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Incomplete markets, transaction costs and liquidity effects

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An agent's optimization problem of the expected terminal wealth utility in a trinomial tree economy is solved. At each transaction date, the agent can trade in a riskless asset, a primitive asset subject to constant proportional transaction costs, and a contingent claim characterized by some parameter κ whose bid and ask price is defined by allowing for different equivalent martingale measures. In addition to the classical portfolio choice problem, the characteristic of the contingent claim κ is determined endogenously in the optimization problem. Under suitable conditions, it is proved that the optimal demand of the agent in the primitive risky asset is zero independently of the choice of the terminal wealth utility function: the agent prefers not to trade in the asset subject to transaction costs, which prevents the market from being complete, rather than trading in both assets. Next, the optimal choice of the contingent claim is characterized and the results are applied to European call and put options with fixed maturity and varying exercise price κ.

Keywords: contingent claim, incomplete markets, transaction costs

1. INTRODUCTION

During the last decade, the exchange volumes in derivative assets have been growing very quickly and tend to be much more important than those in primitive assets. A usual justification of the latter is that transaction costs on primitive assets are much higher than those on contingent claims. In this paper, using a utility maximization argument, we provide a rigorous justification of this empirical fact, in the limit case where there are no transaction costs on the contingent claims.

We will study the limit case of the real situation on the market, assuming that primitive assets are subject to transaction costs, while derivative assets are not. The decision problem of the agent is:

- either not to trade in the primitive risky assets, which are subject to transaction costs, and therefore make decisions in an incomplete market;

- or to trade in the primitive risky assets, which are subject to transaction costs, and therefore form his or her decision in a complete market.

In this paper, we prove that, if the agent can choose the derivative assets optimally, his/her demand in the primitive assets is zero. Such a result is immediate in a financial market with finite degree of incompleteness, if all options are available for trading. Indeed, in such a framework, the underlying asset is clearly redundant and the problem is irrelevant. However, in practice, only a few options are available for trading and, therefore, the underlying asset cannot be considered as a redundant asset. In order to capture these two ideas, we study the simplest framework where there is one degree of incompleteness and only one option available for trading at each transaction date.

In our model, the uncertainty is described by a trinomial multiperiodic event tree. There are two primitive assets available for trading in the economy: a nonrisky asset and a risky one. The latter is subject to constant proportional transaction costs. At any transaction date, besides those primitive assets, there is a nontrivial contingent claim, characterized by a parameter κ, which is also available for trading. The agent chooses optimally his/her investment in the primitive assets and replaces his/her investment in the contingent claim which prevailed in the previous period by that of the current one. The characteristic of the contingent claim is determined endogenously in the optimization problem so that the same option could be traded at two successive transaction dates.

The returns induced by the contingent claim for the agent (whether his/her position is short or long) are defined by the individual optimality conditions. This is an important feature of the model since such conditions are consistent with the existence of a bid–ask spread. The absence of transaction costs on the contingent claim is an important limitation of the model which cannot be overcome in the present analysis. Nevertheless our model can be seen as the limit of a model in which the transaction costs on the risky asset are significantly higher than those on the contingent claims.

The choice of a trinomial model with two primitive assets and a unique contingent claim is only motivated by the simplicity of the presentation. Such a model is the simplest model for which the primitive risky asset, subject to transaction costs, is necessary for the market to be complete and which makes the agent effectively face the choice between incompleteness and transaction costs.

Note that, at each date, the agent can trade only one derivative asset – this captures the idea that only a finite number of derivatives are available for trading. However, this derivative asset is optimal for the agent. Therefore, we are studying a 'best case' scenario, as if the unique derivative asset traded by the agent can be chosen optimally.

The main results of the paper are the following. Under suitable regularity conditions, we prove that the optimal policy of the agent is not to trade in the primitive risky asset which is subject to transaction costs and therefore the agent prefers incompleteness to transaction costs. An important feature of our result is that it holds independently of the choice of the terminal wealth utility function.

Moreover, it turns out that the contingent claims characterized by different parameters κ are not equivalent from the viewpoint of terminal wealth utility maximization. We thus provide existence and uniqueness results of the optimal contingent claim in the power utility case.

The previous results are consistent with empirical observations. Indeed it is well known that the traded volumes are much more important on options than on stocks. Our result is a limit result, in some sense, since the agent does not trade the primitive risky asset in our model. Moreover, it is observed that only a few options are traded in the market although many others are available for trading. A possible interpretation of our model is that the most liquid option corresponds to the optimal one produced by the agent maximization problem.

Our paper deserves also two comments in relation with the security design literature. There are two main reasons which motivate financial market innovation: (i) regulation agency costs and tax codes on one hand and (ii) the use of securities for the hedging of substantive risks – see Duffie and Rahi (1995). In our knowledge, these reasons are always studied separately in the literature. In our model, contingent claims reduce the amount of transaction costs paid by the agent but do not induce any increase of the spanning. It is then difficult to justify the usual approach which decorrelates the two effects and considers them separately. Our paper has also a concrete application: a very important topic in the empirical literature is to know whether financial innovation increases social welfare or not. Empirical studies are based, in general, on the increase of the market volatility, taken as a symptom of reduced social welfare – see Zapatero (1994). Our results lead us to suggest the amount of transaction costs paid by the agents as an alternative measure. Then, quite surprisingly, a reduction of the trading volume on the non-purely financial assets would be the sign of an *increase* in social welfare.

The paper is organized as follows. Section 2 presents the general framework of the paper and describes the agent optimization problem. Section 3 contains our main result concerning the optimal policy of the agent in the primitive risky asset which is subject to transaction costs. Section 4 extends the results of Section 3 to the case where the contingent claim payoff function is not continuously differentiable, as in the important case of a family of European options. The interest of Section 3 is more theoretical, but it is an essential preliminary to Section 4. Section 5 provides an example of 'regular economy' which is a sufficient condition for the results of Sections 3 and 4 to hold. Finally, Section 6 is devoted to the optimal policy of the agent in the contingent claim.

2. THE GENERAL FRAMEWORK

2.1 Tradable assets

The framework of this paper is very close to a classical event tree economy. There are $T+1$ transaction dates $t = 0, \ldots, T$ and a finite number of states.

Tradable assets in the economy are decomposed in primitive assets and contingent claims. There are two primitive assets:

- a nonrisky asset with associated interest rate r, deterministic and normalized to 0;
- a risky asset subject to proportional transaction costs. At any time $t = 0,\ldots,T$, the asset bid price is $S_t(1+h)^{-1}$ and its ask price is $S_t(1+k)$ where h and k are strictly positive constants, independent of time and state.

We introduce the function:

$$\tau(x) = x(1+k)1\{x \geq 0\} + \frac{x}{1+h}1\{x \leq 0\} \tag{2.1}$$

so that the primitive risky asset bid (resp. ask) price process can be written in the form $\{-\tau(-S_t),\ t = 0,\ldots,T\}$ (resp. $\{\tau(S_t),\ t = 0,\ldots,T\}$). The 'average price' $\{S_t,\ 0 \leq t \leq T\}$ dynamics are described by the following event tree. We denote by s_t the state revealed at time t and by $e_t = (s_1,\ldots,s_t)$ the sample path until time t, i.e. the state at time t. Let $\mathcal{E}_t$ be the set of possible states at time t. Given $e_t \in \mathcal{E}_t$, three states can prevail at time $t+1$:

$$s_{t+1}(e_t) \in S(e_t) = \{u_{t+1}(e_t),\ m_{t+1}(e_t),\ d_{t+1}(e_t)\}$$

where $u_{t+1}(e_t) > m_{t+1}(e_t) > d_{t+1}(e_t) > 0$. Therefore there are 3^t possible states in $\mathcal{E}_t$. The risky asset 'average price' at time 0 is $S_0 = 1$ and evolves according to:

$$S_t = s_t S_{t-1} = \prod_{i=1}^{t} s_i,\ t = 1,\ldots,T$$

We also introduce the projection operator:

$$\begin{aligned} \pi_t : \mathcal{E}_T &\to \mathcal{E}_t \\ e_T = (s_1,\ldots,s_T) &\mapsto \pi_t e_T = (s_1,\ldots,s_t) \end{aligned}$$

and we denote by $\mathcal{E}(e_t)$ the set of sample paths $e_T \in \mathcal{E}_T$ which can be attained from the state e_t, i.e.

$$\mathcal{E}(e_t) = \{e_T \in \mathcal{E}_T : \pi_t e_T = e_t\}$$

There is a probability measure P defined on $\mathcal{E}_T$ with $P(e_T) > 0$ for any $e_T \in \mathcal{E}_T$. We denote by P_t the probability measure induced by P on $\mathcal{E}_t$:

$$P_t(e_t) = \sum_{e_T \in \mathcal{E}(e_t)} P(e_T) \qquad \text{for any } e_t \in \mathcal{E}_t$$

The model described above differs from a classical symmetric event tree economy with spanning number 3 (in the sense of Duffie and Huang, 1985) in that the primitive risky asset is subject to transaction costs.

