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# A discrete stochastic model for investment with an application to the transaction costs case

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**Abstract**

This work consists of two parts. In the first one, we study a model where the assets are investment opportunities, which are completely described by their cash-flows. Those cash-flows follow some binomial processes and have the following property called stationarity: it is possible to initiate them at any time and in any state of the world at the same condition. In such a model, we prove that the absence of arbitrage condition implies the existence of a discount rate and a particular probability measure such that the expected value of the net present value of each investment is non-positive if there are short-sales constraints and equal to zero otherwise. This extends the works of Cantor–Lippman [Cantor, D.G., Lippman, S.A., 1983. Investment selection with imperfect capital markets. Econometrica 51, 1121–1144; Cantor, D.G., Lippman, S.A., 1995. Optimal investment selection with a multitude of projects. Econometrica 63 (5) 1231–1241.], Adler–Gale [Alder, I., Gale, D., 1997. Arbitrage and growth rate for riskless investments in a stationary economy. Mathematical Finance 2, 73–81.] and Carassus–Jouini [Carassus, L., Jouini, E., 1998. Arbitrage and investment opportunities with short sales constraints. Mathematical Finance 8 (3) 169–178.], who studied a deterministic setup. In the second part, we apply this result to a financial model in the spirit of Cox–Ross–Rubinstein [Cox, J.C., Ross, S.A., Rubinstein, M., 1979. Option pricing: a simplified approach. Journal of Financial Economics 7, 229–264.] but where there are transaction costs on the assets. This model appears to be stationary. At the equilibrium, the Cox–Ross–Rubinstein's price of a European option is always included between its buying and its selling price. Moreover, if there is transaction

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cost only on the underlying asset, the option price will be equal to the Cox–Ross–Rubinstein's price. Those results give more information than the results of Jouini–Kallal [Jouini, E., Kallal, H., 1995. Martingales and arbitrage in securities markets with transaction costs. Journal of Economic Theory 66 (1) 178–197.], which where working in a finite horizon model. 

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## 1. Introduction

We study a model where investments are completely defined through their generated cash-flows. We assume that the model is stationary, that is, each project is available at every date and in every state of the world at the same conditions. The horizon of the model is then necessarily infinite. This kind of models has been studied in the deterministic case by Cantor and Lippman (1983; 1995), Adler and Gale (1997) and Carassus and Jouini (1998). In the present work, the cash-flows are modeled with stochastic processes, with dynamics described by a binomial tree. First, we generalize the notion of stationarity in a stochastic framework. Then, we prove a no-arbitrage theorem. Recall that, loosely speaking, an arbitrage opportunity is a way of getting something for nothing. The arbitrage assumption is defined thanks to the existence of a strategy leading to a non-negative and non-zero payoff. Under a technical condition, the assumption no-arbitrage implies the existence of an interest rate and a particular probability measure which make the sum of the investments' expected value non-positive if there are short-selling constraints and equal to zero otherwise.

In the second part of this work, we present an economic model with an underlying asset, the price of which follows a binomial process, and a family of options written on this asset. We suppose that there are some buying and selling transaction costs (possibly different) on the options and on the underlying asset. As a matter of fact, we suppose that there exists a bid–ask spread on the option price. We prove that the technical assumption made in the first part of the paper is satisfied in this setting. Recall that in a market without transaction costs, the option price is given by the Black–Scholes' formula (Black and Scholes, 1973) in a continuous framework and similarly by the Cox–Ross–Rubinstein's formula (Cox et al., 1979) in the binomial framework. In our imperfect market, we prove that at the equilibrium the Cox–Ross–Rubinstein's price is always between the buying price and the selling price. Moreover, if the bid–ask spread on the options comes from a constant proportional transaction cost, we give explicit bounds for the option price. Notice that if there are only transaction costs on the underlying asset and not on the options, then the options' price is equal the Cox–Ross–

Rubinstein's price. This is a new result about transaction costs. Recall that in the paper of Jouini and Kallal (1995), it is proved that the absence of arbitrage condition is equivalent to the existence of an equivalent martingale measure, which turns a particular process lying between the bid and the ask price into a martingale. The no-arbitrage condition allows only to situate the option bid–ask prices in some interval. The lower bound (respectively upper bound) of this interval is obtained as the minimum (respectively maximum) of the expected value of the option future values, under all the equivalent martingale measures. Every sub-interval appears then as an interval compatible with the no-arbitrage condition. This means that the lower (respectively upper) bound of every sub-interval corresponds to a selling (respectively buying) price, which is compatible with the no-arbitrage condition. Of course, the Cox–Ross–Rubinstein's price belongs to the maximal interval but there is no reason why it should belong to every sub-interval. In our model, we prove that the only intervals of equilibrium price are the ones containing the Cox–Ross–Rubinstein's price. This can be explained by the infinite horizon of the model and by the interactions between the different assets at different dates (stationarity).

## 2. The model

### 2.1. Uncertainty and information structure

We consider a discrete model with infinite horizon. The evolution of the system is given by a binomial tree over discrete periods. Note that the result holds if we consider a multinomial tree but for sake of simplicity we choose the binomial representation. At time zero, there is only one state of the world. Then, at each date the system has two possibilities: in the propitious case, it will move up, that is follows $u$, and else down, that is follows $d$, where $u$ and $d$ are two real numbers. The infinite set of states of the world is $\Omega = \{(\omega_1, \omega_2, \dots)/\omega_i \in \{u,d\}\} = \{u,d\}^{\mathbb{N}}$. For $\omega \in \Omega$, we denote by $\omega^n = (\omega_1, \dots, \omega_n)$ the $n$ first states of the path. We call $\Omega_n$ the set $\{u,d\}^n$ of this $n$ first states of the path. We work in the discrete probability space $(\Omega, \mathscr{F}, P)$, endowed with the filtration $(\mathscr{F}_n)_{n \in \mathbb{N}}$, i.e., an increasing sequence of $\sigma$-algebra included in $\mathscr{F}$. The $\sigma$-algebra $\mathscr{F}$ is the one generated by the coordinate mappings $(X_n)_{n \in \mathbb{N}^*}$ defined by $X(n,\omega) = \omega_n$ where $\omega = (\omega_1, \omega_2, \dots)$. As usually done, we choose $\mathscr{F}_0 = \{\emptyset, \Omega\}$. From an economic point of view, $\mathscr{F}_n$ represents the available information at time $n$. The $\sigma$-algebra $\mathscr{F}_n$ is generated by the coordinate mappings $(X_p)_{p \in \{1,\dots,n\}}$ and can be identified to $\mathscr{P}(\Omega_n)$. We choose for $P$ the unique probability which turns the mappings $X_n$ to be independent and identically distributed, and such that $P(X_n = u) = P(X_n = d) = 1/2$. This probability displays the following property: $(\omega_1, \dots, \omega_n) \in \Omega_n$, $P(X_1 = \omega_1, \dots, X_n = \omega_n) = P(X_1 = \omega_1) \times \dots \times P(X_n = \omega_n) = 1/2^n = 1/|\Omega_n|$. The projection of $P$ to $\Omega_n$ is the uniform probability on $\Omega_n$.

### 2.2. Investments' set, definitions and properties

#### 2.2.1. The basic investments

The investments are assumed to be completely represented by their cash-flows. Our set of investment, indexed by $I$, is supposed to be infinite. A cash-flow, denoted by $\Phi_i$, is represented thanks to a binomial stochastic process. Thus, $\Phi_i(n,\omega)$ represents the payment of investment $i \in I$ at time $n$ and in state $\omega$. The processes are supposed to be adapted, i.e., for all date $n$, the function $\Phi(n,\cdot): \omega \to \Phi(n,\omega)$ is $\mathscr{F}_n$-measurable. We denote by $\mathscr{A}$ the set of such $\mathscr{F}$-adapted processes. We also define the set $\mathscr{A}_n$ of $\mathscr{F}$-adapted processes such that for all $\omega \in \Omega$, the support of $\Phi(\cdot,\omega): n \to \Phi(n,\omega)$, mapping $\mathbb{N}$ into $\mathbb{R}$, is included in $[0,n]$.

We suppose that every investment has a finite time horizon. Otherwise, assuming the existence of an investment with an infinite horizon, it will always be possible to suspend repayment of the debt to infinity.

**Definition 2.1.** The process $\Phi_i$ is an investment if there exists a finite horizon $T_i$ such that $\Phi_i \in \mathscr{A}_{T_i}$.

We assume that our model contains a lending rate called $r_p$ with $r_p \geq 0$. The lending of \$1 at time $t_0$ if the event $B \in \mathscr{F}_{t_0}$ occurs and its withdrawal at time $t_1$, still if $B$ occurs, generates the following associated cash-flows:

$$TP^{t_0,t_1,B}(t,\omega) = -I_B(\omega) I_{t=t_0}(t) + (1+r_p)^{t-t_0} I_B(\omega) I_{t=t_1}(t),$$

where $I_B(\omega) = 1$ if $\omega \in B$ and zero otherwise and $I_{t=t_0}(t) = 1$ if $t = t_0$ and zero otherwise. We will use the following notation:

$$TP_n = \left\{ TP^{t_0,t_1,B} / t_0, t_1 \in \{0,\dots n\},\ t_0 \leq t_1,\ B \in \mathscr{F}_{t_0} \right\}.$$

Notice that $TP_n$ belongs to $\mathscr{A}_n$. Moreover, if we denote by $TP = \cup_n TP_n$ then $TP \subset \mathscr{A}$.

