Heterogeneous beliefs and asset pricing in discrete time: an analysis of pessimism and doubt

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Abstract

The aim of the paper is to analyze the impact of heterogeneous beliefs in an otherwise standard competitive complete markets discrete time economy. The construction of a consensus belief, as well as a consensus consumer are shown to be valid modulo a predictable aggregation bias, which takes the form of a discount factor. We use our construction of a consensus consumer to investigate the impact of beliefs heterogeneity on the CCAPM and on the expression of the risk free rate. We focus on the pessimism/doubt of the consensus consumer and we study their impact on the equilibrium characteristics (market price of risk, risk free rate). We finally analyze how pessimism and doubt at the aggregate level result from pessimism and doubt at the individual level. We show that it is possible to obtain a higher market price of risk and lower returns for assets with higher beliefs dispersion, which is interesting in light of the risk premium puzzle (Mehra-Prescott, 1985) and of the findings of Diether et al. (2002).

1. Introduction

The aim of the paper is to analyze the impact of heterogeneous beliefs in an otherwise standard competitive complete markets discrete time economy.

We start from a given equilibrium with heterogeneous beliefs in a fairly general setting and we propose to decompose our analysis into three steps.

The first issue deals with the aggregation of these heterogeneous beliefs. The main question is to know if it is possible to analyze the heterogeneous beliefs model in terms of a classical homogeneous beliefs model, in which the common belief may be different from the objective one. In other words, can we define a subjective consensus belief, i.e. a belief which, if held by all individuals, would generate the same equilibrium prices as in the actual heterogeneous economy ?

We show that the answer to the previous question is positive. We are then led to analyze, in a standard homogeneous model, the impact of a subjective belief on the equilibrium characteristics (state price density, risk premium, risk free rate, etc.). This question has been explored by Abel (2002) in the specific setting of power utility functions and i.i.d asset returns. In particular, Abel introduces concepts of pessimism and doubt, and analyzes their impact on the equilibrium characteristics. In the same spirit, we introduce similar concepts that have an unambiguous impact on the equilibrium characteristics in a more general setting (in particular, without specific restrictions on the utility functions or on the distribution of the asset returns).

The last question is to understand how doubt and pessimism at the individual level are captured at the aggregate level, i.e. by the consensus belief. Our definitions of doubt and pessimism appear as particularly adapted to this question. Indeed, roughly speaking, we obtain that the level of pessimism (resp. doubt) at the aggregate level is a weighted average of the level of pessimism (resp. doubt) at the individual level.

In this paper, the different subjective beliefs are considered as given. As in Varian (1985, 1989), Abel (1989) or Harris-Raviv (1993), they reflect difference of opinion among the agents rather than difference of information; indeed, "we assume that investors receive common information, but differ in the way they interpret this information" (Harris-Raviv, 1993). The different subjective beliefs might come from a Bayesian updating of the investors predictive distribution over the uncertain returns on risky securities as in, e.g. Williams (1977), Detemple-Murthy (1994), Zapatero (1998), Gallmeyer (2000), Basak (2000), Gallmeyer and Hollifield (2002), but we do not make such an assumption; we only impose that the subjective probabilities be equivalent to the initial one. Notice that the abovementioned models with learning are not "more endogenous", since the investors' updating rule and the corresponding probabilities can be determined separately from his/her optimization problem (see e.g. Genotte, 1986).

In a companion paper, Jouini and Napp (2003) considered the same problem in continuous time. Even if the approach developed therein is similar to ours, the techniques and concepts are quite diferent. In particular, if we analyze the subjective beliefs in terms of distortions of the objective one, the continuous time setting permits a much smaller set of possible distortions than the discrete time setting. For instance, in continuous time, the volatility of given process is the same under the objective and the different subjective beliefs whereas in discrete time some agents might overestimate or underestimate this volatility leading to doubtfull or overconfident behavior.

The paper is organized as follows. We present in Section 2 a method to aggregate, in a discrete time framework, heterogeneous individual subjective beliefs into a single consensus belief. Given an observed equilibrium with heterogeneous probabilities, we look for a consensus belief, which, if held by all investors would lead to an equivalent equilibrium, in the sense that it would leave invariant the equilibrium market prices. We prove the existence of such a consensus belief modulo the introduction of a predictable discount factor. This means that, modulo this discount factor, the equilibrium price under heterogeneous beliefs is the same as in an economy where all agents would share the same belief, namely the consensus belief. For a large class of utility functions, the consensus belief is given by some weighted average of the individual beliefs, and the discount factor is directly related to the beliefs dispersion. The discount factor might be positive or negative depending on whether the investor is cautious or not. A possible interpretation for the presence of this discount factor consists in considering the dispersion of beliefs as a source of risk. When there is more risk involved, depending on whether the investor is cautious or not, it is well known that the investor will reduce or increase current consumption with respect to future consumption, acting as if his/her utility was discounted by a positive or negative discount rate.

We then derive an adjusted CCAPM formula. We prove that the CCAPM formula under heterogeneous beliefs is given by the CCAPM formula in an economy where all investors would share the same probability belief, namely the consensus belief obtained through the aggregation procedure. It appears that the discount factor does not impact the risk premium but only the risk free rate. Through our aggregation procedure, we have then transformed the study of a model under heterogeneous subjective beliefs into the study of a model under homogeneous subjective beliefs.

The next step consists in the study of the impact of a subjective belief on the equilibrium risk premium and risk free rate, which is the aim of Section 3. More precisely, we try to determine characteristics of the subjective belief that have an unambiguous impact on the market price of risk and on the risk free rate, and, in relation with the risk premium puzzle (see Mehra-Prescott, 1985 or Kocherlakota, 1996 for a survey on the risk premium puzzle) and the risk free rate puzzle (Weil, 1989), which preferrably lead to an increase of the market price of risk and a decrease of the risk free rate. We start by considering the notion of pessimism. The notion we consider was introduced in a static setting in Jouini-Napp (2004), where a probability measure, equivalent to the initial probability, is defined to be first order-pessimistic if its density (with respect to the initial probability) decreases with the aggregate wealth. A natural generalization of this concept to a dynamic setting leads to the following definition. A pessimistic probability is such that its density at date $(t+1)$ decreases with the aggregate wealth conditionally to F_t for all t; this means that conditionally to date t information, it puts more weight on states of nature where the aggregate wealth at date $(t+1)$ is low, which clearly corresponds to a notion of pessimism. We prove that a pessimistic subjective probability belief increases the market price of risk and decreases the risk free rate. We also introduce a notion of doubt. Loosely speaking, we shall say that a probability measure exhibits doubt if conditionally to date t information, its density at date $(t+1)$ puts more (resp. less) weight on the tails and less (resp. more) weight on the center of the distribution of aggregate wealth at date $(t+1)$. We prove that a probability measure which exhibits doubt also increases the market price of risk and decreases the risk free rate for a large class of utility functions.

The aim of Section 4 is to analyze how pessimism and doubt at the aggregate level result from pessimism and doubt at the individual level. We show that with our definitions of pessimism and doubt, the consensus belief level of pessimism (resp. doubt) appears as a weighted average of the individual level of pessimism resp. doubt, where the weights are given by the individual risk tolerances. We prove that there are essentially two effects of beliefs heterogeneity on the equilibrium characteristics. The first effect is given by the equally-weighted average level of pessimism/optimism (resp. doubt/overconfidence). In particular, if agents are on average pessimistic, there is a bias towards a higher market price of risk and a lower risk free rate than in the standard setting, which is interesting in light of the risk premium and risk free rate puzzles (Mehra-Prescott, 1985, Weil, 1989). The second effect is given by the "correlation" between risk tolerance/aversion and pessimism/optimism (resp. doubt/overconfidence). In particular, if risk tolerance and optimism are positively correlated, this induces a lower market price of risk. We also prove that our results are compatible with the findings of Diether et al (2002), i.e. that assets with higher beliefs dispersion yield lower returns.

All proofs are in the appendix.