At any time $t = 0,\ldots,T-1$ and any state $e_t \in \mathcal{E}_t$, in addition to the primitive assets described above, there is a contingent claim chosen in a family of contingent claims indexed by a real parameter κ. For each κ the contingent claim is characterized by its terminal payoff function $\varphi(.,\kappa)$ mapping $\mathcal{E}_T$ into $\mathbb{R}$. The family of contingent claims will be denoted by

$$C(e_t) = \{\varphi(.,\kappa),\ \kappa \in \mathbb{K}(e_t)\}$$

where $\mathbb{K}(e_t)$ is a closed interval of $\mathbb{R}$ with nonempty interior.

In practice, two major types of derivative securities are traded: options and futures. In our framework, any future contracts together with the nonrisky asset are sufficient to duplicate the primitive asset. This implies obviously that the primitive asset will not be traded by the agent and therefore, the futures case is not relevant for our analysis since our results hold trivially.

In the European options example, $C(e_t)$ is indexed by the exercise price, i.e. $C(e_t) = \{\varphi(.,K) = (S_T(.) - K)^+,\ K \in \mathbb{K}(e_t)\}$.

Notice that the terminal payoff function $\varphi(e_T\kappa)$ depends on the state e_T and not only on the primitive asset price S_T as in the case of European options. Therefore our family of contingent claims entails also exotic ones.

In order to avoid the case where the primitive risky asset can be duplicated by the contingent claims and the nonrisky asset, inducing trivially a zero-demand on the risky asset from the considered agent, we will assume:

(i) at any time $t = 0,\ldots,T-1$ and any state $e_t \in \mathcal{E}_t$, only one contingent claim $\varphi(e_T\kappa_t(e_t))$ can be traded and is converted automatically in cash at the following date according to its price (which will be discussed later). Nevertheless the agent can chose optimally $\kappa_t(e_t)$ in the set $\mathbb{K}(e_t)$;

(ii) at any time $t = 0,\ldots,T-1$ and any state $e_t \in \mathcal{E}_t$ the set $\mathbb{K}(e_t)$ is such that for any $\kappa_t(e_t) \in \mathbb{K}(e_t)$ the associated contingent claim and the nonrisky asset are not sufficient to duplicate the primitive risky asset.

For a better understanding of (ii), let us consider again the example of European call options. Suppose that the agent is allowed to trade at any time $t = 0,\ldots,T-1$ and any state $e_t \in \mathcal{E}_t$ in a European call option with exercise price $K(e_t) \leq \inf\{S_T(e_T),\ e_T \in \mathcal{E}(e_t)\}$. Then this provides a strategy in the contingent claims and the nonrisky asset which duplicates the final payoff of the risky primitive asset and avoids the transaction costs. Thus, in the options case, (ii) says that $\mathbb{K}(e_t)$ is restricted to be included in the open interval $(\inf\{S_T(e_T),\ e_T \in \mathcal{E}(e_t)\};\ \sup\{S_T(e_T),\ e_T \in \mathcal{E}(e_t)\})$ for all $t = 0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$.

This condition is essential in our finite state space framework: it would have been innocuous if the state space was unbounded.

2.2 Valuation rule of the contingent claims

The contingent claims valuation raises two problems. The first is related to the presence of transaction costs on the underlying asset price which prevents one from defining the contingent claim price from classical arbitrage arguments. We use an admissible pricing rule in the sense of Jouini and Kallal (1995): consider a process $\{x_t,\ t = 0,\ldots,T\}$ lying between the bid and the ask price processes, then any equivalent probability measure under which $\{x_t,\ t = 0,\ldots,T\}$ is a martingale induces an admissible pricing rule. We define the pricing rule implied by the choice of the average price $\{S_t, t = 0...T\}$ as a process lying between the bid and the ask price.

The second problem is related to market incompleteness: there are two primitive assets and three states of the world at each date $t = 0,...,T-1$. Let $C_t(e_t,\kappa_t(e_t))$ be the price of the contingent claim defined by $\kappa_t(e_t)$ at date t and in the node e_t. Then, at each node e_t, there exist transition probabilities $(q_t^u(e_t), q_t^m(e_t), q_t^d(e_t))$ such that:

$$\begin{cases} q_t^u(e_t) > 0,\ q_t^m(e_t) > 0,\ q_t^d(e_t) > 0 \\ q_t^u(e_t) + q_t^m(e_t) + q_t^d(e_t) = 1 \\ q_t^u(e_t)S_t(e_t)u_{t+1}(e_t) + q_t^m(e_t)S_t(e_t)m_{t+1}(e_t) + q_t^d(e_t)S_t(e_t)d_{t+1}(e_t) = S_t(e_t) \\ q_t^u(e_t)C_{t+1}^u(e_t,\kappa_t(e_t)) + q_t^m(e_t)C_{t+1}^m(e_t,\kappa_t(e_t)) + q_t^d(e_t)C_{t+1}^d(e_t,\kappa_t(e_t)) = C_t(e_t,\kappa_t(e_t)) \end{cases} \tag{2.2}$$

where $C_{t+1}^s(e_t,\kappa_t(e_t))$ is the price of the same contingent claim with exercise price $\kappa_t(e_t)$ at time $t+1$ in the state $(e_t,s) \in \mathcal{E}_{t+1}$ for $s \in \{u_{t+1}(e_t), m_{t+1}(e_t), d_{t+1}(e_t)\}$. Note that this transition probability depends on the contingent claim price process. Conversely, given a system of transition probabilities, satisfying the three first equations of (2.2), the fourth equation together with the terminal condition

$$C_T(e_T,\kappa_T(e_T)) = \varphi(e_T,\kappa_T(e_T))$$

determines the contingent claim price process. However, the three first equations of (2.2) do not determine a unique system of transition probabilities which allows for the existence of a bid–ask spread for the contingent claim. But, in equilibrium and for a given agent, there is only one price for this claim and this unique price is defined by only one transition probability which then satisfies the system (2.2). The price $C_t(e_t,\kappa_t(e_t))$ is an ask price if the agent is a buyer in the contingent claim at this node and a bid price if he/she is a seller.

In the sequel, we will denote by Q the probability on the set of terminal states defined by the transition probabilities $(q_t^u(e_t), q_t^m(e_t), q_t^d(e_t))$. It is clear that under this probability the primitive price process is a martingale. Considering the process $\tilde{C}$ whose returns in state e_t coincide with those of the contingent claim effectively traded by the agent, i.e.

$$\frac{\tilde{C}_{t+1}(e_t,s_{t+1}(e_t))}{\tilde{C}_t(e_t)} = \frac{C_{t+1}((e_t,s_{t+1}(e_t)),\kappa_t(e_t))}{C_t(e_t,\kappa_t(e_t))}$$

then $\tilde{C}$ is also a martingale under the probability measure Q. Note that, at each date and each node, the process $\tilde{C}$ corresponds to a different contingent claim, the best one for the agent for this date and this node. Thus, this process is the relevant one in order to study the optimal decision problem that we will describe now.

2.3 The agent problem

This paper is interested in the optimal portfolio choice of an agent whose preferences are represented by a Von Neuman–Morgenstern utility function with terminal wealth utility function u assumed to be continuously differentiable, increasing and strictly concave. Decisions are taken at the discrete dates $t = 0,1,\ldots,T-1$. The agent is initially endowed by one unit of cash ($W_0 = 1$) and ends up at the final date T with a terminal wealth converted in cash. At each date $t = 0,\ldots, T-1$ and state $e_t \in \varepsilon_t$ the agent

- solds his/her past position in contingent claims,
- and decides to invest an amount $\theta_t^S(e_t)$ in the primitive risky asset, $\theta_t^C(e_t)$ in the contingent claim $\varphi(.,\kappa_t(e_t))$ and the remaining wealth in the nonrisky asset.

Since the strategy adopted by the agent is self-financed and since all the consumption takes place at the final date T, the total gain (or cost) at date t, if e_{t-1} is the state at date $t-1$ and e_t the state at date t is:

$$\theta_{t-1}^C(e_{t-1})\frac{C_t(e_t,\kappa_{t-1}(e_{t-1}))}{C_{t-1}(e_{t-1},\kappa_{t-1}(e_{t-1}))} - \theta_t^C(e_t) - \tau\left(\theta_t^S(e_t) - \theta_{t-1}^S(e_{t-1})\frac{S_t e_t}{S_{t-1}(e_{t-1})}\right)$$

Therefore, the budget constraint of the agent written in all the nodes of the tree provides the following terminal wealth for any final state $e_T \in \varepsilon_T$:

$$\begin{aligned} W_T(e_T) = 1 + &\sum_{t=1}^{T} \theta_{t-1}^C(e_{t-1})\left[\frac{C_t(e_t,\kappa_{t-1}(e_{t-1}))}{C_{t-1}(e_{t-1},\kappa_{t-1}(e_{t-1}))} - 1\right] \\ &- \sum_{t=0}^{T} \tau(\theta_t^S(e_t) - \theta_{t-1}^S(e_{t-1})s_t) \end{aligned} \tag{2.3}$$

where $e_t = \pi e_T$ is the state at date t on the sample path e_T, $s_t = S_t/S_{t-1}$ is the state revealed at date t on the sample path e_T and $(\theta_{-1}^S, s_0) \equiv (0,1)$ by convention. Notice that since all the wealth of the agent is converted in cash at the final date T we have $\theta_T^S \equiv 0$.

