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Production planning and inventories optimization with a general storage cost function

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Abstract

We study the deterministic optimization problem of a profit-maximizing firm which plans its sales/production schedule. The firm knows the revenue associated to a given level of sales, as well as its production and storage costs. The revenue and the production cost are assumed to be, respectively, concave and convex whereas the cost of storage has no particular properties of convexity. First, in spite of this non-convexity, we give an existence result. Second, from the necessary conditions, we derive some precise qualitative description of the optimal plan. In particular, we obtain that inventory accumulation is not optimal.

Keywords: Production planning; Inventory management; Storage costs; Non-convex optimization; Calculus of variations

1. Introduction

We consider a firm which produces and sells a good which can be stored. The firm acts in continuous time on a finite period in order to maximize dynamically its profit. Here, the instantaneous profit of the firm is the revenue entailed by the instantaneous sales, diminished by the cost of the instantaneous production, and by the cost of storage of the current inventory. Our approach of this production planning and inventory management problem is in the same vein as the one launched in 1958 by Arrow et al. [1]. Many contributions to this theory have been brought from the 1950s

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until now, with many different approaches. However, authors generally consider firms that do not have any control on the level of the (possibly stochastic) demand driven sales.

In this paper, we work in a competitive and deterministic context. Following, Arvan and Moses [2], we assume that the firm controls not only its production but also its sales level. It knows the revenue associated to the selling of x units of goods, the cost of producing y units, and the cost of storing S units of the good. We assume that the marginal revenue is non-increasing and that the marginal cost of production is non-decreasing, i.e. the revenue function is concave and the production cost function is convex. Our aim is to study the impact of the shape of the storage cost in this context: does the firm actually use its storage ability to accumulate inventories?

The concavity assumption on the revenue function is quite realistic. A particular case of our model is the following: sales are completely within the company's control but the price is a given function of the level of sales. This price is governed by the demand curve. When the firm is monopolistically competitive, it is a non-increasing function of the quantity of the product that the firm wants to put on the market. By computing the instantaneous revenue as the sales rate times the corresponding price, the concavity of the revenue corresponds to the non-decreasing feature of the demand price. The convexity of the production cost means that producing additional output increases expenditures, without any scale economies.

The sales/production planning problem of the profit-maximizing firm is formulated as a maximization problem on a set of integrable decision variables: the sales and production rates paths. The cumulative sales and production processes are therefore continuous.

The existence issue of an optimal sales/production plan is addressed by considering a relaxed optimization problem where the cumulative sales processes is allowed to jump at time 0: the sales path may have a singular part at time 0. We prove that the relaxed problem has at least one solution and that, if it has a solution with no singular part, this solution provides an optimal sales/production plan.

We also derive from the first-order conditions of optimality some precise qualitative description of the optimal plan. We prove that, whatever the shape of the storage cost is, it is never optimal for the firm to accumulate inventory. In particular, if the company starts with no good in stock, it adopts the static strategy which consists in producing and selling at the same constant rate. This optimal rate is the quantity, a, at which the marginal revenue is equal to the marginal production cost. If the firm is endowed with an initial inventory then, the best sales/production policy consists in selling this initial inventory in an optimal way: the level of sales (resp. production) is always greater (resp. lower) than a, it is non-increasing (resp. non-decreasing) until the exhaustion of the inventory. Production actually starts when the marginal revenue entailed by the sales, $\dot{\pi}(x(t))$, overtakes the lowest marginal cost of production, $\dot{c}(0)$. Then, sales and production rates $(x(t), y(t))$ are chosen so as to equal the marginal revenue and the marginal cost of production, i.e $\dot{\pi}(x(t)) = \dot{c}(y(t))$. Once the firm has cleared all its stock it maximizes its profit over time by maximizing its instantaneous profit, producing and selling at the constant rate a. This leads to a three phases sales/production plan.

The precise formulation of the planning problem of the firm is given in Section 2. The existence result is stated in Section 3. Section 4 is devoted to the characterization of the extremals of both relaxed and initial problems, from which we deduce some qualitative description of the optimal plan. The proofs are collected in Sections 5 and 6.

2. The model formulation

The firm acts in continuous time on a finite planning period $[0,T]$. It is endowed with an initial inventory of $s_0 \in \mathbb{R}^+$ units of the good.

A sales/production plan is represented by a couple (x,y) of functions in $L^1_+[0,T]$, the set of non-negative elements of $L^1[0,T]$, where $x(t)$ (resp. $y(t)$) is the sales (resp. production) rate in units of the good at time t. In other words $\int_0^t x(u)\,\mathrm{d}u$ (resp. $\int_0^t y(u)\,\mathrm{d}u$) is the cumulative quantity of the good sold out (resp. produced) up to time t. We shall say that $(x,y) \in L^1_+[0,T] \times L^1_+[0,T]$ is a sales/production plan if the induced inventory $S^{(x,y)}$ satisfies

$$S^{(x,y)}(t) \triangleq s_0 + \int_0^t y(u)\,\mathrm{d}u - \int_0^t x(u)\,\mathrm{d}u \geqslant 0, \qquad \forall t \in [0,T], \tag{1}$$

which means that the company must never be out of stock. We denote by $\mathscr{A}$ the set of all sales/production plans, i.e.

$$\mathscr{A} \triangleq \{(x,y) \in L^1_+[0,T] \times L^1_+[0,T] \,|\, \text{Eq. (1) holds}\}.$$

When selling out at the rate $x(t)$ at time t, the firm has a revenue rate of $\pi(x(t))$. The cost of producing at the rate $y(t)$ at time t is $c(y(t))$. Both π and c are continuous, non-decreasing functions on $\mathbb{R}^+$. They satisfy $\pi(0)=0$, $c(0)=0$ and $\pi(x) > 0$, $c(x) > 0$, for all positive x. The function π (resp. c) is assumed to be concave (resp. convex).

The cost of storing $S(t)$ units of the good at time t is denoted by $s(S(t))$. The function s is assumed to be continuous, non-decreasing on $\mathbb{R}^+$ and to satisfy $s(0)=0$ and $s(S) > 0$, for all positive S. Observe that, in contrast to the previous literature, we make no assumption on the shape of the storage cost function.

Given the discount rate $\lambda > 0$, the profit over time induced by $(x,y) \in \mathscr{A}$ is defined by

$$J(x,y) \triangleq \int_0^T \mathrm{e}^{-\lambda t}[\pi(x(t)) - c(y(t)) - s(S^{(x,y)}(t))]\,\mathrm{d}t.$$

Observe that by concavity of π and Jensen's inequality

$$\int_0^T \mathrm{e}^{-\lambda t}\pi(x(t))\,\mathrm{d}t < \infty.$$

Since the functions c and s are non-negative, it follows that J is well defined as a map from $\mathscr{A}$ into $\mathbb{R} \cup \{-\infty\}$.

The profit-maximizing company plans its sales/production schedule by solving the following optimization problem

$$\sup_{(x,y)\in\mathscr{A}} J(x,y). \tag{2}$$

3. Relaxed problem and existence result

We first study the case of no starting inventories: $s_0 = 0$. Observe that in this case, the inventory constraint on (x,y) reads

$$S^{(x,y)}(t) = \int_0^t y(u)\,\mathrm{d}u - \int_0^t x(u)\,\mathrm{d}u \geqslant 0, \qquad \forall t \in [0,T]. \tag{3}$$

Define

$$m_x \triangleq \frac{\int_0^T \mathrm{e}^{-\lambda t} x(t)\,\mathrm{d}t}{\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t}, \qquad m_y \triangleq \frac{\int_0^T \mathrm{e}^{-\lambda t} y(t)\,\mathrm{d}t}{\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t} \tag{4}$$

and notice that, by Jensen's inequality

$$J(x,y) \leqslant \left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t\right)[\pi(m_x) - c(m_y)] - \int_0^T \mathrm{e}^{-\lambda t} s(S^{(x,y)}(t))\,\mathrm{d}t.$$

Integrating by parts in (4) and using (3), we see that $m_x \leqslant m_y$. If $S^{(x,y)}$ is not identically null on $[0,T]$, we therefore have

$$J(x,y) < \left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t\right)[\pi(m_y) - c(m_y)] = J(m_y, m_y).$$

As a consequence, we have the following:

Proposition 1. *If $s_0 = 0$ then, Problem (2) has a solution if and only if the function $\pi - c$ admits a maximum. The set of solutions is the set of all $(x,x)\in\mathscr{A}$ with x valued in* $\operatorname{Argmax}_{\mathbb{R}^+}(\pi - c)$.

Proof. It follows from the above discussion that the set of solutions is necessarily included in $\{(x,y)\in\mathscr{A} \mid S^{(x,y)} \equiv 0\} = \{(x,y)\in\mathscr{A} \mid x = y \text{ a.e.}\}$. Writing

$$J(x,x) \leqslant \left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t\right) \sup_{m\in\mathbb{R}^+} (\pi(m) - c(m))$$

concludes the proof. □

The above proposition means that, if the firm has no goods in stock at time 0 then, the concavity/convexity of the revenue/production cost leads to a static optimal plan: the firm produces only for immediate sales. There is no inventory accumulation.

In light of this remark, we make the following:

Assumption 1. The initial inventory s_0 is positive and the function $\pi - c$ admits a maximum at some positive point a.

We shall prove in Proposition 3 that Problem (2) is always finite. However, we know that the maximum may be not attained. This is typically the case when s_0 is too high (see [4]). In order to obtain an existence result, we therefore have to consider a relaxed problem, the sup of which equals the sup in (2), see Proposition 3 below. The relaxed problem consists in allowing the firm to get rid of a certain amount of the good in stock at time 0. The link between the two problems is established in Proposition 3 and Corollary 4. In particular, we shall see that if there is a plan without depletion at time 0 which is optimal for the relaxed problem then, it is optimal for the initial planning problem.

The relaxed problem is constructed as follows. The sales rate is no longer described by an integrable function, but by a non-negative finite Borel measure on $[0,T]$ which has its singular part positively proportional to the Dirac measure at 0. In this framework, for a sales rate equal to $\alpha\delta_0 + x$, where $x \in L^1_+[0,T]$ represents the absolutely continuous part of the considered Borel measure, the cumulative sales process is given by

$$X_{(0)} = 0 \text{ and } X(t) = \alpha + \int_0^t x(u)\,\mathrm{d}u, \qquad \forall t \in (0,T].$$

Here, α is the share of the initial inventory that the firm sells out at time 0.

The production path is still assumed to be integrable. A sales/production plan is now a triplet (α, x, y) in $\mathbb{R}^+ \times L^1_+[0,T] \times L^1_+[0,T]$ which satisfies the inventory constraint

$$S^{(\alpha,x,y)}(t) = s_0 + \int_0^t y(u)\,\mathrm{d}u - \alpha - \int_0^t x(u)\,\mathrm{d}u \geqslant 0, \qquad \forall t \in (0,T]. \tag{5}$$

For $t = 0$, we set $S^{(\alpha,x,y)}(0) = s_0$. The inventory level $S^{(\alpha,x,y)}$ may jump downward ($\alpha \geqslant 0$) at $0+$.

We denote by $\mathscr{B}$ the set of relaxed sales/production plans:

$$\mathscr{B} \triangleq \{(\alpha, x, y) \in \mathbb{R}^+ \times L^1_+[0,T] \times L^1_+[0,T] \mid \text{ Eq (5) holds}\}.$$

The relaxed profit on $\mathscr{B}$ is defined by

$$\mathscr{F}(\alpha, x, y) = \alpha\dot{\pi}(\infty) + \int_0^T \mathrm{e}^{-\lambda t}[\pi(x(t)) - c(y(t)) - s(S^{(\alpha,x,y)}(t))]\,\mathrm{d}t,$$

where we have set

$$\dot{\pi}(\infty) \triangleq \lim_{x\to\infty} \pi(x)/x,$$

which is well defined in $\mathbb{R}^+$ by concavity and non-negativity of π. This is the price at which the firm can sell at an infinite rate. It is also the lowest price accessible for the company. However, since holding inventories has a cost, the firm may take advantage of an immediate depletion, even at this price. Observe that, when π is differentiable, we have

$$\lim_{x\to\infty}\frac{\pi(x)}{x}=\lim_{x\to\infty}\dot{\pi}(x). \tag{6}$$

Remark 2. The plan $(0,x,y)\in\mathscr{B}$ if and only if $(x,y)\in\mathscr{A}$. Moreover, $\mathscr{F}(0,\cdot)\equiv J(\cdot)$ on $\mathscr{A}$.

