ \hat{\pi}_{2,2}^o = \dfrac{1}{2+u} \\ \hat{\pi}_{3,2}^o = \dfrac{u}{2+u} \end{cases}$$

We consider a risky underlying asset such that $\bar{p}_0 = 100$. The following tables provide the pricing intervals given by no-arbitrage (NA) with constraints,

no-arbitrage without constraints and equilibrium conditions respectively, for a European call option with various K, Δt and α. Actually, our method improves the accuracy of the pricing interval by (at least) 20%, in comparison with the no-arbitrage interval without short-sale constraints. In fact, instead of $[BS(0), BS(\alpha)]$, we find an interval close to $[BS(0), BS(\sqrt{\frac{2}{3}}\alpha)]$ when $\Delta t \to 0$.

More precisely, for $\alpha = \sqrt{\frac{3}{2}} \cdot 10\%$ and various Δt, we get:

Table 2. NA and equilibrium prices for a European call at-the-money.

	$\Delta t = 0.1$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$	$\Delta t = 10^{-4}$
NA with constraints	[0;4.762608]	[0;4.870781]	[0;4.881752]	[0;4.882851]
NA without constraints	[0;4.762608]	[0;4.870781]	[0;4.881752]	[0;4.882851]
Equilibrium prices	[0;3.977042]	[0;3.989348]	[0;3.988797]	[0;3.988143]

We consider, in this table, a trinomial reconnecting tree, with a varying number of steps. We compare here the no-arbitrage and the equilibrium bounds for a call at-the-money $\mathrm{K} = \mathrm{p}_0 = 100$. The no-arbitrage situation without constraints is only given as a benchmark, for the reader. For information, Black and Scholes prices are (as a function of the volatility) BS(0%)=0, BS(10%)=3.987769, BS(α) = 4.882981.

Table 3. NA and equilibrium prices for a European call out-of-the-money.

	$\Delta t = 0.1$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$	$\Delta t = 10^{-4}$
NA with constraints	[0;2.995755]	[0;2.885046]	[0;2.897149]	[0;2.896386]
NA without constraints	[0;2.995755]	[0;2.885046]	[0;2.897149]	[0;2.896386]
Equilibrium prices	[0;2.098963]	[0;2.065363]	[0;2.065562]	[0;2.064398]

We consider now the same tree structure, for a call out-of-the-money $\mathrm{K} = 105 > p_0 = 100$. For information, Black and Scholes' model gives: BS(0%) = 0, BS(10%) = 2.064020, BS(α) = 2.896440.

Table 4. NA and equilibrium prices for a European call in-the-money.

	$\Delta t = 0.1$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$	$\Delta t = 10^{-4}$
NA with constraints	[0;7.757625]	[0;7.667523]	[0;7.672704]	[0;7.671739]
NA without constraints	[5;7.757625]	[5;7.667523]	[5;7.672704]	[5;7.671739]
Equilibrium prices	[5;6.927384]	[5;6.892700]	[5;6.889362]	[5;6.888411]

We consider the same tree structure for a call in-the-money $\mathrm{K} = 95 < \mathrm{p}_0 = 100$. Black and Scholes prices are BS(0%) = 5, BS(10%) = 6.888064, BS(α) = 7.671808.

We recall that the true probability is close to a martingale probability. Therefore, we can heuristically see the convergence result of the preceding section (Proposition 5.1). Hence, the European call upper price is not very far from its expectation under the true probability:

Table 5. Convergence of call option price.

	$K = 95$	$K = 100$	$K = 105$
NA upper prices	7.667523	4.870781	2.885046
Equilibrium upper prices	6.892700	3.989348	2.065363
Expectation under P	7.251029	4.250862	2.231820
Discounted Expectation	7.214864	4.229661	2.220689

We consider here a tree structure with 100 steps. We see that the equilibrium upper bound for a call converges to the Black and Scholes expected price. Results using the true probability are given as benchmarks.

Remark 6.2. Note that, since the drift term does not tend to zero, the option expected price under the true probability does not converge. We could expect that it would be the case for the expectation discounted by the true probability drift: $e^{-\mu}$. In fact, it is not the case because the option has a non-linear payoff. Therefore, the price C is not equal to $e^{-\mu} \cdot E^P[(\bar{p}_T - K)^+]$ (even for a so small drift $\mu = 0.5\%$). Hence, the main part of the error comes from the difference in the transition probabilities between the true probability and the martingale ones.

For various values of the volatility α, we get the following graph:

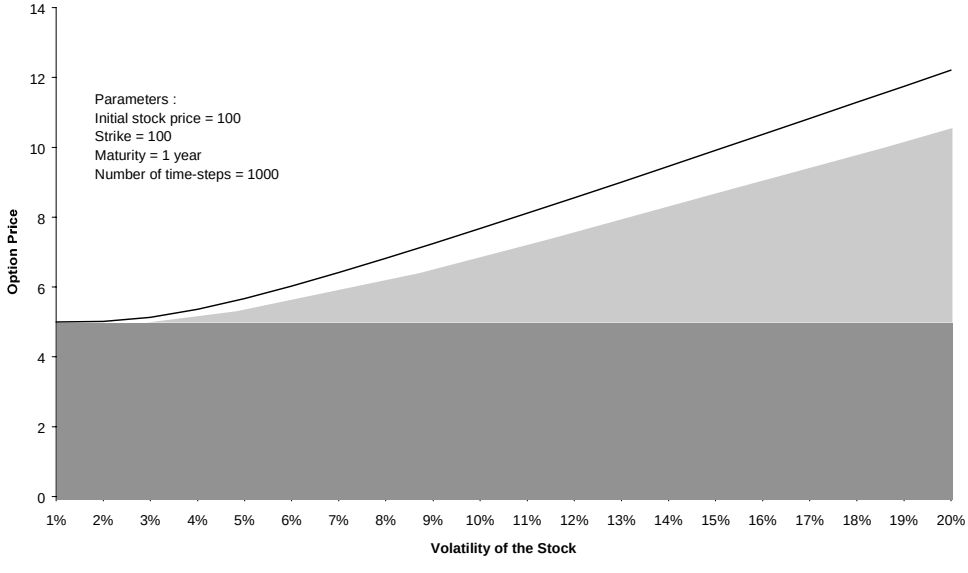

Fig. 1. European call price depending on the volatility. The light grey area represents the pricing interval when using the equilibrium conditions. By adding the white area, we get the no arbitrage pricing interval without short-sales constraints. If we also add the dark grey area, we get the NA pricing interval with short-sales constraints. Note that [equi. cond.] is always included in [NA without const.], which is, itself, included in [NA with const.]. The same graphic conventions hold for the next figures.

Moreover, we immediately see how useful the equilibrium approach is for the pricing of the derivative, when the initial underlying price varies:

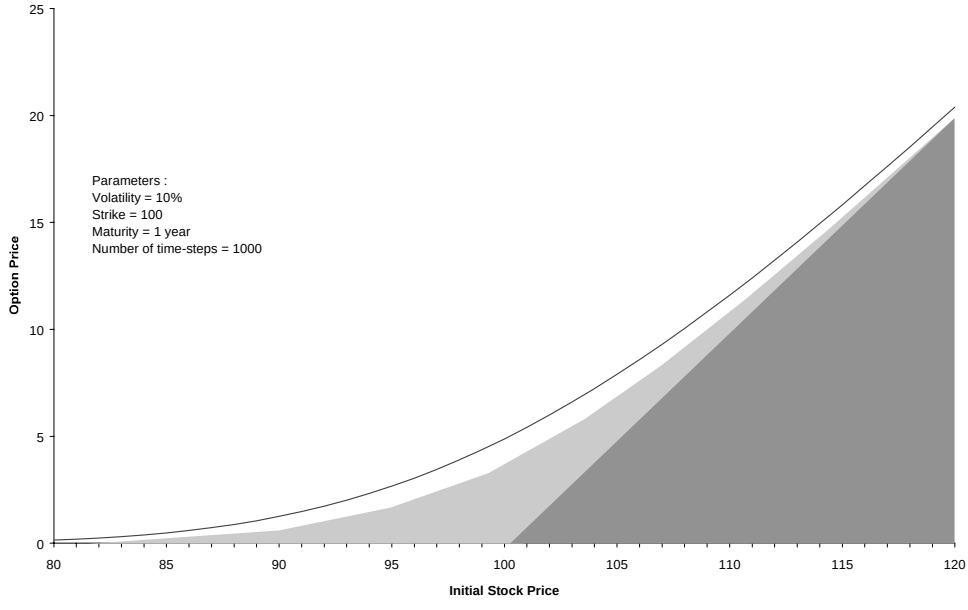

Fig. 2. Price of a European call depending on the initial stock value, with different market assumptions. Note that the lower bound on the pricing interval associated with the equilibrium conditions is the intrinsic value of the option. The three areas are all convex because the option payoff is convex.

We consider now a Call-spread (which is the difference of two European calls with a different strike). More precisely, we have two strikes $K_1 < K_2$ and the payoff of the option is:

$$\begin{aligned} q_T &= 0 && \text{if } \ p_T < K_1 \\ q_T &= p_T - K_1 && \text{if } \ K_1 \leq p_T \leq K_2 \\ q_T &= K_2 - K_1 && \text{if } \ p_T > K_2 \end{aligned}$$

This payoff is a monotonous function of the underlying asset price, so our method can be applied in this situation. Moreover, as it is bounded, we can expect improvements on the accuracy of the pricing interval. Note that the pay-off is not convex; therefore, the price interval (in the NA case without constraints) is not the one associated with the extreme volatilities, but in fact, is larger.

For $\alpha = \sqrt{\frac{3}{2}} \cdot 10\%$ and various Δt, we get:

Table 6. No-arbitrage and equilibrium prices for a call-spread.

	$\Delta t = 0.1$	$\Delta t = 10^{-2}$	$\Delta t = 10^{-3}$
NA with constraints	[0;6.8192]	[0;7.0151]	[0;6.9725]
NA without constraints	[3.03489;6.8192]	[2.8119;7.0151]	[2.7770;6.9725]
Equilibrium prices	[3.03489;4.8388]	[2.8119;4.8307]	[2.7770;4.8249]

We consider, in this table, a trinomial reconnecting tree, with a varying number of steps. We compare the no-arbitrage and the equilibrium bounds. Calculations are made for a call-spread with the following characteristics $K_1 = 95$, $K_2 = 105$ (the underlying price is $p_0 = 100$). The no-arbitrage situation without constraints is only given as a benchmark, for the reader. For information, Black and Scholes prices are (as a function of the volatility) BS(0%) = 5, BS(10%) = 4.824044, BS(α) = 4.775368.