Finally, introducing the set A of all possible controls $(\theta_t^S(e_t), \theta_t^C(e_t), \kappa_t(e_t)) \in \mathbb{R}^2 \times \mathbb{K}(e_t)$ for $t = 0,\ldots,T-1$ and $e_t \in \varepsilon_t$, the optimal decision problem of the agent is

$$\sup_A E[u(W_T)] \tag{2.4}$$

Thus the decision problem of the agent not only determines the optimal demand in the existing assets, but also the 'best' contingent claim through its characteristic $\kappa_t(e_t)$. Such an optimal contingent claim can be seen as the one chosen by an 'invisible hand' which has to fix a unique contingent claim to be traded by the agent. The choice of an optimal contingent claim can be seen as an economic justification to the liquidity effects observed in practice on contingent claims markets.

3. OPTIMAL POLICY IN THE PRIMITIVE RISKY ASSET

In this section, we derive the first-order conditions in the controls of the optimal decision problem (2.4) and we provide a characterization of the optimal investment in the primitive risky asset which holds independently of the agent utility function. In order to simplify the presentation, we assume that the terminal payoff function of the contingent claim $\varphi(e_T,\kappa)$ is continuously differentiable in κ for all possible values of κ. Of course this restriction rules out the typical example of European options; we will show in the next section that our results extend easily to more general terminal payoff functions. We shall use the notation

$$\mu(e_T) = P(e_T)u'(W_T(e_T)) \qquad \text{for all } e_T \in \mathcal{E}_T \tag{3.1}$$

and we introduce the $\mathbb{R}^{3^T}$ vector μ whose components are the $\mu(e_T)$ classified according to the lexicographic order of the final states e_T. We shall denote by e_T^i the ith element $e_T \in \mathcal{E}_T$ in the lexicographic order.

From the terminal wealth expression (2.3) it is clear that, for fixed $\kappa_t(e_t)$, the terminal wealth is linear in the optimal investment in the contingent claim $\theta_t^C(e_t)$ and piecewise linear in the optimal investment of the primitive risky asset $\theta_t^S(e_t)$ for any $t = 0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$. From the concave feature of the utility function u, this shows that for given characteristics of the contingent claim $\kappa_t(e_t)$ the optimal choice in $\theta_t^C(e_t)$ and $\theta_t^S(e_t)$ is characterized by the first-order conditions. Unfortunately these arguments do not extend to variable κ and we therefore consider the following approach. Suppose that the supremum of the terminal wealth expected utility is attained in an interior point of $\mathcal{A}$. Then, from the regularity of the utility function, it satisfies the first-order conditions. Next, we prove under regularity conditions and in the power utility case that the first-order conditions admit a unique solution (see Section 6) and, for the European call options, we will justify our interior point condition (see Section 5).

The first-order conditions in $\theta_t^C(e_t)$ for a given $t = 0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$ are easily obtained by differentiating the objective function:

$$\frac{\partial}{\partial\theta_t^C(e_t)} E[u(W_T)] = \sum_{e_T \in \mathcal{E}(e_t)} \left[\frac{C_{t+1}(\pi_{t+1}e_T,\kappa_t(e_t))}{C_t(e_t,\kappa_t(e_t))} - 1\right]\mu(e_T)$$

$$= \sum_{s_{t+1} \in S(e_t)} \left[\frac{C_{t+1}((e_t,s_{t+1}),\kappa_t(e_t))}{C_t(e_t,\kappa_t(e_t))} - 1\right] \sum_{e_T \in \mathcal{E}(e_t,s_{t+1})} \mu(e_T)$$

Define the $\mathbb{R}^{3^T}$ vector $A^C(e_t)$ whose components are

$$A^C(e_t) = \begin{cases} 0 & \text{if } e_T^i \notin \mathcal{E}(e_t) \\ C_{t+1}(\pi_{t+1}e_T^i,\kappa_t(e_t)) - C_t(e_t,\kappa_t(e_t)) & \text{if } e_T^i \in \mathcal{E}(e_t) \end{cases} \tag{3.2}$$

Then the first-order conditions in the portfolio choice in the contingent claim can be written in:

$$< A^C(e_t), \mu > = 0 \text{ for } t = 0,\ldots,T-1 \text{ and } e_t \in \varepsilon_t \tag{3.3}$$

where $< .,. >$ is the Euclidean vector inner product in $\mathbb{R}^{3^T}$. This provides $(3^T - 1)/2$ first-order conditions.

Next we focus on the first-order conditions in the contingent claim characteristic $\kappa(e_t)$ for $t = 0,\ldots,T-1$ and $e_t \in \varepsilon_t$. By simple differentiation of the expected terminal wealth utility, we get:

$$\frac{\partial}{\partial \kappa_t(e_t)} E\,[u(W_T)] = \theta_t^C(e_t) \sum_{e_T \in \varepsilon(e_t)} \frac{\partial}{\partial \kappa_t(e_t)} \left\{ \frac{C_{t+1}(\pi_{t+1} e_T, \kappa_t(e_t))}{C_t(e_t, \kappa_t(e_t))} \right\} \mu(e_T)$$

$$= \theta_t^C(e_t) \sum_{s_{t+1} \in S(e_t)} \frac{\partial}{\partial \kappa_t(e_t)} \left\{ \frac{C_{t+1}((e_t, s_{t+1}), \kappa_t(e_t))}{C_t(e_t, \kappa_t(e_t))} \right\} \sum_{e_T \in \varepsilon(e_t, s_{t+1})} \mu(e_T)$$

For a given κ, following Jouini and Kallal (1995) the price process $C_t(e_t, \kappa)$ is a martingale under some probability measure Q and we have then $C_t(e_t, \kappa) = E_t^Q[\varphi(e_T, \kappa) \mid e_t]$. It is then clear that $C_t(e_t, \kappa)$ is C^1 relative to κ and we can define:

$$\Delta_t(e_t, \kappa) = \frac{\partial}{\partial \kappa} C_t(e_t, \kappa) = E_t^Q\left[\frac{\partial}{\partial \kappa} \varphi(e_T, \kappa)\right] \tag{3.4}$$

and the $\mathbb{R}^{3^T}$ vector $A^\kappa(e_t)$ whose components are

$$A^\kappa(e_t) = \begin{cases} 0 & \text{if } e_T^i \notin \varepsilon(e_t) \\ \Delta_{t+1}(\pi_{t+1} e_T^i, \kappa_t(e_t)) - \Delta_T(e_t, \kappa_t(e_t)) & \text{if } e_T^i \in \varepsilon(e_t) \end{cases} \tag{3.5}$$

Then, using the first-order conditions in the portfolio choice in the contingent claim (3.3), the first-order conditions in the contingent claim characteristic can be written in:

$$\theta_t^C(e_t) < A^\kappa(e_t), \mu > = 0 \text{ for } t = 0,\ldots,T-1 \text{ and } e_t \in \varepsilon_t \tag{3.6}$$

which provides $(3^T - 1)/2$ additional first-order conditions. Now suppose that the optimal portfolio choice in the contingent claim $\theta_t^C(e_t) = 0$ for some $t \in \{0,\ldots,T-1\}$ and $e_t \in \varepsilon_t$. Then $\theta_t^C(e_t) = 0$ solves the first-order condition in $\theta_t^C(e_t)$ for any contingent claim in the family $C(e_t)$, i.e.

$$\sum_{s_{t+1} \in S(e_t)} \left[\frac{C_{t+1}((e_t, s_{t+1}), \kappa(e_t))}{C_t(e_t, \kappa(e_t))} - 1\right] \left[\sum_{e_T \in \varepsilon(e_t, s_{t+1})} \mu(e_T)\right]_{\theta_t^C(e_t) = 0} = 0$$

for any $\kappa(e_t) \in \mathbb{K}(e_t)$. But from the definition of $\mu(e_T)$ in (3.1) and the expression of the terminal wealth in (2.3), $[\mu(e_T)]_{\theta_t^C(e_t) = 0}$ is independent of $\kappa(e_t)$. Therefore, differentiating the last equation with respect to $\kappa(e_t)$ we get

$$< A^\kappa(e_t), \mu > = 0 \text{ for any } \kappa(e_t) \in \mathbb{K}(e_t)$$

This proves that the first-order condition in the contingent claim characteristic (3.6) can be written equivalently:

$$< A^{\kappa}(e_t),\mu > = 0 \text{ for } t = 0,\ldots,T-1 \text{ and } e_t \in \varepsilon_t \tag{3.7}$$

Proposition 3.1 *Suppose that the* $3^T - 1$ *vectors* $A^C(e_t)$, $A^{\kappa}(e_t)$, $t = 0,\ldots,T-1$ *and* $e_t \in \varepsilon_t$ *in* $\mathbb{R}^{3^T}$ *are linearly independent. Then, for any choice of the agent utility function u the optimal demand in the primitive risky asset is zero, i.e.*

$$\theta_t^S(e_t) = 0 \text{ for all } t = 0,\ldots,T \text{ and } e_t \in \varepsilon_t$$

Remark. Recall that we have considered only optimal interior points because, as will be justified in Section 5 (in the case of European call options), the case of extremal points can be treated easily and directly.