Let $\mathscr{R}_n = \mathbb{R} \times \mathbb{R}^2 \times \dots \times \mathbb{R}^{2^n} \times \{0\} \times \{0\} \times \dots$, then $\mathscr{R}_n \simeq \mathbb{R}^{2^{n+1}-1}$ and let $\mathscr{R} = \cup_n \mathscr{R}_n$. First, we show that the set of adapted processes $\mathscr{A}$ and the set $\mathscr{R}$ are isomorphic. To do that, we remark that each cash-flow $\Phi \in \mathscr{A}$, observed in state $\omega$ and at time $n$ only depends on the information available at time $n$. Let $\Upsilon_n$ be the function mapping $\mathscr{A}$ into $\mathbb{R}^{\Omega_n} \simeq \mathbb{R}^{2^n}$ and associating $\Upsilon_n(\Phi) \in \mathbb{R}^{\Omega_n}$ to every $\Phi \in \mathscr{A}$, where $\Upsilon_n(\Phi)(\omega^n) = \Phi(n,\omega)$. This function is uniquely defined, because $\Phi$ is $\mathscr{F}$-adapted. Indeed we have $\Phi(n,\omega) = \Phi(n,\omega')$ if $\omega'^n = \omega^n$. We also denote by $\Upsilon$, the mapping from $\mathscr{A}$ into $\mathscr{R}$ defined by $\Upsilon(\Phi)_n = \Upsilon_n(\Phi)$, for $\Phi \in \mathscr{A}$. It is straightforward to see that $\Upsilon$ is one-to-one and then the set of linear mappings from $\mathscr{A}$ into $\mathscr{A}$ is in one-to-one correspondence with the set of linear mappings from $\mathscr{R}$ into $\mathscr{R}$. An investment $i$ will be represented by an element $\Phi_i$

of $\mathscr{A}_{T_i}$ or equivalently by $\Upsilon(\Phi_i) \in \mathscr{R}_{T_i} \subset \mathscr{R}$, where $\Upsilon(\Phi_i)$ is the countable sequence $(\Upsilon_0(\Phi_i), \Upsilon_1(\Phi_i), \dots, \Upsilon_{T_i}(\Phi_i), 0, \dots)$ where only the first $T_i + 1$th are non-zero. We call the length of an investment its last non-zero index. For sake of simplicity, we use the notation $\Upsilon_n(\Phi)(\omega^n) = \Phi(\omega^n)$.

We specify the topological structure on $\mathscr{A}$ and $\mathscr{R}$. If we denote by $E$ the expected value under $P$, set

$$L^2_{\mathscr{A}} = \left\{ \Phi \in \mathscr{A} / \sum_{n \in \mathbb{N}} 2^n E\left[ \Phi^2(n,\cdot) \right] < \infty \right\}.$$

The set $L^2_{\mathscr{A}}$ is then an Hilbert space endowed with the following scalar product:

$$\begin{aligned} \langle \Phi, \Phi' \rangle_{\mathscr{A}} &= \sum_{n \in \mathbb{N}} 2^n E\left[ \Phi(n,\cdot)\Phi'(n,\cdot) \right] \\ &= \sum_{n \in \mathbb{N}} 2^n \int_{\omega \in \Omega} \Phi(n,\omega)\Phi'(n,\omega) \mathrm{d}P(\omega). \end{aligned}$$

If we also define

$$L^2_{\mathscr{R}} = \left\{ \Upsilon(\Phi) \in \mathscr{R} / \sum_{n \in \mathbb{N}} \sum_{\omega^n \in \Omega_n} \Phi^2(\omega^n) < \infty \right\},$$

we get that $L^2_{\mathscr{R}}$ is an Hilbert's space endowed with the following scalar product

$$\langle \Upsilon(\Phi), \Upsilon(\Phi') \rangle_{\mathscr{R}} = \sum_{n \in \mathbb{N}} \sum_{\omega^n \in \Omega_n} \Phi(\omega^n)\Phi'(\omega^n).$$

Furthermore, it is easily checked that

$$\langle \Phi, \Phi' \rangle_{\mathscr{A}} = \langle \Upsilon(\Phi), \Upsilon(\Phi') \rangle_{\mathscr{R}}.$$

This proves that $L^2_{\mathscr{A}}$ and $L^2_{\mathscr{R}}$ are isometric for the distances associated with the scalar products.

Notice that the set of linear mappings from $\mathscr{A}$ into $\mathscr{A}$ is also isometric to the set of linear mappings from $\mathscr{R}$ into $\mathscr{R}$.

Next, notice that if $\Phi \in \mathscr{A}_n$,

$$\langle \Phi, \Phi \rangle_{\mathscr{A}} = \langle \Upsilon(\Phi), \Upsilon(\Phi) \rangle_{\mathscr{R}} = \sum_{t \in \mathbb{N}} \sum_{\omega^t \in \Omega_t} \Phi^2(\omega^t) = \sum_{t=0}^{n} \sum_{\omega^t \in \Omega_t} \Phi^2(\omega^t).$$

This sum is finite and therefore every element of $\mathscr{A}_n$, and in particular every investment, belongs to $L^2_{\mathscr{A}}$ and we will work in this space or in its isometric $L^2_{\mathscr{R}}$.

We now introduce the following notations. A process $\Phi \in L^2_{\mathscr{A}}$ is non-negative (respectively positive) if for all $n$ and $\omega$, $\Phi(n,\omega) \geq 0$ (respectively $\Phi(n,\omega) > 0$). We denote by $L^2_{\mathscr{A}_+}$ (respectively $L^2_{\mathscr{A}_{++}}$) the set of adapted non-negative (respectively positive) processes. We will also use the following notation: $L^2_{\mathscr{A}_{n+}}$, $L^2_{\mathscr{A}_{n++}}$, $L^2_{\mathscr{R}_+}$, $L^2_{\mathscr{R}_{++}}$, $\mathscr{R}_{n_+}$ and $\mathscr{R}_{n_{++}}$, which definitions are straightforward.

### 2.2.2. Translated investments: stationarity

Let us consider a set $\{\Phi_i / i \in I\}$ of basic investments indexed by some set $I$. We assume that this set contains a lending rate $r_p$ or following our notation $TP \subset \{\Phi_i / i \in I\}$. In the second part of the paper, we will propose different kinds of investment sets. We assume that the investments display the stationarity property. In the stochastic setup, this assumption implies that an investor can initiate every cash-flow at every date and in every state of the world. To formalize this assumption, we use the following function $T^u$. This function associates to the tree defining the investment the sub-tree conditioned by the realization of $u$ between time zero and time one. We denote by $(u,\omega)$ (respectively $(d,\omega)$), the state $\omega'$ such that $\omega'^1 = u$ (respectively $\omega'^1 = d$) and $\omega'^k = \omega^{k-1}$ for all $k \geq 2$. With this notation the function $T^u$ maps $L^2_{\mathscr{A}}$ into $L^2_{\mathscr{A}}$ and is defined for all $\Phi \in \mathscr{A}$, by:

$$T^u(\Phi)(0) = 0$$

$$T^u(\Phi)(n,(d,\omega)) = 0$$

$$T^u(\Phi)(n,(u,\omega)) = \Phi(n-1,\omega).$$

We define similarly the function $T^d$ by,

$$T^d(\Phi)(0) = 0$$

$$T^d(\Phi)(n,(u,\omega)) = 0$$

$$T^d(\Phi)(n,(d,\omega)) = \Phi(n-1,\omega).$$

If $\omega^n = (\omega_1, \dots, \omega_n) \in \Omega_n$, we will denote by $T^{\omega^n}$ the map $T^{\omega_n} \circ \dots \circ T^{\omega_1}$. Notice that $T^{\omega^n}$ is a linear function.

Now, we define the notion of stationarity.

**Definition 2.2.** A set of investment $\mathscr{I} \subset L^2_{\mathscr{A}}$ is stationary if for all $\Psi \in \mathscr{I}$, $n \in \mathbb{N}$, $\omega^n \in \Omega_n$, $T^{\omega^n}(\Psi)$ belongs to $\mathscr{I}$.

Let $\mathscr{I}_i = \{T^{\omega^n}(\Phi_i)/n \in \mathbb{N}, \omega^n \in \Omega_n\}$. The set $\mathscr{I}_i$ represents the set of investments generated by translations from the asset $i$. It is easy to see that $\mathscr{I}_i$ is stationary. More generally, the stationary set of investments generated by $\{\Phi_i\}_{i \in \mathscr{I}}$ is $\mathscr{I} = \cup_{i \in I} \mathscr{I}_i$.

In particular, the set of cash-flows generated by the lending rate, $TP$, is a stationary set. Indeed for all $n \in \mathbb{N}$, $\omega^n \in \Omega_n$, $t_0$, $t_1 \in \mathbb{N}$ with $t_0 \leq t_1$ and $B \in \mathscr{F}_{t_0}$, $T^{\omega^n}(TP^{t_0,t_1,B}) = TP^{t_0+n,t_1+n,B_{\omega^n}}$ with $B_{\omega^n} = \{(\omega^n,\omega)/\omega \in B\} \in \mathscr{F}_{t_0+n}$.

### 2.3. Strategies and payoffs

First, we define the set of admissible strategies. We assume that the investments cannot be sold. This is not really a restriction since it suffices to add $-\Phi_i$ to our investment model in order to have a possibility to sell $\Phi_i$. An investor must choose a finite number of investments, indexed by $J$ in the infinite set of investment indexed by $I$. We introduce the following notations, $T_J = \sup_{j \in J} T_j$ and for $N \geq T_J$:

$$\mathscr{I}_N^J = \left\{ T^{\omega^n}\left(\Phi_j\right)/n \in \{0,\dots,N-T_J\},\ \omega^n \in \Omega_n,\ j \in J \right\}.$$

Notice that the support of $T^{\omega^n}(\Phi_j)(\cdot,\omega)$ is a subset of $[0,T_j+n]$. In fact, $N$ is the last date when the investor receives a non-zero cash-flow and we call it the investment horizon. Then the set $\mathscr{I}_N^J$ represents the set of investments generated by translations from the investments indexed by $J$ and having an investment horizon less than or equal to $N$. We call $\Lambda_N^J$ set of admissible strategies generated by $\mathscr{I}_N^J$. The investor should choose an investment in the set $\mathscr{I}_N^J$ for $N \geq T_J$. We call $\lambda_\iota$ the non-negative number of investment $\Psi_\iota \in \mathscr{I}_N^J$ chosen by the investor.