If we vary the strike interval, we get (for 1000 steps):

Table 7. No-arbitrage and equilibrium prices for a call-spread.

$[K_1; K_2]$	[90; 100]	[95; 105]	[100; 110]
NA with constraints	[0;10]	[0;6.9725]	[0;4.5385]
NA without constraints	[5.3372;10]	[2.7770;6.9725]	[0;4.5385]
Equilibrium prices	[5.3372;6.7259]	[2.7770;4.8249]	[0;3.0344]

We compute the expectation of the payoff of the call-spread under the true probability, as we did for the European call. As the payoff is also an increasing function of the underlying price, the same "convergence result" of the upper bound of the equilibrium pricing interval holds:

Table 8. Convergence of call-spread price (1000 steps).

$[K_1; K_2]$	[90;100]	[95;105]	[100;110]
NA upper prices	10	6.972480	4.538512
Equilibrium upper prices	6.725890	4.824868	3.034399
Expectation under P	6.895562	5.015274	3.203801
Discounted Expectation	6.861170	4.990260	3.187822

We consider a trinomial tree with 1000 steps. We give here the upper bound of different pricing intervals for different calls-spread. The equilibrium boundary converges well to the Black and Scholes price (even better than the expectation under the true probability).

When we vary the volatility of the underlying asset, we get the following graph:

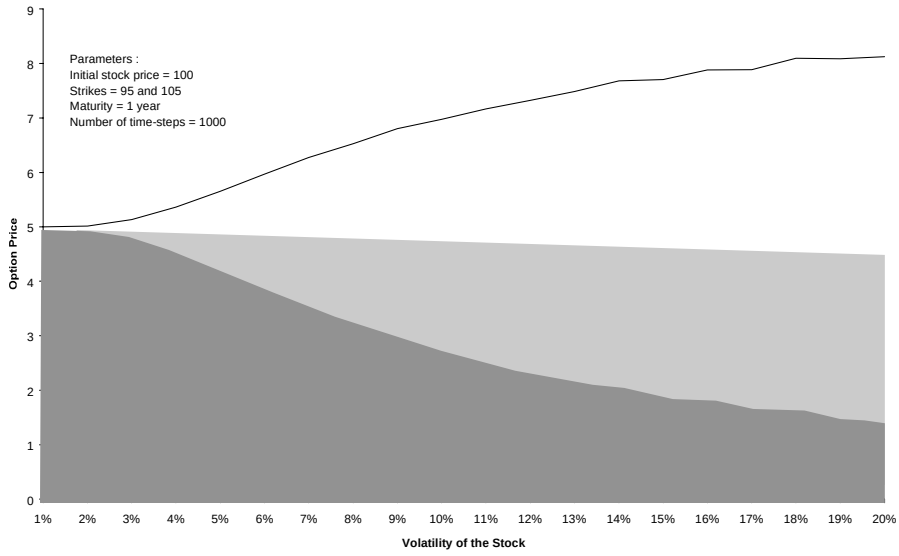

Fig. 3. Call Spread Price depending on the Volatility.

Moreover, when one considers varying initial stock values, the conclusions are the same as in the case of a standard call (see next graph).

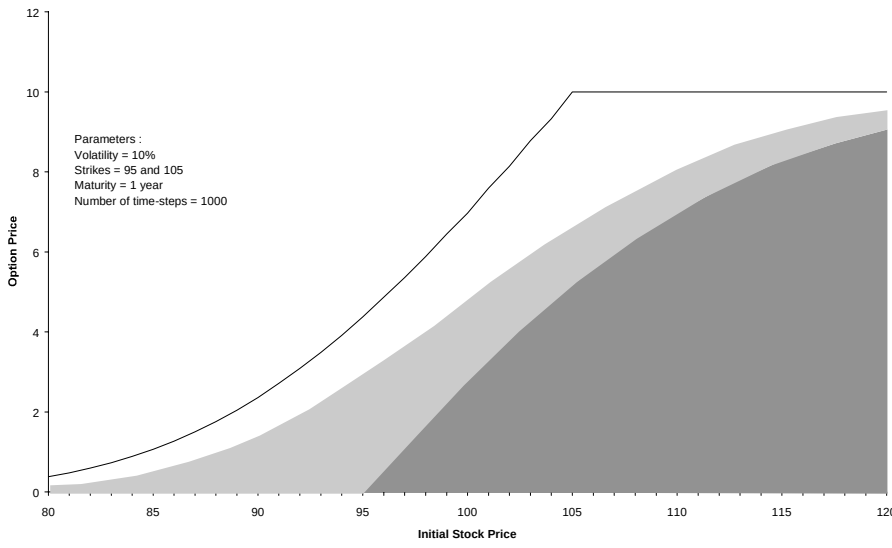

Fig. 4. Price of a European call spread depending on the initial stock value, with different market assumptions. We loose the Black and Scholes bounds with respect to the case of the call because the payoff of the call spread is not convex.