Proof. See Appendix.

Proposition 3.1 provides the optimal policy of the agent in the primitive risky asset under suitable conditions on the contingent claims available in the economy. Thus, in an economy satisfying the assumption of Proposition 3.1, the optimal demand of the agent in the primitive risky asset, which is subject to transaction costs, is always zero.

An important feature of the last result is that it does not depend on the nature of the terminal wealth utility function. The assumption of Proposition 3.1 seems to be artificial and does not have any economic justification. The following result provides a simple sufficient condition, with a nice economic interpretation, for an economy to satisfy this assumption.

Proposition 3.2 *For any* $t = 0,\ldots, T-1$ *and* $e_t \in \varepsilon_t$ *define the matrix:*

$$B(e_t) = \begin{pmatrix} 1 & C_{t+1}(u_{t+1}(e_t),\kappa(e_t)) & \Delta_{t+1}(u_{t+1}(e_t),\kappa(e_t)) \\ 1 & C_{t+1}(m_{t+1}(e_t),\kappa(e_t)) & \Delta_{t+1}(m_{t+1}(e_t),\kappa(e_t)) \\ 1 & C_{t+1}(d_{t+1}(e_t),\kappa(e_t)) & \Delta_{t+1}(d_{t+1}(e_t),\kappa(e_t)) \end{pmatrix} \tag{3.8}$$

If $B(e_t)$ *is invertible for all* $t = 0,\ldots,T-1$ *and* $e_t \in \varepsilon_t$ *then the assumption of Proposition 3.1 is satisfied and therefore the optimal demand in the primitive risky asset is zero.*

Proof. See Appendix.

Let us provide an economic justification of the assumption of Proposition 3.2. Suppose that $B(e_t)$ is singular for some $t \in \{0,\ldots,T-1\}$ and $e_t \in \varepsilon_t$. Then the contingent claims characterized by $\kappa_t(e_t)$ and $\kappa_t(e_t) + \varepsilon$ (for a sufficiently small ε) are redundant up to the first order: if date $t+1$ price of the contingent claim $\varphi(e_T,\kappa_t(e_t) + \varepsilon)$ is approximated by its first-order expansion around $\varepsilon = 0$, then date $t+1$ payoff matrix of the nonrisky asset and the two contingent claims is singular. Therefore optimizing over the characteristic of the contingent claim does not improve the terminal wealth utility of the agent.

Notice that Proposition 3.2 provides only a sufficient condition for the assumption of Proposition 3.1 to be satisfied. However this condition is still difficult to check in practice since $B(e_t)$ requires knowledge of the optimal choice of the contingent claim characteristic $\kappa(e_t)$. The following result provides a sufficient condition which can be verified in practice.

Proposition 3.3 *Consider some $t \in \{0,\ldots,T-1\}$ and $e_t \in \mathcal{E}_t$. A sufficient condition for $B(e_t)$ to be invertible is that the function*

$$\kappa \mapsto \frac{C_{t+1}((e_t,m),\kappa) - C_{t+1}((e_t,d),\kappa)}{C_{t+1}((e_t,u),\kappa) - C_{t+1}((e_t,m),\kappa)}$$

admits no singularities, i.e. its derivative does not vanish. An economy satisfying this condition will be called regular.

Proof. The result follows directly from the fact that the determinant of $B(e_t)$ is given by

$$\frac{\det B(e_t)}{[C_{t+1}((e_t,u),\kappa_t(e_t)) - C_{t+1}((e_t,m_t),\kappa_t(e_t))]^2}$$

$$= \frac{\partial}{\partial\kappa}\left\{\frac{C_{t+1}((e_t,m),\kappa) - C_{t+1}((e_t,d),\kappa)}{C_{t+1}((e_t,u),\kappa) - C_{t+1}((e_t,m),\kappa)}\right\}_{\kappa=\kappa_t(e_t)}$$

since the derivative of the function given in the proposition is nonzero for any κ.

In Section 5, we shall provide a simple example of regular economy. But let us first extend the previous results to the case where the terminal payoff function is only piecewise continuously differentiable as in the standard example of European options.

4. THE NONREGULAR PAYOFF FUNCTION CASE

In this section we follow the arguments of the previous section except that the terminal payoff function is not continuously differentiable. Instead we assume it to admit both right and left derivatives on its domain.

For any $t = 0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$, the first-order condition (3.3) in the optimal investment in the contingent claim $\theta_t^c(e_t)$ is unchanged since it involves only the continuity of φ.

For any $t = 0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$, the first-order condition in the contingent claim characteristic $\kappa_t(e_t)$ is now

$$\frac{\partial^+}{\partial\kappa_t(e_t)}E[u(W_T)] \leq 0 \text{ and } \frac{\partial^-}{\partial\kappa_t(e_t)}E[u(W_T)] \geq 0$$

where $\partial^+/\partial\kappa$ and $\partial^-/\partial\kappa$ are, respectively, the right and the left derivative operators. As in the previous section, using the first-order conditions (3.3), these conditions can be written

$$\theta_t^c(e_t) < A^{\kappa+}(e_t),\mu> \leq 0 \text{ and } \theta_t^c(e_t) < A^{\kappa-}(e_t),\mu> \geq 0 \quad (4.1)$$

where $A^{\kappa+}(e_t)$ (resp. $A^{\kappa-}(e_t)$) are defined in the same way as $A^{\kappa}(e_t)$ in (3.5), replacing the derivatives by the right (resp. left) derivatives. Now if $\theta_t^C(e_t)=0$ then it is easily seen that $<A^{\kappa+}(e_t),\mu> = <A^{\kappa-}(e_t),\mu> = 0$. If $\theta_t^C(e_t) \neq 0$ then it can be simplified in (4.1). In both cases the first-order conditions (4.1) imply that

$$<A^{\kappa+}(e_t),\mu><A^{\kappa-}(e_t),\mu> \leq 0 \tag{4.2}$$

for any $t=0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$. This proves the existence of $\lambda(e_t) \in [0,1]$ such that

$$<\lambda(e_t)A^{\kappa+}(e_t) + (1-\lambda(e_t))A^{\kappa-}(e_t),\mu> = 0 \tag{4.3}$$

Now define

$$A^{\kappa}(e_t) = \lambda(e_t)A^{\kappa+}(e_t) + (1-\lambda(e_t))A^{\kappa-}(e_t) \tag{4.4}$$

for any $t=0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$. With this more general definition of the $\mathbb{R}^{3^T}$ vectors $A^{\kappa}(e_t)$, one can check easily that Proposition 3.1 still holds. Next we define a regular economy in this more general framework exactly as in the previous section except that a singularity is now a point for which 0 lies in the interval defined by the left and the right derivative. Notice that if the terminal payoff function is continuously differentiable, this definition coincides with that of the previous section.

Proposition 4.1 *Assume that the economy is regular. Then the optimal demand of the agent in the primitive risky asset is zero.*

Proof. Define $B(e_t)$ as in the previous section by replacing its third column by

$$\lambda(e_t)\begin{pmatrix}\Delta_{t+1}^{+}(u_{t+1}(e_t),\kappa(e_t))\\ \Delta_{t+1}^{+}(m_{t+1}(e_t),\kappa(e_t))\\ \Delta_{t+1}^{+}(d_{t+1}(e_t),\kappa(e_t))\end{pmatrix} + (1-\lambda(e_t)) \begin{pmatrix}\Delta_{t+1}^{-}(u_{t+1}(e_t),\kappa(e_t))\\ \Delta_{t+1}^{-}(m_{t+1}(e_t),\kappa(e_t))\\ \Delta_{t+1}^{-}(d_{t+1}(e_t),\kappa(e_t))\end{pmatrix}$$

If the 3^T-1 vectors $A^C(e_t)$, $A^{\kappa}(e_t)$ for $t=0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$ are linearly dependent then, following the proof of Proposition 3.2, there exists some $t \in \{0,\ldots,T-1\}$ and $e_t \in \mathcal{E}_t$ such that $B(e_t)$ is not invertible. This implies obviously that the function

$$\kappa \mapsto \frac{C_{t+1}((e_t,m),\kappa) - C_{t+1}((e_t,d),\kappa)}{C_{t+1}((e_t,u),\kappa) - C_{t+1}((e_t,m),\kappa)}$$

admits a singularity since there exists a convex combination of the right and the left derivative which vanishes.