From now on, we shall work on the relaxed optimization problem

$$\sup_{(\alpha,x,y)\in\mathscr{B}}\mathscr{F}(\alpha,x,y). \tag{7}$$

This relaxation is justified by the

Proposition 3. *Let $\mathscr{A}^*$ (resp. $\mathscr{B}^*$) be the subset of $\mathscr{A}$* (resp. $\mathscr{B}$) *given by*

$$\mathscr{A}^*\triangleq\{(x,y)\in\mathscr{A}\mid x\geqslant a\geqslant y \text{ a.e., } x \text{ is non-increasing, } y \text{ is non-decreasing}\},$$

$$\mathscr{B}^*\triangleq\{(\alpha,x,y)\in\mathscr{B}\mid x\geqslant a\geqslant y \text{ a.e., } x \text{ is non-increasing, } y \text{ is non-decreasing}\}.$$

Then, we have

$$\sup_{(x,y)\in\mathscr{A}}J(x,y)=\sup_{(x,y)\in\mathscr{A}^*}J(x,y)=\sup_{(\alpha,x,y)\in\mathscr{B}^*}\mathscr{F}(\alpha,x,y)=\sup_{(\alpha,x,y)\in\mathscr{B}}\mathscr{F}(\alpha,x,y).$$

Moreover, if (x,y) (resp. (α,x,y)) is a maximum of J on $\mathscr{A}$ (resp. of $\mathscr{F}$ on $\mathscr{B}$) then $x\geqslant y$ a.e. on $[0,T]$.

As a direct consequence of the above proposition and Remark 2, we obtain the following relation between the initial and relaxed problems.

Corollary 4. *(x,y) solves $\sup_{\mathscr{A}}J$ if and only if $(0,x,y)$ solves $\sup_{\mathscr{B}}\mathscr{F}$.*

It also results from the right-hand side equality of Proposition 3, that we can replace $\mathscr{B}$ by $\mathscr{B}^*$ in (7). This optimization domain restriction will play an essential role in the proof of our existence result.

Theorem 5. *The optimization problem (7) has at least one solution which is in $\mathscr{B}^*$.*

Remark 6. The lack of uniqueness of the respective solutions of Problem (2) and Problem (7) prevent us from asserting that solving Problem (2) is equivalent to solving Problem (7). Typically, we cannot affirm that if Problem (7) has a solution with a singular part ($\alpha>0$) then, Problem (2) has no solution. This question is answered in [4] in the case of a convex storage cost function. The authors establish that Problem (2) has a solution if and only if Problem (7) has a solution with $\alpha=0$. Moreover, they characterized the situations (depending on s_0) where the solution of Problem (7) does or does not have a singular part.

4. The characterization of extremals

We turn now to our main purpose: to give a precise qualitative description of the optimal plans. To do this, we derive some characterizations of the extremals of (2) and (7). We shall require the following additional regularity assumptions on π, c and s.

Assumption 2. The function $\pi - c$ admits a unique maximum at the positive point a. The function π is differentiable on $(0, \infty)$, its derivative $\dot{\pi}$ is continuous and one to one on $[a, \infty)$. The function c is differentiable on $\mathbb{R}^+$, its derivative $\dot{c}$ is continuous and one to one on $[0, a]$. The function s is continuously differentiable on $\mathbb{R}^+$.

We start with characterizing the extremals of (7).

Theorem 7. *Let* $(\alpha, x, y) \in \mathscr{B}$.

If (α, x, y) *is a solution of Problem* (7) *then*,

1. $(\alpha, x, y) \in \mathscr{B}^*$,
2. $S^{(\alpha,x,y)}(T) = 0$, $T_0 \triangleq \inf\{t \in (0, T] \mid S^{(\alpha,x,y)}(t) = 0\} \in (0, T]$,
3. *if* $T_0 < T$ *then*, $S^{(\alpha,x,y)} \equiv 0$ *on* $[T_0, T]$ *and* $x = y = a$ *a.e. on* $[T_0, T]$,
4. *the function* $S^{(\alpha,x,y)}$ *is decreasing on* $[0, T_0]$,
5. *the functions* x *and* y *are both continuous on* $(0, T_0)$ *and they satisfy the following system*:

$$\forall t \in (0, T_0), \qquad \begin{cases} e^{-\lambda t}\dot{\pi}(x(t)) - \displaystyle\int_0^t e^{-\lambda u}\dot{s}(S^{(\alpha,x,y)}(u))\,du = \dot{\pi}(x(0+)), \\ y(t) = g(x(t)), \end{cases} \tag{8}$$

$$\textit{where } g(z) = \begin{cases} \dot{c}^{-1}(\dot{\pi}(z)) & \textit{if } \dot{\pi}(z) > \dot{c}(0) \\ 0 & \textit{elsewhere} \end{cases}, \forall z \in [a, \infty),$$

6. *if* $T_0 < T$ *then*, $x(T_0-) = a$,
7. *if* $\alpha > 0$ *then*, $x(0+) = \infty$.

Conversely, suppose that (α, x, y) *satisfies the above conditions then, for every* $(\beta, h, k) \in \mathbb{R} \times L^1[0, T] \times L^1[0, T]$ *such that* $(\alpha, x, y) + \varepsilon_0(\beta, h, k) \in \mathscr{B}$ *for some* $\varepsilon_0 > 0$, *we have*

$$\limsup_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left\{ \frac{\mathscr{F}((\alpha, x, y) + \varepsilon(\beta, h, k)) - \mathscr{F}(\alpha, x, y)}{\varepsilon} \right\} \leqslant 0.$$

We postpone the proof of this result to Section 6 and turn to the characterization of the extremal sales/production plans of our initial problem (2). This is an immediate corollary of Proposition 3 and Theorem 7.

Corollary 8. *Let* $(x, y) \in \mathscr{A}$.
If (x, y) *is a solution of Problem* (2) *then,*

1. $(x, y) \in \mathscr{A}^*$,
2. $S^{(x,y)}(T) = 0$ *and* $T_0 \triangleq \inf\{t \in (0, T] \mid S^{(x,y)}(t) = 0\} \in (0, T]$,
3. *if* $T_0 < T$ *then,* $S^{(x,y)} \equiv 0$ *on* $[T_0, T]$ *and* $x = y = a$ *a.e. on* $[T_0, T]$,
4. *the function* $S^{(x,y)}$ *is decreasing on* $[0, T_0]$,
5. *the functions* x *and* y *are both continuous on* $(0, T_0)$ *and they satisfy the following system*:

$$\forall t \in (0, T_0), \begin{cases} e^{-\lambda t}\dot{\pi}(x(t)) - \displaystyle\int_0^t e^{-\lambda u}\dot{s}(S^{(x,y)}(u))\,du = \dot{\pi}(x(0+)), \\ y(t) = g(x(t)), \end{cases} \tag{9}$$

where g *is defined as in Theorem* 7,

6. *if* $T_0 < T$ *then,* $x(T_0-) = a$.

Conversely, suppose that (x, y) *satisfies the above conditions then, for every* $(h, k) \in L^1[0, T] \times L^1[0, T]$ *such that* $(x, y) + \varepsilon_0(h, k) \in \mathscr{A}$ *for some* $\varepsilon_0 > 0$, *we have*

$$\limsup_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left\{ \frac{J((x, y) + \varepsilon(h, k)) - J(x, y)}{\varepsilon} \right\} \leqslant 0.$$

The above corollary states that, if the revenue is concave and if the production cost is convex then, the behavior of the company towards inventories is qualitatively the same for any kind of storage cost function: there is no inventory accumulation. The sales rate is always greater than the production rate. The optimal way to deplete the initial inventory is in two phases. This leads to a three phases sales/production plan. The first phase is devoted to the selling activity. The sales rate is non-increasing or equivalently the marginal revenue is non-decreasing. If the marginal revenue overtakes the lowest marginal cost of production then, the production activity actually starts. During this second phase, the sales rate and the production rate are such that the marginal revenue equals the marginal production cost (see the definition of g). The sales level is still non-increasing, the production rate non-decreasing. The third phase starts when the stock is all cleared. Production and sales are at the same constant rate, the one that maximizes the instantaneous profit.

5. Maximization domain restriction, relaxed problem justification and existence result: proofs

5.1. Proof of Proposition 3

We split the proof in several steps. The first three steps are dedicated to the proof of the equality

$$\sup_{\mathscr{B}} \mathscr{F} = \sup_{\mathscr{B}^*} \mathscr{F}. \tag{10}$$

Since $\mathscr{B}^* \subset \mathscr{B}$, we only have to prove that $\sup_{\mathscr{B}} \mathscr{F} \leqslant \sup_{\mathscr{B}^*} \mathscr{F}$. We do it by showing that

$$\forall (\alpha, x, y) \in \mathscr{B},\ \exists (\alpha, x^*, y_*) \in \mathscr{B}^* : \mathscr{F}(\alpha, x^*, y_*) \geqslant \mathscr{F}(\alpha, x, y). \tag{11}$$

We will then prove that $\sup_{\mathscr{B}^*} \mathscr{F} < \infty$, and the last step will end the proof.

Observe that, since $(s_0, a, a) \in \mathscr{B}^*$ and satisfies $\mathscr{F}(s_0, a, a) \geqslant 0$, we may restrict our attention to plans such that $\mathscr{F}(\alpha, x, y) > -\infty$ and we redefine $\mathscr{B}$ and $\mathscr{B}^*$ accordingly.

In the following, for a measurable set $E \subset [0, T]$, we shall denote by $|E|$ its Lebesgue measure.

Step 1: *For all* $(\alpha, x, y) \in \mathscr{B}$, *there exists some* $(\hat{x}, \hat{y})$ *such that* $(\alpha, \hat{x}, \hat{y}) \in \mathscr{B}$, $\hat{x} \geqslant \hat{y}$ *a.e. and* $\mathscr{F}(\alpha, \hat{x}, \hat{y}) \geqslant \mathscr{F}(\alpha, x, y)$. *Moreover, if* $|\{y > x\}| > 0$, *then* $\mathscr{F}(\alpha, \hat{x}, \hat{y}) > \mathscr{F}(\alpha, x, y)$.

For ease of notation, we shall write

$$S(t) \triangleq s_0 - \alpha + \int_0^t [y(u) - x(u)]\,\mathrm{d}u,\ \ \forall t \in [0, T].$$

By definition, S is absolutely continuous on $[0, T]$. Let us define

$$M(t) \triangleq \min_{[0,t]} S(u),\ \ \forall t \in [0, T].$$

We will prove that the triplet $(\alpha, \hat{x}, \hat{y})$ defined by

$$\hat{x} = \hat{y} = a \quad \text{on } \{M < S\} \triangleq \{t \in [0, T] \mid M(t) < S(t)\},$$

$$\begin{cases} \hat{x} = x \\ \hat{y} = y \end{cases} \quad \text{on} \quad \{M = S\} \triangleq \{t \in [0, T] \mid M(t) = S(t)\}$$

satisfies the requirements of Step 1.

We shall make use of the following technical result whose proof is reported at the end of this section.

Lemma 1. 1. *The function* M *is absolutely continuous on* $[0, T]$.

2. *The derivative of* M, *denoted by* $\dot{M}$, *exists almost everywhere and is integrable. Moreover,* $\dot{M} \equiv -[y - x]^- \mathbf{1}_{\{M=S\}}$, *where, for any real number* z, *we denote* $[z]^- = \max\{0, -z\}$. *More precisely,*

(a) $\dot{M} = 0$ *a.e. on* $\{M < S\}$,
(b) $\dot{M} = \dot{S} = y - x$ *a.e. on* $\{M = S\}$ *and*
(c) *the set* $\{y > x\} \cap \{M = S\}$ *has zero measure.*

3. $0 \leqslant M(t) = M(0) + \int_0^t \dot{M}(u)\,\mathrm{d}u = S^{(\alpha, \hat{x}, \hat{y})}(t)$, $\forall t \in (0, T]$.

By construction $(\hat{x}, \hat{y}) \in L^1_+[0, T] \times L^1_+[0, T]$. From property 3 of the above lemma, we deduce that $(\alpha, \hat{x}, \hat{y})$ satisfies (5), and therefore lies in $\mathscr{B}$. From property 2(c) and by construction of $(\hat{x}, \hat{y})$, we also see that $\hat{x} \geqslant \hat{y}$ a.e.

It remains to prove that $\mathscr{F}(\alpha,x,y) \leqslant \mathscr{F}(\alpha,\hat{x},\hat{y})$, with strict inequality if $|\{y>x\}|>0$. Writing

$$\{y > x\} = (\{y > x\} \cap \{M = S\}) \cup (\{y > x\} \cap \{M < S\})$$

we see from property 2(c) of Lemma 1 that $\{y > x\}$ has positive measure if and only if $\{M < S\}$ does. The above assertion is then equivalent to: $\mathscr{F}(\alpha,x,y) \leqslant \mathscr{F}(\alpha,\hat{x},\hat{y})$, with strict inequality if $|\{M < S\}| > 0$.