In the next sections we shall need the following notations. For given families of positive real numbers $(x_i)_{i=1,\cdots,n}$ and $(\theta_i)_{i=1,\cdots,n}$ we denote by $\bar{\theta}$ the sum $\sum_{i=1}^n \theta_i$ by $\mathcal{E}_k^{\theta.}(x.)$ the weighted $k-$average defined for $k \in \mathbb{Z} \backslash \{0\}$ by $\mathcal{E}_k^{\theta.}(x.) \equiv \left[\sum_{i=1}^n \frac{\theta_i}{\bar{\theta}} x_i^k\right]^{\frac{1}{k}}$ and we let $\mathcal{E}_0^{\theta.}(x.) \equiv \prod_{i=1}^n x_i^{\frac{\theta_i}{\bar{\theta}}}$ denote the geometric average. We denote by $\mathbf{1}$ the vector of R^n whose coordinates are all equal to 1. With these notations, $\mathcal{E}_k^{\mathbf{1}}(x.)$ is then the equally weighted $k-$average of the x_i. We also denote by $Var^{\theta.}(x.)$ the weighted variance $Var^{\theta.}(x.) \equiv \sum_{i=1}^n \frac{\theta_i}{\bar{\theta}} \left[x_i - \mathcal{E}_1^{\theta}(x_i)\right]^2$. Note that straightforward Taylor expansions gives us for $Var^{\theta.}(x.)$ small enough, the following approximations $\mathcal{E}_k^{\theta.}(x.) \sim \mathcal{E}_1^{\theta.}(x.) + \frac{k-1}{2} \frac{Var^{\theta.}(x.)}{\mathcal{E}_1^{\theta}(x.)}$.

2. Consensus belief, adjusted CCAPM and risk free rate

In the classical representative agent approach, all investors are taken as having the same subjective beliefs and in this section, we analyze to which extent this approach can be extended to heterogeneous subjective beliefs. More precisely, we start from a given equilibrium with heterogeneous beliefs in an otherwise standard complete discrete time market model, and we explore to which extent it is possible 1) to define a consensus belief, i.e. a belief, which, if held by all individuals would generate the same equilibrium prices as in the actual heterogeneous economy and 2) to define a representative agent (or a consensus consumer). We shall then analyze the impact of heterogeneity of beliefs on asset pricing, and more precisely on the CCAPM formula and on the expression of the risk free rate.

The model is standard, except that we allow the agents to have distinct subjective probabilities. It is essentially the same as in Jouini-Napp (2003), apart that we now deal with a discrete time setting. We fix a finite time horizon T on which we are going to treat our problem. We consider a filtrated probability space $\left(\Omega, (F_t)_{t\in\{0,\ldots,T\}}, P\right)$, where the filtration $(F_t)_{t\in\{0,\ldots,T\}}$ satisfies the usual conditions. Each investor indexed by $i = 1, \cdots, N$ solves a standard dynamic utility maximization problem. He has a current income at date t denoted by $e_t^{*^i}$ and a von Neumann-Morgenstern utility function for consumption of the form $E^{Q^i}\left[\sum_{t=0}^T u_i(t, c_t)\right]$, where Q^i is a probability measure equivalent to P which corresponds to the subjective belief of individual i. If we denote by $(M_t^i)_{t\in\{0,\ldots,T\}}$ the positive density process of Q^i with respect to P, then the utility function can be rewritten as $E^P\left[\sum_{t=0}^T M_t^i u_i(t, c_t)\right]$.

We make the following classical assumptions on the utility functions and on the subjective beliefs.

Assumption

- for all $t = 1, ..., T$, $u_i(t, \cdot) : [k_i, \infty) \to \mathbb{R} \cup \{-\infty\}$ is of class C^1 on (k_i, ∞), strictly increasing and strictly concave[1],
- $u_i(., c)$ and $u_i'(t, .) = \frac{\partial u_i}{\partial c}(t, .)$ are continuous on $[0, T]$,
- for $i = 1, \cdots, N$, $P\{e_t^i > k_i\} > 0$ and $P\{e_t^i \geq k_i\} = 1$,
- there exists $\varepsilon > 0$ such that $e^* > \sum_{i=1}^N k_i + \varepsilon$, P a.s.,
- $E^P\left[\sum_{t=0}^T e_t^*\right] < \infty$,
- the density process M^i is uniformly bounded for $i = 1, \cdots, N$.

The third condition can be seen as a survival assumption at the individual level and the fourth one as a survival assumption at the aggregate level. All the remaining conditions are very classical ones.

We do not specify the utility functions u_i, although we shall focus on the classical cases of linear risk tolerance utility functions (which include logarithmic, power as well as exponential utility functions).

2.1. Consensus belief

We start from an equilibrium $\left(q^*, \left(y^{*^i}\right)\right)$ relatively to the beliefs (M^i) and the income processes e^i. We recall that an equilibrium relatively to the beliefs (M^i) and the income processes (e^i) is defined by a positive, uniformly bounded price process q^* and a family of optimal admissible consumption plans $\left(y^{*^i}\right)$ such that markets clear i.e. a family of adapted $[k_i, \infty)$-valued processes y^{*^i} such that $E^P\left[\sum_0^T \left|y_t^{*^i}\right|\right] < \infty$ and satisfying

$$\begin{cases} y^{*^i} = y^i(q^*, M^i, e^i) \\ \sum_{i=1}^N y^{*^i} = \sum_{i=1}^N e^i \equiv e^* \end{cases}$$

[1]Note that we could easily generalize the results of this paper to the case where k_i is a function of t.

where

$$y^i(q,M,e) \equiv \arg \max_{E^P\left[\sum_0^T q_t\left(y_t^i - e_t\right)\right] \leq 0} E^P\left[\sum_0^T M_t u_i(t,c_t)dt\right].$$

Such an equilibrium, when it exists, can be characterized by the first order necessary conditions for individual optimality and the market clearing condition. These conditions can be written as follows

$$\begin{cases} M_t^i u_i'(t, y_t^{*^i}) \leq \lambda_i q_t^*, & \text{on } \left\{y^{*^i} = k_i\right\} \\ M_t^i u_i'(t, y_t^{*^i}) = \lambda_i q_t^*, & \text{on } \left\{y^{*^i} > k_i\right\} \\ E^P\left[\sum_0^T q_t^* \left(y_t^{*^i} - e_t^i\right)\right] = 0 & \\ \sum_{i=1}^N y^{*^i} = e^* & \end{cases} \quad (2.1)$$

for some set of positive Lagrange multipliers (λ_i).

In the next, we will say that $(q^*, (y^{*^i}))$ is an interior equilibrium relatively to the beliefs (M^i) and the income processes (e^i) if $y^{*^i} > k_i$, P a.s. for $i = 1, \cdots, N$[2].

Our first aim is to find an "equivalent equilibrium" in which the heterogeneous subjective beliefs would be aggregated into a common characteristics M. Following the approach of Calvet et al. (2002), we shall define an "equivalent equilibrium" by two requirements. First, the "equivalent equilibrium" should generate the same equilibrium trading volumes $\left(y^{*^i} - e^i\right)$ and price process q^* as in the original equilibrium with heterogeneous beliefs. Second, every investor should be indifferent at the margin between investing one additional unit of income in the original equilibrium with heterogeneous beliefs and in the "equivalent equilibrium", so that each asset gets the same marginal valuation by each investor (in terms of his marginal utility) in both equilibria[3]. The existence of such an "equivalent equilibrium" is given by the following proposition.

Proposition 2.1. *Consider an interior equilibrium $(q^*, (y^{*^i}))$ relatively to the beliefs (M^i) and the income processes (e^i). There exists a unique positive and*

[2]Note that under the following additional condition

$$u_i'(t,k_i) = \infty \text{ for } t \in \{0,...,T\} \text{ and } i = 1, \cdots, N,$$

all the equilibria are interior ones.

[3]This condition was introduced by Calvet et al. (2002). It would be equivalent to impose that the Lagrange multipliers are the same for each investor in both equilibria. As a consequence, the representative agent utility function will also be the same in both equilibria.

adapted process $(M_t)_{t\in\{0,...,T\}}$ *with* $M_0 = 1$, *there exists a unique family of income processes* $(\bar{e}^i)$ *with* $\sum_{i=1}^N \bar{e}^i = e^*$ *and a unique family of individual consumption processes* $(\bar{y}^i)$ *such that* $(q^*, (\bar{y}^i))$ *is an equilibrium relatively to the common characteristics* M *and the income processes* $(\bar{e}^i)$ *and such that trading volumes and individual marginal valuation remain the same, i.e.*

$$y^{*^i} - e^i = \overline{y}^i - \overline{e}^i \qquad i = 1, \cdots, N$$
$$M_t^i u_i'(t, y^{*^i}) = M_t u_i'(t, \bar{y}^i), \qquad t \in \{0, ..., T\}, \, i = 1, \cdots, N.$$

This means that $(q^*, (\bar{y}^i))$ is an equilibrium with income transfers relatively to the common characteristics M and the income processes (e^i) such that individual marginal valuation is the same as in the original equilibrium with heterogeneous beliefs. In other words, we proved that modulo a feasible modification of the individual incomes (i.e. $\sum_{i=1}^N \bar{e}^i = \sum_{i=1}^N e^i$) the initial equilibrium price process and trading volumes remain equilibrium price process and trading volumes in an homogeneous beliefs setting. The positive process M can then be interpreted as a consensus characteristics. In particular, if there is no heterogeneity, i.e. if all the investors have the same belief represented by $M^i = \tilde{M}$ for all i, we obtain $M = \tilde{M}$ and there is no transfer nor optimal allocations modification (i.e. $\bar{e}^i = e^i$ and $\bar{y}^i = y^{*^i}$ for all i).