**Definition 2.3.** A strategy $\lambda \in \Lambda_N^J$ is defined by:
–a finite subset of investments $J$,
–an investment horizon $N \geq T_J$,
–a non-negative and adapted buying strategy, $\lambda = (\lambda_\iota)_{\Psi_\iota \in \mathscr{I}_N^J}$.

The set of payoffs associated to the strategy $\Lambda_N^J$ is called $S_N^J$. Thus, $p$ belongs to $S_N^J$ if there exists $\lambda = ((\lambda_\iota)_{\Psi_\iota \in \mathscr{I}_N^J}) \subset \Lambda_N^J$, such that

$$p = \sum_{\Psi_\iota \in \mathscr{I}_N^J} \lambda_\iota \Psi_\iota.$$

Recalling that $\Psi_\iota \in \mathscr{I}_N^J$, the set $S_N^J$ represents the payoffs generated by $J$ and ending before $N$. This set belongs to $L^2_{\mathscr{A}_N}$. Moreover, there exists $j \in J$, $p \in \{0,\dots,N-T_j\}$ and $\omega^p \in \Omega_p$ such that $\Psi_\iota = T^{\omega^p}(\Phi_j)$. Writing $\lambda_\iota = \lambda_j(\omega^p)$, we find that

$$p = \sum_{j \in J} \sum_{p=0}^{N-T_j} \sum_{\omega^p \in \Omega_p} \lambda_j(\omega^p) T^{\omega^p}\left(\Phi_j\right).$$

These sums are finite so $p$ is well defined.

Furthermore, notice that $S_N^J$ is a convex cone.

### 2.4. No-arbitrage condition

**Definition 2.4.** An arbitrage opportunity consists in a finite subset $J$ of $I$, a horizon investment $N \geq T_J$ and a strategy in $\Lambda_N^J$ leading to a non-negative and non-zero payoff.

Notice that we can assume without loss of generality that the subset $J$ contains the lending rate $r_p$ and, thus, we can consider that the set of investment $\mathscr{I}_N^J$ contains $TP_N$. Indeed, if the arbitrage opportunity implies the existence of a finite subset indexed by $J$ that does not contain $TP_N$, we just have to add it (if there exists an arbitrage opportunity with $\mathscr{I}_N^J$, then there will also exist one with $\mathscr{I}_N^J \cup TP_N$).

### 2.5. Main results

We present our main theorem under the following postulate: there is no arbitrage opportunity and, if there is a positive element in the negative polar of the payoff set then it will also contains a positive element which is independent of the path. Notice that using a hyperplane separating result, the no-arbitrage condition implies that the negative polar of the payoff set contains a positive element. Our assumption is obviously satisfied if every element of the negative polar of the payoff set is independent of the path. We will see in the next section that this condition is satisfied under some conditions in a model taking into account transaction costs.

Let us state precisely the assumption about the negative polar of the payoff set. To do that, we use the following notation. Let $H_N^J$ be the negative polar of the payoff set $S_N^J$:

$$H_N^J = \left(S_N^J\right)^0 = \left\{h \in L^2_{\mathscr{A}} / \langle h, p \rangle_{\mathscr{A}} \leq 0,\ \forall p \in S_N^J\right\}.$$

Note that

$$H_N^J = \left\{h \in L^2_{\mathscr{A}} / \langle h, \Psi \rangle_{\mathscr{A}} \leq 0,\ \forall \Psi \in \mathscr{I}_N^J\right\} = \left(\mathscr{I}_N^J\right)^0.$$

Now, we distinguish the elements of $H_N^J$ which are independent of the path. To do that, we define new processes, which are independent of the path and called them harmonized processes. Thus, a harmonized process is such that for each date $n$ and each state of the world $\omega$, $\Phi(n,\omega)$ only depends on the product $\omega_1 \ldots \omega_n$. Let $X^n = \prod_{i=1}^n X_i$, then the definition of the harmonized process is stated as follows:

**Definition 2.5.** The harmonized process associated to $\Phi$ is defined by:

$$\tilde{\Phi}(n,\cdot) = E\left[\Phi_n | X^n\right].$$

The element $H_N^J$ which are independent of the path are called $\tilde{H}_N^J$ and are defined by:

$$\tilde{H}_N^J := \left\{ h \in H_N^J / h = \tilde{h} \right\}.$$

**Assumption 2.1.** *For every finite subset J of I and for every investment horizon N, if the set $H_N^J$ contains a positive element up to date N, then $\tilde{H}_N^J$ also contains such an element:*

$$H_N^J \cap L^2_{\mathscr{A}_{N++}} \neq \emptyset \Rightarrow \tilde{H}_N^J \cap L^2_{\mathscr{A}_{N++}} \neq \emptyset.$$

The main theorem states as follows:

**Theorem 2.1.** *Suppose that the model contains a lending rate $r_p \geq 0$. Under Assumption 2.1, the absence of arbitrage opportunity implies the existence of two real numbers r and $\pi^*$, with $r \geq r_p$ and $0 \leq \pi^* \leq 1$, such that for every investment $i \in I$,*

$$\sum_{n=0}^{T_i} \frac{1}{(1+r)^n} E^* \left[ \Phi_i(n,\cdot) \right] \leq 0,$$

*where $E^*$ is the expectation under $P^*$, which is the unique probability making the coordinate mappings independent and identically distributed and such that $P^*(X_n = u) = \pi^*$ and $P^*(X_n = d) = 1 - \pi^*$.*

Theorem 2.1 can be explained as follows: the discounted sum at the rate $r$ of the investments expected value under $P^*$ is non-positive.

If we denote by $|\omega|_n$ the number of up in $\omega^n$, the preceding expectation under $P^*$ is computed as follows,

$$\begin{aligned} E^* \left[ \Phi_i(n,\cdot) \right] &= \sum_{\omega^n = (\omega_1,\ldots,\omega_n) \in \Omega_n} \Phi_i(\omega^n) P^*(X_1 = \omega_1, \ldots, X_n = \omega_n) \\ &= \sum_{\omega^n = (\omega_1,\ldots,\omega_n) \in \Omega_n} \Phi_i(\omega^n) P^*(X_1 = \omega_1) \ldots P^*(X_n = \omega_n) \\ &= \sum_{\omega^n \in \Omega_n} \pi^{*|\omega|_n} (1-\pi^*)^{n-|\omega|_n} \Phi_i(\omega^n). \end{aligned}$$

Notice that if there is short selling constraints we can apply the preceding results to the cash-flows $\Phi_i$ and $-\Phi_i$, and we obtain that the discounted sum of the expected value is equal to zero.

Under some additional assumption, we are able to prove that $\pi^* \in ]0,1[$.

**Assumption 2.2.** *There exists two investments k and l, such that for all $x \in ]0,1[$,*

$$\sum_{n=0}^{T_k} x^n \Phi_k(u^n) > 0 \text{ and } \sum_{n=0}^{T_l} x^n \Phi_l(d^n) > 0.$$

We have denoted by $u^n$ (respectively $d^n$) the path with length $n$ and which contains only $u$ (respectively $d$). We will use this Assumption only for $x \leq \frac{1}{1+r_p}$.

Under this additional assumption, we can prove the following:

**Theorem 2.2.** *Suppose that the model contains a lending rate $r_p \geq 0$. Under Assumptions 2.1 and 2.2, the absence of arbitrage opportunity is equivalent the existence of two real numbers $r$ and $\pi^*$, with $r \geq r_p$ and $0 < \pi^* < 1$, such that for every investment $i \in I$,*

$$\sum_{n=0}^{T_i} \frac{1}{(1+r)^n} E^*[\Phi_i(n,\cdot)] \leq 0,$$

*where $E^*$ is the expectation under $P^*$, which is the unique probability making the coordinate mappings independent and identically distributed and such that $P^*(X_n = u) = \pi^*$ and $P^*(X_n = d) = 1 - \pi^*$.*

We present an example under which Assumption 2.2 is satisfied. Assume that our model contains an asset following a binomial price process called $S$ and consider a call and a put with exercise date equal to $T$ and exercise date equal to $K$, such that $S(0)d^T < K < S(0)u^T$. Recall that a call option (respectively a put option) is the right to buy (respectively to sell) the underlying at the exercise time $T$ for the exercise price $K$. The condition $S(0)d^T < K < S(0)u^T$ is in particular satisfied for options at the money ($K = S(0)$). Note that if this condition is not satisfied, the option is obviously redundant since it can be duplicated by a simple buy and hold strategy. Remarking that $(S(0)d^T - K)^+ = 0$ (respectively $(K - S(0)d^T)^+ = 0$), the cash-flow generated by the selling of the call option (respectively the put option) is an example of asset $l$ (respectively $k$).

### 2.6. Proof of the main results

#### 2.6.1. Proof of Theorem 2.1

We first prove the implication of the absence of arbitrage opportunity.

***Step 1:*** *use of a hyperplane separating result.*

First, we recall the following proposition, which is proved for example in Carassus and Jouini (1997):

**Proposition 2.1.** *If $\mathscr{Z}$ is a set of vectors in $\mathbb{R}^{\alpha}$ then exactly one of the following two alternatives must occur.*

1. *There is a linear combination of vectors of $\mathscr{Z}$ with non-negative coefficients, which belongs to $\mathbb{R}^{\alpha}_{+}$ and is not equal to zero.*
2. *There exists a vector of $\mathbb{R}^{\alpha}_{++}$ which makes a non-positive scalar product with all elements of $\mathscr{Z}$.*

Notice that property 2 is equivalent to $\mathscr{Z}^0 \cap \mathbb{R}^{\alpha}_{++} \neq \emptyset$.