Since by construction $M \leqslant S$, it follows from property 3 of Lemma 1 that $S^{(\alpha,\hat{x},\hat{y})} \leqslant S$. Since the function s is non-decreasing, we therefore have by definition of $(\hat{x},\hat{y})$,

$$\begin{aligned}
&\mathscr{F}(\alpha,x,y) - \mathscr{F}(\alpha,\hat{x},\hat{y})\\
&\quad = \int_0^T \mathrm{e}^{-\lambda t}\pi(x(t))\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t - \pi(a)\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t\\
&\qquad - \int_0^T \mathrm{e}^{-\lambda t}c(y(t))\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t + c(a)\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t\\
&\qquad - \int_0^T \mathrm{e}^{-\lambda t}s(S(t))\,\mathrm{d}t + \int_0^T \mathrm{e}^{-\lambda t}s(S^{(\alpha,\hat{x},\hat{y})}(t))\,\mathrm{d}t\\
&\quad \leqslant \int_0^T \mathrm{e}^{-\lambda t}\pi(x(t))\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t - \pi(a)\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t\\
&\qquad - \int_0^T \mathrm{e}^{-\lambda t}c(y(t))\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t + c(a)\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t. \qquad (12)
\end{aligned}$$

If $|\{M < S\}| = 0$ then, the right-hand side is null and hence $\mathscr{F}(\alpha,x,y) \leqslant \mathscr{F}(\alpha,\hat{x},\hat{y})$. In order to conclude the proof, we only have to show that $\mathscr{F}(\alpha,x,y) < \mathscr{F}(\alpha,\hat{x},\hat{y})$ whenever $|\{M < S\}| > 0$. We therefore assume from now on that $|\{M < S\}| > 0$. By Jensen's inequality, we deduce from (12) that

$$\begin{aligned}
&\mathscr{F}(\alpha,x,y) - \mathscr{F}(\alpha,\hat{x},\hat{y})\\
&\quad \leqslant [\pi(m_x) - c(m_y) - (\pi(a) - c(a))]\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t, \qquad (13)
\end{aligned}$$

where

$$m_x \triangleq \frac{\int_0^T \mathrm{e}^{-\lambda t}x(t)\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t}{\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t} \quad \text{and} \quad m_y \triangleq \frac{\int_0^T \mathrm{e}^{-\lambda t}y(t)\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t}{\int_0^T \mathrm{e}^{-\lambda t}\mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t}. \qquad (14)$$

Now, observe that, by definition of S and the characterization of $\dot{M}$ in 2 of Lemma 1,

$$
\begin{aligned}
S(t) - M(t) &= \int_0^t (y(u) - x(u))\,\mathrm{d}u + \int_0^t [y(u) - x(u)]^- \mathbf{1}_{\{M=S\}}(u)\,\mathrm{d}u \\
&= \int_0^t (y(u) - x(u))\mathbf{1}_{\{M<S\}}(u)\,\mathrm{d}u \\
&\quad + \int_0^t [y(u) - x(u)]^+ \mathbf{1}_{\{M=S\}}(u)\,\mathrm{d}u \\
&= \int_0^t (y(u) - x(u))\mathbf{1}_{\{M<S\}}(u)\,\mathrm{d}u, \quad \forall t \in [0, T],
\end{aligned}
$$

where the last equality is obtained by 2(c) of Lemma 1, and we used the notation $[z]^+ = \max\{0, z\}$. We deduce that

$$
\begin{aligned}
&\int_0^t (y(u) - x(u))\mathbf{1}_{\{M<S\}}(u)\,\mathrm{d}u = 0 \quad \text{on } \{M = S\} \\
&\int_0^t (y(u) - x(u))\mathbf{1}_{\{M<S\}}(u)\,\mathrm{d}u > 0 \quad \text{on } \{M < S\}.
\end{aligned}
\tag{15}
$$

Integrating by parts in (14) and using (15), it is easy to see that $m_x < m_y$. Since c is convex, non-decreasing, and satisfies $c(0) = 0$ and $c(y) > 0$, $\forall y > 0$, it is indeed increasing. It follows that $c(m_x) < c(m_y)$. Then, using (13) and recalling that $\pi - c$ reaches its maximum at a, we obtain

$$
\begin{aligned}
&\mathscr{F}(\alpha, x, y) - \mathscr{F}(\alpha, \hat{x}, \hat{y}) \\
&\quad < [\pi(m_x) - c(m_x) - (\pi(a) - c(a))] \int_0^T \mathrm{e}^{-\lambda t} \mathbf{1}_{\{M<S\}}(t)\,\mathrm{d}t \leqslant 0.
\end{aligned}
$$

This completes the proof of Step 1.

Remark 9. The last assertion of Proposition 3 is an immediate consequence of Step 1.

Step 2: *For all* $(\alpha, x, y) \in \mathscr{B}$ *such that* $x \geqslant y$ *a.e., there exists some* $(\hat{x}, \hat{y})$ *such that* $(\alpha, \hat{x}, \hat{y}) \in \mathscr{B}$, $\hat{x} \geqslant a \geqslant \hat{y}$ *a.e. and* $\mathscr{F}(\alpha, \hat{x}, \hat{y}) \geqslant \mathscr{F}(\alpha, x, y)$.

Let us define

$$
(\hat{x}(t), \hat{y}(t)) \triangleq \begin{cases} (x(t) - (y(t) - a), a) & \text{if } x(t) \geqslant y(t) > a, \\ (a, y(t) + (a - x(t))) & \text{if } a > x(t) \geqslant y(t), \\ (x(t), y(t)) & \text{if } x(t) \geqslant a \geqslant y(t). \end{cases}
$$

By construction $\hat{x} \geqslant a \geqslant \hat{y}$ a.e. Moreover, $\hat{x}$ and $\hat{y}$ are in $L^1_+[0, T]$ and

$$
\hat{x} - \hat{y} = x - y \text{ a.e. and therefore } S^{(\alpha, \hat{x}, \hat{y})} \equiv S^{(\alpha, x, y)}, \tag{16}
$$

so that $(\alpha,\hat{x},\hat{y})\in\mathscr{B}$. In order to conclude, we shall show that $\mathscr{F}(\alpha,\hat{x},\hat{y})\geqslant\mathscr{F}(\alpha,x,y)$. By (16), we have

$$\mathscr{F}(\alpha,\hat{x},\hat{y})-\mathscr{F}(\alpha,x,y) = \int_0^T e^{-\lambda t}[\pi(\hat{x}(t))-\pi(x(t))-(c(\hat{y}(t))-c(y(t)))]\,dt. \tag{17}$$

We shall prove that for almost every $t\in[0,T]$,

$$\pi(\hat{x}(t))-\pi(x(t))-[c(\hat{y}(t))-c(y(t))] \geqslant [\dot{\pi}_l(a)-\dot{c}_l(a)](a-x(t))\mathbf{1}_{\{x\leqslant a\}}(t) + [\dot{\pi}_r(a)-\dot{c}_r(a)](a-y(t))\mathbf{1}_{\{y\geqslant a\}}(t), \tag{18}$$

where $\dot{\pi}_l$ and $\dot{c}_l$ (resp. $\dot{\pi}_r$ and $\dot{c}_r$) are the left (resp. right) derivatives of π and c. Since $\pi-c$ reaches its maximum at a, we have

$$[\dot{\pi}_l(a)-\dot{c}_l(a)](a-x(t))\mathbf{1}_{\{x\leqslant a\}}(t)+[\dot{\pi}_r(a)-\dot{c}_r(a)](a-y(t))\mathbf{1}_{\{y\geqslant a\}}(t)\geqslant 0. \tag{19}$$

Hence, (18) will imply that $\mathscr{F}(\alpha,\hat{x},\hat{y})\geqslant\mathscr{F}(\alpha,x,y)$. Let us denote $\delta(t)=\pi(\hat{x}(t))-\pi(x(t))-[c(\hat{y}(t))-c(y(t))]$. On $\{t\in[0,T]\,|\,x(t)\geqslant a\geqslant y(t)\}$ inequality (18) holds because both sides are null. On $\{t\in[0,T]\,|\,a>x(t)\geqslant y(t)\}$, we have by construction $\hat{x}(t)=a$ and $\hat{y}(t)=y(t)+a-x(t)$. It follows from the concavity of π and $-c$ that

$$\delta(t)=\pi(a)-\pi(x(t))-[c(y(t)+a-x(t))-c(y(t))]$$
$$\geqslant \dot{\pi}_l(a)(a-x(t))-\dot{c}_l(y(t)+a-x(t))(a-x(t)) \tag{20}$$
$$\geqslant \dot{\pi}_l(a)(a-x(t))-\dot{c}_l(a)(a-x(t)), \tag{21}$$

where inequality (20) (resp. 21) holds because $x(t)<a$ and $y(t)<y(t)+a-x(t)$ (resp. $y(t)+a-x(t)\leqslant a$). The same kind of arguments shows that on $\{t\in[0,T]\,|\,x(t)\geqslant y(t)>a\}$, $\delta(t)\geqslant\dot{\pi}_r(a)(a-y(t))-\dot{c}_r(a)(a-y(t))$. Recalling that $x\geqslant y$ a.e. by assumption, this proves that inequality (18) holds a.e. on $[0,T]$. This concludes Step 2.

For later purpose, we make the following.

Remark 10. Assume that π is strictly concave on $[a,\infty)$ and that c is strictly convex on $[0,a]$. If one of the sets $\{t\in[0,T]\,|\,x(t)<a\}$ or $\{t\in[0,T],y(t)>a\}$ has positive measure then, $\mathscr{F}(\alpha,\hat{x},\hat{y})>\mathscr{F}(\alpha,x,y)$. Indeed, for every $t\in[0,T]$ such that $x(t)<a$, we have $y(t)\leqslant x(t)<a$ and $y(t)+a-x(t)<a$. Since c is strictly convex on $[0,a]$, it follows that inequality (20) is strict. Hence, if the set $\{t\in[0,T]\,|\,x(t)<a\}$ has positive measure, inequality (18) is strict on a set of positive measure and therefore, by (17), $\mathscr{F}(\alpha,\hat{x},\hat{y})>\mathscr{F}(\alpha,x,y)$. Analogously, we check that if π is strictly concave on $[a,\infty)$ and if the set $\{t\in[0,T]\,|\,y(t)>a\}$ has positive measure then, $\mathscr{F}(\alpha,\hat{x},\hat{y})>\mathscr{F}(\alpha,x,y)$.

Step 3. For all $(\alpha,x,y)\in\mathscr{B}$ such that $x\geqslant a\geqslant y$ a.e., there exists some (x^*,y_*) such that $(\alpha,x^*,y_*)\in\mathscr{B}^*$ and $\mathscr{F}(\alpha,x^*,y_*)\geqslant\mathscr{F}(\alpha,x,y)$.

Let x^* be the non-increasing rearrangement of x and y_* be the non-decreasing rearrangement of y (see [7] for the definitions of the rearrangement operators and their basic properties). From the increasing feature of the rearrangement operators and the assumption $x \geqslant a \geqslant y \geqslant 0$ a.e., we have $x^* \geqslant a \geqslant y_* \geqslant 0$. Moreover, the equi-measurability property provides the equalities

$$\int_0^T y_*(u)\,\mathrm{d}u = \int_0^T y(u)\,\mathrm{d}u \quad \text{and} \quad \int_0^T x^*(u)\,\mathrm{d}u = \int_0^T x(u)\,\mathrm{d}u.$$

Therefore,

$$\begin{aligned} S^{(\alpha,x^*,y_*)}(t) &= s_0 - \alpha + \int_0^t (y_*(u) - x^*(u))\,\mathrm{d}u \\ &\geqslant s_0 - \alpha + \int_0^T (y_*(u) - x^*(u))\,\mathrm{d}u = S^{(\alpha,x,y)}(T) \geqslant 0, \quad \forall t \in (0,T], \end{aligned}$$

where the inequality holds because $x^* \geqslant y_*$. Recalling that $x^* \geqslant a \geqslant y_* \geqslant 0$, and that x^* (resp. y_*) is non-increasing (resp. non-decreasing), it follows that $(\alpha, x^*, y_*) \in \mathscr{B}^*$. It remains to prove that $\mathscr{F}(\alpha, x^*, y_*) \geqslant \mathscr{F}(\alpha, x, y)$.

By property of the rearrangement operators, we have

$$\int_0^t y_*(u)\,\mathrm{d}u \leqslant \int_0^t y(u)\,\mathrm{d}u \quad \text{and} \quad \int_0^t x^*(u)\,\mathrm{d}u \geqslant \int_0^t x(u)\,\mathrm{d}u, \ \forall t \in [0,T].$$

Consequently,

$$\begin{aligned} S^{(\alpha,x^*,y_*)}(t) &= s_0 - \alpha + \int_0^t y_*(u)\,\mathrm{d}u - \int_0^t x^*(u)\,\mathrm{d}u \\ &\leqslant s_0 - \alpha + \int_0^t y(u)\mathrm{d}u - \int_0^t x(u)\,\mathrm{d}u = S^{(\alpha,x,y)}(t). \end{aligned}$$

From the Hardy–Littlewood inequality and the increasing feature of s, it follows that

$$\begin{aligned} \mathscr{F}(\alpha, x^*, y_*) - \mathscr{F}(\alpha, x, y) = &\int_0^T \mathrm{e}^{-\lambda t}\pi(x^*(t))\,\mathrm{d}t - \int_0^T \mathrm{e}^{-\lambda t}\pi(x(t))\,\mathrm{d}t \\ &- \left[\int_0^T \mathrm{e}^{-\lambda t}c(y_*(t))\,\mathrm{d}t - \int_0^T \mathrm{e}^{-\lambda t}c(y(t))\,\mathrm{d}t\right] \\ &- \int_0^T \mathrm{e}^{-\lambda t}[s(S^{(\alpha,x^*,y_*)}(t)) - s(S^{(\alpha,x,y)}(t))]\,\mathrm{d}t \\ \geqslant\ &0. \end{aligned}$$

This concludes the proof of Step 3.