Once the result on beliefs aggregation achieved, it is easy to construct as in the standard case, a representative agent, i.e. an expected utility maximizing aggregate investor, representing the economy in equilibrium. As in the standard case, for $\alpha \in \left(\mathbb{R}_+^*\right)^N$, we introduce the function

$$u_\alpha(t, x) = \max_{\sum_{i=1}^N x_i \le x} \sum_{i=1}^N \frac{1}{\alpha_i} u_i(t, x_i).$$

Proposition 2.2. *Consider an interior equilibrium* $(q^*, (y^{*^i}))$ *relatively to the beliefs* (M^i) *and the income processes* (e^i) *and let* (λ_i) *denote the Lagrange multipliers associated to this equilibrium. There exists a consensus investor defined by the normalized von Neumann-Morgenstern utility function* u_λ *and the consensus characteristics* M *of Proposition 2.1, in the sense that the portfolio* e^* *maximizes his expected utility* $E^P\left[\sum_{t=0}^T M_t u_\lambda(c_t)\right]$ *under the market budget constraint* $E^P\left[\sum_{t=0}^T q_t^* \left(c_t - e_t^*\right)\right] \le 0.$

Remark that the consensus investor utility function is constructed as usual with the Lagrange multipliers of the initial equilibrium. In the specific case of linear risk tolerance utility functions we obtain the following explicit expressions for the consensus characteristics. Recall that the risk-tolerance of agent i is defined by $T_i(t,y) = -\frac{u_i'}{u_i''}(t,y)$.

Example 2.3. *1. If the individual utility functions are of exponential type, i.e. if $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$, then $u'(t,x) = ae^{-x/\bar{\theta}}$ for $a = e^{e_0^*/\bar{\theta}}$ and*

$$M = \prod_{i=1}^{N} \left(M^i\right)^{\theta_i/\bar{\theta}} = \mathcal{E}_0^{\theta_\cdot}(M^\cdot).$$

2. If the individual utility functions are of logarithmic or power type, i.e. such that $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ for $\eta \neq 0$, then $u'(e) = b(\bar{\theta}+\eta e)^{-\frac{1}{\eta}}$ for $b = (\bar{\theta}+\eta e_0)^{\frac{1}{\eta}}$ and

$$\begin{aligned} M &= \left[\sum_{i=1}^{N} \gamma_i (M^i)^\eta\right]^{\frac{1}{\eta}} = \mathcal{E}_\eta^{\lambda_\cdot^{-\eta}}(M^\cdot) \\ &= \left[\sum_{i=1}^{N} \tau_i (M^i)^{-\eta}\right]^{-\frac{1}{\eta}} = \mathcal{E}_{-\eta}^{T_\cdot(t,y^{*^\cdot})}(M^\cdot) \end{aligned}$$

for $\gamma_i = \frac{\lambda_i^{-\eta}}{\sum_{j=1}^N \lambda_j^{-\eta}}$ and $\tau_i = \frac{T_i(t,y^{*^i})}{\sum_{j=1}^N T_j(t,y^{*^i})}$.

Notice that the consensus belief M is a martingale (i.e. the density process of a given probability) only when $\eta = 1$ (logarithmic case). It is a supermartingale when $\eta < 1$, and a submartingale when $\eta > 1$.

It is interesting to notice in Example 2.3 that for all utility functions in the classical class of linear risk tolerance utility functions, the consensus characteristics is obtained as a weighted average of the individual subjective beliefs, the weights being given by the individual risk tolerances. It is easy to see that in the general case, the consensus characteristics can still be considered as an average of the individual beliefs.

The process M represents a consensus characteristics, however, except in the logarithmic case, it fails to be a martingale. Consequently, it can not be interpreted as a belief, i.e. the density process of a given probability measure. It is easy to see that it is not possible in general to recover the consensus characteristics as a martingale, as soon as we want the equilibrium price to remain the same and the optimal allocations in the equivalent equilibrium to be feasible, in the sense that they still add up to e^* (even if we do not impose the invariance of individual marginal valuation).

This means that in the general case, there is a bias induced by the aggregation of the individual probabilities into a consensus probability.

Proposition 2.4. *Consider an interior equilibrium price process q^* relatively to the beliefs (M^i), and the income processes (e^i). There exists a positive martingale process $\bar{M}$ with $\bar{M}_0 = 1$, and a predictable positive process B with $B_0 = 1$ such that*

$$\bar{M}_t B_t u'(t, e_t^*) = q^*.$$

The process B is given by

$$B_t = B_{t-1} \frac{E_{t-1}[M_t]}{M_{t-1}}, \qquad B_0 = 1$$

and can be thought of as a discount factor.

The process B represents an aggregation bias. It satisfies $\frac{B_t}{B_{t-1}} = \frac{E_{t-1}[M_t]}{M_{t-1}}$ and measures the default of martingality of the consensus characteristics M. If all investors share the same beliefs, then $B \equiv 1$. Otherwise, the process B leads to a (possibly negative) discount of utility from future consumption. We shall interpret the presence and the properties of this "discount factor" below, after Example 2.5.

We shall see now that the process $\overline{M}$ corresponds to a (weighted) average of the individual beliefs and that the discount factor B is directly related to the dispersion of the individual beliefs. We first consider two simple and enlightening examples.

Suppose that $\log \frac{e_{t+1}^*}{e_t^*}$ has a normal conditional distribution under P and under Q_i, more precisely suppose that, under P, $\log \frac{e_{t+1}^*}{e_t^*} \sim_{F_t} \mathcal{N}(m, \sigma)$ and that under Q_i, $\log \frac{e_{t+1}^*}{e_t^*} \sim_{F_t} \mathcal{N}(\mu_i, \sigma)$ and that all the utility functions are exponential, i.e. $u_i'(t, x) = e^{-\frac{x}{\theta_i}}$. We have

$$\frac{M_{t+1}^i}{M_t^i} = \exp - \frac{\mu_i^2 - m^2 + 2 \log \frac{e_{t+1}^*}{e_t^*} (m - \mu_i)}{2\sigma^2}$$

and consequently

$$
\begin{aligned}
\frac{M_{t+1}}{M_t} &= \exp - \frac{\sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \mu_i^2 - m^2 + 2 \log \frac{e^*_{t+1}}{e^*_t} \left(m - \sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \mu_i \right)}{2\sigma^2} \\
&= \frac{\bar{M}_{t+1}}{\bar{M}_t} \frac{B_{t+1}}{B_t}
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\bar{M}_{t+1}}{\bar{M}_t} &= \exp - \frac{\left[\mathcal{E}_1^{\theta.} (\mu.) \right]^2 - m^2 + 2 \log \frac{e^*_{t+1}}{e^*_t} \left(m - \mathcal{E}_1^{\theta.} (\mu.) \right)}{2\sigma^2} \\
\frac{B_{t+1}}{B_t} &= \exp - \frac{\sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \mu_i^2 - \left(\sum_{i=1}^{N} \frac{\theta_i}{\overline{\theta}} \mu_i \right)^2}{2\sigma^2} \\
&= \exp - \frac{Var^{\theta.} (\mu.)}{2\sigma^2}.
\end{aligned}
$$

If we let Q denote the probability measure with density process $\bar{M}$ with respect to P, we have under Q, $\log \frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N} \left(\mathcal{E}_1^{\theta.} (\mu.) , \sigma \right)$. The aggregate belief $\bar{M}$ corresponds then clearly to a weighted average belief, the weights being given by the individual risk tolerances. The discount factor $\frac{B(t+1)}{B(t)}$ is directly related to $Var^{\theta.} (\mu.)$ the weighted variance of the μ_i with the same weights and the process B can be interpreted as a measure of beliefs dispersion.