For every $N \in \mathbb{N}$, every $J$ included in $I$, we apply Proposition 2.1 with $\mathscr{Z} = \Upsilon(S_N^J) = \{\Upsilon(p)/p \in S_N^J\} \subset \mathscr{R}_N$. The no-arbitrage condition implies that there does not exist a non-negative and non-zero linear combination, with non-negative coefficients, of elements of $S_N^J$. By isometry, there does not exist any non-negative and non-zero linear combination, with non-negative coefficients of elements of $Z = \Upsilon(S_N^J)$. Thus, we get that $(\Upsilon(S_N^J))^0 \cap \mathscr{R}_{N_{++}} \neq \emptyset$. From the isometry between $\mathscr{A}$ and $\mathscr{R}$, this is equivalent to $(S_N^J)^0 \cap L^2_{\mathscr{A}_{N_{++}}} \neq \emptyset$ and applying Assumption 2.1, $\tilde{H}_N^J \cap L^2_{\mathscr{A}_{N_{++}}} \neq \emptyset$. Let $k \in \tilde{H}_N^J \cap L^2_{\mathscr{A}_{N_{++}}}$, we normalize it by the condition $k(0) = 1$.

Consider now the following subset of $L^2_{\mathscr{A}}$:

$$\tilde{K} = \left\{ h \in L^2_{\mathscr{A}_+} / \tilde{h} = h,\, h(0) = 1 \text{ and } \langle h, \Psi \rangle_{\mathscr{A}} \leq 0,\, \forall \Psi \in \mathscr{I} \right\}.$$

Let $\tilde{L}^2_{\mathscr{A}} := \{\mathrm{h} \in \mathrm{L}^2_{\mathscr{A}} / \tilde{h} = h\}$, we get that $\tilde{K} = \{h \in \tilde{L}^2_{\mathscr{A}_+} / h(0) = 1\} \cap \mathscr{I}^0$.

First, we prove that $\tilde{K}$ is a non-empty compact set for the topology $\sigma(L^2_{\mathscr{A}}, L^2_{\mathscr{A}})$. To do that, we define the following subset of $L^2_{\mathscr{A}}$:

$$\begin{aligned} \tilde{K}_N^J &= \left\{ h \in L^2_{\mathscr{A}_+} / \tilde{h} = h,\, h(0) = 1 \text{ and } \langle h, \Psi \rangle_{\mathscr{A}} \leq 0,\, \forall \Psi \in \mathscr{I}_N^J \right\} \\ &= \left\{ h \in \tilde{L}^2_{\mathscr{A}_+} / h(0) = 1 \right\} \cap \left( \mathscr{I}_N^J \right)^0. \end{aligned}$$

Notice that $\tilde{K}_N^J = \{h \in L^2_{\mathscr{A}_+} / h(0) = 1\} \cap \tilde{H}_N^J$. Recalling that for every $J$ and $N$, $\mathscr{I}_N^J \subset \mathscr{I}$ and that $\mathscr{I} = \cup_{J,N} \mathscr{I}_N^J$, we find that $\tilde{K} \subset \tilde{K}_N^J$ and $\tilde{K} = \cap_{J,N} \tilde{K}_N^J$. First, we show that $\tilde{K}_N^J$ and thus $\tilde{K}$ are included in some weak-compact subspace of $L^2_{\mathscr{A}}$.

Recall that without loss of generality we can assume that $\mathscr{I}_N^J$ contains each $TP_N$ and that for every $h \in \tilde{K}_N^J$, and every $t = 0, \ldots, N$, $T^{0,t,\Omega} \in \mathscr{I}_N^J$, and $\langle h, T^{0,t,\Omega} \rangle_{\mathscr{A}} = \langle h, T^{0,t,\Omega} \rangle_{\mathscr{R}} \le 0$. Now,

$$\langle h, T^{0,t,\Omega} \rangle_{\mathscr{R}} = -h(0) + \sum_{\omega^t \in \Omega_t} h(\omega^t)(1 + r_p)^t.$$

Thus, $\sum_{\omega^t \in \Omega_t} h(\omega^t) \le (1 + r_p)^{-t}$. As for all $t \in \mathbb{N}$ and for all $\omega^t \in \Omega_t$, $h(\omega^t) \ge 0$, we get that,

$$\sum_{\omega^t \in \Omega_t} h^2(\omega^t) \le \left( \sum_{\omega^t \in \Omega_t} h(\omega^t) \right)^2 \le (1 + r_p)^{-2t}.$$

Furthermore,

$$\|h\|_{\mathscr{A}}^2 = \|h\|_{\mathscr{R}}^2 = \sum_{t=0}^{N} \sum_{\omega^t \in \Omega_t} h^2(\omega^t) \le \sum_{t=0}^{N} (1 + r_p)^{-2t} \le \frac{(1 + r_p)^2}{r_p^2 + 2 r_p}.$$

Let $M = \sqrt{(1 + r_p)^2 / (r_p^2 + 2 r_p)}$ and $\mathscr{B} = \{h \in L^2_{\mathscr{A}_+} / \|h\|_{\mathscr{A}} \le M\}$. Recalling that $L^2_{\mathscr{A}}$ is a reflexive set, the Theorem of Kakutani applies (see, for example, Brézis, 1983, p. 44), and $\mathscr{B}$ is a weakly compact set.

Let $J \subset J'$, and $N \in IN$, as $\mathscr{I}_N^J \subset \mathscr{I}_{N+1}^{J'}$, we find that $\tilde{K}_{N+1}^{J'} \subset \tilde{K}_N^J$. We next show that for every $J$ and $N$, $\tilde{K}_N^J$ is a non-empty and weakly closed set. First, $\tilde{K}_N^J$ is a non-empty set because there exists $k$ in $\tilde{H}_N^J \cap L^2_{\mathscr{A}_{N++}}$ such that $k(0) = 1$. Next, we prove that $\tilde{K}_N^J$ is a weakly closed set.

Let $(h_p)_p \subset \tilde{K}_N^J$ be a sequence converging weakly to $h$. First, we fix some $t \in \mathbb{N}$ and $\omega \in \Omega$, and let $\psi$ be a function mapping $L^2_{\mathscr{A}}$ into itself and defined by $\psi(t, \omega) = 1$ and zero otherwise. From the weak convergence, we get that $h_p(t, \omega)$ goes simply to $h(t, \omega)$. It is easy to see that $h \in L^2_{\mathscr{A}_+}$. Now, if we fixed again some $t$, we get that $\tilde{h}_p(t, \cdot) = E[h_p | X^t]$ goes to $\tilde{h}(t, \cdot) = E[h | X^t]$ as $p$ goes to infinity. Recalling that $\tilde{h}_p = h_p$, we get that $\tilde{h}(t, \cdot) = h(t, \cdot)$ and hence $h = \tilde{h}$. Then, we choose for $\psi$ the function defined by $\psi(0) = 1$ and $\psi$ equal to zero elsewhere, this function belongs to $L^2_{\mathscr{A}}$ and so $\langle h_p, \psi \rangle_{\mathscr{A}} = h_p(0) = 1$ converges simply to $\langle h, \psi \rangle_{\mathscr{A}} = h(0) = 1$. Finally, let $\Psi \in \mathscr{I}_N^J \subset L^2_{\mathscr{A}}$, $\langle h_p, \Psi \rangle_{\mathscr{A}} \le 0$ converges simply to $\langle h, \Psi \rangle_{\mathscr{A}}$ and $\langle h, \Psi \rangle_{\mathscr{A}} \le 0$. We conclude that $h \in \tilde{K}_N^J$.

The sequence $(\tilde{K}_N^J)_{J,N}$ of $\mathscr{B}$, which is a weakly compact set, is a decreasing sequence of non-empty, weakly closed set. From a compactness argument, their intersection $\tilde{K}$ is a non-empty set, furthermore $\tilde{K}$ does not reduced to zero. Remarking that each $\tilde{K}_N^J$ is a convex and weakly closed set, $\tilde{K}$ is also a convex and weakly closed set. As $\tilde{K}$ is subset in the weakly compact set $\mathscr{B}$, $\tilde{K}$ is also weakly compact.

***Step 2:*** *use of a fixed point theorem.*

Let us denote by $T^{*u}$ the adjoint of $T^u$. Using the adjoint property, that is for all $\Phi$ and $h$ included in $L^2_{\mathscr{A}}$, $\langle T^u(\Phi),h\rangle_{\mathscr{A}} = \langle \Phi, T^{*u}(h)\rangle$, it is easily checked that

$$T^{*u}(h)(n,\omega) = h(n+1,(u,\omega)).$$

In the same way, we prove that the adjoint of $T^d$, denoted by $T^{*d}$, is defined by

$$T^{*d}(h)(n,\omega) = h(n+1,(d,\omega)).$$

Let $\psi^u$ be the mapping from $\tilde{K}$ into $\tilde{K}$ and associating to $h$, $\psi^u(h) = (T^{*u}(h) + h)/(T^{*u}(h)(0) + h(0))$. First, we justify that if $h$ belongs to $\tilde{K}$ then $\psi^u(h)$ also belongs to $\tilde{K}$. We see immediately that $\psi^u(h) \in L^2_{\mathscr{A}_+}$ and $\psi^u(h)(0) = 1$. It remains to show that for every $N \in \mathbb{N}$, for every finite subset $J$ of $I$, $\psi^u(h) \in \tilde{H}^J_N$ or equivalently $T^{*u}(h) \in \tilde{H}^J_N$. As $h$ belongs to $\tilde{K}$, $h \in \tilde{K}^J_{N+1} \subset \tilde{H}^J_{N+1}$. Let $p \in S^J_N$, then $T^u(p) \in S^J_{N+1}$. Hence, we find that $\langle T^u(p),h\rangle_{\mathscr{A}} \leq 0$, and from the adjoint property $\langle p, T^{*u}(h)\rangle_{\mathscr{A}} \leq 0$. This prove that $T^{*u}(h) \in H^J_N$. From the linearity of $T^{*u}$ and of the conditional expectation, we get that

$$\begin{aligned}\widetilde{T^{*u}}(h)(n,\cdot) &= E\left[T^{*u}(h)(n,\cdot)|X^n\right] = T^{*u}\left(E\left[h(n,\cdot)|X^n\right]\right)\\ &= T^{*u}(\tilde{h})(n,\cdot).\end{aligned}$$

Recalling that $h \in \tilde{H}^J_{N+1}$, and thus $\tilde{h} = h$, $\widetilde{T^{*u}}(h) = T^{*u}(\tilde{h}) = T^{*u}(h)$. Accordingly, we proved that $T^{*u}(h) \in \tilde{H}^J_N$ for all $N$ and for all finite subset $J$ of $I$.