Appendix A

Proof of Proposition 3.1. For sake of simplicity, we denote $K_t(\cdot)$ the function $E_t[V_{t+1}(.\,; p_{t+1} + d_{t+1})]$ and $X = (W - C_t + \theta_{t+1}\delta_{t+1}p_t)(1 + r_{t+1})$. The date t program is then:

$$V_t(W; p_t + d_t) = \max_{C_t \geq 0, \theta_{t+1} \geq 0} u_t(C_t) + K_t[(W - C_t + \theta_{t+1}\delta_{t+1}p_t)(1 + r_{t+1})] \tag{A.1}$$

If $V_{t+1}(.\,; p_{t+1} + d_{t+1})$ is increasing and differentiable, then $K_t(\cdot)$ inherits these properties.

It is clear that $V_{T'}(\cdot)$ satisfies these properties. Suppose now that for $0 \leq t \leq T' - 1$, V_{t+1} is also strictly concave, increasing and differentiable. Then, for C_t and θ_{t+1} fixed, and because K_t is increasing, we have

$$\forall\, W^1 \leq W^2, X^1 \leq X^2 \tag{A.2}$$

and

$$u_t(C_t) + K_t(X^1) \leq u_t(C_t) + K_t(X^2) \tag{A.3}$$

Thus, by maximization

$$\forall\, W^1 \leq W^2, V_t(W^1) \leq V_t(W^2) \tag{A.4}$$

and V_t is increasing.

Consider now (C_t^1, θ_{t+1}^1) (resp. (C_t^2, θ_{t+1}^2)) an optimum for the wealth W^1 (resp. W^2). By assumption and because E_t is linear,

$$\frac{K_t(X^1) + K_t(X^2)}{2} \leq K_t\left(\frac{X^1 + X^2}{2}\right). \tag{A.5}$$

As u_t and K_t are increasing and concave, we get

$$\begin{aligned} &\frac{1}{2}\left(u_t(C_t^1) + K_t(X^1)\right) + \frac{1}{2}\left(u_t(C_t^2) + K_t(X^2)\right) \\ &\quad \leq u_t\left(\frac{1}{2}C_t^1 + \frac{1}{2}C_t^2\right) + K_t\left(\frac{1}{2}X^1 + \frac{1}{2}X^2\right) \end{aligned} \tag{A.6}$$

i.e.

$$\frac{V_t(W^1; p_t + d_t) + V_t(W^2; p_t + d_t)}{2} \leq u_t\left(\frac{C_t^1 + C_t^2}{2}\right) + K_t\left(\frac{X^1 + X^2}{2}\right). \tag{A.7}$$

By maximizing the second term of the inequality, we clearly have

$$\frac{V_t(W^1; p_t + d_t) + V_t(W^2; p_t + d_t)}{2} \leq V_t\left(\frac{W^1 + W^2}{2}; p_t + d_t\right). \tag{A.8}$$

The concavity of V_t with respect to its first argument is then proved.

The optimal strategy (C_t^*, θ_{t+1}^*) is the solution of the following maximization program:

$$\max_{\substack{C_t \geq 0 \\ \theta_{t+1} \geq 0}} u_t(C_t) + K_t((W_t^* - C_t + \theta_{t+1}\delta_{t+1}p_t)(1 + r_{t+1}))\,. \tag{A.9}$$

Recall that, by definition of V_t, we have

$$\max_{\substack{C_t \geq 0 \\ \theta_{t+1} \geq 0}} u_t(C_t) + K_t((W - C_t + \theta_{t+1}\delta_{t+1}p_t)(1 + r_{t+1})) = V_t(W; p_t + d_t)\,. \tag{A.10}$$

It is clear then that, for all positive W:

$$V_t(W; p_t + d_t) \geq u_t(C_t^*) + K_t((W - C_t^* + \theta_{t+1}^*\delta_{t+1}p_t)(1 + r_{t+1})) \tag{A.11}$$

and the equality holds for $W = W_t^*$. The function in the second term of the previous inequality is concave and differentiable with respect to W. According to the Benveniste-Scheinkman Theorem, this implies the differentiability of V_t at W_t^*.

By using the same kind of arguments, we can prove the differentiability of V_0 for the optimal strategy $(C_0^*, \theta_1^*, \alpha^*)$. □

Proof of Theorem 4.1. Let us consider a given model $M_{\Delta t}$.

Remember that, for each agent n, $1 \leq n \leq N$, we have built in the previous section a probability P^n defined by its density of transition (see Eq. (3.20)). The demonstration is divided in two parts:

- Firstly, we prove that there exists an agent n_0 such that $P_{\sigma_t}^{n_0}$ is equal to $P_{\sigma_t}^*$.
- Secondly, we build the probability Q^* node by node using for each node the density of transition of a well-chosen P^n.