With the same utility functions and if we now assume that under P, $\log \frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N} (\mu, \sigma)$ and that under Q_i, $\log \frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N} (\mu, \sigma_i)$, we obtain with the same notations that under Q, $\log \frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N} \left(\mu, \mathcal{E}_{-2}^{\theta.} (\sigma.) \right)$ which still can be interpreted as an average belief (the variance under Q is an average of the variances under Q_i). Furthermore, $\frac{B(t+1)}{B(t)} = \frac{\mathcal{E}^{\theta}_{-2}(\sigma)}{\mathcal{E}^{\theta}_{0}(\sigma)}$ and a simple second order Taylor expansion gives us $\frac{B(t+1)}{B(t)} \sim 1 - \frac{Var^{\theta.}(\sigma.)}{\mathcal{E}^{\theta.}(\sigma.)^2}$ when the dispersion of the σ_i is sufficiently small. Once again, the process B is direcly related to beliefs dispersion and its distance to 1 increases with beliefs dispersion.

More generally, with HARA utility functions, it is easy to see using the approximations provided in the introduction that when beliefs dispersion is sufficiently small, $\bar{M}$ is equal, at the first order, to a risk-tolerance weighted arithmetic average of the M^i and that the growth rate of B is, at the second order, proportional to the dispersion of the M^i.

As in the continuous time setting (see Jouini-Napp, 2003), for linear risk tolerance utility functions, we are able to explicitly compute the process B and to determine whether it is increasing, decreasing, smaller or greater than one. Although the techniques used are different, the results are similar to the ones obtained in the continuous time setting.

Example 2.5. *For general power utility functions, the process B satisfies*

$$\frac{B_t}{B_{t-1}} = \frac{E_{t-1}\left[\mathcal{E}_\eta^{\gamma_\cdot}\left(M_t^{\cdot}\right)\right]}{\mathcal{E}_\eta^{\gamma_\cdot}\left(M_{t-1}^{\cdot}\right)}$$

so that by Minkovski's Lemma, B is nondecreasing, greater than 1 *(resp. nonincreasing, lower than* 1*) if $\eta \geq 1$ (resp. $\eta \leq 1$).*

For logarithmic utility functions, $B \equiv 1$.

In the exponential case, the process B satisfies

$$\frac{B_t}{B_{t-1}} = \frac{E_{t-1}\left[\mathcal{E}_0^{\theta_\cdot}\left(M_t^{\cdot}\right)\right]}{\mathcal{E}_0^{\theta_\cdot}\left(M_{t-1}^{\cdot}\right)}$$

so that by Hölder's Lemma, B is nondecreasing, greater than 1.

The interpretation is the following. The parameter η is a cautiousness parameter. When there is more risk involved, depending on whether the investor is cautious or not, that is to say, depending on whether $\eta < 1$ or $\eta > 1$, it can be shown that the investor will reduce or increase current consumption with respect to future consumption. For instance, for $\eta < 1$, the investor is cautious and increases current consumption acting as if his/her utility was discounted by a positive discount rate. The converse reasoning leads to a negative discount rate if $\eta > 1$. Now in our context with heterogeneous beliefs, a possible interpretation consists in considering the dispersion of beliefs as a source of risk thereby leading for the representative agent to a discount factor associated to a positive or negative discount rate depending on whether $\eta < 1$ or $\eta > 1$. This interpretation of beliefs heterogeneity as a source of risk is compatible with the empirical findings of Cragg and Malkiel (1982) who studied the relationship between ex post returns and various measures of risk and found that the measure that performed best was a measure of divergence of opinion about the asset returns.

To summarize, we have pointed out through previous propositions two distinct effects of the introduction of beliefs heterogeneity on the equilibrium price.

There is first a change of probability effect from P to the new common probability Q, whose density is given by $\bar{M}$. This aggregate probability can be seen (at least in the classical utility functions cases) as a weighted average of the individual subjective probabilities. The weights of this average are given by the individual risk tolerances. The second effect is represented by an "aggregation bias", which is predictable and takes the form of a discount factor. This discount factor is, at least for classical utility functions, directly related to the dispersion of the beliefs. We are able, for linear risk tolerance utility functions, to determine if it is associated to a positive or negative discount rate. Modulo this discount factor, the equilibrium (state) price (density) is the same as in an economy in which all agents (or the representative agent) would share the same subjective belief Q. Remark that the interpretation of beliefs heterogeneity as a source of risk is only related to the second effect. The relative importance of these effects is discussed in detail in Section 4.

We shall now use our construction of a representative consumer to analyze the impact of heterogeneity of beliefs on the equilibrium properties.

2.2. Adjusted CCAPM, Risk Premium and Risk Free Rate

In this subsection, we wish to compare the equilibrium characteristics (CCAPM formula, risk premium, risk free rate) in the heterogeneous beliefs setting and in the standard setting. We shall consider as the standard setting an equilibrium where the beliefs are homomogeneous and given by the objective probability P, and where the representative agent utility function is given by[4] u_λ with the same (λ_i) as in our heterogeneous beliefs setting[5].

We suppose the existence of a riskless asset with price process S^0 such that $S_0^0 = 1$ and $S_t^0 = \prod_{s=1}^{t}\left(1 + r_s^f\right)$ for some predictable risk free rate process r^f. We consider a risky asset with positive price process S and associated rate of return between date t and $(t+1)$ denoted by $R_{t+1} \equiv \frac{S_{t+1}}{S_t} - 1$. In such a context, since q^*S is a P^*-martingale, we obtain, as in the classical case (see the Appendix),

$$E_t^P\left[R_{t+1}\right] - r_{t+1}^f = -cov_t^P\left[\frac{q_{t+1}^*}{E_t^P\left[q_{t+1}^*\right]}, R_{t+1}\right]. \tag{2.2}$$

[4]We recall that for $\alpha \in \left(\mathbb{R}_+^*\right)^N$, $u_\alpha(t,x) \equiv \max_{\sum_{i=1}^N x_i \leq x} \sum_{i=1}^N \frac{1}{\alpha_i} u_i(t,x_i)$.

[5]This is in particular the case when the standard setting equilibrium has the same Lagrange multipliers (or equivalently the same marginal valuations at date 0 or the same initial individual equilibrium consumptions) as in the heterogeneous beliefs framework, or without restriction on the Lagrange multipliers when investors have linear risk tolerance utility functions.

Now, since $q_t^* = \overline{M}_t B_t u'(t, e_t^*)$, with B predictable, Equation (2.2) can be written

$$\begin{aligned} E_t^P[R_{t+1}] - r_{t+1}^f &= -cov_t^P\left[\frac{\overline{M}_{t+1}B_{t+1}u'\left(t+1, e_{t+1}^*\right)}{E_t^P\left[\overline{M}_{t+1}B_{t+1}u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \\ &= -cov_t^P\left[\frac{\overline{M}_{t+1}u'\left(t+1, e_{t+1}^*\right)}{E_t^P\left[\overline{M}_{t+1}u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \qquad (2.3) \end{aligned}$$

so that the adjustment process B plays no role in the CCAPM formula with heterogeneous beliefs. This adjusted CCAPM formula differs from the following standard formula

$$E_t^P[R_{t+1}] - r_{t+1}^f = -cov_t^P\left[\frac{u'\left(t+1, e_{t+1}^*\right)}{E_t^P\left[u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \qquad (2.4)$$

only through the change of probability from P to the consensus probability Q. This implies that only the change of probability has an impact on the difference of the risk premium in the standard and in the heterogeneous beliefs setting.

The risk free rate r^f is given, as in the classical case, by (see the Appendix)

$$1 + r_{t+1}^f = \frac{q_t^*}{E_t^P\left[q_{t+1}^*\right]}.$$

Now, since $q_t^* = \overline{M}_t B_t u'(t, e_t^*)$, with B predictable, we obtain

$$\begin{aligned} 1 + r_{t+1}^f[\text{het.}] &= \left(\frac{B_t}{B_{t+1}}\right)\frac{\overline{M}_t u'(t, e_t^*)}{E_t^P\left[\overline{M}_{t+1}u'\left(t+1, e_{t+1}^*\right)\right]} \qquad (2.5) \\ &= \left(\frac{B_t}{B_{t+1}}\right)\left[1 + r_{t+1}^f[\text{homogeneous under } Q]\right] \qquad (2.6) \end{aligned}$$

where r^f [homogeneous under Q] denotes the equilibrium risk free rate in a model where all investors share the same subjective probability belief Q. Both the change of probability and the discount factor have an impact on the risk free rate. The impact of the discount factor leads to a decrease (resp. increase) of the riskfree rate if B is nondecreasing (resp. nonincreasing). This effect has a clear interpretation. A nonincreasing (resp. nondecreasing) discount factor B corresponds to the fact that the representative agent "discounts" future consumption, which means that future consumption is less (resp. more) important for the representative agent, it is then natural to obtain a higher (resp. lower) equilibrium risk free rate. For instance, for power utility functions with $\eta \leq 1$, as well as for exponential utility functions, we have seen that the "discount factor" B is nonincreasing and therefore contributes to a higher risk free rate.