Next, we prove that $\psi^u$ or equivalently $T^{*u}$ is weakly continuous. Let $(h_p)_p \subset L^2_{\mathscr{A}}$ converging weakly to $h$. Let $\psi \in L^2_{\mathscr{A}}$, $\langle T^{*u}(h_p),\psi\rangle_{\mathscr{A}} = \langle h_p, T^u(\psi)\rangle_{\mathscr{A}}$, as $T^u(\psi) \in L^2_{\mathscr{A}}$, $\langle h_p, T^u(\psi)\rangle_{\mathscr{A}}$ converges simply to $\langle h, T^u(\psi)\rangle_{\mathscr{A}} = \langle T^{*u}(h),\psi\rangle_{\mathscr{A}}$.

The mapping $\psi^u$ is weakly continuous from $\tilde{K}$, which is a convex, weakly compact set into itself and Theorem of Schauder–Tychonoff (see Dugundji and Granas, 1982) applies: there exists a fixed point $l$ for $\psi^u$. Notice that $l(0) = 1$ and thus $l$ is not equal to zero. Moreover, we have that,

$$l = \frac{T^{*u}(l) + l}{T^{*u}(l)(0) + 1}.$$

Denoting by $\alpha_u = T^{*u}(l)(0)$, we get that $T^{*u}(l) = \alpha_u l$.

Let $\psi^d$ be the function mapping $\tilde{L} = \tilde{K} \cap \{h \in L^2_{\mathscr{A}} / T^{*u}(h) = \alpha_u h\}$ into $\tilde{L}$, defined by $\psi^d(h) = (T^{*d}(h) + h)/(T^{*d}(h)(0) + h(0))$. First, we justify that if $h \in \tilde{L}$, then $\psi^d(h) \in \tilde{L}$. As previously done, we prove that $\psi^d(h) \in \tilde{K}$, it remains to prove that $\psi^d(h) \in \{h \in L^2_{\mathscr{A}} / T^{*u}(h) = \alpha_u h\}$. To do that, we first prove that for all $h \in \tilde{L}$, $T^{*u} \circ T^{*d}(h) = T^{*d} \circ T^{*u}(h)$.

$$\begin{aligned}T^{*u} \circ T^{*d}(h)(n,\omega) \quad &= T^{*u}(\omega \to h(n+1,(d,\omega)))(n,\omega)\\ &= h(n+2,(u,d,\omega))\\ &= h(n+2,(d,u,\omega))\\ &= T^{*d}(\omega \to h(n+1,(u,\omega)))(n,\omega)\\ &= T^{*d} \circ T^{*u}(h)(n,\omega).\end{aligned}$$

We used that $h$ belongs to $\tilde{K}$, and therefore is independent of the path.

If we denote by $id$ the function mapping $L^2_{\mathscr{A}}$ into itself and such that $id(\Phi) = \Phi$, we get that for all $h \in \tilde{L}$,

$$\begin{aligned} T^{*u}(\psi^d(h)) &= T^{*u}\left(\frac{T^{*d}(h) + h}{T^{*d}(h)(0) + h(0)}\right) = \frac{T^{*d} \circ T^{*u}(h) + id \circ T^{*u}(h)}{T^{*d}(h)(0) + h(0)} \\ &= \frac{(T^{*d} + id) \circ T^{*u}(h)}{T^{*d}(h)(0) + h(0)} = \frac{(T^{*d} + id)(\alpha_u h)}{T^{*d}(h)(0) + h(0)} \\ &= \alpha_u \frac{T^{*d}(h) + h}{T^{*d}(h)(0) + h(0)} = \alpha_u \psi^d(h). \end{aligned}$$

As previously done, we prove that $\psi^d$ is a weakly continuous function mapping $\tilde{L}$, which is a convex and weakly compact set, into itself. The theorem of Schauder–Tychonoff applies and we get that there exists a fixed point $f$ for $\psi^d$. Notice that $f(0) = 1$ and thus the fixed point is not equal to zero. Let $\alpha_d = T^{*d}(f)(0)$, we get that $T^{*d}(f) = \alpha_d f$. Recalling that $f \in \tilde{L}$: we get that $T^{*u}(f) = \alpha_u f$. Recalling that $|\omega|_n$ is the number of up in $\omega^n$ and using the fixed point properties, we prove by induction that

$$f(n,\omega) = \alpha_u^{|\omega|_n} \alpha_d^{n-|\omega|_n}.$$

Notice that $T^{*u}(f)(0) = \alpha_u f(0) = \alpha_u$ and hence $\alpha_u + \alpha_d = T^{*u}(f)(0) + T^{*d}(f)(0)$. Recalling that $f \in \tilde{K}$, and that $\mathscr{I}$ contains $T^{0,1,\Omega}$, we get that $-f(0) + [\Upsilon(f)(u) + \Upsilon(f)(d)](1 + r_p) \leq 0$. As $\Upsilon(f)(u) = T^{*u}(f)(0)$ and $\Upsilon(f)(d) = T^{*d}(f)(0)$, we find that $\alpha_u + \alpha_d \leq 1/(1 + r_p)$. Remark also that $f \in L^2_{\mathscr{A}_+}$ and therefore $(\alpha_u, \alpha_d) \in [0,1/(1 + r_p)]^2$.

Suppose that $\alpha_u = \alpha_d = 0$ then for all investments $i$, $\Phi_i(0) \leq 0$. If every investment is equal to zero at time zero, it is sufficient to shift the zero of the time origin. Now, if every investment is negative, recalling that the investor has no money to begin with, she/he cannot begin any strategy, and it is obvious that there is no-arbitrage opportunity. We conclude that $\alpha_u$ and $\alpha_d$ cannot be simultaneously equal to zero, in a model where trade occurs even without an initial amount of money.

Let $r = [1/(\alpha_u + \alpha_d)] - 1$, and $\pi^* = [\alpha_u/(\alpha_u + \alpha_d)]$, it is straightforward to prove that $0 \leq \pi^* \leq 1$, $r \geq r_p$, and that $f(n,\omega) = \frac{1}{(1+r)^n} \pi^{*|\omega|_n}(1 - \pi^*)^{n-|\omega|_n}$, we get that

$$\sum_{n=0}^{T_i} \frac{1}{(1+r)^n} E^*[\Phi_i(n,\cdot)] \leq 0.$$

**Remark.** We use Assumption 2.1 and harmonized investments in order to prove that $T^{*u} \circ T^{*d} = T^{*d} \circ T^{*u}$ and thus to find a common fixed point to $\psi^u$ and $\psi^d$.

*2.6.2. Proof of Theorem 2.2*

To prove the first implication, it remains to show that under Assumption 2.2 $\pi^* \in ]0,1[$. Suppose that $\pi^* = 0$, then there exists $r \geq r_p$ such that for every investment $i$, $\sum_{n=0}^{T_i} \leq 0[1/(1+r)^n]\Phi_i(d^n) \leq 0$. Indeed, we remark that the unique $\omega^n \in \Omega_n$ such that $|\omega|_n = 0$ is $d^n$. This contradicts Assumption 2.2 for investment $l$, with $x = 1/(1+r) \in ]0,1/(1+r_p)[ \subset ]0,1[$. Now if $\pi^* = 1$, there exists $r \geq r_p$ such that for every investment $i$, we have $\sum_{n=0}^{T_i}[1/(1+r)^n]\Phi_i(u^n) \leq 0$. This contradicts Assumption 2.2 for investment $k$, with $x = 1/(1+r) \in ]0,1/(1+r_p)[ \subset ]0,1[$.

Conversely, suppose that there exists two real numbers $r$ and $\pi^*$, with $0 < \pi^* < 1$, such that for all $i \in I$, we have

$$\sum_{n=0}^{T_i} \frac{1}{(1+r)^n} E^*\left[\Phi_i(n,\cdot)\right] \leq 0.$$

Thus

$$\sum_{n=0}^{T_i} \left(\frac{2}{1+r}\right)^n \int_{\omega \in \Omega} \pi^{*|\omega|_n}(1-\pi^*)^{n-|\omega|_n} \Phi_i(n,\omega) \mathrm{d}P(\omega) = \langle f, \Phi_i \rangle_{\mathscr{A}} \leq 0, \tag{2.1}$$

where we use the notation, $f(n,\omega) = \frac{1}{(1+r)^n} \pi^{*|\omega|_n}(1-\pi^*)^{n-|\omega|_n}$.

Suppose that there exists an arbitrage in the sense of Definition 2.4. Then there exist an investment horizon $N$, a finite subset $J$ of $I$ and a strategy $\lambda \in \Lambda_N^J$ leading to a non-negative and non-zero payoff $p_\lambda$. If we multiply Eq. (2.1) applied to the assets $T^{\omega^p}(\Phi_j)$, by $\lambda_j(\omega^p)$ which is non-negative, and if we sum on every $\omega^p$ and for all $j \in J$, we get that $\langle f, p_\lambda \rangle_{\mathscr{A}} \leq 0$. But the previous inequality is the scalar product between the positive function $f$ and the non-negative and non-zero function $p_\lambda$. This lefthand-side should be positive and we get a contradiction. ■

## 3. Application to the transaction cost case

### *3.1. The model*

We study a very simple model, which displays the following properties.

The model contains an underlying asset, which price process follows a binomial process denoted by $S = (S_t)_{t \in \mathbb{N}}$. This means that $S(t,\omega) = S_0 u^{|\omega|_t} d^{t-|\omega|_t}$. This asset represents the productive power of the economy. We assume that the price $S$ is a strictly increasing function of the produced quantity. We assume that there are some proportional transaction costs on the underlying asset. We denote by $c$ the buying transaction cost, where $c \geq 0$ and by $c'$ the selling transaction cost, where $0 \leq c' < 1$.