Let t between 0 and $T - \Delta t$ and $\sigma_t \in \Sigma_t$. We want to show that there exists an agent n_0 such that

$$\forall\,(\sigma_1, \sigma_2) \in f^2(\sigma_t)\,, \quad p_{t+\Delta t} + d_{t+\Delta t}(\sigma_1) \geq p_{t+\Delta t} + d_{t+\Delta t}(\sigma_2) \tag{A.12}$$

the following holds:

$$\begin{aligned}(V_{t+\Delta t}^{1,0})^{n_0}(W_{t+\Delta t}^{*,n_0}(\sigma_1); p_{t+\Delta t} + d_{t+\Delta t}(\sigma_1)) \\ \leq (V_{t+\Delta t}^{1,0})^{n_0}(W_{t+\Delta t}^{*,n_0}(\sigma_2); p_{t+\Delta t} + d_{t+\Delta t}(\sigma_2))\end{aligned} \tag{A.13}$$

or equivalently that

$$\frac{P_{\sigma_t}^{n_0}}{P_{\sigma_t}}(\sigma_1) \leq \frac{P_{\sigma_t}^{n_0}}{P_{\sigma_t}}(\sigma_2)\,. \tag{A.14}$$

As short-selling is forbidden, for every agent, we have $\theta_{t+\Delta t}^{n_0,*} \geq 0$, and

$$\theta_{t+\Delta t}^{n_0,*}(p_{t+\Delta t} + d_{t+\Delta t}) \underset{\sigma_t}{\sim} p_{t+\Delta t} + d_{t+\Delta t} \tag{A.15}$$

By adding terms depending on the information available at date t only, we obtain that

$$W_{t+\Delta t}^{n_0,*} \underset{\sigma_t}{\sim} p_{t+\Delta t} + d_{t+\Delta t} \tag{A.16}$$

or

$$W_{t+\Delta t}^{n_0,*} \underset{\sigma_t}{\sim} \bar{p}_{t+\Delta t} \tag{A.17}$$

Therefore, the wealth and the asset price are comonotonic and we only have to prove that the function $(V_{t+\Delta t}^{1,0})^{n_0}$ is decreasing in W and in $p+d$.

For an agent n_0, let us consider the differential of $(V_t^{1,0})^{n_0}$ for a given σ_t:

$$\begin{aligned} dV_t^{1,0} = ((1+r_{t+\Delta t})(E_t[V_{t+\Delta t}^{2,0}](dW-dc) \\ + p_t E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}]\, d\theta + \theta_{t+\Delta t} E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}] dp) \\ + E_t[\rho_{t+\Delta t} V_{t+\Delta t}^{1,1}] dp)(1+r_{t+\Delta t}) \end{aligned} \tag{A.18}$$

Note that, by hypothesis, the model is such that $\delta_{t+\Delta t}$ and $\rho_{t+\Delta t}$ do not depend on p_t. This (strong) hypothesis is only used here and forces us to avoid models with a stochastic relative volatility. However, it can be relaxed but we need cumbersome assumptions on $E_t[\frac{\partial \delta_{t+\Delta t}}{\partial p} V_{t+\Delta t}^{2,0}]$ and $E_t[\frac{\partial \delta_{t+\Delta t}}{\partial p} V_{t+\Delta t}^{1,1}]$.

Let us recall the first order conditions with respect to the consumption and to the quantity in the underlying asset:

$$u_t^1(c) - (1+r_{t+\Delta t}) E_t[V_{t+\Delta t}^{1,0}] = 0 \tag{A.19}$$

$$(1+r_{t+\Delta t}) E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{1,0}] = 0 \tag{A.20}$$

If we differentiate the first equation, we get:

$$\begin{aligned} u_t^2(c)dc = ((1+r_{t+\Delta t})(E_t[V_{t+\Delta t}^{2,0}](dW-dc) \\ + p_t E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}]\, d\theta + \theta_{t+\Delta t} E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}]\, dp) \\ + E_t[\rho_{t+\Delta t} V_{t+\Delta t}^{1,1}]\, dp)(1+r_{t+\Delta t}) \end{aligned} \tag{A.21}$$

Then

$$d(V_t^{1,0})^{n_0} = u_t^2(c)dc \tag{A.22}$$

As $u_t^2(c)$ is always negative (because the utility function is concave), we only have to prove that the consumption c_t and the price $p_t + d_t$ are comonotonic. In order to obtain this result, let us differentiate the first-order condition on θ; then, using the condition on c, we simplify the equation by $d\theta$:

$$\begin{aligned} dc \left(u_t^2(c) + (1+r_{t+\Delta t})^2 \left(E_t[V_{t+\Delta t}^{2,0}] - \frac{(E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}])^2}{E_t[(\delta_{t+\Delta t})^2 V_{t+\Delta t}^{2,0}]} \right) \right) \\ = dW \cdot (1+r_{t+\Delta t})^2 \left(E_t[V_{t+\Delta t}^{2,0}] - \frac{(E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}])^2}{E_t[(\delta_{t+\Delta t})^2 V_{t+\Delta t}^{2,0}]} \right) + dp \cdot (1+r_{t+\Delta t}) \\ \cdot \left(E_t[\rho_{t+\Delta t} V_{t+\Delta t}^{1,1}] - E_t[\delta_{t+\Delta t} \rho_{t+\Delta t} V_{t+\Delta t}^{1,1}] \frac{E_t[\delta_{t+\Delta t} V_{t+\Delta t}^{2,0}]}{E_t[(\delta_{t+\Delta t})^2 V_{t+\Delta t}^{2,0}]} \right) \end{aligned} \tag{A.23}$$

Using Cauchy–Schwartz, we have:

$$E_t[V^{2,0}_{t+\Delta t}]E_t[(\delta_{t+\Delta t})^2 V^{2,0}_{t+\Delta t}] \geq (E_t[\delta_{t+\Delta t} V^{2,0}_{t+\Delta t}])^2 \tag{A.24}$$