These three first steps prove (11) and therefore (10).

Remark 11. The same arguments show that $\sup_{\mathscr{A}} J = \sup_{\mathscr{A}^*} J$. The proof is therefore omitted.

Step 4. $\sup_{\mathscr{B}^*} \mathscr{F} < \infty$.

Fix $(\alpha, x, y) \in \mathscr{B}^*$. It follows from the non-negativity of x, (5) and the inequality $0 \leqslant y \leqslant a$ a.e. that

$$\int_0^t \mathrm{e}^{-\lambda u} x(u)\,\mathrm{d}u \leqslant \int_0^t x(u)\,\mathrm{d}u \leqslant s_0 - \alpha + \int_0^t y(u)\,\mathrm{d}u \leqslant s_0 - \alpha + Ta,\ \ \forall t \in (0, T].$$

Observe that

$$\mathscr{F}(\alpha, x, y) \leqslant \dot{\pi}(\infty)\alpha + \left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t\right) \pi \left(\frac{1}{\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t} \int_0^T \mathrm{e}^{-\lambda t} x(t)\,\mathrm{d}t\right)$$

by Jensen's inequality and non-negativity of c and s. Sending t to 0 in (5), we also see that $0 \leqslant S^{(\alpha,x,y)}(0+) = s_0 - \alpha$. We then deduce from the two previous inequalities and the increasing feature of π that $\mathscr{F}$ is uniformly bounded from above on $\mathscr{B}^*$ by

$$\dot{\pi}(\infty)s_0 + \left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t\right) \pi \left(\frac{1}{\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}t}(s_0 + Ta)\right).$$

This concludes the proof of Step 4.

The following step ends the proof of Proposition 3.

Step 5. $\sup_{\mathscr{A}} J = \sup_{\mathscr{B}} \mathscr{F}$.

Since $\{0\} \times \mathscr{A} \subset \mathscr{B}$ and $\mathscr{F}(0, \cdot) = J(\cdot)$ on $\mathscr{A}$, we have $\sup_{\mathscr{A}} J \leqslant \sup_{\mathscr{B}} \mathscr{F}$ and thus, we only have to establish the converse inequality. In virtue of the equalities $\sup_{\mathscr{B}} \mathscr{F} = \sup_{\mathscr{B}^*} \mathscr{F}$ and $\sup_{\mathscr{A}} J = \sup_{\mathscr{A}^*} J$ that comes from the above steps, it is enough to prove that $\sup_{\mathscr{A}} J \geqslant \sup_{\mathscr{B}^*} \mathscr{F}$. We do it by proving that for every $(\alpha, x, y) \in \mathscr{B}^*$, there exists a sequence $(x_n)_n$ such that $(x_n, y) \in \mathscr{A}$ and $\lim_{n\to\infty} J(x_n, y) = \mathscr{F}(\alpha, x, y)$.

Fix $(\alpha, x, y) \in \mathscr{B}^*$, we shall prove that $x_n \triangleq n\alpha \mathbf{1}_{[0,1/n]} + x\mathbf{1}_{[1/n,T]}$ satisfies the above requirements. We have

$$\begin{aligned} S^{(x_n,y)}(t) = s_0 + \int_0^t y(u)\,\mathrm{d}u - \alpha n t &\geqslant s_0 + \int_0^t y(u)\,\mathrm{d}u - \alpha \\ &\geqslant S^{(\alpha,x,y)}(t)\ \text{ if }\ t \leqslant \frac{1}{n}, \end{aligned}$$

$$S^{(x_n,y)}(t) = s_0 + \int_0^t y(u)\,\mathrm{d}u - \alpha - \int_{1/n}^t x(u)\,\mathrm{d}u \geqslant S^{(\alpha,x,y)}(t)\ \text{ if }\ t > \frac{1}{n}.$$

Since $S^{(\alpha,x,y)} \geqslant 0$, $(x_n, y) \in \mathscr{A}$. Moreover, $S^{(x_n,y)}$ is bounded by $s_0 + Ta$ and it converges simply towards $S^{(\alpha,x,y)}$. Therefore, since s is continuous, it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^T \mathrm{e}^{-\lambda t} s(S^{(x_n,y)}(t))\,\mathrm{d}t = \int_0^T \mathrm{e}^{-\lambda t} s(S^{(\alpha,x,y)}(t))\,\mathrm{d}t.$$

Observing that

$$\int_0^T e^{-\lambda t}\pi(x_n(t))\,dt = \int_0^{1/n} e^{-\lambda t}\pi(\alpha n)\,dt + \int_{1/n}^T e^{-\lambda t}\pi(x(t))\,dt$$
$$= \frac{\pi(\alpha n)}{n}\left(\frac{1-e^{-\lambda 1/n}}{\lambda 1/n}\right) + \int_{1/n}^T e^{-\lambda t}\pi(x(t))\,dt$$

we also deduce that

$$\lim_{n\to\infty}\int_0^T e^{-\lambda t}\pi(x_n(t))\,dt = \dot{\pi}(\infty)\alpha + \int_0^T e^{-\lambda t}\pi(x(t))\,dt.$$

We have proved the convergence $\lim_{n\to\infty} J(x_n,y) = \mathscr{F}(\alpha,x,y)$. The proof of Proposition 3 is completed. □

Proof of Lemma 1. By construction, M is non-increasing and satisfies $0 \leqslant M \leqslant S$ on $[0,T]$. In order to prove that M is absolutely continuous, we first establish that

$$\forall\ 0 \leqslant u < v \leqslant T,\ \exists \bar{v}\in[u,v] \text{ such that } |M(u)-M(v)| \leqslant |S(u)-S(\bar{v})|. \tag{22}$$

Let $0\leqslant u<v\leqslant T$. The case where $M(u)=M(v)$ is obvious. Assume that $M(v)<M(u)$. Then, we have $M(v)=\min_{t\in[0,v]} S(t)=\min_{t\in[u,v]} S(t)$, i.e. S reaches its minimum on $[0,v]$ at some point $\bar{v}\in(u,v]$ and we have $S(\bar{v})=M(v)<M(u)\leqslant S(u)$, so that (22) holds.

Now, fix $\varepsilon>0, n\in\mathbb{N}$, and let $\{u_i,v_i\}_{i=1}^n$ be ends of n non-overlapping (possibly abutting) subintervals $[u_i,v_i]$ of $[0,T]$. Let $\bar{v}_i$ be defined as in (22) with $(u,v)=(u_i,v_i)$. The intervals $[u_i,\bar{v}_i]$, are non-overlapping and satisfy

$$\sum_{i=1}^n |u_i-\bar{v}_i| \leqslant \sum_{i=1}^n |u_i-v_i| \quad\text{and}\quad \sum_{i=1}^n |M(u_i)-M(v_i)| \leqslant \sum_{i=1}^n |S(u_i)-S(\bar{v}_i)|.$$

Therefore, since S is absolutely continuous on $[0,T]$, there exists some $\delta>0$ such that

$$\sum_{i=1}^n |u_i-v_i| < \delta \Rightarrow \sum_{i=1}^n |M(u_i)-M(v_i)| < \varepsilon.$$

This proves 1.

2. Since M is absolutely continuous, $\dot{M}$ exists almost everywhere and is integrable.

2(a). Since the functions M and S are both continuous on $[0,T]$, the set $\{M<S\}$ is open. Hence, it is enough to prove that M is constant on each connected component of $\{M<S\}$. Let I be such a (non-empty) component. It suffices to prove that $M(t)=S(\inf I)$, for all $t\in I$. Let $t\in I$. If $M(t)=\min_{u\in[0,t]} S(u)<S(\inf I)$ then S reaches its minimum on $[0,t]$ at some $\hat{u}\in(\inf I,t]$. We therefore have $M(\hat{u})=S(\hat{u})$ and $\hat{u}\in I\subset\{M<S\}$. This is a contradiction. Hence, $M(t)=S(\inf I)$.

2(b). By property of Lebesgue measurable sets, if $|\{M=S\}|>0$ then there exists $E\subset\{M=S\}$ such that $|\{M=S\}\backslash E|=0$ and

$$\lim_{\varepsilon\to 0}\frac{|(t-\varepsilon,t+\varepsilon)\cap\{M=S\}|}{2\varepsilon}=1,\ \forall t\in E.$$

It follows that, for every $t\in E$, the limits $\lim_{u\to t}(M(u)-M(t))/(u-t)$ and $\lim_{u\to t}(S(u)-S(t))/(u-t)$, when they exist, can be computed along some sequence $(t_n)_n$ in $\{M=S\}\backslash\{t\}$. We therefore have

$$\dot M(t)=\lim_{n\to\infty}\frac{M(t_n)-M(t)}{t_n-t}=\lim_{n\to\infty}\frac{S(t_n)-S(t)}{t_n-t}=\dot S(t)=y(t)-x(t).$$

2(c). Assume that

$$|\{y-x>0\}\cap\{M=S\}|>0.$$

Then, by 2(b), there exists some $t\in\{y-x>0\}\cap\{M=S\}$ such that $\dot S(t)=y(t)-x(t)>0$ and

$$\lim_{\varepsilon\to 0}\frac{|(t-\varepsilon,t+\varepsilon)\cap\{y-x>0\}\cap\{M=S\}|}{2\varepsilon}=1,$$

so that, there exists a sequence $(t_n)_n$ in $\{y-x>0\}\cap\{M=S\}\backslash\{t\}$ which converges towards t. Hence, for every sufficiently large $n\in\mathbb{N}$, we have

$$\frac{M(t_n)-M(t)}{t_n-t}=\frac{S(t_n)-S(t)}{t_n-t}>0,$$

which contradicts the decreasing feature of M.

Combining properties 2(a)–(c), we obtain $\dot M=-[y-x]^-\mathbf{1}_{\{M=S\}}$.

3. Finally, by absolute continuity of M, property 2, 2(c) and definition of $(\hat x,\hat y)$, we have

$$\begin{aligned}0\leqslant M(t)&=M(0)+\int_0^t\dot M(u)\,\mathrm{d}u\\&=s_0-\alpha-\int_0^t[y(u)-x(u)]^-\mathbf{1}_{\{M=S\}}(u)\,\mathrm{d}u\\&=s_0-\alpha+\int_0^t(y(u)-x(u))\mathbf{1}_{\{M=S\}}(u)\,\mathrm{d}u\\&=S^{(\alpha,\hat x,\hat y)}(t),\ \forall t\in(0,T].\end{aligned}$$

This concludes the proof of Lemma 1. □

5.2. *Proof of Theorem 5*

By Proposition 3, there exists a sequence $(\alpha_n,x_n,y_n)_n$ in $\mathscr{B}^*$ such that

$$\lim_{n\to\infty}\mathscr{F}(\alpha_n,x_n,y_n)=\sup_{\mathscr{B}}\mathscr{F}.$$

Step 1. There exist some $\alpha, \beta \in \mathbb{R}^+$ and some functions $x, y \in L^1_+[0,T]$ such that, possibly along some subsequence of (α_n, x_n, y_n),

1. *the sequence $(\alpha_n)_n$ converges to α,*
2. *the sequence $(y_n)_n$ converges a.e. to y,*
3. *the sequence $(x_n)_n$ converges a.e. to x and satisfies*

$$\lim_{n\to\infty} \int_0^T x_n(t) g(t)\,\mathrm{d}t = \beta g(0) + \int_0^T x(t) g(t)\,\mathrm{d}t$$

for every bounded function g on $[0,T]$, continuous in 0. In particular,

$$\lim_{n\to\infty} \int_0^t x_n(u)\,\mathrm{d}u = \beta + \int_0^t x(t)\,\mathrm{d}u, \ \forall t \in (0,T],$$

4. *$(\alpha+\beta, x, y) \in \mathscr{B}^*$ and the sequence $(S^{(\alpha_n, x_n, y_n)})_n$ converges pointwise to $S^{(\alpha+\beta,x,y)}$.*

Since the sequence $(\alpha_n)_n$ is valued in $[0, s_0]$, it is clear that up to a subsequence it converges towards some $\alpha \in [0, s_0]$.

The sequence $(y_n)_n$ is in the class of non-decreasing functions with values in $[0,a]$. Then, from the Helly compactness theorem (see [6] or [3]), it has a subsequence, which converges a.e. to some non-decreasing function y which satisfies $0 \leqslant y \leqslant a$.