The impact of the change of probability effect from P to the consensus probability Q depends on the sign of $E_t^Q \left[\frac{u'\left(t+1, e^*_{t+1}\right)}{u'(t, e^*_t)} \right] - E_t^P \left[\frac{u'\left(t+1, e^*_{t+1}\right)}{u'(t, e^*_t)} \right]$.

Through our aggregation procedure, we have then transformed the study of a model under heterogeneous subjective beliefs into the study of a model under homogeneous subjective beliefs. Indeed, the equilibrium risk premium formula under heterogeneous beliefs (Equation 2.3) is the same as in an economy where all investors would share the same probability belief, namely the consensus belief obtained through the aggregation procedure. The same is true of the risk free rate, modulo the "discount factor", and we have seen that we are able to determine the impact of the discount factor, at least for classical utility functions. The next step consists in the study of the impact of a subjective belief on the equilibrium risk premium and risk free rate, which is the aim of the next section.

3. Pessimism/doubt of the consensus belief

We consider in this section a standard homogeneous beliefs model, the agents common subjective belief $\widehat{P}$ being possibly different from the objective probability P. We suppose that $\widehat{P}$ is equivalent to the objective probability P and we denote by $\widehat{M}$ its density. We want to determine characteristics of the subjective belief $\widehat{P}$ that have an unambiguous impact on the risk premium and on the risk free rate, and, in relation with the risk premium puzzle (see Mehra-Prescott, 1985 or Kocherlakota, 1996 for a survey on the risk premium puzzle) and the risk free rate puzzle (Weil, 1989), which preferrably lead to an increase of the risk premium and a decrease of the risk free rate. In particular, we wish to analyze how pessimism /doubt of the subjective probability may affect the equilibrium risk premium and risk free rate.

In order to analyze the impact of the subjective belief on the value of the risk premium, we need to be aware of the fact that the possible returns $R = (R_t)_{t=1}^T$ obtained in equilibrium are not the same in the standard (objective belief) setting and in the subjective belief settings. What exactly is to be compared?

As in the classical CCAPM, we consider an asset, whose returns R_{t+1} between date t and date $t+1$ are perfectly correlated with e^*_{t+1} from date t point of view.

More precisely, we consider a return process such that $R_{t+1} = k_t e^*_{t+1} + b_t$ for some F_t-measurable positive random variables k_t and b_t. Notice that such an asset exists by market (dynamic) completeness. The risk premium for this asset in the objective belief setting is given by

$$E_t^P\left[R_{t+1}\right] - r^f_{t+1} = -cov_t^P\left[\frac{u'\left(e^*_{t+1}\right)}{E_t^P\left[u'\left(e^*_{t+1}\right)\right]}, k_t e^*_{t+1}\right].$$

For any positively homogeneous risk measure ρ, the market price of risk i.e., the ratio between the risk premium and the "level of risk" is given by

$$MPR_t^{obj}\left(R_{t+1}\right) = -\frac{1}{\rho\left(e^*_{t+1}\right)} cov_t^P\left[\frac{u'\left(e^*_{t+1}\right)}{E_t^P\left[u'\left(e^*_{t+1}\right)\right]}, e^*_{t+1}\right].$$

Since this quantity does not depend upon k_t and b_t, we shall denote it by $MPR_t^{obj}\left(e^*_{t+1}\right)$. We obtain analogously in the subjective belief setting under $\widehat{P}$ that

$$MPR_t^{subj(\widehat{P})}\left(e^*_{t+1}\right) = -\frac{1}{\rho\left(e^*_{t+1}\right)} cov_t^P\left[\frac{\widehat{M}_{t+1} u'\left(e^*_{t+1}\right)}{E_t^P\left[\widehat{M}_{t+1} u'\left(e^*_{t+1}\right)\right]}, e^*_{t+1}\right].$$

In order to compare the market price of risk in both settings, we are then led to compare $-cov_t^P\left[\frac{u'\left(e^*_{t+1}\right)}{E_t^P\left[u'\left(e^*_{t+1}\right)\right]}, e^*_{t+1}\right]$ with $-cov_t^P\left[\frac{\widehat{M}_{t+1} u'\left(e^*_{t+1}\right)}{E_t^P\left[\widehat{M}_{t+1} u'\left(e^*_{t+1}\right)\right]}, e^*_{t+1}\right]$. The subjective belief setting leads to a higher market price of risk if and only if

$$\frac{E_t^{\widehat{P}}\left[u'\left(e^*_{t+1}\right) e^*_{t+1}\right]}{E_t^{\widehat{P}}\left[u'\left(e^*_{t+1}\right)\right]} \leq \frac{E_t^P\left[u'\left(e^*_{t+1}\right) e^*_{t+1}\right]}{E_t^P\left[u'\left(e^*_{t+1}\right)\right]}. \tag{3.1}$$

Abel (2002) studies a similar problem. He considers a return process R_t generated by an asset whose dividends are given by e^*_t. In Abel's framework ($\frac{e^*_{t+1}}{e^*_t}$ i.i.d. and isoelastic utility functions), such a return process satisfies our condition $R_{t+1} = k_t e^*_{t+1} + b_t$ for some F_t-measurable positive random variables k_t and b_t. Abel's definition of the risk premium is slightly different, since it is equal to the ratio between the expected return of the considered asset and the riskfree return, instead of the difference between these two quantities. However, the probability measures leading to a higher risk premium (in Abel's sense) are those satisfying Inequality (3.1).

To summarize what we have just seen, the subjective probability measures $\widehat{P}$ leading to an increase of the market price of risk (compared to the standard objective belief setting) are characterized by Inequality (3.1) or equivalently by the fact that

$$cov_t^{P_u}\left(\widehat{M}_{t+1}, e_{t+1}^*\right) \leq 0 \tag{3.2}$$

where $\frac{dP_u}{dP}$ is given (up to a constant) by $u'\left(t+1, e_{t+1}^*\right)$ and $\widehat{M}_{t+1} = E\left[\frac{d\widehat{P}}{dP} \mid F_{t+1}\right]$. The subjective belief setting leads to a higher market price of risk when the subjective probability is negatively correlated with the total wealth under some probability and it is natural to interpret this property as a form of pessimism. Moreover, as seen in the previous section, the subjective probability measures $\widehat{P}$ leading to a decrease of the risk free rate are characterized by

$$E_t^{\widehat{P}}\left[\frac{u'\left(t+1, e_{t+1}^*\right)}{u'\left(t, e_t^*\right)}\right] \geq E_t^{P}\left[\frac{u'\left(t+1, e_{t+1}^*\right)}{u'\left(t, e_t^*\right)}\right]$$

or equivalently by

$$cov_t^{P}\left[\widehat{M}_{t+1}, u'\left(e_{t+1}^*\right)\right] \geq 0,$$

which also corresponds to some form of pessimism. We shall introduce more formally different notions of pessimism in the next section. Note that even if this section is in the same spirit as Abel (2002), our framework and definitions are different. Indeed, Abel (2002) first characterizes pessimism by the the first-order stochastic dominance and then introduces the concept of uniform pessimism characterized by a stronger form of dominance[6]. Abel (2002) proves then that uniform pessimism increases the risk premium, when agents have power utility functions. Jouini-Napp (2004) showed that this is nomore the case when uniform pessimism is replaced by pessimism (in Abel's sense) or when the class of utility functions is enlarged to the whole class of concave and nondecreasing functions. They proposed then, in a static setting, another definition for pessimism that leads to a risk-premium increase for all nondecreasing utility functions and that is characterized by this last property. In the next section we generalize this concept to a dynamic framework. Furthermore, as we shall see in Section 4, this concept

[6] Abel (2002) defines a pessimistic (resp. uniformly pessimistic) probability measure (with respect to e) as a probability measure $\widehat{P}$ such that, for all x, $\widehat{P}\left[e \leq x\right] \geq P\left[e \leq x\right]$ (resp. $\widehat{P}\left[e \leq x\right] = P\left[e \leq kx\right]$ for some $k > 1$).

is particularly adapted to the beliefs aggregation problem since we shall describe how individual pessimism is captured at the aggregate level. Abel (2002) also introduces the concept of doubt based on the second-order stochastic dominance. In the next we shall propose another definition that will appear in Section 4 as much more tractable in order to analyze how individual doubt is captured at the aggregate level.