Moreover, because of the strict concavity of the value function with respect to the wealth:

$$E_t[(\delta_{t+\Delta t})^2 V^{2,0}_{t+\Delta t}] < 0 \tag{A.25}$$

Then

$$E_t[V^{2,0}_{t+\Delta t}] - \frac{(E_t[\delta_{t+\Delta t} V^{2,0}_{t+\Delta t}])^2}{E_t[(\delta_{t+\Delta t})^2 V^{2,0}_{t+\Delta t}]} \leq 0 \tag{A.26}$$

Because of the short sales constraint for the agent (and because the wealth and the price process are comonotonic), we only have to prove that, for a time-step Δt that is small enough,

$$E_t[\rho_{t+\Delta t} V^{1,1}_{t+\Delta t}] - E_t[\delta_{t+\Delta t}\rho_{t+\Delta t} V^{1,1}_{t+\Delta t}] \frac{E_t[\delta_{t+\Delta t} V^{2,0}_{t+\Delta t}]}{E_t[(\delta_{t+\Delta t})^2 V^{2,0}_{t+\Delta t}]} \leq 0 \tag{A.27}$$

Or equivalently, that

$$E_t[\rho_{t+\Delta t} V^{1,1}_{t+\Delta t}]E_t[(\delta_{t+\Delta t})^2 V^{2,0}_{t+\Delta t}] - E_t[\delta_{t+\Delta t}\rho_{t+\Delta t} V^{1,1}_{t+\Delta t}]E_t[\delta_{t+\Delta t} V^{2,0}_{t+\Delta t}] \geq 0 \tag{A.28}$$

As the sequence of discrete-time models converges, the following hold:

$$\begin{aligned} (V^{1,1}_{t+\Delta t}(W^*_{t+\Delta t}, p_{t+\Delta t} + d_{t+\Delta t}))^n_{\Delta t} &\to (\bar{V}^{1,1}_t(W^*_t, p_t + d_t))^n \\ (V^{2,0}_{t+\Delta t}(W^*_{t+\Delta t}, p_{t+\Delta t} + d_{t+\Delta t}))^n_{\Delta t} &\to (\bar{V}^{2,0}_t(W^*_t, p_t + d_t))^n \\ (u_{t+\Delta t}(C^*_{t+\Delta t}))^n_{\Delta t} &\to (\bar{u}_t(C^*_t))^n \end{aligned} \tag{A.29}$$

Recall that

$$\rho_{t+\Delta t} = (\delta_{t+\Delta t} + 1)(1 + r_{t+\Delta t}) \tag{A.30}$$

Then, the quadratic form

$$Q_{\Delta t}(x) = E_t[(1+x)V^{1,1}_{t+\Delta t}]E_t[x^2 V^{2,0}_{t+\Delta t}] - E_t[(x^2+x)V^{1,1}_{t+\Delta t}]E_t[x V^{2,0}_{t+\Delta t}] \tag{A.31}$$

converges uniformly to

$$\bar{V}^{2,0}_t \cdot \bar{V}^{1,1}_t \cdot (E_t[x^2] - (E_t[x])^2) = \bar{V}^{2,0}_t \cdot \bar{V}^{1,1}_t \cdot Var_t[x] \tag{A.32}$$

on the unit sphere. As the model is non-degenerate, we have

$$\begin{aligned} Var_t\left[\frac{\delta_{t+\Delta t}}{\|\delta_{t+\Delta t}\|}\right] &= \frac{Var_t[\delta_{t+\Delta t}]}{\|\delta_{t+\Delta t}\|^2} = \frac{Var_t[\delta_{t+\Delta t}]}{E_t[(\delta_{t+\Delta t})^2]} \\ &= \frac{Var_t[\delta_{t+\Delta t}]}{Var_t[\delta_{t+\Delta t}] + (E_t[\delta_{t+\Delta t}])^2} \\ &\approx \frac{v_t^2 \cdot \Delta t}{v_t^2 \cdot \Delta t + (\mu_t \cdot \Delta t)^2} \to 1 \quad (>0)\,. \end{aligned} \tag{A.33}$$

Then the variance of $\frac{\delta_{t+\Delta t}}{\|\delta_{t+\Delta t}\|}$ remains positive and does not tend to zero when $\Delta t \to 0$:

$$Var_t\left[\frac{\delta_{t+\Delta t}}{\|\delta_{t+\Delta t}\|}\right] \underset{\Delta t \to 0}{\not\longrightarrow} 0\,. \tag{A.34}$$

And, for Δt that is small enough, $Q_{\Delta t}(\delta_{t+\Delta t})$ has the sign of $\bar{V}_t^{2,0} \cdot \bar{V}_t^{1,1}$. As for each model, and each agent, V_t is concave in the wealth, we get $\bar{V}_t^{2,0} \leq 0$. Moreover, for each discrete-time model, at the final date T, we have:

$$V_T^{1,1} = \alpha \frac{\partial h(p_T + d_T)}{\partial(p+d)} V_T^{2,0}\,. \tag{A.35}$$