Recalling that $y_n \leqslant a$ and $\alpha_n \geqslant 0$, for all $n \in \mathbb{N}$, we obtain from (5) that

$$\sup_{n\in\mathbb{N}} \int_0^T x_n(t)\,\mathrm{d}t \leqslant s_0 + Ta. \tag{23}$$

Since the sequence $(x_n)_n$ is in the class of non-increasing functions of $L^1_+[0,T]$, it follows that, for every $\delta \in (0,T]$ and $n \in \mathbb{N}$

$$0 \leqslant x_n(t) \leqslant \frac{1}{\delta}(s_0 + Ta), \ \forall t \in [\delta, T]. \tag{24}$$

Then, by the Helly compactness theorem, and by a classical diagonal extraction process, we can construct a subsequence, still denoted $(x_n)_n$, that converges a.e. to some x on $(0,T]$, which is non-negative, non-increasing and satisfies $x \geqslant a$ a.e. By Fatou's lemma and (23), we have

$$\int_0^T x(t)\,\mathrm{d}t \leqslant \liminf_n \int_0^T x_n(t)\,\mathrm{d}t \leqslant s_0 + Ta. \tag{25}$$

Since $x \geqslant 0$, it follows that $x \in L^1_+[0,T]$. Observe that, possibly after passing to a subsequence, we may assume that

$$\lim_n \int_0^T x_n(t)\,\mathrm{d}t = \liminf_n \int_0^T x_n(t)\,\mathrm{d}t.$$

We still denote by (α_n, x_n, y_n) the induced subsequence.

We now prove item 3. Let g be a bounded function on $[0,T]$, continuous in 0. Define $\tilde{g} \triangleq g - g(0)$ on $[0,T]$. Fix $\varepsilon > 0$. By continuity of $\tilde{g}$ in 0, there exists some $\delta > 0$

such that $0 \leqslant t \leqslant \delta \Rightarrow |\tilde{g}(t)| \leqslant \varepsilon/4(s_0 + Ta)$. Since $x \geqslant 0$ and $x_n \geqslant 0$, it follows from (23) that

$$\left|\int_0^T x_n(t)\tilde{g}(t)\,\mathrm{d}t - \int_0^T x(t)\tilde{g}(t)\,\mathrm{d}t\right| \leqslant \int_0^\delta |x_n(t) - x(t)||\tilde{g}(t)|\,\mathrm{d}t + \int_\delta^T |x_n(t)\tilde{g}(t) - x(t)\tilde{g}(t)|\,\mathrm{d}t \leqslant \frac{\varepsilon}{2} + \int_\delta^T |x_n(t)\tilde{g}(t) - x(t)\tilde{g}(t)|\,\mathrm{d}t.$$

By (24), the sequence $(x_n\tilde{g})_n$ is uniformly bounded by $(1/\delta)(s_0 + Ta)\|\tilde{g}\|_\infty$ on $[\delta, T]$. As it converges a.e. towards $x\tilde{g}$, we get from the dominated convergence theorem, that there exists some $n_0 \in \mathbb{N}$ for which

$$n \geqslant n_0 \Rightarrow \int_\delta^T |x_n(t)\tilde{g}(t) - x(t)\tilde{g}(t)|\,\mathrm{d}t \leqslant \frac{\varepsilon}{2}.$$

We have proved that $\lim_{n\to\infty} \int_0^T x_n(t)\tilde{g}(t)\,\mathrm{d}t = \int_0^T x(t)\tilde{g}(t)\,\mathrm{d}t$, i.e.

$$\lim_{n\to\infty} \int_0^T x_n(t)g(t)\,\mathrm{d}t = \beta g(0) + \int_0^T x(t)g(t)\,\mathrm{d}t, \tag{26}$$

where

$$\beta \triangleq \lim_n \int_0^T x_n(t)\,\mathrm{d}t - \int_0^T x(t)\,\mathrm{d}t.$$

Taking $g \equiv \mathbf{1}_{[0,t]}$ for $t > 0$ in (26), we obtain

$$\lim_{n\to\infty} \int_0^t x_n(u)\,\mathrm{d}u = \beta + \int_0^t x(u)\,\mathrm{d}u, \quad \forall t \in (0, T]. \tag{27}$$

This concludes the proof of item 3.

We finally prove item 4. As the sequence $(S^{(\alpha_n,x_n,y_n)})_n$ is in the class of non-increasing functions with values in $[0, s_0]$, it follows from the Helly compactness theorem, that it has a subsequence, still denoted $(S^{(\alpha_n,x_n,y_n)})_n$, which converges pointwise to some non-increasing function S with values in $[0, s_0]$. In order to conclude the proof of Step 1, it remains to check that $S \equiv S^{(\alpha+\beta,x,y)}$. Notice that this will show that $(\alpha+\beta, x, y) \in \mathscr{B}^*$. Indeed, we already know that the functions x and y are in $L^1_+[0, T]$, satisfy $x \geqslant a \geqslant y$ a.e., are, respectively, non-increasing and non-decreasing, and that S is non-negative.

By definition, $S^{(\alpha_n,x_n,y_n)}(0) = s_0$, $\forall n \in \mathbb{N}$, and therefore $S(0) = s_0$. Finally, since $(y_n)_n$ is bounded, we obtain, by the dominated convergence theorem and Eq. (27), that for every $t \in (0, T]$,

$$S(t) = \lim_{n\to\infty} \left(s_0 - \alpha_n + \int_0^t y_n(u)\,\mathrm{d}u - \int_0^t x_n(u)\,\mathrm{d}u\right)$$
$$= s_0 - \alpha + \lim_{n\to\infty} \left(\int_0^t y_n(u)\,\mathrm{d}u\right) - \lim_{n\to\infty} \left(\int_0^t x_n(u)\,\mathrm{d}u\right)$$

$$= s_0 - \alpha + \int_0^t y(u)\,\mathrm{d}u - \beta - \int_0^t x(u)\,\mathrm{d}u$$
$$= S^{(\alpha+\beta,x,y)}(t).$$

Step 2. The sequence $(\pi(x_n) - \dot{\pi}(\infty)x_n)_n$ is uniformly integrable and therefore

$$\lim_{n\to\infty} \int_0^T \mathrm{e}^{-\lambda t}[\pi(x_n(t)) - \dot{\pi}(\infty)x_n(t)]\,\mathrm{d}t = \int_0^T \mathrm{e}^{-\lambda t}[\pi(x(t)) - \dot{\pi}(\infty)x(t)]\,\mathrm{d}t.$$

Remind that $\sup_n \int_0^T x_n(t)\,\mathrm{d}t < \infty$ by (25). The function f defined by $f(z) \triangleq \pi(z) - \dot{\pi}(\infty)z$, $\forall z \in \mathbb{R}^+$, is continuous, non-decreasing, concave and satisfies: $f(0) = 0$ and $\lim_{z\to\infty} f(z)/z = 0$. Since the case where f is bounded is obvious, we assume that $\lim_{z\to\infty} f(z) = \infty$. The function f is concave, non-decreasing and unbounded, then, it is increasing and it admits an inverse G on $\mathbb{R}^+$ which satisfies $\lim_{z\to\infty} G(z)/z = \infty$. The proof is concluded by using the equality

$$\sup_n \int_0^T G(f(x_n(t)))\,\mathrm{d}t = \sup_n \int_0^T x_n(t)\,\mathrm{d}t < \infty$$

and applying the la Vallée–Poussin's criterion of uniform integrability (see [5]).

Step 3. $\sup_{\mathscr{B}} \mathscr{F} = \lim_{n\to\infty} \mathscr{F}(\alpha_n, x_n, y_n) = \mathscr{F}(\alpha+\beta, x, y)$.

By Steps 1 and 2, the uniform bounds on (y_n) and $(S^{(\alpha_n,x_n,y_n)})$, and by the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{F}(\alpha_n, x_n, y_n) &= \lim_{n\to\infty} \Bigg[\dot{\pi}(\infty)\alpha_n + \int_0^T \mathrm{e}^{-\lambda t}\pi(x_n(t))\,\mathrm{d}t \\
&\qquad - \int_0^T \mathrm{e}^{-\lambda t} c(y_n(t))\,\mathrm{d}t - \int_0^T \mathrm{e}^{-\lambda t} s(S^{(\alpha_n,x_n,y_n)}(t))\,\mathrm{d}t\Bigg] \\
&= \dot{\pi}(\infty)\alpha + \int_0^T \mathrm{e}^{-\lambda t}[\pi(x(t)) - \dot{\pi}(\infty)x(t)]\,\mathrm{d}t \\
&\qquad + \dot{\pi}(\infty)\lim_{n\to\infty}\left(\int_0^T \mathrm{e}^{-\lambda t} x_n(t)\,\mathrm{d}t\right) \\
&\qquad - \int_0^T \mathrm{e}^{-\lambda t} c(y(t))\,\mathrm{d}t - \int_0^T \mathrm{e}^{-\lambda t} s(S^{(\alpha+\beta,x,y)}(t))\,\mathrm{d}t \\
&= \dot{\pi}(\infty)\alpha + \int_0^T \mathrm{e}^{-\lambda t}\pi(x(t))\,\mathrm{d}t - \dot{\pi}(\infty)\int_0^T \mathrm{e}^{-\lambda t} x(t)\,\mathrm{d}t \\
&\qquad + \dot{\pi}(\infty)\left(\beta + \int_0^T \mathrm{e}^{-\lambda t} x(t)\,\mathrm{d}t\right)
\end{aligned}$$

$$- \int_0^T e^{-\lambda t} c(y(t))\,dt - \int_0^T e^{-\lambda t} s(S^{(\alpha+\beta,x,y)}(t))\,dt$$
$$= \mathscr{F}(\alpha+\beta,x,y).$$

This concludes the proof of Theorem 5. □

6. Proof of Theorem 7

Let $(\alpha,x,y)\in\mathscr{B}$. For ease of notation, we now denote by S the function $S^{(\alpha,x,y)}$.

We first assume that (α,x,y) is a solution of Problem (7) and establish the optimality conditions 1–7.

1. Let us prove that $(\alpha,x,y)\in\mathscr{B}^*$. We know, from Proposition 3, that $x \geqslant y$ a.e. on $[0,T]$. Moreover under Assumption 2, π is strictly concave on $[a,\infty)$ and c is strictly convex on $[0,a]$. Then, by Remark 10, we have

$$x \geqslant a \geqslant y \text{ a.e. on}[0,T].$$

Let us prove that x (resp. y) is non-increasing (resp. non-decreasing) on $[0,T]$. We have already seen, in the proof of Proposition 3, that by considering x^* the non-increasing rearrangement of x and y_* the non-decreasing rearrangement of y, we have $S^{(\alpha,x^*,y_*)} \geqslant S$ and $\mathscr{F}(\alpha,x^*,y_*) - \mathscr{F}(\alpha,x,y) \geqslant 0$. Then, by optimality, $\mathscr{F}(\alpha,x^*,y_*) - \mathscr{F}(\alpha,x,y) = 0$ i.e.

$$0 = \int_0^T e^{-\lambda t}\pi(x^*(t))\,dt - \int_0^T e^{-\lambda t}\pi(x(t))\,dt$$
$$- \left[\int_0^T e^{-\lambda t} c(y_*(t))\,dt - \int_0^T e^{-\lambda t} c(y(t))\,dt\right]$$
$$- \int_0^T e^{-\lambda t}[s(S^{(\alpha,x^*,y_*)}(t)) - s(S(t))]\,dt.$$

By the increasing feature of s and the Hardy–Littlewood inequality, the right-hand side is a sum of non-negative terms and therefore

$$\int_0^T e^{-\lambda t}[s(S^{(\alpha,x^*,y_*)}(t)) - s(S(t))]\,dt = \int_0^T e^{-\lambda t}\pi(x^*(t))\,dt - \int_0^T e^{-\lambda t}\pi(x(t))\,dt$$
$$= \int_0^T e^{-\lambda t} c(y_*(t))\,dt - \int_0^T e^{-\lambda t} c(y(t))\,dt$$
$$= 0.$$

Observe that by the increasing feature of π and $-c$, the non-increasing rearrangements of $\pi\circ x$ and $-c\circ y$ coincide with $\pi(x^*)$ and $-c(y_*)$. As the Hardy–Littlewood inequality is strict if the function which is rearranged is not monotone, we deduce from the above equalities that $\pi(x^*) = \pi(x)$ and $c(y_*) = c(y)$ a.e. Under Assumption 2, the functions π and c are increasing, so that $x = x^*$ and $y = y_*$ a.e. and therefore x is non-increasing and y is non-decreasing.

2. Let us prove that $S(T)=0$. Assume to the contrary that $S(T)>0$. Recalling that S is non-increasing by item 1, there is some interval $[\hat{t},T]$ on which S is bounded from below by some $\sigma>0$. Let $\hat{x} \triangleq x+(\sigma/(T-\hat{t}))\mathbf{1}_{[\hat{t},T]}$, then it is easy to check that $(\alpha,\hat{x},y)\in\mathscr{B}$ and $\mathscr{F}(\alpha,\hat{x},y)>\mathscr{F}(\alpha,x,y)$, which contradicts the optimal feature of (α,x,y). This proves the first part of item 2 and insures the existence of $T_0 \triangleq \inf\{t\in(0,T]\,|\,S(t)=0\}$. We shall prove in 3. that $T_0>0$.