3.1. Doubt and first order pessimism

In a static framework, Jouini-Napp (2004) established that Inequality (3.1) is satisfied for all nondecreasing utility functions u if and only if $\widehat{M}$ decreases with e^*. This anticomonotony property between $\widehat{M}$ and e^* is referred to as first order pessimism. We propose the following natural generalization of this concept to a dynamic framework.

Definition 3.1. *We say that a probability $\widehat{P}$ on (Ω, F, P) equivalent to P, with density process $\left(\widehat{M_t}\right)$ is pessimistic (with respect to e^*) if for all t, $\widehat{M}_{t+1}$ and $-e^*_{t+1}$ are comonotonic*[7] *conditionally to F_t.*

A pessimistic probability is such that its density $\widehat{M}_{t+1}$ at date $(t+1)$ decreases with e^*_{t+1} conditionally to F_t for all t; this means that conditionally to date t information, it puts more weight on states of nature where e^*_{t+1} is low, which clearly corresponds to a notion of pessimism.

For instance, if the information structure is described by a tree and if, for a given node ϖ, the transition probabilities of $\widehat{P}$ (resp. P) between ϖ and its immediate successors is given by $\widehat{\pi}_\varpi$ (resp. π_ϖ), then $\widehat{P}$ is pessimistic (with respect to e^*) if, for each t and each date$-t$ node ϖ, the transition density $\frac{\widehat{\pi}_\varpi}{\pi_\varpi}$ is comonotonic with $-e^*_{t+1}$ on the set of successors of ϖ. In particular this will be the case if $\frac{\widehat{\pi}_\varpi}{\pi_\varpi}$ decreases with e^*_{t+1}.

If F_{t+1} is generated by e^*_{t+1} and $\frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N}(\mu, \sigma)$ (resp. $\mathcal{N}(\widehat{\mu}, \widehat{\sigma})$) under P (resp. $\widehat{P}$) then $\widehat{P}$ is pessimistic (with respect to e^*) between t and $(t+1)$ if and

[7] Two random variables x and z are said to be comonotonic conditionally to F_t if there exists a random variable ξ and two functions f and g on $\Omega \times R$ such that

$$x(\omega) = f(\omega, \xi(\omega)) \text{ and } z(\omega) = g(\omega, \xi(\omega)), \qquad P \ a.s.$$

and such that f and g are F_t with respect to their first argument and nondecreasing with respect to their second argument.

only if $\mu \geq \widehat{\mu}$ and $\sigma = \widehat{\sigma}$. The same result holds for lognormal distributions. This result remains valid if μ and σ are F_t-measurable random variables and if the distributions are replaced by conditional distributions.

If (e_t^*) follows a Cox-Ross-Rubinstein binomial process with "returns" at each period denoted by u and d and associated transition probabilities π_u and $\pi_d = 1 - \pi_u$ (resp. $\widehat{\pi}_u$ and $\widehat{\pi}_d = 1 - \widehat{\pi}_u$) under P (resp. $\widehat{P}$), then $\widehat{P}$ is pessimistic if and only if $\pi_u \geq \widehat{\pi}_u$. As previously, the result remains valid if we introduce a time and state dependence for u, d, π_u and $\widehat{\pi}_u$ as long as $u(t)$ and $d(t)$, the returns between t and $(t+1)$ are F_t-measurable.

In Jouini-Napp (2004) is also introduced (still in a static setting) a concept of second order pessimism, characterized by the fact that Inequality (3.1) is satisfied for all nondecreasing and concave utility functions.

In particular, it is shown that if $\widehat{P}$ is second-order pessimistic with respect to e^* then $E^{\widehat{P}}\left[e^* \left| e^* \leq x\right.\right] \leq E^P\left[e^* \left| e^* \leq x\right.\right]$ for all x, which means that the notion of second order pessimism is another way of expressing the fact that $\widehat{P}$ "puts more weight" than P on the bad states of the world and corresponds then naturally to a concept of pessimism.

It is also shown that if a second order pessimistic probability measure $\widehat{P}$ satisfies $E^{\widehat{P}}\left[e^*\right] = E^P\left[e^*\right]$, then $Var^{\widehat{P}}\left[e^*\right] \geq Var^P\left[e^*\right]$. This permits to relate this concept of pessimism to the concept of doubt introduced by Abel (2002).

This notion of second order pessimism appears as hard to characterize from a practical point of view. Jouini-Napp (2004) introduced in a static setting an interesting subset of the second order pessimistic probability measures that are simple to describe and that clearly exhibit doubt[8]. A natural generalization of this class of measures in a dynamic framework leads to the following definition in the case where the conditional distribution of e_{t+1}^* is symmetric.

Definition 3.2. *Suppose that for all* t, *the* F_t-*conditional distribution of* e_{t+1}^* *under* P *is symmetric with respect to* $E_t\left[e_{t+1}^*\right]$. *We say that a probability* $\widehat{P}$ *on* (Ω, F, P) *equivalent to* P, *with density process* $\left(\widehat{M_t}\right)$ *exhibits doubt (resp. overconfidence) between date* t *and* $(t+1)$ *(with respect to* e^**) if for all* t, $\widehat{M_{t+1}}(\omega) = f_t(\omega, e_{t+1}^*(\omega) - E_t\left[e_{t+1}^*\right](\omega))$ *where* f_t *is* F_t-*measurable with respect to its first*

[8] Suppose that e^* has a symmetric distribution under P with respect to $E^P\left[e^*\right]$. A probability P' on (Ω, F) equivalent to P, with density M' is said to exhibit doubt with respect to e^* if M' is a function of e^*, symmetric with respect to $E^P\left[e^*\right]$ nonincreasing before $E^P\left[e^*\right]$ (and nondecreasing after).

variable, even and nondecreasing (resp. nonincreasing) on $\mathbb{R}_+$ with respect to its second variable.

This means that a probability measure, equivalent to P, exhibits doubt (resp. overconfidence) between date t and $(t+1)$ if conditionally to date t information, its density puts more (resp. less) weight on the tails and less (resp. more) weight on the center of the distribution. If F_{t+1} is generated by e^*_{t+1} and $\frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N}(\mu, \sigma)$ (resp. $\mathcal{N}(\widehat{\mu}, \widehat{\sigma})$) under P (resp. $\widehat{P}$), then $\widehat{P}$ exhibits doubt (with respect to e^*) between t and $(t+1)$ if and only if $\mu = \widehat{\mu}$ and $\sigma \leq \widehat{\sigma}$.

If (e^*_t) follows a trinomial process with "returns" at each period u, m and d and associated transition probabilities $\pi, 1-2\pi$, and π (resp. $\widehat{\pi}$, $1-2\widehat{\pi}$ and $\widehat{\pi}$) under P (resp. $\widehat{P}$), then $\widehat{P}$ exhibits doubt if and only if $\pi \leq \widehat{\pi}$.

3.2. Impact on the market price of risk and on the interest rate

We prove that pessimism and doubt lead to a higher market price of risk.

Proposition 3.3. *If the probability $\widehat{P}$ is pessimistic or exhibits doubt in the sense of the previous definitions, then for all t,*

$$MPR_t^{subj(\widehat{P})}\left(e^*_{t+1}\right) \geq MPR_t^{obj}\left(e^*_{t+1}\right)$$

i.e., the market price of risk between t and $(t+1)$ in the subjective belief setting under $\widehat{P}$ is greater than in the standard objective belief setting.

The interpretation is the following. In fact, the market price of risk subjectively expected is not modified by some pessimism. The reason why pessimism increases the objective expectation of the market price of risk is not that a pessimistic representative agent requires a higher market price of risk. He/She requires the same market price of risk but his/her pessimism leads him/her to underestimate the average rate of return of equity (leaving unchanged his/her estimation of the risk free rate). Thus the objective expectation of the equilibrium market price of risk is greater than the representative agent's subjective expectation, hence is greater than the standard market price of risk. The same interpretation holds for doubt. These results are consistent with the empirical findings of Cecchetti, Lam and Mark (2000) and Giordani and Söderlind (2003). In the first reference the authors prove that a model in which consumers exhibit pessimistic beliefs can better match sample moments of asset returns than can a rational expectations model. In the second reference the authors provide evidence of pessimism in investors forecasts.