5. AN EXAMPLE: EUROPEAN CALL OPTIONS AND i.i.d. RETURNS

In this section we provide an example of regular economy. The primitive risky asset returns are independent and identically distributed in the sense that:

$$S(e_t) = \{u, m, d\} \text{ with } m = ud = 1 \text{ for any } t = 0, \ldots, T-1 \text{ and } e_t \in \varepsilon_t \tag{5.1}$$

Equivalent martingale measures are characterized by (2.2). In our economy this condition induces the following set of e.m.m.:

$$(q_t^u(e_t), q_t^m(e_t), q_t^d(e_t)) = (q_t^u(e_t), q_t(e_t), u q_t^u(e_t)) \text{ with } q_t^u(e_t) = \frac{1 - q_t(e_t)}{1 + u} \tag{5.2}$$

where the middle state probability $q_t(e_t)$ varies in $(0,1)$. Moreover we assume that the transition probabilities of the e.m.m. are time and state independent, i.e.

$$q_t(e_t) = q \text{ for any } t = 0, \ldots, T-1 \text{ and } e_t \in \varepsilon_t \tag{5.3}$$

and then $q_t^u(e_t) = q^u$ is also time and state independent. The family of contingent claims consists of European call options, with common maturity T, indexed by their exercise price K. Thus, we have

$$C(e_t) = \{(S_T - K)^+, K \in \mathbb{K}(e_t)\}$$

Note that if the optimal exercise price $K^*(e_t)$ is such that $K^*(e_t) \leq \min_{e_T \in \varepsilon(e_t)} S(e_T)$ then the agent does not use the primitive asset since he can replicate without transaction costs using the contingent claim and the nonrisky asset. Furthermore, if $K^*(e_t) \geq \max_{e_T \in \varepsilon(e_t)} S(e_T)$ then the optimal contingent claim has a zero payoff. This is equivalent to saying that the agent does not use the entire family of contingent claims. In particular, the agent does not use the contingent claim with zero exercise price which is the same thing as the risky asset but without transaction costs. Then, it is obvious that again the agent does not use the primitive asset. It is justified now to restrict our attention to the case where $K^*(e_t) \in (\min_{e_T \in \varepsilon(e_t)} S(e_T), \max_{e_T \in \varepsilon(e_t)} S(e_T))$. Since this interval is open we can use the first-order conditions for an interior point developed in the previous sections.

Proposition 5.1 *Suppose that $q \geq q^u \sqrt{u}$. Then for any $t = 0, \ldots, T-1$ and $e_t \in \varepsilon_t$ the function*

$$K \mapsto \frac{C_{t+1}((e_t, m), K) - C_{t+1}((e_t, d), K)}{C_{t+1}((e_t, u), K) - C_{t+1}((e_t, m), K)}$$

is strictly monotonic and therefore the economy described above is regular. Hence, the agent does not use the primitive risky asset.

Proof. It is clear that it is sufficient to prove the required result for $t = 0$ and for any number of periods T in the economy. We define the function:

$$\xi(K) = \frac{C_1(m, K) - C_1(d, K)}{C_1(u, K) - C_1(m, K)}$$

First notice that on each interval lying between two terminal nodes (u^i, u^{i+1}) for $i = -T, \ldots, T-1$, the function ζ is differentiable with

$$\zeta'(K) = \frac{\beta(K)}{[C_1(u,K) - C_1(m,K)]^2} \qquad u^i < K < u^{i+1}$$

and

$$\beta(K) = \det\begin{pmatrix} 1 & E^Q[S_T 1\{(S_T \geq K)\} \mid u] & Q[S_T \geq K \mid u] \\ 1 & E^Q[S_T 1\{(S_T \geq K)\} \mid m] & Q[S_T \geq K \mid m] \\ 1 & E^Q[S_T 1\{(S_T \geq K)\} \mid d] & Q[S_T \geq K \mid d] \end{pmatrix}$$

where $E^Q[.\mid s]$ is the expectation under Q conditionally to $s_1 = s$, $s \in \{u, m, d\}$ and $Q[.\mid s]$ is the probability under Q conditionally to $s_1 = s$, $s \in \{u, m, d\}$. Now it is easily seen that the function β is constant on any semi-closed interval $[u^i, u^{i+1})$ for any $i = -T, \ldots, T-1$. Therefore, in order to prove that the function ζ admits no singularities, it is sufficient to prove that the sequence

$$\{\zeta(u^i),\ i = -T+1, \ldots, T-1\}$$

is strictly monotonic. More precisely we intend to prove that it is strictly decreasing. Define the probabilities:

$$q_i = Q[S_T = u^i \mid m] \qquad i = -T+1, \ldots, T-1$$

Then, from (5.3), it is easily seen that

$$Q[S_T = u^i \mid u] = q_{i-1} \qquad i = -T+2, \ldots, T$$

and

$$Q[S_T = u^i \mid d] = q_{i+1} \qquad i = -T, \ldots, T-2$$

Substituting the options prices we get

$$\zeta(u^i) = \frac{1}{u}\left(1 + \frac{q_i u^i}{\sum_{j \geq i+1} q_j u^j}\right)^{-1} \qquad \text{for } i = -T+1, \ldots, T-2$$

$$\zeta(u^{T-1}) = 0$$

Define the sequences:

$$\psi(i) = \frac{q_i}{\sum_{j \leq i-1} q_j} \quad \text{for } i = -T+2, \ldots, T-1$$

$$\psi^*(i) = \frac{q_i}{\sum_{j \leq i} q_j} \quad \text{for } i = -T+1, \ldots, T-1$$

then using the fact that $q_i = u^i q_i$ for $i = -T+1, \ldots, T-1$ (this can easily be shown by induction on the number of periods T), direct computation shows that

$$\zeta(u^i) < \zeta(u^{i-1}) \qquad \forall i = -T+2, \ldots, T-2$$

$$\Leftrightarrow \psi(i) > \frac{q_{i+1}}{q_i} - 1 \qquad \forall i = -T+2, \ldots, T-2$$

and for a fixed i we have

$$\psi(i) > \frac{q_{i+1}}{q_i} - 1 \Leftrightarrow \psi^*(i) > 1 - \frac{q_i}{q_{i-1}}$$

For the rest of the proof, we need a technical lemma, the proof of which is given in the Appendix.

Lemma 5.1 *Suppose that* $q \geq q^u \sqrt{u}$. *Then* $q_{i+1}/q_i \geq q_{i+2}/q_{i+1}$ *for any* $i = -T+1, \ldots, T-3$.

Now we prove by induction that $\psi^*(i) > 1 - (q_i/q_{i-1})$ for any $i = -T+1, \ldots, T-2$.

- For $i = -T+1$

$$\psi^*(-T+1) = 1 > 1 - \frac{q^{-T+1}}{q^{-T+2}}$$

- For $i \leq T-3$, assume $\psi^*(i) > 1 - (q_i/q_{i-1})$. Then

$$\psi(i+1) = \frac{q_{i+1}}{\sum_{j\leq i} q_j} = \frac{q_{i+1}}{q_i}\psi^*(i)$$

$$> \frac{q_{i+1}}{q_i}\left(1 - \frac{q_i}{q_{i-1}}\right) = \frac{q_{i+1}}{q_i} - 1 \geq \frac{q_{i+2}}{q_{i+1}} - 1$$

where the last inequality follows from Lemma 5.1.

6. OPTIMAL CHOICE OF THE CONTINGENT CLAIM IN A REGULAR ECONOMY

Our main result (Proposition 3.1) states that the agent does not trade in the primitive risky asset when the economy is regular. In this section we focus on the optimal choice of the contingent claim. We will prove that if there exists an optimal contingent claim then it is unique. If the family of contingent claims is composed by European call options we provide necessary and sufficient conditions for the existence of such an optimal contingent claim.

We first establish the result that, in a regular economy, the first-order conditions make a direct connection between the equivalent martingale measure Q and the marginal rate of substitution of the agent at the optimum.

Proposition 6.1 *Suppose that the economy is regular. Then there exists a positive scalar* λ *such that:*

$$Q(e_T) = \lambda\mu(e_T), \text{ for any } e_T \in \mathcal{E}_T$$

Proof. The result follows directly from the proof of Proposition 3.1. Recall that the $\mathbb{R}^{3^T}$ vectors μ and q are both orthogonal to the linear space spanned by the $\mathbb{R}^{3^T}$ vectors $A^C(e_t), A^\kappa(e_t)$, $t = 0, \ldots, T-1$ and $e_t \in \mathcal{E}_t$. Since the economy is regular, the dimension of the last linear space is $3^T - 1$ and therefore the vectors μ and q are linearly dependent.