We use the same probability space $(\Omega, \mathscr{F}, P)$ as in the previous section, and also the same filtration. Notice that in this model the filtration can also be defined by $\mathscr{F}_t = \sigma(S_1, \dots, S_t)$. The horizon of dates is infinite and we choose for $P$ the unique probability which turns the coordinate mapping $X_n$ to be independent and identically distributed, and such that $P(X_n = u) = P(X_n = d) = 1/2$.

The model contains an infinite number of consumers $i$, which are assumed to be infinitely small. Each consumer $i$ has a finite life time equal to $T_i$. Notice that the horizon of dates is infinite. The consumers have preferences over their consumption all over the time. More precisely, we assume that at each date $t$, the consumer $i$ specifies her/his preferences thanks to the function $u_{i,t}$. We assume that each function $u_{i,t}$ is strictly increasing and strictly concave. The utility of an adapted consumption process $c_i = \{c_i(t,\omega)/t = 0, \dots, T_i, \omega \in \Omega\} \in L^2_{\mathscr{A}}$ is given by:

$$U_i(c_i) = \sum_{t=0}^{T_i} \beta_i^t \int_{\omega \in \Omega} u_{i,t}(c_i(t,\omega)) \mathrm{d}P(\omega) = \sum_{t=0}^{T_i} \beta_i^t E\left[u_{i,t}(c_i(t,\cdot))\right].$$

This utility is in fact the discounted sum of the expected value of the consumptions' utility. The discounted rate $\beta_i/2$ is called the psychological rate and it is positive. Notice that we have included the $2^n$ of the scalar product in the psychological rate.

The model includes a lending rate equal to $r_p$ and a borrowing rate equal to $r_e$.

We consider a family of options (call and put options) written on the underlying asset They are indexed by their common exercise date $T$ and a real number $k$, which allows to compute the exercise price. We assume that at each date $t$, there exists an option with exercise time equal to $t+T$ and exercise price equal to $kS_t$. Recall that a call (respectively put) option is the right to buy (sell) at the exercise date, the option at the exercise price We say that an option is not trivial if $d^T \le k \le u^T$. Otherwise, assume that $k > u^T$, $kS_t > S_t u^T \ge S_{t+T}$, then $(S_{t+T} - kS_t)^+ = 0$, and the call option is not an interesting asset. Moreover, the put option will have a payoff equal to $(kS_t - S_{t+T})^+$, which is strictly positive and the put option is a static combination of the underlying asset and of an obligation. In the same way, if $d^T > k$, $S_{t+T} \ge S_t d^T > kS_t$, the call option will be such a combination and the put option will have a zero payoff. We assume that there exists a bid–ask spread on the option price. This includes the transaction costs case. The buying price for a call option at time $t$ will be denoted by $C(T,k,S_t)$, and its selling price at time $t$ by $C'(T,k,S_t)$. We assume that the prices are homogeneous, that is to say $C(T,k,\lambda S_t) = \lambda C(T,k,S_t)$, (respectively $C'(T,k,\lambda S_t) = \lambda C'(T,k,S_t)$). This means that if there is a change of scale on the asset price, the same one will apply on the options price. We can also think this property as an invariance against the monetary unit.

Finally, we assume that there exist some assets called exchange assets which allow to transfer money from two particular states against an initial cost. For every date $t$, and every states $\omega$, $\omega' \in \Omega$ such that $|\omega'|_t = |\omega|_t$, if the consumer pays at time $t=0$ (initial state) $p_0^{\omega^t,\omega'^t}$, then she/he can exchange at time $t$ \$1 from state $\omega$ to state $\omega'$. This asset is denoted by $E_0^{\omega^t,\omega'^t}$. This possibility is given not only from the initial state but from every state of the world having $\omega^t$ and $\omega'^t$ as successor. We also allow some transaction costs on the exchange assets. Notice that the consumer receives (algebraically) at time zero $-p_0^{\omega'^t,\omega^t}$ in order to exchange at time $t$, \$1 from state $\omega'$ to state $\omega$. Thus, $-p_0^{\omega'^t,\omega^t}$ is the selling price of $E_0^{\omega^t,\omega'^t}$ and from the transaction costs this number is in general different from its buying price $p_0^{\omega^t,\omega'^t}$.

Each consumer $i$ maximizes her/his individual utility $U_i$, and has a demand function called $D_i$. The total demand of the model is the sum of the individual demands, which makes sense because the consumers are assumed to be infinitely small. As the single production possibility is the underlying asset, the supply is defined by the production in this asset. We will say that the model is at equilibrium when every consumer maximizes its individual utility function and when the demand is equal to the supply.

### 3.2. Main results

Suppose that the lending rate is equal to the buying rate and denote by $r = r_p = r_e$ this common value. When there is no transaction costs, the price at time $t$ of an option with exercise time equal to $t+T$, and exercise price equal to $kS_t$, is

$$\mathrm{CRR}(T,k,S_t) = \frac{1}{(1+r)^T}\sum_{j=0}^{T} C_T^j \pi^{*j}(1-\pi^*)^{T-j}\left(S_t u^j d^{T-j} - kS_t\right)^+,$$

where $u\pi^* + d(1-\pi^*) = (1+r)$.

This is the main result of the model of Cox et al. (1979). Recall that the limit of this formula in continuous time is the well-known formula of Black and Scholes (1973). The option price is the expected value under the probability, defined by the transition probability $\pi^*$, of the terminal value of the option discounted at the rate $r$. We mention that when there are transaction costs, the no-arbitrage condition allows only to situate the option price in an interval. The lower (respectively upper) bound of this interval corresponds to the lower (respectively upper) option price that does not allow arbitrage. Moreover, as showed by Jouini and Kallal (1995), the bounds of every sub-interval also define no arbitrage bid and ask prices. Of course, the Black–Scholes' price, or the Cox–Ross–Rubinstein's price in discrete time setting, belong to the maximal interval, but they do not belong to every sub-interval of the maximal one. We show that with an infinite time horizon, the Cox–Ross–Rubinstein's price is always between the selling and buying price.

**Theorem 3.1.** *At the equilibrium, there exists an equilibrium rate r, between* $r_p$ *and* $r_e$*, and a real number* $\pi^* \in ]0,1[$*, such that* $u\pi^* + d(1-\pi^*) = 1 + r$*, and for every date t,*

$$C'(T,k,S_t) \leq \mathrm{CRR}(T,k,S_t) \leq C(T,k,S_t).$$

If we consider the case of a buying (respectively selling) transaction cost on the option equals to $\mu$ (respectively $\mu'$). We denote by $C_\mu$ the option price process, that is $C(T,k,S_t) = (1+\mu)C_\mu(T,k,S_t)$ and $C'(T,k,S_t) = (1-\mu')C_\mu(T,k,S_t)$. In this case, we find explicit bounds on the option price.

**Corollary 3.1.** *At the equilibrium, there exists an equilibrium rate r, between* $r_p$ *and* $r_e$*, and a real* $\pi^* \in ]0,1[$*, such that* $u\pi^* + d(1-\pi^*) = 1 + r$*, and for every date t,*

$$\frac{1}{1+\mu}\mathrm{CRR}(T,k,S_t) \leq C_\mu(T,k,S_t) \leq \frac{1}{1-\mu'}\mathrm{CRR}(T,k,S_t).$$

If there are transaction costs on the underlying asset but not on the options, we find that the option price is the Cox–Ross–Rubinstein's one.

**Corollary 3.2.** *At the equilibrium, there exists an equilibrium rate r, between* $r_p$ *and* $r_e$*, and a real number* $\pi^* \in ]0,1[$*, such that* $u\pi^* + d(1-\pi^*) = 1 + r$*, and for every date t,*

$$C(T,k,S_t) = \mathrm{CRR}(T,k,S_t).$$

## *3.3. Proof of Theorem 3.1*

We first prove that our model satisfies the stationarity property.

### *3.3.1. Stationarity*

The first step is to give the representation of the considered assets in term of cash-flows. We first express the cash-flow $AV^{t_0,t_1,B}$, which corresponds to the buying at time $t_0$ and the selling at time $t_1$ of one unit of underlying asset if event the $B \in \mathscr{F}_{t_0}$ occurs:

$$\begin{aligned} AV^{t_0,t_1,B}(t,\omega) = & -S_0 u^{|\omega|_t} d^{t-|\omega|_t}(1+c) I_{t=t_0}(t) I_{\omega\in B}(\omega) \\ & + S_0 u^{|\omega|_t} d^{t-|\omega|_t}(1-c') I_{t=t_1}(t) I_{\omega\in B}(\omega). \end{aligned}$$

Next, we express the cash-flow $VA^{t_0,t_1,B}$, which corresponds to the selling at time $t_0$ and the buying at time $t_1$ of one unit of underlying asset still if the event the $B \in \mathscr{F}_{t_0}$ occurs:.

$$VA^{t_0,t_1,B}(t,\omega) = S_0 u^{|\omega|_t} d^{t-|\omega|_t}(1-c') I_{t=t_0}(t) I_{\omega \in B}(\omega) - S_0 u^{|\omega|_t} d^{t-|\omega|_t}(1+c) I_{t=t_1}(t) I_{\omega \in B}(\omega).$$

Next, we represent the cash-flows generated by the lending and the borrowing rates. The cash-flow $TP^{t_0,t_1,B}$ represents the lending of \$1 at time $t_0$ if the event $B \in \mathscr{F}_{t_0}$ occurs and its withdrawal at time $t_1$, still if the event $B$ occurs,

$$TP^{t_0,t_1,B}(t,\cdot) = -I_B I_{t=t_0} + \left(1+r_p\right)^{t-t_0} I_B I_{t=t_1}.$$