3. Let us prove that if $T_0<T$ then, $S\equiv 0$ on $(T_0,T]$ and $x=y=a$ a.e. on $[T_0,T]$. From item 1, S is non-increasing on $[0,T]$. Since it is non-negative, it follows from the definition of T_0 that it is identically null on $(T_0,T]$. Since, by item 1, $x\geqslant a\geqslant y$ a.e. on $[0,T]$, this shows that $x=y=a$ a.e. on $[T_0,T]$.

We now prove that T_0 is positive. Since S is continuous on $(0,T]$, this will imply $S(T_0)=0$ and therefore conclude the proof. From the above discussion we know that if $T_0<T$ then, $S(T_0+)=0$. Then, assuming that $T_0=0$, leads to $s_0-\alpha=S(0+)=0$ and hence $\alpha>0$. By property 7 (which will be proved later) this implies that $x(0+)=\infty$ which contradicts the fact that $x=y=a$ a.e. on $[T_0,T]$.

4. Let us prove that S is decreasing on $[0,T_0]$. Since by item 1 $x\geqslant a\geqslant y$ a.e., it is enough to establish that the set $\{y=x=a\}\cap[0,T_0]$ has zero measure. Let us assume that it has positive measure. We shall construct a variation in order to end up with a contradiction. Define

$$\eta \triangleq \frac{|\{y=x=a\}\cap[0,T_0]|}{2}.$$

Writing that $|\{y=x=a\}\cap[T_0-\eta,T_0]|\leqslant\eta$, we see that $|\{y=x=a\}\cap[0,T_0-\eta)|>0$. Besides, since, by item 1, definition of T_0 and item 3, $S>0$ on $[0,T_0)$, and $S(T_0)=0$, it follows that $|\{x>y\}\cap(T_0-\eta/2,T_0)|>0$ and therefore $|\{x>y\}\cap(T_0-\eta/2,T_0)\cap\{x\geqslant a\}|>0$, by item 1. We can now apply Lusin's theorem to the sets $\{y=x=a\}\cap[0,T_0-\eta)$ and $\{x>y\}\cap(T_0-\eta/2,T_0)\cap\{x\geqslant a\}$, to find two compact sets $F\subset\{y=x=a\}\cap[0,T_0-\eta)$ and $K\subset\{x>y\}\cap(T_0-\eta/2,T_0)\cap\{x\geqslant a\}$ which both have positive measures and such that the restrictions $x_{|F}$ and $x_{|K}$ are continuous. Observe that, by construction, we have

$$\max F<\min K, \min_K x\geqslant a \quad \text{and} \quad \sigma\triangleq\min_{\mathrm{conv}(F\cup K)} S>0, \tag{28}$$

where the last inequality holds by continuity of S, item 1, definition of T_0 and the fact that the compact $\mathrm{conv}(F\cup K)\subset[0,T_0)$. Now, since $|F|>0$ and $|K|>0$, we can define $q=|F|/|K|$ and find ε_0 such that

$$0<\varepsilon_0<\min\left(\frac{a}{q},\frac{\sigma}{|F|}\right).$$

We are now in position to construct a sequence of variations that will lead to the required contradiction.

For $0<\varepsilon\leqslant\varepsilon_0$, let

$$x_\varepsilon \triangleq x+\varepsilon\mathbf{1}_F-\varepsilon q\mathbf{1}_K.$$

Using (28) and the definition of ε_0, it is easy to check that $x_\varepsilon\in L^1_+[0,T]$. Then, writing successively $S^{(\alpha,x_\varepsilon,y)}$ on $[0,\max F]$, $(\max F,\min K)$ and $[\min K,T]$, we immediately see from (28) and the definition of ε_0 again, that $0 \leqslant S^{(\alpha,x_\varepsilon,y)} \leqslant S$ on $[0,T]$, and in particular that $(\alpha,x_\varepsilon,y)\in\mathscr{B}$. Now, since s is non-decreasing and $x=a$ on F, we have

$$\frac{\mathscr{F}(\alpha,x_\varepsilon,y)-\mathscr{F}(\alpha,x,y)}{\varepsilon} \geqslant \int_0^T e^{-\lambda t}\left(\frac{\pi(a+\varepsilon)-\pi(a)}{\varepsilon}\right)\mathbf{1}_F(t)\,dt + \int_0^T e^{-\lambda t}\left(\frac{\pi(x(t)-\varepsilon q)-\pi(x(t))}{\varepsilon}\right)\mathbf{1}_K(t)\,dt.$$

By concavity of π and the inequality $x\geqslant a$ on K, this implies that

$$\frac{\mathscr{F}(\alpha,x_\varepsilon,y)-\mathscr{F}(\alpha,x,y)}{\varepsilon} \geqslant e^{-\lambda \max F}\dot{\pi}(a+\varepsilon)|F| - e^{-\lambda \min K} q\dot{\pi}(a-\varepsilon q)|K|.$$

Recalling that $\dot{\pi}$ is continuous in a, $q=|F|/|K|$ and that $\max F<\min K$, we deduce that

$$\liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \left\{\frac{\mathscr{F}(\alpha,x_\varepsilon,y)-\mathscr{F}(\alpha,x,y)}{\varepsilon}\right\} \geqslant (e^{-\lambda\max F}-e^{-\lambda\min K})\dot{\pi}(a)|F|>0,$$

which contradicts the optimal feature of (α,x,y) and concludes the proof of item 4.

5. Let us prove that the functions x and y are both continuous on $(0,T_0)$ and that they satisfy

$$\forall t\in(0,T_0),\quad \begin{cases} e^{-\lambda t}\dot{\pi}(x(t)) - \int_0^t e^{-\lambda u}\dot{s}(S(u))\,du = \dot{\pi}(x(0+)), \\ y(t)=g(x(t)), \end{cases} \tag{29}$$

where

$$g(z)=\begin{cases} \dot{c}^{-1}(\dot{\pi}(z)) & \text{if } \dot{\pi}(z)>\dot{c}(0), \\ 0 & \text{elsewhere}, \end{cases} \qquad \forall z\in[a,\infty).$$

We split the proof in two steps.

Step 1. The function x is continuous on $(0,T_0)$ and satisfies

$$\forall t\in(0,T_0),\ e^{-\lambda t}\dot{\pi}(x(t)) - \int_0^t e^{-\lambda u}\dot{s}(S(u))\,du = \dot{\pi}(x(0+))\in[\dot{\pi}(\infty),\dot{\pi}(a)].$$

We first prove that the left-hand side is constant a.e. Since, by items 3 and 4, S is decreasing on $[0,T_0]$ and $S(T_0)=0$, for small enough $\delta>0$, we have $S(t)\geqslant S(T_0-\delta)>0$, for every $t\in[0,T_0-\delta]$. Let $h\in C_c^\infty(0,T_0-\delta)$ be such that $\int_0^{T_0-\delta} h(t)\,dt=0$, and fix $\varepsilon_0>0$ satisfying

$$\varepsilon_0\|h\|_\infty < \min\left\{a, \frac{S(T_0-\delta)}{T_0-\delta}\right\}.$$

Observe that, for every $\varepsilon\in(0,\varepsilon_0]$, $(\alpha,x_\varepsilon,y)\in\mathscr{B}$ where we have set

$$x_\varepsilon \triangleq x+\varepsilon h\mathbf{1}_{[0,T_0-\delta]}.$$

Thus, by the optimal feature of (α, x, y) we have

$$0 \geqslant \frac{\mathscr{F}(\alpha, x_\varepsilon, y) - \mathscr{F}(\alpha, x, y)}{\varepsilon}$$

$$= \int_0^{T_0-\delta} e^{-\lambda t} \left(\frac{\pi(x(t) + \varepsilon h(t)) - \pi(x(t))}{\varepsilon} \right) dt$$

$$- \int_0^{T_0-\delta} e^{-\lambda t} \left(\frac{s(S(t) - \varepsilon \int_0^t h(u)\, du) - s(S(t))}{\varepsilon} \right) dt. \tag{30}$$

Observing that, for every $\varepsilon \in (0, \varepsilon_0]$ we have

$$\left| \frac{\pi(x(t) + \varepsilon h(t)) - \pi(x(t))}{\varepsilon} \right| \leqslant \dot{\pi}(a - \varepsilon_0 \|h\|_\infty) \|h\|_\infty \text{ a.e. on } [0, T_0 - \delta]$$

and

$$\left| \frac{s(S(t) - \varepsilon \int_0^t h(u)\, du) - s(S(t))}{\varepsilon} \right| \leqslant (T_0 - \delta) \|h\|_\infty \max_I \dot{s} \text{ a.e. on } [0, T_0 - \delta],$$

where $I = [0, s_0 + \varepsilon_0 \|h\|_\infty (T_0 - \delta)]$, we can let ε tend to 0 in (30) to get by the dominated convergence theorem

$$0 \geqslant \int_0^{T_0-\delta} e^{-\lambda t} \dot{\pi}(x(t)) h(t)\, dt + \int_0^{T_0-\delta} e^{-\lambda t} \dot{s}(S(t)) \left(\int_0^t h(u)\, du \right) dt.$$

Noticing that, by item 1,

$$0 \leqslant e^{-\lambda t} \dot{s}(S(t)) \leqslant \max_{[0, s_0]} \dot{s} \text{ for all } t \in [0, T], \tag{31}$$

we can integrate by parts the second term in the previous inequality to get

$$0 \geqslant \int_0^{T_0-\delta} e^{-\lambda t} \dot{\pi}(x(t)) h(t)\, dt - \int_0^{T_0-\delta} \left[\int_0^t e^{-\lambda u} \dot{s}(S(u))\, du \right] h(t)\, dt.$$

By homogeneity, the inequality is indeed an equality. This proves that, for every $h \in C_c^\infty(0, T_0 - \delta)$ satisfying $\int_0^{T_0-\delta} h(t)\, dt = 0$, we have

$$0 = \int_0^{T_0-\delta} \left[e^{-\lambda t} \dot{\pi}(x(t)) - \int_0^t e^{-\lambda u} \dot{s}(S(u))\, du \right] h(t)\, dt.$$

Since the function $t \to f(t) \triangleq e^{-\lambda t} \dot{\pi}(x(t)) - \int_0^t e^{-\lambda u} \dot{s}(S(u))\, du$ is in $L^\infty(0, T_0 - \delta)$, it follows from the previous result that there exists some constant k_δ for which

$$e^{-\lambda t} \dot{\pi}(x(t)) - \int_0^t e^{-\lambda u} \dot{s}(S(u))\, du = k_\delta \text{ a.e. on } [0, T_0 - \delta]. \tag{32}$$

It remains to prove that $k_\delta = \dot{\pi}(x(0+))$, for all small $\delta > 0$, where $x(0+) \triangleq \lim_{\substack{t\to 0\\ t>0}} x(t)$ is well defined in $\mathbb{R}^+ \cup \{\infty\}$ by the non-increasing feature of x (item 1), and $\lim_{\substack{t\to 0\\ t>0}} \dot{\pi}(x(t)) = \dot{\pi}(x(0+)) \in [\dot{\pi}(\infty), \dot{\pi}(a)] \subset \mathbb{R}$, by continuity of $\dot{\pi}$ on $[a,\infty)$ and by (6). We do it by letting t tend to 0 in (32) and using (31). This leads to

$$e^{-\lambda t}\dot{\pi}(x(t)) - \int_0^t e^{-\lambda u}\dot{s}(S(u))\,du = \dot{\pi}(x(0+)) \text{ a.e. on } \bigcup_{\delta>0}[0, T_0-\delta] = [0, T_0).$$

By continuity of the function $t \to e^{\lambda t}(\dot{\pi}(x(0+)) + \int_0^t e^{-\lambda u}\dot{s}(S(u))\,du)$ on $(0, T_0)$, and since $\dot{\pi}$ is continuous and one to one on $[a,\infty)$, this also shows that x is continuous on $(0, T_0)$.