There exists (at least) one agent n_0 such that:

$$\alpha \frac{\partial h(p_T + d_T)}{\partial(p+d)} \leq 0\,. \tag{A.36}$$

Indeed, at the equilibrium, there exist agents with a positive (resp. negative) α (i.e. purchasing and selling the derivative). If the payoff is increasing with the underlying price, we consider an agent n_0 who buys the derivative at date 0 and

$$(V_T^{1,1})^{n_0} \leq 0\,. \tag{A.37}$$

Otherwise, we consider an agent selling the derivative at date 0. By backward induction, the following result is true for all dates:

$$\forall\, t \leq T\,, \quad (V_t^{1,1})^{n_0} = \alpha E_t\left[\frac{\partial h(p_T + d_T)}{\partial(p+d)}\,(V_T^{2,0})^{n_0}\right]\,. \tag{A.38}$$

Then, there exists an agent such that:

$$\bar{V}_t^{1,1} \leq 0\,. \tag{A.39}$$

We obtain that the quadratic form $Q_{\Delta t}(\delta_{t+\Delta t})$ is negative (for agent n_0). Then the density of transition probability of P^{n_0} with respect to P at the node σ_t defined as follows:

$$P_{\sigma_t}^{n_0}(\sigma_{t+\Delta t}) = \frac{V_{t+\Delta t}^{1,0}(W_{t+\Delta t}^{n_0,*}(\sigma_{t+\Delta t}); (p_{t+\Delta t} + d_{t+\Delta t})(\sigma_{t+\Delta t}))}{E_t[V_{t+\Delta t}^{1,0}(W_{t+\Delta t}^{n_0,*}; p_{t+\Delta t} + d_{t+\Delta t})|\sigma_t]} P_{\sigma_t}(\sigma_{t+\Delta t}) \tag{A.40}$$

satisfies the comonotonicity property. Moreover, $(\bar{p}_t)_{t=0,\ldots,T}$ is a supermartingale under the probability P^{n_0}. We choose $P_{\sigma_t}^* = P_{\sigma_t}^{n_0}$.

Note that, because of the uniform continuity, the preceding equation holds at each date of the model and we can apply the backward induction for the pricing of the contingent claim. Note also that we only need *one* agent (holding a non-null quantity α of the claim) for all the periods in order to get this result.

Similarly, for each node σ_t, there exists at least one agent n_1 such that $\theta^{n_1,*}_{t+\Delta t} > 0$ (because the total supply of the underlying asset is equal to one). Then we define the density of transition probability of Q^* with respect to P at the node σ_t by

$$Q^*_{\sigma_t}(\sigma_{t+\Delta t}) = \frac{V^{1,0}_{t+\Delta t}(W^{n_1,*}_{t+\Delta t}(\sigma_{t+\Delta t}); (p_{t+\Delta t} + d_{t+\Delta t})(\sigma_{t+\Delta t}))}{E_t[V^{1,0}_{t+\Delta t}(W^{n_1,*}_{t+\Delta t}; p_{t+\Delta t} + d_{t+\Delta t})|\sigma_t]} P_{\sigma_t}(\sigma_{t+\Delta t})\,. \tag{A.41}$$

The martingale property is inherited from the properties of P^{n_1}, and from the fact that the constraint $\theta^{n_1,*}_{t+\Delta t} \geq 0$ is non-binding. This ends the construction of Q^*. □

Proof of Lemma 4.1. At date t, agent n prefers $W^{n,*}_{t+\Delta t}$ to $W^n_{t+\Delta t} = (W^{n,*}_t - C^{n,*}_t)(1 + r_{t+\Delta t})$, which depends only on date t information. It implies that

$$\begin{aligned} &E_t[V^n_{t+\Delta t}(W^{n,*}_{t+\Delta t}; p_{t+\Delta t} + d_{t+\Delta t})] \\ &\qquad \geq E_t[V^n_{t+\Delta t}(W^n_{t+\Delta t}; p_{t+\Delta t} + d_{t+\Delta t})]\,. \end{aligned} \tag{A.42}$$

For Δt that is small enough, we have that:

$$\forall\, \sigma_{t+\Delta t} \in f(\sigma_t),\ V^n_{t+\Delta t}(X(\sigma_{t+\Delta t}); p_{t+\Delta t} + d_{t+\Delta t}) \to \bar{V}^n_t(X(\sigma_{t+\Delta t}); p_t + d_t)\,. \tag{A.43}$$

And $\bar{V}^n_t$ is an increasing concave function, as it inherits these properties from $V^n_{t+\Delta t}$. Then

$$\begin{aligned} &E_t[V^n_{t+\Delta t}(W^n_{t+\Delta t}; p_{t+\Delta t} + d_{t+\Delta t})] \to \bar{V}^n_t(W^n_{t+\Delta t}; p_t + d_t) \\ &E_t[V^n_{t+\Delta t}(W^{n,*}_{t+\Delta t}; p_{t+\Delta t} + d_{t+\Delta t})] \to E_t[\bar{V}^n_t(W^{n,*}_{t+\Delta t}; p_t + d_t)]\,. \end{aligned} \tag{A.44}$$

Because $\bar{V}^n_t$ is a concave function of the wealth, we get by Jensen's Inequality:

$$\bar{V}^n_t(E_t[W^{n,*}_{t+\Delta t}]; p_t + d_t) \geq E_t[\bar{V}^n_t(W^{n,*}_{t+\Delta t}; p_t + d_t)] \tag{A.45}$$

and since $\bar{V}^n_t$ is increasing with the wealth, we obtain that

$$E_t[W^{n,*}_{t+\Delta t}] \geq W^n_{t+\Delta t}\,, \tag{A.46}$$

or equivalently

$$E_t[\theta^{n,*}_{t+\Delta t}(p_{t+\Delta t} + d_{t+\Delta t} - p_t(1 + r_{t+\Delta t}))] \geq 0\,. \tag{A.47}$$