The borrowing of \$1 at time $t_0$ if the event $B \in \mathscr{F}_{t_0}$ occurs and its refunding at time $t_1$, still if the event $B$ occurs generate the following cash-flow, called $TE^{t_0,t_1,B}$:

$$TE^{t_0,t_1,B}(t,\cdot) = I_B I_{t=t_0} - (1+r_e)^{t-t_0} I_B I_{t=t_1}.$$

Now, we express the cash-flow generated by the options. Let $AO^{t_0,T,k,B}$ be the cash-flow associated to the buying at time $t_0$ of a call option of exercise date $t_0+T$ and of exercise price $kS_{t_0}$, if the event $B \in \mathscr{F}_{t_0}$ occurs.

$$AO^{t_0,T,k,B}(t,\cdot) = -C(T,k,S_t) I_{t=t_0} I_B + \left(S_t - kS_{t_0}\right)^+ I_{t=t_0+T} I_B.$$

Now, if we consider the selling of a call option of exercise date $t_0+T$ and of exercise price $kS_{t_0}$, if the event $B \in \mathscr{F}_{t_0}$ occurs and denote by $VO^{t_0,T,k,B}$ the associated cash-flow, we get that:

$$VO^{t_0,T,k,B}(t,\cdot) = C'(T,k,S_t) I_{t=t_0} I_B - \left(S_t - kS_{t_0}\right)^+ I_{t=t_0+T} I_B.$$

In order to prove that the stationarity assumption holds, we can check by induction that for every $n$ and every $\omega^n \in \Omega_n$:

$$T^{\omega^n}\left(AV^{t_0,t_1,B}\right) = u^{-|\omega|_n} d^{|\omega|_n - n} AV^{t_0+n,t_1+n,B_{\omega^n}}, \tag{3.1}$$

where $B_{\omega^n} = \{(\omega^n,\omega)/\omega \in B\} \in \mathscr{F}_{t_0+n}$. The same formula holds for $VA^{t_0,t_1,B}$. We can also check that

$$T^{\omega^n}\left(AO^{t_0,T,k,B}\right) = u^{-|\omega|_n} d^{|\omega|_n - n} AO^{t_0,T,k,B}, \tag{3.2}$$

and again the same formula holds for $VO^{t_0,T,k,B}$.

This prove that the cone generated by the cash-flows $AV^{t_0,t_1,B}$, $VA^{t_0,t_1,B}$, $AO^{t_0,T,k,B}$ and $VO^{t_0,T,k,B}$ is a stationary set of investments. Notice that the same results also apply for the put option, and therefore the cone generated by the put option cash-flows is also stationary.

Moreover, it is straightforward to see that $T^{\omega^n}(TP^{t_0,t_1,B}) = TP^{t_0+n,t_1+n,B_{\omega^n}}$, and as well for $TE^{t_0,t_1,B}$.

Finally, the exchange assets are stationary by definition, because we have assumed that we can transfer at each date and in each state of the world. Thus, $T^{\alpha^n}(E_0^{\omega^n,\omega'^n}) = E_{\alpha^n}^{(\alpha^n,\omega^n),(\alpha^n,\omega'^n)}$, and this one exists by assumption.

In order to prove Theorem 3.1, we want to use the results of the second part. To do that, we prove that Assumptions 2.1 and 2.2 are satisfied.

### *3.3.2. Assumptions 2.1 and 2.2*

We have already seen that Assumption 2.2 is satisfied if the model contains a non-trivial call and put option, which is assumed to be true by construction of this model. For example, consider the case of a call (respectively put) option at the money. As the selling price of call option $C'(T,1,S_0)$ (respectively the put option $P'(T,1,S_0)$), is positive and $(S_0 d^T - S_0)^+ = 0$ (respectively $(S_0 - S_0 u^T)^+ = 0$), Assumption 2.2 is trivially satisfied. We have already seen that the considered model is stationary.

First, we prove that at the equilibrium, there is no arbitrage opportunity. Otherwise, assume that there exists an arbitrage opportunity in the sense of Definition 2.4. Then there exists a finite subset $J$ of $I$, an investment horizon $N \geq T_J$ and a strategy $\lambda \in \Lambda_N^J$ leading to a non-negative and non-zero payoff $p_\lambda$. Without any excess cost, the consumer $i$ can add $\varepsilon\lambda$ to her/his portfolio and therefore consumes $\varepsilon p_\lambda$ more. Let $c_i$ be the initial consumption of consumer $i$. Her/his utility at time $t$ and in the state of the world $\omega$ varies of $\varepsilon p_\lambda(t,\omega)u'_{i,t}(c_i(t,\omega)) + o(\varepsilon)$. Recalling that all the sums are finite, the variation of her/his global utility is equal to

$$\varepsilon \sum_{t=0}^{T_i} \beta_i^t \int_{\omega\in\Omega} \mathrm{d}P(\omega)\, p_\lambda(t,\omega)\, u'_{i,t}\big(c_i(t,\omega)\big) + o(\varepsilon).$$

Notice that the utility $u_{i,t}$ is a strictly increasing function and the function $u'_{i,t}$ is strictly positive. Moreover, the payoff $p_\lambda$ is a non-negative and non-zero function. The total variation of the utility is then positive. The individual optimization problem of agent $i$ has no solution and there is no possible equilibrium.

At the equilibrium, there is no arbitrage opportunity and from Section 2.6.1, step 1, we conclude that for every finite subset $J$ of $I$, for every investment horizon $N \geq T_J$, the set

$$H_N^J = \left(S_N^J\right)^0 = \left\{h \in L^2_{\mathscr{A}} / \langle h,p\rangle_{\mathscr{A}} \leq 0,\ \forall p \in S_N^J\right\},$$

of separating forms (often called numéraire) contains a positive element denoted by $k$ and normalized to $k(0) = 1$. We put $K_N^J = H_N^J \cap O$, where $O := \{h \in L^2_{\mathscr{A}_+}/h(0) = 1\}$. This set is non-empty and also weakly closed. Recall that considering $TP_N \cup \mathscr{I}_N^J$, the set $K_N^J$ is included in the weakly compact set $\mathscr{B} = \{h \in L^2_{\mathscr{A}_+}/\|h\|_{\mathscr{A}} \leq M\}$. Thus, the set $K_N^J$ is weakly compact.

The next step consists in showing that every investment of $H_N^J$ is independent of the path, that is $\tilde{H}_N^J = H_N^J$. To do that, we fix an investment horizon $N$, and a finite subset $J$ of $I$ containing the exchange assets occurring before $N$. Let us consider two states of the world $\omega$ and $\omega'$, such that $|\omega'|_N = |\omega|_N$. First, we prove that there exists $h^* \in K_N^J$ such that $h^*(N,\omega) = h^*(N,\omega')$.

To see this point, we first show that the mapping $h \to h(N,\omega)$ is weakly continuous on $L^2_{\mathscr{A}}$, where $\omega$ is fixed in $\Omega$. Let $\psi$ be the function equal to 1 in $(N, \omega)$ and zero otherwise, $\psi$ belongs obviously to $L^2_{\mathscr{A}}$. Let $(h_p)_p$ be a sequence converging weakly to some $h$, then $\langle h_p, \psi \rangle_{\mathscr{A}} = 2^N h_p(N,\omega)$ converges to $\langle h, \psi \rangle_{\mathscr{A}} = 2^N h(N,\omega)$. Recalling that the set $K_N^J$ is weakly compact and non-empty, $q_N^J = \inf_{h \in K_N^J} \frac{h(N,\omega)}{h(N,\omega')}$ exists and is reached. We will prove that $q_N^J = 1$. Otherwise, suppose that $q_N^J > 1$. We denote by $\varphi_N^J$ the asset paying at time $N$, $q_N^J$ in state $\omega'$ and $-1$ in state $\omega$, and nothing else. We do not assume that such an asset is feasible and belongs to the investment set.

Let $h \in K_N^J$, $\langle h, \varphi_N^J \rangle_A = 2^N(h(N,\omega')q_N^J - h(N,\omega)) \leq 0$, by definition of $q_N^J$; thus, we get that $\varphi_N^J \in (K_N^J)^0$. In order to work in the finite dimension space $\mathscr{R}_N$, we consider the isometrics by $\Upsilon$ of our sets. Applying well-known results on negative polar sets, we find that,

$$\begin{aligned}\left(\Upsilon\left(K_N^J\right)\right)^0 &= \left(\left(\Upsilon\left(S_N^J\right)\right)^0 \cap \Upsilon(O)\right)^0 = \left(\left(\Upsilon\left(S_N^J\right)\right)^0\right)^0 + (\Upsilon(O))^0 \\ &= \Upsilon\left(S_N^J\right) + (\Upsilon(O))^0.\end{aligned}$$

Hence, there exists $x_N$ such that $\Upsilon(x_N) \in (\Upsilon(O))^0$ satisfies $\varphi_N^J - x_N \in S_N^J$, that is to say $\varphi_N^J - x_N$ belongs to the set of investments build from $J$ ending before $N$. We consider a consumer $i$ with life time $T_i$ greater than $N$, and equilibrium consumption equal to $c_i^*$. Suppose that she/he adds $\varepsilon(\varphi_N^J - x_N)$ to her/his equilibrium portfolio.