From the strict concave feature of the utility function u, the functions

$$W \mapsto E[u'(W_T) \mid e_t, W(e_t) = W]$$

are strictly decreasing for any $t = 0, \ldots, T-1$ and $e_t \in \mathcal{E}_t$. This shows that it is one-to-one since it is clearly continuous. We denote by I_{e_t} its inverse function. The result of the last proposition can be written in terms of this function as

$$W_{t+1}(e_t, u_{t+1}(e_t)) = I_{(e_t u_{t+1}(e_t))}\left(\frac{q_t^u(e_t)}{\lambda p_t^u(e_t)}\right) \tag{6.1}$$

$$W_{t+1}(e_t, m_{t+1}(e_t)) = I_{(e_t m_{t+1}(e_t))}\left(\frac{q_t^m(e_t)}{\lambda p_t^m(e_t)}\right) \tag{6.2}$$

$$W_{t+1}(e_t, d_{t+1}(e_t)) = I_{(e_t d_{t+1}(e_t))}\left(\frac{q_t^d(e_t)}{\lambda p_t^d(e_t)}\right) \tag{6.3}$$

where $W_t(e_t)$ is the optimal wealth at date t in the state e_t. This allows one to prove the following uniqueness result.

Proposition 6.2 *If the utility function is a power function then the optimal contingent claim (if it exists) is characterized by* $\kappa_t(e_t)$ *for some* $t \in \{0, \ldots, T-1\}$ *and* $e_t \in \mathcal{E}_t$ *is unique.*

Proof. Expressing the optimal wealth at date $t+1$ in terms of that of date t, equations (6.1), (6.2) and (6.3) can be written:

$$W_t(e_t) + \theta_t^c(e_t)\left[\frac{C_{t+1}((e_t, u_{t+1}(e_t)), \kappa_t(e_t))}{C_t(e_t, \kappa_t(e_t))} - 1\right] = I_{(e_t, u_{t+1}(e_t))}\left(\frac{q_t^u(e_t)}{\lambda p_t^u(e_t)}\right)$$

$$W_t(e_t) + \theta_t^c(e_t)\left[\frac{C_{t+1}((e_t, m_{t+1}(e_t)), \kappa_t(e_t))}{C_t(e_t, \kappa_t(e_t))} - 1\right] = I_{(e_t, m_{t+1}(e_t))}\left(\frac{q_t^m(e_t)}{\lambda p_t^m(e_t)}\right)$$

$$W_t(e_t) + \theta_t^c(e_t)\left[\frac{C_{t+1}((e_t, d_{t+1}(e_t)), \kappa_t(e_t))}{C_t(e_t, \kappa_t(e_t))} - 1\right] = I_{(e_t, d_{t+1}(e_t))}\left(\frac{q_t^d(e_t)}{\lambda p_t^d(e_t)}\right)$$

Combining these equations we get

$$\frac{C_{t+1}((e_t,m_{t+1}(e_t)),\kappa_t(e_t)) - C_{t+1}((e_t,d_{t+1}(e_t)),\kappa_t(e_t))}{C_{t+1}((e_t,u_{t+1}(e_t)),\kappa_t(e_t)) - C_{t+1}((e_t,m_{t+1}(e_t)),\kappa_t(e_t))}$$

$$= \frac{I_{(e_t,m_{t+1}(e_t))}\left(\frac{q_t^m(e_t)}{\lambda p_t^m(e_t)}\right) - I_{(e_t,d_{t+1}(e_t))}\left(\frac{q_t^d(e_t)}{\lambda p_t^d(e_t)}\right)}{I_{(e_t,u_{t+1}(e_t))}\left(\frac{q_t^u(e_t)}{\lambda p_t^u(e_t)}\right) - I_{(e_t,m_{t+1}(e_t))}\left(\frac{q_t^m(e_t)}{\lambda p_t^m(e_t)}\right)}$$

$$= \frac{I_{(e_t,m_{t+1}(e_t))}\left(\frac{q_t^m(e_t)}{p_t^m(e_t)}\right) - I_{(e_t,d_{t+1}(e_t))}\left(\frac{q_t^d(e_t)}{p_t^d(e_t)}\right)}{I_{(e_t,u_{t+1}(e_t))}\left(\frac{q_t^u(e_t)}{p_t^u(e_t)}\right) - I_{(e_t,m_{t+1}(e_t))}\left(\frac{q_t^m(e_t)}{p_t^m(e_t)}\right)}$$

where the right-hand side term does not depend on $\kappa_t(e_t)$ and where the last equality is obtained under the power utility assumption. The left-hand side term is a monotonic function of $\kappa_t(e_t)$ since the economy is regular. This shows that if $\kappa_t(e_t)$ exists then it is unique.

In the last equation we noted that the left-hand side is monotonic. Therefore, to prove the existence of an optimal contingent claim it is necessary and sufficient to compare the right-hand side of the equation with the extreme values of the monotonic function. In the next proposition we apply this result to the example studied in Section 5. More precisely, we characterize the existence of an optimal exercise price in $(\min_{e_T\in\mathcal{E}(e_t)}S(e_T), \max_{e_T\in\mathcal{E}(e_t)}S(e_T))$*. It is easy to show, in this case, that the extreme values are 0 and $(m-d)/(u-m)$ and we therefore have the following.

Proposition 6.3 *In the power utility case and under the assumption of Proposition 5.1, there exists an optimal exercise price $K^*(e_t)$ in $(\min_{e_T\in\mathcal{E}(e_t)}S(e_T), \max_{e_T\in\mathcal{E}(e_t)} S(e_T))$ if and only if*

$$\frac{I_{(e_t,m)}\left(\frac{q_t^m(e_t)}{p_t^m(e_t)}\right) - I_{(e_t,d)}\left(\frac{q_t^d(e_t)}{p_t^d(e_t)}\right)}{I_{(e_t,u)}\left(\frac{q_t^u(e_t)}{p_t^u(e_t)}\right) - I_{(e_t,m)}\left(\frac{q_t^m(e_t)}{p_t^m(e_t)}\right)} \in \left(0, \frac{m-d}{u-m}\right)$$

Proof. This is a direct consequence of the monotonicity.

7. CONCLUSION

In this paper we have proved that the optimal strategy for an investor faced with transaction costs and market incompleteness is not to trade in an asset subject to transaction costs. This is in agreement with empirical evidence which

* As shown in Section 5, the case of an optimal exercise price outside this interval is a degenerated one.

shows much higher trading volumes on contingent claims markets than in primitive assets ones. Our model allows for a bid–ask spread for the contingent claim. The absence of transaction costs on the contingent claim can be seen as an (unrealistic) limit case of the real situation on the market: in practice transaction costs on primitive assets are much higher than those on contingent claims ones.

Our analysis extends easily to the case of recursive utility $u_t = f(u^u_{t+1}, u^m_{t+1}, u^d_{t+1})$. The only difference is that the probabilities $p(e_T)$ in (3.1) are replaced by an appropriate combination of partial derivatives of the certainty equivalence function f, whose positivity is ensured by the usual assumptions on f. Therefore the definition of a regular economy remains unchanged and the results of Section 5 are valid in this larger class of utility functions.

In a multinomial model, introducing sufficient assets to make the choice between market incompleteness and transaction costs relevant, it is easily seen that the results of Section 4 still hold under the regular economy assumption. However, in this more general framework, such an assumption is very difficult to check for a particular economy (as in Section 5).

APPENDIX

Proof of proposition 3.1

Consider any sample path $e_T \in \mathcal{E}_T$. We intend to prove that the optimal demand of the agent in the risky asset is zero along the sample path e_T, i.e. $\theta^S_t(\pi_t e_T) = 0$ for any $t = 0, \ldots, T-1$. Let

$$\chi_0(e_T) = \inf[\{u \geq 0 : \theta^S_u(\pi_u e_T) \neq 0\} \cup \{T\}]$$

be the first date of nonzero optimal demand in the primitive risky asset. We also define for $t = 1, \ldots, T$:

$$\chi_t(e_T) = \inf\left[\left\{u \geq \chi_{t-1}(e_T) : \theta^S_u(\pi_u e_T) \neq \theta^S_{\chi_{t-1}(e_T)}(\pi_{\chi_{t-1}(e_T)} e_T) \frac{S_u(e_T)}{S_{\chi_{t-1}(e_T)}(e_T)}\right\} \cup \{T\}\right]$$

to be the date of the tth (nonzero) trading in the primitive risky asset on the sample path e_T (if $\chi_t(e_T) < T$). In the rest of the proof, we show by induction that if the agent has a long (resp. short) position in the primitive risky asset at some date, then his/her position in the risky asset is always long (resp. short) until the final date T, i.e.

$$\text{if } \chi_t(e_T) < T \text{ and } \theta^S_{\chi_t(e_T)}(\pi_{\chi_t(e_T)} e_T) \neq 0$$
$$\text{then for any } t = 0, \ldots, T$$

$$\theta^S_{\chi_t(e_T)}(\pi_{\chi_t(e_T)} e_T)\left[\theta^S_{\chi_{t+1}(e_T)}(\pi_{\chi_{t+1}(e_T)} e_T) - \theta^S_{\chi_t(e_T)}(\pi_{\chi_t(e_T)} e_T) \frac{S_{\chi_{t+1}(e_T)}(e_T)}{S_{\chi_t(e_T)}(e_T)}\right] > 0$$

which clearly implies the result stated in the proposition (recall that at the end the agent converts his/her position in cash and $\theta^S_T = 0$). Fix a sample path $\bar{e}_T \in \mathcal{E}_T$.