Step 2. The function y is continuous on $(0, T_0)$ and satisfies

$$y(t) = \begin{cases} 0 & \text{if } \dot{\pi}(x(t)) < \dot{c}(0), \\ \dot{c}^{-1}(\dot{\pi}(x(t))) & \text{if } \dot{\pi}(x(t)) \geqslant \dot{c}(0). \end{cases}$$

Recalling that, by item 1, y is non-decreasing on $(0, T_0)$, by defining

$$T_y \triangleq \sup\{t \in (0, T_0) \mid y(t) = 0\}$$

with the convention $\sup\emptyset = 0$, we have $y(t) = 0, \forall t \in (0, T_y)$ and $y(t) > 0, \forall t \in (T_y, T_0)$. We first prove that

$$\dot{c}(y(t)) = \dot{\pi}(x(t)) \text{ for a.e. } t \text{ in } (T_y, T_0). \tag{33}$$

Let $h \in C_c^\infty(T_y, T_0)$, with support denoted by K. By definition of T_y and since y is non-decreasing, y is bounded from below by $y(\min K) > 0$ on K. Fix $\varepsilon_0 > 0$ such that $\varepsilon_0\|h\|_\infty < y(\min K)$. For every $\varepsilon \in (0, \varepsilon_0]$, set

$$(x_\varepsilon, y_\varepsilon) \triangleq (x, y) + \varepsilon(h, h)\mathbf{1}_{(T_y, T_0)}.$$

Then, clearly, $(\alpha, x_\varepsilon, y_\varepsilon) \in \mathscr{B}$ and $S^{(\alpha,x_\varepsilon,y_\varepsilon)} \equiv S^{(\alpha,x,y)}$. Therefore, from the optimal feature of (α, x, y) we have

$$\begin{aligned} 0 &\geqslant \frac{\mathscr{F}(\alpha, x_\varepsilon, y_\varepsilon) - \mathscr{F}(\alpha, x, y)}{\varepsilon} \\ &= \int_{T_y}^{T_0} e^{-\lambda t}\left(\frac{\pi(x(t) + \varepsilon h(t)) - \pi(x(t))}{\varepsilon}\right) dt \\ &\quad - \int_{T_y}^{T_0} e^{-\lambda t}\left(\frac{c(y(t) + \varepsilon h(t)) - c(y(t))}{\varepsilon}\right) dt. \end{aligned} \tag{34}$$

Noticing that, by construction, for every $\varepsilon \in (0, \varepsilon_0]$

$$\left|\frac{\pi(x(t) + \varepsilon h(t)) - \pi(x(t))}{\varepsilon}\right| \leqslant \dot{\pi}(a - \varepsilon_0\|h\|_\infty)\|h\|_\infty \text{ a.e. on } (T_y, T_0)$$

and

$$\left|\frac{c(y(t)+\varepsilon h(t))-c(y(t))}{\varepsilon}\right| \leqslant \dot{c}(a+\varepsilon_0\|h\|_\infty)\|h\|_\infty \text{ a.e. on } (T_y,T_0),$$

we can let ε tend to 0 in (34) to get, by the dominated convergence theorem,

$$0 \geqslant \int_{T_y}^{T_0} \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))h(t)\,\mathrm{d}t - \int_{T_y}^{T_0} \mathrm{e}^{-\lambda t}\dot{c}(y(t))h(t)\,\mathrm{d}t.$$

Since by homogeneity the inequality is in actual fact an equality, this proves that, for every $h \in C_c^\infty(T_y,T_0)$,

$$0 = \int_{T_y}^{T_0} \mathrm{e}^{-\lambda t}[\dot{\pi}(x(t)) - \dot{c}(y(t))]h(t)\,\mathrm{d}t.$$

Since the function $t \mapsto |\dot{\pi}(x(t)) - \dot{c}(y(t))|$ is in $L^\infty(T_y,T_0)$, we then have

$$\dot{\pi}(x(t)) = \dot{c}(y(t)) \quad \text{for a.e } t \text{ in } (T_y,T_0). \tag{35}$$

We have proved (33). Observe that, since by Step 1, the function $\dot{\pi}\circ x$ is continuous on (T_y,T_0) and $\dot{c}$ is continuous and one to one on $[0,a]$, this shows that

$$y \text{ is continuous on } (T_y,T_0). \tag{36}$$

We now concentrate on $(0,T_y)$. Considering variations of the form

$$(x_\varepsilon,y_\varepsilon) \triangleq (x,y) + \varepsilon(h,h)\mathbf{1}_{(0,T_y)}, \qquad h \geqslant 0,$$

we can establish in an analogous way that

$$\dot{\pi}(x(t)) \leqslant \dot{c}(y(t)) = \dot{c}(0) \quad \text{for a.e. } t \text{ in } (0,T_y). \tag{37}$$

Since y is non-decreasing, the limit $y(T_y+)$ exists, and by (35), Step 1, (37), it satisfies $\dot{c}(y(T_y+)) = \dot{\pi}(x(T_y+)) = \dot{\pi}(x(T_y-)) \leqslant \dot{c}(y(T_y-)) = \dot{c}(0)$. Hence, since $\dot{c}$ is non-decreasing, we get $y(T_y+) = 0 = y(T_y-)$. This shows that y is continuous in T_y and equal to 0 on $[0,T_y]$. In particular, it follows from (36) that y is continuous on $(0,T_0)$.

Finally, since by item 1 $x \geqslant a \geqslant y$, since $\dot{\pi}([a,\infty)) = (\dot{\pi}(\infty),\dot{\pi}(a)] = (\dot{\pi}(\infty),\dot{c}(a)] \subset [0,\dot{c}(a)]$ and since $\dot{c}^{-1}$ is well defined on $[0,\dot{c}(a)]$ (see Assumption 2), we deduce from (35) and (37) that, for every $t \in (0,T_0)$,

$$y(t) = \begin{cases} 0 & \text{if } \dot{\pi}(x(t)) < \dot{c}(0), \\ \dot{c}^{-1}(\dot{\pi}(x(t))) & \text{if } \dot{\pi}(x(t)) \geqslant \dot{c}(0). \end{cases}$$

This concludes the proof of item 5.

6. Let us prove that if $T_0 < T$ then, $x(T_0-)=a$, where $x(T_0-)$ is well defined because x is non-increasing. By item 1, we know that $x(T_0-) \geqslant a$. Let $0 < \delta < \min\{T-T_0, T_0\}$. For every $\varepsilon \in (0,a/2]$, we define

$$x_\varepsilon \triangleq x - \varepsilon\mathbf{1}_{(T_0-\delta,T_0)} + \varepsilon\mathbf{1}_{(T_0,T_0+\delta)}.$$

Then, clearly, $(\alpha, x_\varepsilon, y) \in \mathscr{B}$ and therefore, by optimality of (α, x, y),

$$
\begin{aligned}
0 &\geqslant \frac{\mathscr{F}(\alpha, x_\varepsilon, y) - \mathscr{F}(\alpha, x, y)}{\delta\varepsilon} \\
&= \frac{1}{\delta}\int_{T_0-\delta}^{T_0} \mathrm{e}^{-\lambda t}\left(\frac{\pi(x(t)-\varepsilon) - \pi(x(t))}{\varepsilon}\right)\mathrm{d}t \\
&\quad + \frac{1}{\delta}\left(\frac{\pi(a+\varepsilon)-\pi(a)}{\varepsilon}\right)\int_{T_0}^{T_0+\delta} \mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&\quad - \frac{1}{\delta}\int_{T_0-\delta}^{T_0} \mathrm{e}^{-\lambda t}\left(\frac{s(S(t)+\varepsilon(t-(T_0-\delta))) - s(S(t))}{\varepsilon}\right)\mathrm{d}t \\
&\quad - \frac{1}{\delta}\int_{T_0}^{T_0+\delta} \mathrm{e}^{-\lambda t}\left(\frac{s(\varepsilon\delta - \varepsilon(t-T_0)) - s(0)}{\varepsilon}\right)\mathrm{d}t. \qquad (38)
\end{aligned}
$$

Since, by construction, for every $\varepsilon \in (0, a/2]$, we have

$$
\left|\frac{\pi(x(t)-\varepsilon)-\pi(x(t))}{\varepsilon}\right| \leqslant \dot{\pi}\left(\frac{a}{2}\right) \quad \text{a.e. on } (T_0-\delta, T_0),
$$

$$
\left|\frac{s(S(t)+\varepsilon(t-(T_0-\delta)))-s(S(t))}{\varepsilon}\right| \leqslant \delta \max_{[0,\, s_0+(a/2)\delta]} \dot{s} \quad \text{a.e. on } (T_0-\delta, T_0)
$$

and

$$
\left|\frac{s(\varepsilon\delta-\varepsilon(t-T_0))-s(0)}{\varepsilon}\right| \leqslant \delta \max_{[0,(a/2)\delta]} \dot{s} \quad \text{a.e. on } (T_0, T_0+\delta),
$$

we can let ε tend to 0 in (38) to get, by the dominated convergence theorem,

$$
\begin{aligned}
0 &\geqslant -\frac{1}{\delta}\int_{T_0-\delta}^{T_0} \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))\,\mathrm{d}t + \frac{1}{\delta}\,\dot{\pi}(a)\int_{T_0}^{T_0+\delta} \mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&\quad - \frac{1}{\delta}\int_{T_0-\delta}^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))(t-(T_0-\delta))\,\mathrm{d}t \\
&\quad - \dot{s}(0)\,\frac{1}{\delta}\int_{T_0}^{T_0+\delta} \mathrm{e}^{-\lambda t}(\delta-(t-T_0))\,\mathrm{d}t.
\end{aligned}
$$

Recall that $x(T_0-)$ is well defined in $[a, \infty)$. Also notice that $\dot{\pi}$ is continuous on $[a, \infty)$, S is continuous in T_0 and $\dot{s}$ is continuous on $\mathbb{R}^+$. Then, by sending δ tend to 0 in the above inequality, we get $\dot{\pi}(x(T_0-)) \geqslant \dot{\pi}(a)$, which concludes the proof since $x(T_0-) \geqslant a$ and $\dot{\pi}$ is decreasing on $[a, \infty)$.

7. Let us prove that if $\alpha > 0$ then, $x(0+) = \infty$, where $x(0+)$ is well defined as an element of $[a, \infty]$, by item 1. Assume that $\alpha > 0$. Let $\varepsilon > 0$ and $\delta > 0$ be such that

$\varepsilon\delta \leqslant \alpha$, and define

$$x_\varepsilon \triangleq x + \varepsilon \mathbf{1}_{[0,\delta]}.$$

Then, $(\alpha - \delta\varepsilon, x_\varepsilon, y) \in \mathscr{B}$. Moreover, $S^{(\alpha-\delta\varepsilon,x_\varepsilon,y)}(t) = S(t) + \varepsilon(\delta - t)$ for $t \in [0,\delta]$, and $S^{(\alpha-\delta\varepsilon,x_\varepsilon,y)}(t) = S(t)$ for $t \in [\delta, T]$. By optimality of (α, x, y) it follows that

$$\begin{aligned} 0 &\geqslant \frac{\mathscr{F}(\alpha - \delta\varepsilon, x^\varepsilon, y) - \mathscr{F}(\alpha, x, y)}{\delta\varepsilon} \\ &= -\dot{\pi}(\infty) + \frac{1}{\delta}\int_0^\delta \mathrm{e}^{-\lambda t}\left(\frac{\pi(x(t)+\varepsilon) - \pi(x(t))}{\varepsilon}\right)\mathrm{d}t \\ &\quad - \frac{1}{\delta}\int_0^\delta \mathrm{e}^{-\lambda t}\left(\frac{s(S(t)+\varepsilon(\delta - t)) - s(S(t))}{\varepsilon}\right)\mathrm{d}t. \end{aligned}$$

Observe that, by item 1,

$$\left|\frac{s(S(t)+\varepsilon(\delta-t)) - s(S(t))}{\varepsilon}\right| \leqslant \delta \max_{[s_0-\alpha, s_0+\alpha]} \dot{s}.$$

Then, letting ε tend to 0, we get, from the monotone convergence theorem for the first term and from the dominated convergence theorem for the last one, that

$$\begin{aligned} 0 &\geqslant -\dot{\pi}(\infty) + \frac{1}{\delta}\int_0^\delta \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))\,\mathrm{d}t - \frac{1}{\delta}\int_0^\delta \mathrm{e}^{-\lambda t}\dot{s}(S(t))(\delta - t)\,\mathrm{d}t \\ &\geqslant -\dot{\pi}(\infty) + \frac{1}{\delta}\int_0^\delta \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))\,\mathrm{d}t - \int_0^\delta \mathrm{e}^{-\lambda t}\dot{s}(S(t))\,\mathrm{d}t. \end{aligned}$$

Recall that $x(0+)$ is well defined in $[a,\infty]$. Also recall that $\dot{\pi}$ is continuous on $[a,\infty)$ and satisfies $\lim_{x\to\infty}\dot{\pi}(x) = \dot{\pi}(\infty)$, see (6). Since, by item 1, the functions $t \to \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))$ and $t \to \mathrm{e}^{-\lambda t}\dot{s}(S(t))$ are bounded (respectively, by $\dot{\pi}(a)$ and $\max_{[0,s_0]}\dot{s}$), letting δ tend to 0 in the above inequality leads to $0 \geqslant -\dot{\pi}(\infty) + \dot{\pi}(x(0+))$. Since $x(0+) \in [a,\infty]$ and $\dot{\pi}$ is decreasing on $[a,\infty)$, this implies that $x(0+) = \infty$.

Observe that we only used item 1 to derive this result.

We have proved the first claim of Theorem 7. We turn to the proof of the converse assertion.