As far as the risk free rate is concerned, we obtain the following result.

Proposition 3.4. *1. If the subjective probability belief $\widehat{P}$ exhibits pessimism and if the representative agent's utility function is concave and nondecreasing, then the equilibrium risk free rate is lower in the subjective belief setting than in the standard objective belief setting.*

2. If the subjective probability belief $\widehat{P}$ exhibits doubt and if the representative agent's utility function is nondecreasing, concave, with a convex derivative, then the equilibrium risk free rate is lower in the subjective belief setting than in the standard objective belief setting.

Hence, for a large class of utility functions including HARA utility functions, if there is doubt, the change of probability decreases the risk free rate. This result holds true if there is pessimism without restriction on the utility functions.

3.3. Application to the heterogeneous beliefs model

We adopt the same notations as above and we denote by $MPR_t^{het}\left(e_{t+1}^*\right)$ the market price of risk in the heterogeneous beliefs setting given by

$$MPR_t^{het}\left(e_{t+1}^*\right) = -\frac{1}{\rho\left(e_{t+1}^*\right)} cov_t^P \left[\frac{\overline{M}_{t+1} u'\left(e_{t+1}^*\right)}{E_t^P\left[\overline{M}_{t+1} u'\left(e_{t+1}^*\right)\right]}, e_{t+1}^*\right].$$

We want to compare this quantity to the market price of risk in the standard setting, as well as the risk free rate in both settings. We have seen in the preceding section that the heterogeneous beliefs model can be analyzed in terms of an equivalent homogeneous beliefs model, the homogeneous belief being given by the consensus probability belief. In particular, we have seen that $MPR_t^{het}\left(e_{t+1}^*\right) = MPR_t^{subj(Q)}\left(e_{t+1}^*\right)$ where Q denotes the consensus probability belief and $1 + r_{t+1}^f\,[\text{heterogeneous}] = \left(\frac{B_t}{B_{t+1}}\right)\left[1 + r_{t+1}^f\,[\text{homogeneous under } Q]\right]$. An immediate application of our results on pessimism and doubt leads then to the following proposition.

Proposition 3.5. *1. If the consensus probability belief Q exhibits pessimism and/or doubt then the market price of risk in the heterogeneous beliefs setting is lower than in the standard objective belief setting.*

2. *If the representative agent utility function is in the HARA class (i.e. $-\frac{u'(t,x)}{u''(t,x)} = \theta + \eta x$) with a cautiousness parameter $\eta > 1$ and if Q exhibits pessimism and/or doubt then the equilibrium interest rate in the heterogeneous beliefs setting is lower than in the standard objective belief setting.*

The last step of our analysis consists now in determining how pessimism and doubt at the aggregate level (i.e. on the consensus belief Q) can be inherited from the pessimism and doubt at the individual level (i.e. on the Q_i).

4. From individual to aggregate pessimism/doubt

The aim of this subsection is to analyze how the properties of the individual beliefs are transferred to the consensus belief. In particular, since we have seen in the preceding section that a pessimistic consensus probability (as well as a consensus probability which exhibits doubt for a large class of utility functions) leads to a higher market price of risk and a lower risk free rate, we wish to explore which properties of the individual beliefs lead to a pessimistic consensus belief or to a consensus belief which exhibits doubt.

We start with the analysis of the notion of pessimism. Notice that there is no need for all investors to be pessimistic in order to obtain a pessimistic consensus belief, hence an increase of the market price of risk and a decrease of the risk free rate. Indeed, since the consensus probability corresponds to some average of the individual beliefs, it suffices that some average of the individual beliefs be pessimistic.

If all the agents have exponential utility functions, we have seen that $M = \mathcal{E}_0^{\theta_\cdot}(M^\cdot)$. The aggregate characteristics M overweights the beliefs M^i for which θ_i is greater than the average and underweights the beliefs M^i for which θ_i is smaller than the average. In order to enlighten our analysis, let us consider the case where $\frac{e_{t+1}^*}{e_t^*} \sim_{F_t} \mathcal{N}(\mu, \sigma)$ under P, and $\frac{e_{t+1}^*}{e_t^*} \sim_{F_t} \mathcal{N}(\mu_i, \sigma)$ under Q_i. As in Section 2, we easily obtain that $\frac{e_{t+1}^*}{e_t^*} \sim_{F_t} \mathcal{N}\left(\mathcal{E}_1^{\theta_\cdot}(\mu_\cdot), \sigma\right)$ under Q.

If all θ_i are equal, then $M = \mathcal{E}_0^{\mathbf{1}}(M^\cdot)$, which means that the consensus characteristics M is an equally-weighted geometric average of the individual beliefs and the expected growth rate under the consensus characteristics appears as the equal weighted average of the individual subjective expected growth rates. The impact on the market price of risk is then simply given by the pessimism/optimism of the "equal-weighted average" investor. If the investors are on average pessimistic i.e. if $\mathcal{E}_1^{\mathbf{1}}(\mu_\cdot) \leq \mu$ (resp. optimistic, i.e. if $\mathcal{E}_1^{\mathbf{1}}(\mu_\cdot) \geq \mu$), then the market price of risk is higher (resp. lower) than in the standard setting - and the risk free rate is lower (resp. higher).

If the θ_i are different, the expected return under the consensus belief can be written as $\mathcal{E}_1^{\boldsymbol{\theta}_\cdot}(\mu_\cdot) = \mathcal{E}_1^{\mathbf{1}}(\mu_\cdot) + \sum_{i=1}^N (\mu_i - \mathcal{E}_1^{\mathbf{1}}(\mu_\cdot)) \left(\frac{\theta_i}{\overline{\theta}} - \frac{1}{N}\right)$. There are then two effects of beliefs heterogeneity on the equilibrium characteristics. The first effect is given as in the previous case by the average level of optimism/pessimism and the second effect is given by the "correlation" between risk tolerance/aversion and optimism/pessimism. If we assume for instance that the more risk tolerant investors are pessimistic (resp. optimistic) and the less risk tolerant investors are optimistic (resp. pessimistic), then the second effect increases (resp. reduces) the market price of risk. The intuition leads us to think that risk tolerance and optimism are positively correlated, which would induce a lower risk premium for assets with a higher dispersion. It remains to prove that this intuition is correct (or not). This could be done through behavioral or psychological empirical studies, and to our knowledge, this question is still open. This could also be done through the introduction in our model of a specific learning process which would lead to such a correlation, and this is left for future research.

Let us now assume that the utility functions are nomore exponential but of the form $u'(x) = (\theta_i + \eta x)^{-\frac{1}{\eta}}$. The consensus characteristics M is now given by $\mathcal{E}_\eta^{\gamma_\cdot}(M^\cdot)$. Our aim is to exhibit a third effect and for this purpose, let us assume that all the agents are equally weighted in the definition of M (no weights effect) and that there is no systematic bias in the average returns individual estimations (i.e. $\mathcal{E}_1^{\mathbf{1}}(\mu_\cdot) = \mu$). In order to simplify the analysis, we will even assume that the μ_i are symmetrically distributed around μ. The equally weighted geometric average belief is then equal to the objective belief and the consensus belief is an equally weighted average of the individual beliefs. However, this last average is an $\eta-$average and not a geometric average and, contrarily to the exponential case, the consensus belief is not equal to the objective one. The consensus characteristics M is such that $\frac{M_{t+1}}{M_t} = \left(\frac{1}{N}\sum_{i=1}^N \exp\frac{\eta}{2\sigma^2}\left[-(\mu_i-\mu)^2 - 2(\mu-\mu_i)\left(\frac{e^*_{t+1}}{e^*_t} - \mu\right)\right]\right)^{\frac{1}{\eta}}$. This function is clearly symmetric with respect to μ, increasing after μ. Then the consensus probability Q exhibits doubt and the market price of risk is higher than in the standard setting. It is easy to check that this doubt effect increases with the cautiousness parameter η. More precisely, if we denote by Q^η (resp. $Q^{\eta'}$) the consensus probability when all the agents have a cautiousness parameter η (resp. η') then, for $\eta' \geq \eta$, the density of $Q^{\eta'}$ with respect to Q^η is symmetric with respect to μ, and increases after μ. The probability $Q^{\eta'}$ exhibits more doubt than Q^{η} and the market price of risk is higher in the η' cautiousness framework than in the η cautiousness framework.