We shall use the notation $\bar{\chi}_t = \chi_t(\bar{e}_T)$ and $\bar{e}_{\bar{\chi}_t} = \pi_{\bar{\chi}_t}\bar{e}_T$. Suppose that $\bar{\chi}_t < T$ and $\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t}) \neq 0$. Then the first-order condition in $\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t})$ is

$$\frac{\partial}{\partial\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t})} E[u(W_T)] = <A^S_{\bar{\chi}_t}, \mu> = 0$$

where the components of the $\mathbb{R}^{3^T}$ vector $A^S_{\bar{\chi}_t}$ are

$$\{A^S_{\bar{\chi}_t}\}_i = \begin{cases} 0 \quad \text{if } e^i_T \notin \varepsilon(\bar{e}_{\bar{\chi}_t}) \\ \tau'\left(\theta^S_{\chi_{t+1}(e^i_T)}(\pi_{\chi_{t+1}(e^i_T)}e^i_T) - \theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t})\dfrac{S_{\chi_{t+1}(e^i_T)}(e^i_T)}{S_{\bar{\chi}_t}(\bar{e}_T)}\right)\dfrac{S_{\chi_{t+1}(e^i_T)}(e^i_T)}{S_{\bar{\chi}_t}(\bar{e}_T)} \\ -\tau'\left(\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t}) - \theta^S_{\bar{\chi}_{t-1}}(\bar{e}_{\bar{\chi}_{t-1}})\dfrac{S_{\bar{\chi}_t}(\bar{e}_T)}{S_{\bar{\chi}_{t-1}}(\bar{e}_T)}\right) \quad \text{if } e^i_T \in \varepsilon(\bar{e}_{\bar{\chi}_t}) \end{cases}$$

The last expression is well defined since τ is differentiable in $\mathbb{R}\setminus\{0\}$. Next define the matrix A whose columns are $A^S_{\bar{\chi}_t}$, $A^C(e_t)$, $A^\kappa(e_t)$ for $t = 0, \ldots, T-1$ and $e_t \in \varepsilon_t$, then the first-order conditions in $\kappa_t(e_t)$, $\theta^C_t(e_t)$ and $\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t})$ can be written

$$A\mu = 0 \text{ with } \mu_i > 0 \text{ for all } i = 1, \ldots, 3^T$$

This proves that the matrix A is singular and the assumption of the proposition ensures that $A^S_{\bar{\chi}_t}$ is a linear combination of the $A^C(e_t)$ and the $A^\kappa(e_t)$ for $t = 0, \ldots, T-1$ and $e_t \in \varepsilon_t$. Now remember that the contingent claim price process is a martingale under the e.m.m. Q (with a fixed characteristic κ) and notice that this property is inherited by the process Δ_t defined in (3.4). We therefore have

$$<A^\kappa(e_t), q> = 0 \text{ and } <A^C(e_t), q> = 0$$

for $t = 0, \ldots, T-1$ and $e_t \in \varepsilon_t$, where q is the $\mathbb{R}^{3^T}$ vector whose components are the $Q(e_T)$ classified according to the lexicographic order of the final states e_T. This shows that

$$<A^\kappa(e_t), q - \mu> = <A^C(e_t), q - \mu> = 0 \text{ for } t = 0, \ldots, T-1 \text{ and } e_t \in \varepsilon_t$$

and since $A^S_{\bar{\chi}_t}$ is a linear combination of the $A^C(e_t)$ and the $A^\kappa(e_t)$, we have

$$<A^S_{\bar{\chi}_t}, q> = <A^S_{\bar{\chi}_t}, \mu>$$

The first-order condition in $\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t})$ can thus be written

$$\tau'\left(\theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t}) - \theta^S_{\bar{\chi}_{t-1}}(\bar{e}_{\bar{\chi}_{t-1}})\frac{S_{\bar{\chi}_t}(\bar{e}_T)}{S_{\bar{\chi}_{t-1}}(\bar{e}_T)}\right) S_{\bar{\chi}_t}(\bar{e}_T)$$

$$= \sum\left[\tau'\left(\theta^S_{\chi_{t+1}(e_T)}(\pi_{\chi_{t+1}(e_T)}e_T) - \theta^S_{\bar{\chi}_t}(\bar{e}_{\bar{\chi}_t})\frac{S_{\chi_{t+1}(e_T)}(e_T)}{S_{\bar{\chi}_t}(\bar{e}_T)}\right)S_{\chi_{t+1}(e_T)}(e_T)\right] \frac{Q(e_T)}{Q(\bar{e}_{\bar{\chi}_t})}$$

But $\tau'(x)$ can only take the values $(1+k)$ or $(1+h)^{-1}$ for $x\neq 0$. Therefore since the average price process $\{S_t, t=0,\ldots,T\}$ is a strong martingale under the e.m.m. Q, the previous equation implies that

$$\theta^S_{\bar{\chi}_t}(\bar{e}_{\chi_t}) - \theta^S_{\bar{\chi}_{t-1}}(\bar{e}_{\chi_{t-1}}) \frac{S_{\bar{\chi}_t}(\bar{e}_T)}{S_{\bar{\chi}_{t-1}}(\bar{e}_T)}\left[\theta^S_{\chi_{t+1}(e_T)}(\pi_{\chi_{t+1}(e_T)}e_T) - \theta^S_{\bar{\chi}_t}(\bar{e}_{\chi_t}) \frac{S_{\chi_{t+1}(e_T)}(e_T)}{S_{\bar{\chi}_t}(\bar{e}_T)}\right] > 0$$

for all $e_T \in \mathcal{E}(\bar{e}_{\chi_t})$, which ends the proof.

Proof of Proposition 3.2

The result is obtained by backward induction on the number of periods T; we therefore add a T subscript to the notations used in this proof. Let A^0_T be the vector in $\mathbb{R}^{3^T}$ whose components are all 1 and consider the matrix A_T whose columns are A^0_T, $A^\kappa_T(e_t)$ and $A^C_T(e_t)$ for $t=0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$. Suppose that the 3^T-1 vectors $A^\kappa_T(e_t)$ and $A^C_T(e_t)$ for $t=0,\ldots,T-1$ and $e_t \in \mathcal{E}_t$ are not linearly independent then the matrix A_T is singular. We now prove that this implies that $B(e_t)$ is singular for some $t \in \{0,\ldots,T-1\}$ and $e_t \in \mathcal{E}_t$ which will provide the required result.

First suppose that the vectors A^0_T, $A^\kappa_T(e_0)$ and $A^C_T(e_0)$ are linearly dependent. Then, from the definition of these vectors in (3.2) and (3.5), this is equivalent to the fact that the matrix $B(e_0)$ is singular which ends the proof. Next we examine the case where the vectors A^0_T, $A^\kappa_T(e_0)$ and $A^C_T(e_0)$ are linearly independent. Since A_T is singular there exists a vector $(\alpha_0, \alpha(e_t), \beta(e_t), t=0,\ldots,T-1, e_t \in \mathcal{E}_t)$ in $\mathbb{R}^{3^T}\backslash\{0\}$ such that

$$\alpha_0 A^0_T + \sum_{e_T \in \mathcal{E}_T} \sum_{t=0}^{T-1} \{\alpha(\pi_t e_T) A^C_T(\pi_t e_T) + \beta(\pi_t e_T) A^\kappa_T(\pi_t e_T)\} = 0 \tag{A.1}$$