As $\sum_{n=1}^N \theta^{n,*}_{t+\Delta t} = 1$, there exists an agent such that $\theta^{n,*}_{t+\Delta t} > 0$ (recall that this quantity is fixed at date t). Then, we get

$$E_t[p_{t+\Delta t} + d_{t+\Delta t} - p_t(1 + r_{t+\Delta t})] \geq 0 \tag{A.48}$$

i.e.

$$E_t[\bar{p}_{t+\Delta t}] \geq \bar{p}_t\,. \tag{A.49}$$

which ends the proof. □

Proof of Theorem 4.2. We only consider the case where $\forall\, \sigma_t \in \Sigma_t,\ \bar{p}_{t+\Delta t} \underset{\sigma_t}{\sim} \bar{q}_{t+\Delta t}$. The proof can be easily adapted to the case where $\forall\, \sigma_t \in \Sigma_t,\ \bar{p}_{t+\Delta t} \underset{\sigma_t}{\sim} -\bar{q}_{t+\Delta t}$.

(i) Since the states-of-the-world space is finite, the supremum is reached for a probability $\hat{P}$ in $\mathcal{P}^*$, which is not necessarily equivalent to P. Suppose that $(\bar{p}_t)_{t=0,\ldots,T}$ is not a martingale under $\hat{P}$. Since $\hat{P}$ is a supermartingale probability, there exists one particular node σ_t such that

$$\bar{p}_t(\sigma_t) > E_t^{\hat{P}}[\bar{p}_{t+\Delta t}|\sigma_t]\,. \tag{A.50}$$

At this node, the density of transition probabilities of $\hat{P}$ with respect to P is not constant because of Lemma 4.1:

$$\bar{p}_t(\sigma_t) \leq E_t[\bar{p}_{t+\Delta t}|\sigma_t]\,. \tag{A.51}$$

Therefore, from Theorem 4.1, there exist $\sigma_{t+\Delta t}$ and $\sigma'_{t+\Delta t}$ in $f(\sigma_t)$ such that

$$\bar{p}_{t+\Delta t}(\sigma_{t+\Delta t}) \geq \bar{p}_{t+\Delta t}(\sigma'_{t+\Delta t}) \tag{A.52}$$

and

$$\frac{\hat{P}_{\sigma t}(\sigma_{t+\Delta t})}{P_{\sigma t}(\sigma_{t+\Delta t})} < \frac{\hat{P}_{\sigma t}(\sigma'_{t+\Delta t})}{P_{\sigma_{t+\Delta t}}(\sigma'_{t+\Delta t})}\,. \tag{A.53}$$

We can assume, without loss of generality, that there is no σ'' such that

$$\bar{p}_{t+\Delta t}(\sigma) \geq \bar{p}_{t+\Delta t}(\sigma'') \geq \bar{p}_{t+\Delta t}(\sigma')\,. \tag{A.54}$$

Let us construct a new probability $\hat{P}^\varepsilon \in \mathcal{P}^*$ by adding (resp. subtracting) to $\frac{\hat{P}_{\sigma_t}(\sigma_{t+\Delta t})}{P_{\sigma_t}(\sigma_{t+\Delta t})}$ (resp. $\frac{\hat{P}_{\sigma_t}(\sigma'_{t+\Delta t})}{P_{\sigma_t}(\sigma'_{t+\Delta t})}$) an $\varepsilon > 0$ sufficiently small, such that $\hat{P}^\varepsilon$ makes also the discounted process $(\bar{p}_t)_{t=0,\ldots,T}$ supermartingale, and that condition (ii) of Theorem 4.1 holds. The conditional expectation under the new probability $E_t^{\hat{P}^\varepsilon}[\bar{q}_{t+\Delta t}|\sigma_t]$ is higher than or equal to $E_t^{\hat{P}}[\bar{q}_{t+\Delta t}|\sigma_t]$ since $\forall\, t,\ \forall\, \sigma_t,\ \bar{q}_{t+\Delta t} \underset{\sigma_t}{\sim} \bar{p}_{t+\Delta t}$. By repeating such a transformation, we finally obtain either a martingale maximizing measure or a probability such that $\frac{\hat{P}_{\sigma_t}(\sigma_{t+\Delta t})}{P_{\sigma_t}(\sigma_{t+\Delta t})}$ is constant in $f(\sigma_t)$. In both cases, since P is a submartingale and $\hat{P}$ a supermartingale, we have a martingale maximizing measure.

(ii) Note that the probability such that the transition probability is always equal to one for the lowest value of the underlying asset, is always in $\mathcal{P}^*$, since

$$\forall\, \sigma_t \in \Sigma_t,\ \bar{q}_{t+\Delta t} \underset{\sigma_t}{\sim} \bar{p}_{t+\Delta t}\,.$$

Therefore, $\inf_{P^* \in \mathcal{P}^*} q_0(P^*) = \inf_{\sigma_T} \bar{q}_T \leq \inf_{Q^* \in \mathcal{M}} q_0(Q^*)$. □

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