Thanks to $\varepsilon\varphi_N^J$, the consumer $i$ consumes at time $t = N$, $c_i^*(N,\omega) - \varepsilon$ in state $\omega$, $c_i^*(N,\omega') + \varepsilon q_N^J$ and her/his consumption does not change in the other state of the world. Thus, the utility of consumer $i$ at time $N$ will vary of $-\varepsilon u'_{i,N}(c_i^*(N,\omega)) + o(\varepsilon)$ in state $\omega$, and of $+\varepsilon q_N^J u'_{i,N}(c_i^*(N,\omega')) + o(\varepsilon)$ in state $\omega'$, and zero otherwise. The global variation of the utility is then equal to:

$$\varepsilon\left(\frac{\beta}{4}\right)^N \left[-u'_{i,N}\left(c_i^*(N,\omega)\right) + q_N^J u'_{i,N}\left(c_i^*(N,\omega')\right)\right] + o(\varepsilon).$$

Now, thanks to $-\varepsilon x_N$ the utility of consumer $i$ varies at time $t$ and in state $\omega$ from $-\varepsilon x_N(t,\omega) u'_{i,t}(c_i^*(t,\omega)) + o(\varepsilon)$. The global variation from the utility is then equal to:

$$\begin{aligned}&-\varepsilon \sum_{t=0}^{T_i} \beta_i^t \int_{\omega \in \Omega} \mathrm{d}P(\omega)\, x_N(t,\omega)\, u'_{i,t}\left(c_i^*(t,\omega)\right) + o(\varepsilon) \\ &= -\varepsilon \langle z, x_N \rangle_{\mathscr{A}} + o(\varepsilon),\end{aligned}$$

where $z(t,\omega)=(\beta_i/2)^t u'_{i,t}(c_i^*(t,\omega))$. Notice that the utility $u_{i,0}$ is strictly increasing and concave, so that the function $u'_{i,0}$ is positive. Hence, we get that $z_0/u'_{i,0}(c_i^*(0))=1$, and thus $[z/u'_{i,0}(c_i^*(0))]\in O$. We deduce that $\langle z,x_N\rangle_{\mathscr{A}}=\langle \Upsilon(z),\Upsilon(x_N)\rangle_{\mathscr{R}}\le 0$.

As $c_i^*$ is the optimal consumption, the total variation of the utility should be non-positive. Thus,

$$\varepsilon\left(\frac{\beta}{4}\right)^N\left[-u'_{i,N}\left(c_i^*(N,\omega)\right)+q_N^J u'_{i,N}\left(c_i^*(N,\omega')\right)\right]$$
$$-\varepsilon\langle z,x_N\rangle_{\mathscr{A}}+o(\varepsilon)\le 0.$$

As $q_N^J$ is strictly greater than 1 and $\langle z,x_N\rangle_{\mathscr{A}}$ is non-positive, we obtain that,

$$u'_{i,N}\left(c_i^*(N,\omega')\right)<u'_{i,N}\left(c_i^*(N,\omega)\right).$$

Moreover, the function $u_{i,N}$ is strictly concave and

$$c_i^*(N,\omega')>c_i^*(N,\omega).$$

This last inequality is true for every agent $i$, so that summing over all consumers living at this date (this makes sense because the consumers are assumed to be infinitely small), we find that the total consumption at date $N$ and in state $\omega$ is strictly greater than the demand at time $N$ and in state $\omega$. Thus, the economy should produce at time $N$, strictly more in state $\omega'$ than in state $\omega$, that is $S_N(\omega')>S_N(\omega)$. This contradicts the binomial evolution of the underlying asset and proves that $\inf_{h\in K_N^J}\frac{h(N,\omega)}{h(N,\omega')}\le 1$. Using the same line of arguments, we show the converse inequality and finally, $\inf_{h\in K_N^J}\frac{h(N,\omega)}{h(N,\omega')}=1$. Hence, with $J$, $N$, $\omega$, such that $|\omega'|_N=|\omega|_N$, fixed, there exists $h^*\in K_N^J$, such that $h^*(N,\omega)=h^*(N,\omega')$. Next, we prove that the equality $H_N^J=\tilde{H}_N^J$ holds.

Recalling that $h^*\in K_N^J$, we get that $h^*\in H_N^J$. Moreover, the exchange securities $E_0^{\omega_N,\omega'_N}$ and $E_0^{\omega'_N,\omega_N}$ belong to $S_N^J$. We obtain that,

$$-p_0^{\omega_N,\omega'_N}h^*(0)-2^N h^*(N,\omega)+2^N h^*(N,\omega')\le 0$$

and

$$-p_0^{\omega'_N,\omega_N}h^*(0)+2^N h^*(N,\omega)-2^N h^*(N,\omega')\le 0.$$

As $h^*(N,\omega)=h^*(N,\omega')$, and $h^*(0)=1$, we get that $p_0^{\omega_N,\omega'_N}$ and $p_0^{\omega'_N,\omega_N}$ are non-negative. We call $c_1$ (respectively $c'_1$) the buying (respectively selling) transaction costs on the exchange assets. Hence, there exists $x$ such that paying $x[(1+c_1)/(1-c_1)]$, we can exchange \$1 at time $N$ from state $\omega$ to state $\omega'$ and $p_0^{\omega_N,\omega'_N}=x[(1+c_1)/(1-c_1)]$. Now, receiving $x[(1-c'_1)/(1+c_1)]$ at time zero, the consumer exchanges \$1 at time $N$ from state $\omega'$ to state $\omega$, and thus $-p_0^{\omega'_N,\omega_N}=x[(1-c'_1)/(1+c_1)]$. We obtain that $p_0^{\omega_N,\omega'_N}=-[(1+c_1)/(1-c_1)]^2 p_0^{\omega'_N,\omega_N}$. As $p_0^{\omega_N,\omega'_N}$ and $p_0^{\omega'_N,\omega_N}$ are non-negative numbers, we get that $p_0^{\omega_N,\omega'_N}$ and $p_0^{\omega'_N,\omega_N}$ are equal to zero.

Let $h \in H_N^J$, recalling that $E_0^{\omega_N,\omega'_N}$ and $E_0^{\omega'_N,\omega_N}$ belongs to $S_N^J$, we get that,

$$-2^N h(N,\omega) + 2^N h(N,\omega') \leq 0 \text{ and } 2^N h(N,\omega) - 2^N h(N,\omega') \leq 0.$$

Thus, for every $J$, $N$, $\omega$, such that $|\omega'|_N = |\omega|_N$ and $h \in H_N^J$, we obtain that $h(N,\omega) = h(N,\omega')$. We proved then that for every finite subset $J$ containing the exchange assets and for every investment horizon $N$, the equality $H_N^J = \tilde{H}_N^J$ holds. Recalling that the set $\tilde{H}_N^J$ is non-empty, we get that Assumption 2.1 is satisfied.

*3.3.3. End of proof of Theorem 3.1*

We now can apply the result of Theorem 2.2 to our model and there exists two real numbers $r$ and $\pi^*$, with $r \geq r^p$ and $0 < \pi^* < 1$, such that for every investment $i \in I$,

$$\sum_{n=0}^{T_i} \frac{1}{(1+r)^n} E^*\left[\Phi_i(n,\cdot)\right] \leq 0$$

where $E^*$ is the expectation under $P^*$, which is the unique probability making the coordinate mappings independent and identically distributed and such that $P^*(X_n = u) = \pi^*$ and $P^*(X_n = d) = 1 - \pi^*$. Applying this result to the underlying asset, and more precisely to $AV^{0,t,\Omega_t}$ and $VA^{0,t,\Omega_t}$, we get that,

$$\frac{1-c'}{1+c} \leq \frac{1}{(1+r)^t} \sum_{j=0}^{t} C_t^j (u\pi^*)^j (d(1-\pi^*))^{t-j} \leq \frac{1+c}{1-c'}.$$

Thus, we obtain that,

$$\frac{1-c'}{1+c} \leq \left(\frac{u\pi^* + d(1-\pi^*)}{1+r}\right)^t \leq \frac{1+c}{1-c'}.$$

We deduce that, $\frac{1}{1+r}(u\pi^* + d(1-\pi^*)) = 1$. Otherwise, if $0 \leq 1/(1+r)(u\pi^* + d(1-\pi^*)) < 1$, taking the limit, we will get that $c' \geq 1$, which is impossible by assumption. The probability defined by the transition probability $\pi^*$ is the Cox–Ross–Rubinstein's probability.

Applying Theorem 2.1 to the interest rate $r_e$, and more precisely to $TE^{0,1,\Omega_1}$, we get that $1 - \frac{1}{1+r}(\pi^* + (1-\pi^*))(1+r_e) \leq 0$. Finally, we obtain that $r_p \leq r \leq r_e$.

Now applying Theorem 2.2 to the options $AO^{t,T,k,\Omega_t}$ and $VO^{t,T,k,\Omega_t}$, we get that,

$$-\frac{1}{(1+r)^t} C(T,k,S_t) + \frac{1}{(1+r)^{t+T}} \sum_{\omega^{t+T} \in \Omega_{t+T}} \pi^{*|\omega|_{t+T}} (1-\pi^*)^{t+T-|\omega|_{t+T}}$$

$$\times (S_{t+T} - kS_t)^+ I_{\omega^t \in \Omega_t} \leq 0,$$

$$\frac{1}{(1+r)^t}C'(T,k,S_t) - \frac{1}{(1+r)^{t+T}} \sum_{\omega^{t+T}\in\Omega_{t+T}} \pi^{*|\omega|_{t+T}}(1-\pi^*)^{t+T-|\omega|_{t+T}}$$
$$\times (S_{t+T}-kS_t)^+ I_{\omega^t\in\Omega_t} \leq 0.$$

Remarking that, $\sum_{\omega^t\in\Omega_t}\pi^{*|\omega|_t}(1-\pi^*)^{t-|\omega|_t} = \sum_{j=0}^{t} C_t^j \pi^{*j}(1-\pi^*)^{t-j} = (\pi^* + (1-\pi^*))^t = 1$, we obtain that,

$$\sum_{\omega^{t+T}\in\Omega_{t+T}} \pi^{*|\omega|_{t+T}}(1-\pi^*)^{t+T-|\omega|_{t+T}}(S_{t+T}-kS_t)^+ I_{\omega^t\in\Omega_t}$$
$$= \sum_{j=0}^{T} \pi^{*j}(1-\pi^*)^{T-j} C_T^j \left(S_t u^j d^{T-j} - kS_t\right)^+.$$

Thus, denoting by:

$$\mathrm{CRR}(T,k,S_t) = \frac{1}{(1+r)^T}\sum_{j=0}^{T} C_T^j \pi^{*j}(1-\pi^*)^{T-j}\left(S_t u^j d^{T-j} - kS_t\right)^+,$$

we get that,

$$C'(T,k,S_t) \leq \mathrm{CRR}(T,k,S_t) \leq C(T,k,S_t).$$

■

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