Case A. If $\alpha_0 A^0_T + \alpha(e_0)A^C_T(e_0) + \beta(e_0)A^\kappa_T(e_0) = 0$. Then since A^0_T, $A^C_T(e_0)$ and $A^\kappa_T(e_0)$ are linearly independent we have that $\alpha_0 = \alpha(e_0) = \beta(e_0) = 0$ and therefore equation (3.9) can be written

$$\sum_{S_1 \in S(e_0)} \sum_{e_T \in \mathcal{E}(s_1)} \sum_{t=1}^{T-1} \{\alpha(\pi_t e_T) A^C_T(\pi_t e_T) + \beta(\pi_t e_T) A^\kappa_T(\pi_t e_T)\} = 0 \tag{A.2}$$

where $(\alpha(\pi_t e_T), \beta(\pi_t e_T), t=1,\ldots,T-1, e_T \in \mathcal{E}_T)$ is a nonzero vector in $\mathbb{R}^{3^T-3}$. Now notice that, from the definition of the vectors $A^\kappa_T(e_t)$ and $A^C_T(e_t)$ in (3.2) and (3.5), given a final state $e_T \in \mathcal{E}_T$ there exists a unique $s_1 \in S(e_0)$ for which $A^\kappa_T(\pi_t e_T) \neq 0$ and $A^C_T(\pi_t e_T) \neq 0$ for any $t=1,\ldots,T-1$, and therefore the previous equation is equivalent to

$$\forall_{S_1} \in S(e_0), \sum_{e_T \in \mathcal{E}(s_1)} \sum_{t=1}^{T-1} \{\alpha(\pi_t e_T) A^C_T(\pi_t e_T) + \beta(\pi_t e_T) A^\kappa_T(\pi_t e_T)\} = 0$$

with a nonzero vector $(\alpha(\pi_t e_T), \beta(\pi_t e_T), t=1,\ldots,T-1, e_T \in \mathcal{E}_T)$ in $\mathbb{R}^{3^T-3}$. This proves the existence of a state $s_1^* \in S(e_0)$ such that vectors $A^C_T(\pi_t e_T)$ and $A^\kappa_T(\pi_t e_T)$, $t=1,\ldots,T-1$, $e_T \in \mathcal{E}(s_1^*)$, are not linearly independent. Finally, define the vectors

$A^{\zeta}_{T-1}(\pi,e_T)$ and $A^{\kappa}_{T-1}(\pi,e_T)$ in $\mathbb{R}^{3^{(T-1)}}$ from the vectors $A^{\zeta}_{T}(\pi,e_T)$ and $A^{\kappa}_{T}(\pi,e_T)$ by dropping all the components i for which $e^i_T \notin \mathcal{E}(s^*_1)$. Then the $3^{T-1}-1$ vectors $A^{\zeta}_{T-1}(\pi,e_T)$ and $A^{\kappa}_{T-1}(\pi,e_T)$, $t = 1,\ldots,T-1$ and $e_T \in \mathcal{E}(s_1)$, are linearly dependent. Defining the matrix $A_{T-1}(s^*_1)$ whose columns are A^0_{T-1}, the $A^{\zeta}_{T-1}(\pi,e_T)$'s and the $A^{\kappa}_{T-1}(\pi,e_T)$'s we see that the matrix $A_{T-1}(s^*_1)$ is singular.

Case B. If $\alpha_0 A^0_T + \alpha(e_0)A^{\zeta}_T(e_0) + \beta(e_0)A^{\kappa}_T(e_0) \neq 0$. Then $(\alpha_0,\alpha(e_0),\beta(e_0)) \neq (0,0,0)$. From the definition of the $A^{\kappa}(e_t)$ and the $A^{\zeta}(e_t)$ in (3.2) and (3.5), separating the $\mathbb{R}^{3^T}$ vectors in three blocks of $\mathbb{R}^{3^{T-1}}$ vectors, equation (3.9) can be written equivalently:

$$\forall S_1 \in S(e_0) : \lambda(s_1)A^0_{T-1} + \sum_{e_T\in \mathcal{E}(s_1)} \sum_{t=1}^{T-1} \{\alpha(\pi,e_T)A^{\zeta}_{T-1}(\pi,e_T) + \beta(\pi,e_T)A^{\kappa}_{T-1}(\pi,e_T)\} = 0$$

where $(\lambda(u_1(e_0)),\lambda(m_1(e_0)),\lambda(d_1(e_0))) \neq (0,0,0)$ and the $\mathbb{R}^{3^{T-1}}$ vectors $A^{\zeta}_{T-1}(\pi,e_T)$ and $A^{\kappa}_{T-1}(\pi,e_T)$ are defined as in case A of this proof. Chose s^*_1 such that $\lambda(s^*_1) \neq 0$, then the matrix $A_{T-1}(s^*_1)$, defined as in case A, is singular.

Iterating the previous arguments $T-1$ times we have that

- either $B(e_t)$ is singular for some $t = 0,\ldots,T-2$ and $e_t \in \mathcal{E}_t$,
- or $A_1(e^*_{T-1}) = B(e^*_{T-1})$ is singular for some $e^*_{T-1} \in \mathcal{E}_{T-1}$,

which ends the proof.

Proof of Lemma 5.1

The result is proved by induction on the number of periods T and we therefore add a superscript T in the notation of the terminal node probabilities $q_i^{(T)}$, $i = -T,\ldots,T$. We shall prove in the sequel that

$$\forall i = -T,\ldots,T-2 \left(q^{(T)}_{i+1}\right)^2 \geq q^{(T)}_i q^{(T)}_{i+2}$$

for any $T \geq 1$.

For $T = 1$ the result follows directly from the fact that $\left(q_0^{(1)}\right)^2 \geq u$ $(q^d)^2 = q^{(1)}_{-1}q^{(1)}_1$.

Suppose that the result holds for $T-1$, i.e.

$$\forall i = -T+1,\ldots,T-3 \left(q^{(T-1)}_{i-1}\right)^2 \geq q^{(T-1)}_i q^{(T-1)}_{i-2}$$

In order to simplify the presentation, we extend the definition of the $q_i^{(T-1)}$ as follows:

$$q^{(T-1)}_{-T-1} = q^{(T-1)}_{-T} = q^{(T-1)}_T = q^{(T-1)}_{T+1} = 0$$

so that the induction assumption can be written

$$\forall i = -T-1,\ldots,T-1 \left(q^{(T-1)}_{i+1}\right)^2 \geq q^{(T-1)}_i q^{(T-1)}_{i+2}$$

The terminal node probabilities $q_i^{(T)}$ for date T are obtained from the $q_i^{(T-1)}$ as

$$q_i^{(T)} = q_{i-1}^{(T-1)}q^u + q_i^{(T-1)}q + q_{i+1}^{(T-1)}uq^u \qquad i = -T, \ldots, T \tag{A.3}$$

We intend to prove that

$$\forall i = -T, \ldots, T-2 \ \left(q_{i+1}^{(T)}\right)^2 \geq q_i^{(T)}q_{i+2}^{(T)}$$

Using (A.1) and rearranging the terms, the previous requirement is equivalent to

$$\begin{aligned} 0 \leq & \left(q^2 - u(q^u)^2\right)\left[\left(q_{i+1}^{(T-1)}\right)^2 - q_i^{(T-1)}q_{i+2}^{(T-1)}\right] \\ & + (q^u)^2\left[\left(q_i^{(T-1)}\right)^2 - q_{i-1}^{(T-1)}q_{i+1}^{(T-1)}\right] \\ & + (uq^u)^2\left[\left(q_{i+2}^{(T-1)}\right)^2 - q_{i+1}^{(T-1)}q_{i+3}^{(T-1)}\right] \\ & + u(q^u)^2\left[q_i^{(T-1)}q_{i+2}^{(T-1)} - q_{i-1}^{(T-1)}q_{i+3}^{(T-1)}\right] \\ & + qq^u\left[q_i^{(T-1)}q_{i+1}^{(T-1)} - q_{i-1}^{(T-1)}q_{i+2}^{(T-1)}\right] \\ & + uqq^u\left[q_{i+1}^{(T-1)}q_{i+2}^{(T-1)} - q_i^{(T-1)}q_{i+3}^{(T-1)}\right] \end{aligned}$$

The first three terms are non-negative from the induction assumption. We now show that the final three terms are also positive. We present the arguments only for the fourth term since the other terms can be treated in the same way. From the induction assumption we have

$$\left(q_i^{(T-1)}\right)^2 \geq q_{i-1}^{(T-1)}q_{i+1}^{(T-1)} \text{ and } \left(q_{i+2}^{(T-1)}\right)^2 \geq q_{i+1}^{(T-1)}q_{i+3}^{(T-1)}$$

and therefore

$$\begin{aligned} \left(q_i^{(T-1)}q_{i+2}^{(T-1)}\right)^2 & \geq q_{i-1}^{(T-1)}\left(q_{i+1}^{(T-1)}\right)^2 q_{i+3}^{(T-1)} \\ & \geq q_{i-1}^{(T-1)}q_i^{(T-1)}q_{i+2}^{(T-1)}q_{i+3}^{(T-1)} \end{aligned}$$

where the final inequality follows from the induction assumption. This provides the required result:

$$q_i^{(T-1)}q_{i+2}^{(T-1)} \geq q_{i-1}^{(T-1)}q_{i+3}^{(T-1)}$$

since:

- if $-T+1 \leq i \leq T-3$, then the result is obvious from the fact that $q_i^{(T-1)}q_{i+2}^{(T-1)} \neq 0$;
- if $i \leq -T$, then $q_i^{(T-1)}q_{i-1}^{(T-1)} = 0$ and the result follows;
- if $i \geq T-2$, then $q_{i+2}^{(T-1)}q_{i+3}^{(T-1)} = 0$ and the result follows.

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