Converse assertion: We now prove that if (α, x, y) has properties 1–7 of Theorem 7 then, for every $(\beta, h, k) \in \mathbb{R} \times L^1[0,T] \times L^1[0,T]$ such that $(\alpha,x,y) + \varepsilon_0(\beta,h,k) \in \mathscr{B}$ for some $\varepsilon_0 > 0$, we have

$$\limsup_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} \left\{\frac{\mathscr{F}((\alpha,x,y)+\varepsilon(\beta,h,k)) - \mathscr{F}(\alpha,x,y)}{\varepsilon}\right\} \leqslant 0.$$

Let $(\beta,h,k) \in \mathbb{R} \times L^1[0,T] \times L^1[0,T]$ be such that $(\alpha,x,y) + \varepsilon_0(\beta,h,k) \in \mathscr{B}$ for some $\varepsilon_0 > 0$. We introduce the functions

$$H(t) \triangleq \beta + \int_0^t h(u)\,\mathrm{d}u \quad \text{and} \quad K(t) \triangleq \int_0^t k(u)\,\mathrm{d}u \quad \text{on } [0,T].$$

Define

$$\delta^\varepsilon \mathscr{F} \triangleq \frac{\mathscr{F}((\alpha,x,y)+\varepsilon(\beta,h,k)) - \mathscr{F}(\alpha,x,y)}{\varepsilon} \quad \text{and} \quad \delta\mathscr{F} \triangleq \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \delta^\varepsilon \mathscr{F}.$$

By concavity of π and $-c$ we have

$$\begin{aligned}
\delta^\varepsilon \mathscr{F} &= \dot{\pi}(\infty)\beta + \int_0^T e^{-\lambda t}\left(\frac{\pi(x(t)+\varepsilon h(t)) - \pi(x(t))}{\varepsilon}\right) dt \\
&\quad - \int_0^T e^{-\lambda t}\left(\frac{c(y(t)+\varepsilon k(t)) - c(y(t))}{\varepsilon}\right) dt \\
&\quad - \int_0^T e^{-\lambda t}\left(\frac{s(S(t)+\varepsilon(K(t)-H(t))) - s(S(t))}{\varepsilon}\right) dt \\
&\leqslant \dot{\pi}(\infty)\beta + \int_0^T e^{-\lambda t}\dot{\pi}(x(t))h(t)\,dt - \int_0^T e^{-\lambda t}\dot{c}(y(t))k(t)\,dt \\
&\quad - \int_0^T e^{-\lambda t}\left(\frac{s(S(t)+\varepsilon(K(t)-H(t))) - s(S(t))}{\varepsilon}\right) dt.
\end{aligned}$$

Observe that for $\varepsilon\in(0,\varepsilon_0]$ and $t\in[0,T]$, we have

$$\left|\frac{s(S(t)+\varepsilon(K(t)-H(t))) - s(S(t))}{\varepsilon}\right| \leqslant [\|K\|_\infty + \|H\|_\infty] \max_{[0,s_0+\varepsilon_0(\|K\|_\infty+\|H\|_\infty)]} \dot{s}.$$

Then, it follows from the dominated convergence theorem that

$$\begin{aligned}
\delta\mathscr{F} &\leqslant \dot{\pi}(\infty)\beta + \int_0^T e^{-\lambda t}\dot{\pi}(x(t))h(t)\,dt - \int_0^T e^{-\lambda t}\dot{c}(y(t))k(t)\,dt \\
&\quad - \int_0^T e^{-\lambda t}\dot{s}(S(t))\,[K(t)-H(t)]\,dt.
\end{aligned} \tag{39}$$

We now consider the right-hand side terms separately. First, observe that, by item 5, we have $\dot{\pi}(x) = \dot{c}(y)$ a.e. on $\{t\in[0,T_0] \mid y(t)>0\}$ and then

$$\begin{aligned}
&\int_0^{T_0} e^{-\lambda t}[\dot{\pi}(x(t))h(t) - \dot{c}(y(t))k(t)]\,dt \\
&\quad = \int_0^{T_0} e^{-\lambda t}\dot{\pi}(x(t))(h(t)-k(t))\mathbf{1}_{\{y>0\}}(t)\,dt \\
&\qquad + \int_0^{T_0} e^{-\lambda t}[\dot{\pi}(x(t))h(t) - \dot{c}(y(t))k(t)]\mathbf{1}_{\{y=0\}}(t)\,dt.
\end{aligned}$$

Moreover, since $(\alpha,x,y)+\varepsilon_0(\beta,h,k)\in\mathscr{B}$, we have $k\geqslant 0$ a.e. on $\{t\in[0,T_0] \mid y(t)=0\}$. By item 5, we also know that $\dot{\pi}(x)\leqslant\dot{c}(y)$ a.e. on $\{t\in[0,T_0] \mid y(t)=0\}$. It follows that $-\dot{c}(y)k \leqslant -\dot{\pi}(x)k$ a.e. on $\{t\in[0,T_0]\mid y(t)=0\}$. We then deduce from the above equality that

$$\int_0^{T_0} \mathrm{e}^{-\lambda t}[\dot{\pi}(x(t))h(t)-\dot{c}(y(t))k(t)]\,\mathrm{d}t \leqslant \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))(h(t)-k(t))\,\mathrm{d}t. \quad (40)$$

If $T_0 < T$, we know from item 3 that $\dot{\pi}(x(t))=\dot{c}(y(t))=\dot{\pi}(a)$ on $[T_0,T]$, and therefore

$$\int_{T_0}^{T} \mathrm{e}^{-\lambda t}[\dot{\pi}(x(t))h(t)-\dot{c}(y(t))k(t)]\,\mathrm{d}t = \dot{\pi}(a)\int_{T_0}^{T} \mathrm{e}^{-\lambda t}(h(t)-k(t))\,\mathrm{d}t. \quad (41)$$

Since, by item 3 again, $S=0$ on $[T_0,T]$, we get after plugging (40) and (41) into (39)

$$\begin{aligned}\delta\mathscr{F} \leqslant\ & \dot{\pi}(\infty)\beta + \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))[h(t)-k(t)]\,\mathrm{d}t \\ & - \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))[K(t)-H(t)]\,\mathrm{d}t \\ & + \dot{\pi}(a)\int_{T_0}^{T} \mathrm{e}^{-\lambda t}(h(t)-k(t))\,\mathrm{d}t - \dot{s}(0)\int_{T_0}^{T} \mathrm{e}^{-\lambda t}[K(t)-H(t)]\,\mathrm{d}t. \end{aligned} \quad (42)$$

Let us now consider the term

$$\begin{aligned} & \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))(h(t)-k(t))\,\mathrm{d}t - \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))[K(t)-H(t)]\,\mathrm{d}t \\ & = \int_0^{T_0}\left[\dot{\pi}(x(0+)) + \int_0^{t} \mathrm{e}^{-\lambda u}\dot{s}(S(u))\,\mathrm{d}u\right](h(t)-k(t))\,\mathrm{d}t \\ & \quad + \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))[H(t)-K(t)]\,\mathrm{d}t, \end{aligned}$$

where the equality holds by (8). Then, integrating by parts the last term, we obtain

$$\begin{aligned} & \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{\pi}(x(t))(h(t)-k(t))\,\mathrm{d}t - \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))[K(t)-H(t)]\,\mathrm{d}t \\ & = \dot{\pi}(x(0+))\int_0^{T_0}(h(t)-k(t))\,\mathrm{d}t + \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))\,\mathrm{d}t[H(T_0)-K(T_0)] \\ & = -\dot{\pi}(x(0+))\beta + \left[\dot{\pi}(x(0+)) + \int_0^{T_0} \mathrm{e}^{-\lambda t}\dot{s}(S(t))\,\mathrm{d}t\right][H(T_0)-K(T_0)] \\ & = -\dot{\pi}(x(0+))\beta + \mathrm{e}^{-\lambda T_0}\dot{\pi}(x(T_0-))[H(T_0)-K(T_0)], \end{aligned}$$

where the second equality follows from the definition of H and K, and the last one from (8). Plugging the last equality into (42) leads to

$$\begin{aligned}\delta\mathscr{F} \leqslant\ & \dot{\pi}(\infty)\beta - \dot{\pi}(x(0+))\beta + \mathrm{e}^{-\lambda T_0}\dot{\pi}(x(T_0-))[H(T_0)-K(T_0)] \\ & + \dot{\pi}(a)\int_{T_0}^{T} \mathrm{e}^{-\lambda t}(h(t)-k(t))\,\mathrm{d}t - \dot{s}(0)\int_{T_0}^{T} \mathrm{e}^{-\lambda t}[K(t)-H(t)]\,\mathrm{d}t. \end{aligned} \quad (43)$$

We now consider two different cases. First, we deduce from the above inequality that

$$\delta\mathscr{F} \leqslant \dot{\pi}(\infty)\beta - \dot{\pi}(x(0+))\beta + e^{-\lambda T}\dot{\pi}(x(T-))[H(T) - K(T)] \quad \text{if } T_0 = T. \tag{44}$$

We now consider the case $T_0 < T$. Then, by item 6, $x(T_0-) = a$ and therefore

$$\begin{aligned} & e^{-\lambda T_0}\dot{\pi}(x(T_0-))[H(T_0) - K(T_0)] + \dot{\pi}(a)\int_{T_0}^{T} e^{-\lambda t}(h(t) - k(t))\,dt \\ &\quad = \dot{\pi}(a)\left(e^{-\lambda T_0}[H(T_0) - K(T_0)] + \int_{T_0}^{T} e^{-\lambda t}(h(t) - k(t))\,dt\right) \\ &\quad = \dot{\pi}(a)\left(e^{-\lambda T}[H(T) - K(T)] + \lambda\int_{T_0}^{T} e^{-\lambda t}(H(t) - K(t))\,dt\right), \end{aligned}$$

where the last equality is obtain by integrating by parts. Plugging this into (43), we get

$$\begin{aligned} \delta\mathscr{F} \leqslant{} & \dot{\pi}(\infty)\beta - \dot{\pi}(x(0+))\beta \\ & + \dot{\pi}(a)\left(e^{-\lambda T}[H(T) - K(T)] + \lambda\int_{T_0}^{T} e^{-\lambda t}(H(t) - K(t))\,dt\right) \\ & - \dot{s}(0)\int_{T_0}^{T} e^{-\lambda t}[K(t) - H(t)]\,dt \quad \text{if } T_0 < T. \end{aligned} \tag{45}$$

We shall now prove that

$$\dot{\pi}(\infty)\beta - \dot{\pi}(x(0+))\beta \leqslant 0 \tag{46}$$

$$K - H \geqslant 0 \quad \text{on } [T_0, T]. \tag{47}$$

Plugging these inequalities in (44) and (45) will complete the proof. First, since $\dot{\pi}(\infty) = \inf \dot{\pi} \leqslant \dot{\pi}(x(0+))$, (46) holds for $\beta \geqslant 0$. Since, $(\alpha, x, y) + \varepsilon_0(\beta, h, k) \in \mathscr{B}$, we have $\alpha + \varepsilon_0\beta \geqslant 0$. Therefore, we see that β can be negative only if $\alpha > 0$, which implies $\dot{\pi}(x(0+)) = \dot{\pi}(\infty)$ by item 7. This proves that (46) also holds for $\beta < 0$. We now turn to the proof of (47). From $(\alpha, x, y) + \varepsilon_0(\beta, h, k) \in \mathscr{B}$ again, we deduce that

$$\begin{aligned} 0 &\leqslant s_0 - \alpha + \int_0^t [y(s) - x(s)]\,ds + \varepsilon_0(K(t) - H(t)) \\ &= S^{(\alpha,x,y)}(t) + \varepsilon_0(K(t) - H(t)) \quad \text{for all } t \in (0, T]. \end{aligned}$$

Since by item 3, $S^{(\alpha,x,y)} = 0$ on $[T_0, T]$, we see that (47) holds by taking $t \in [T_0, T]$ in the above inequality. □

7. Concluding remarks

In this paper, we have shown that, if the revenue is concave and if the production cost is convex then, the behavior of the company towards inventories is qualitatively the same for any kind of storage cost function: the firm does not use its storage ability for inventory accumulation. One possible explanation of this phenomenon is the following. Since the firm must have cleared all its stock at the end of the period, an inventory accumulation would lead it to let variate its sales rate from below a to above a and vice versa for the production rate. [1] But, the concavity of the revenue and the convexity of the production cost urges the company to reduce the variations of the sales and production rates. Our model is more suited for determining the optimal way to sell an inventory than for explaining inventory accumulation. In order to carry out this last task, it would be of interest to introduce non-convexities in the production cost function. There are some empirical results showing evidence of firms facing decreasing marginal cost on some range of output (see [8]). The variation of the production rate should be greater under these conditions than under increasing marginal cost. Finally, our model suffers from time independence. Time plays an obvious role in production planning mixed with inventory management: inventories constitute an alternative to future production. Hence, time-dependent production cost and/or time-dependent revenue should be considered. These are directions for future research.

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[1] The optimality condition: $\dot{\pi}(x) = \dot{c}(y)$ implies that $x < a < y$ when $x < y$ and $y < a < x$ when $y < x$.