We have shown on these examples that there are three possible effects when we aggregate individual heterogeneous beliefs into a consensus belief:

- an average effect: if the (equally weighted) average belief is pessimistic or optimistic, then the consensus belief will be influenced accordingly,
- a cautiousness effect: when the consensus belief is an $\eta'-$average with given weights[9] and the objective probability is an $\eta-$average with the same weights, then there is a bias towards doubt if $\eta' \geq \eta$ and a bias towards overconfidence if $\eta \geq \eta'$.
- a relative weights effect: if the more risk tolerant investors are pessimistic (resp. optimistic) and the less risk tolerant investors are optimistic (resp. pessimistic), then the consensus belief will present a bias toward pessimism (resp. optimism).

The first effect is natural and does not necessitate specific developments. The second effect has been studied in a different framework by Gollier (2003) where the author analyzes the social impact of different exogeneous aggregation of beliefs procedures. With our notations, Gollier (2003) states in particular for some specific values of the cautiousness parameter η that the $\eta'-$average aggregation procedure is socially efficient if and only if $\eta' = \eta$. When this condition is not satisfied, Gollier (2003) analyzes the impact of a disagreement increase depending on the location of this increase (in the tails or in the center of the distribution).

However, the Taylor expansions provided in the introduction permit to show that these distinct averages differ only by second order terms, when the beliefs dispersion is sufficiently small. Hence the second effect deserves specific attention only when the two other effects cancel out.

Abel (1989) studied a problem that is similar to ours in a specific framework. He imposes a normal distribution for the aggregate wealth and considers exponential utility functions. Furthermore, it is assimed that the "average investor is rational", i.e. the geometric average belief does not exhibit any bias and is equal to the objective probability. Under these assumptions our three effects vanish and there is no impact on the market price of risk as defined in this paper. Abel

[9] i.e. when we have CRRA utility functions with a cautiousness parameter η.

(1989) is not interested in the impact of beliefs heterogeneity on the market price of risk but on the risk premium. The impact of beliefs heterogeneity on the risk premium is directly related to the impact on the risk free rate and Abel finds that beliefs heterogeneity leads to an increase of the risk premium.

In the next, we focus on the third and last effect and, for this purpose, we suppose that the equally-weighted average of the individual beliefs corresponds to the objective probability P. We obtain the following result, without any specific assumption on the distributions of the growth rate of aggregate wealth.

Proposition 4.1. *Assume that* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i > 0$ *and that*

$$\frac{\mathcal{E}_0^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)}{E_t\left[\mathcal{E}_0^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)\right]} = 1 \qquad t = 0, .., T.$$

If the pessimistic (resp. optimistic) agents have a risk tolerance higher (resp. lower) than the average, then the market price of risk in the heterogeneous beliefs setting is greater than in the standard setting.

This result can be generalized to the other utility functions in the HARA class. We have

Proposition 4.2. *Assume that* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ *for* $\eta \neq 0$, *and that*

$$\frac{\mathcal{E}_\eta^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)}{E_t\left[\mathcal{E}_\eta^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)\right]} = 1 \qquad t = 0, .., T.$$

If the pessimistic (resp. optimistic) agents have a risk tolerance higher (resp. lower) than the average, then the market price of risk in the heterogeneous beliefs setting is greater than in the standard setting.

Remark that the condition on the (M^i) still means that the equally-weighted average of the individual beliefs corresponds to the objective probability P. However, the geometric average is replaced by the power η average. It is possible to replace this condition by a geometric average condition (or an $\eta'-$average with $\eta' < \eta$) modulo an additional "cautiousness" effect. As mentioned previously, this effect is of the second order for small dispersions. Furthermore, this effect should, as in the examples above, lead to an additional increase of the market price of risk.

We can adopt the same approach to analyze how doubt/overconfidence at the aggregate level can be inherited from the doubt/overconfidence at the individual level.

In the case of exponential utility functions, we have seen that $M = \mathcal{E}_0^{\boldsymbol{\theta}_\cdot}(M^\cdot)$ and M overweights the beliefs M^i for which θ_i is greater than the average and underweights the beliefs M^i for which θ_i is smaller than the average. For instance, suppose that under P, $\frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N}(\mu, \sigma)$ and that under Q_i, $\frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N}(\mu, \sigma_i)$. It is then easy to see that, as in Section 2, $\frac{e^*_{t+1}}{e^*_t} \sim_{F_t} \mathcal{N}\left(\mu, \mathcal{E}^{\boldsymbol{\theta}_\cdot}_{-2}(\sigma_\cdot)\right)$ under Q. There are then two effects of beliefs heterogeneity on the market price of risk. The first effect (the average effect) is given by the average level of doubt/overconfidence, measured by an equally weighted average of the individual levels of overconfidence $\frac{1}{\sigma_i^2}$. The second effect (relative weights effect) is given by the covariance between individual risk tolerance/aversion and the individual level of overconfidence. If we assume for instance that the more risk tolerant investors exhibit doubt and that the less risk tolerant investors exhibit overconfidence, then we get a higher (resp. lower) market price of risk. We could, as in the analysis of pessimism, exhibit a third "cautiousness" effect if we consider power utility functions.

As above, if we focus on the "relative weights effect" and suppose that the equally-weighted average of the individual beliefs corresponds to the objective probability P, we obtain the analog of Propositions 4.1 and 4.2.

Proposition 4.3. *Assume that* $-\frac{u_i'(t,x)}{u_i''(t,x)} = \theta_i + \eta x$ *(with η possibly equal to zero) and for all t,* $\frac{\mathcal{E}^{\mathbf{1}}_\eta(M^\cdot_{t+1})}{E_t\left[\mathcal{E}^{\mathbf{1}}_\eta(M^\cdot_{t+1})\right]} = 1$.

If the agents that exhibit doubt (resp. overconfidence) have a risk tolerance higher (resp. lower) than the average, then the market price of risk in the heterogeneous beliefs setting is greater than in the standard setting.

Pessimism and doubt can then be seen as possible explanations for the risk-premium puzzle as well as for the risk-free rate puzzle as underlined by Abel (1989). However, it is not necessary to assume pessimism or doubt at the individual level nor on (equally weighted) average. A potential important source of pessimism or doubt at the aggreagate level is the correlation between the individual risk tolerance and the individual level of pessimism/doubt.

5. Conclusion

In this paper, we provided an aggregation procedure which permits to rewrite in a simple way the equilibrium characteristics (state price density, market price of risk, risk premium, risk-free rate) in a heterogeneous beliefs framework and to compare them with an otherwise similar standard setting. This procedure permits to analyze in detail the impact of beliefs heterogeneity on the equilibrium chracteristics.

In particular, we introduced concepts of pessimism and doubt and we proved, in a fairly general setting, that pessimism and doubt at the aggregate level lead to an increase of the market price of risk. We also have shown how pessimism and doubt are transmitted from the individual to the aggregate level.

In particular, it appears that pessimism and doubt lead to an increase of the market price of risk and, under some additional conditions on the cautiousness level, a decrease of the market price of risk. Furthermore, the aggregate level of pessimism and/or doubt is, almost for utility functions in the HARA class, a weighted average of the individual levels of pessimism and doubt. These weights are proportional to the individual risk tolerances and, at the equilibrium, there is a bias toward the beliefs of the more risk tolerant agents. The intuition leads us to think that risk tolerant agents are more optimistic. It remains to prove that this intuition is correct and this could be done through behavioral experimental studies or through the introduction of a theoretical model of learning and beliefs construction. This is left for future research.

Appendix

Proof of Proposition 2.1 Since q^* is an interior equilibrium price process relatively to the beliefs (M^i), and the income processes e^i, we know that $\sum_{i=1}^N y^{*^i} = e^*$ and that there exist positive Lagrange multipliers (λ_i) such that for all i and for all t,

$$M_t^i u_i'\left(t, y_t^{*^i}\right) = \lambda_i q_t^*.$$

We consider the maximization problem

$$\left(\mathcal{P}^\lambda\right) : \max \sum_{i=1}^N \frac{1}{\lambda_i} U_i\left(y^i\right) \text{ under the constraint } \sum_{i=1}^N y^i \leq e^*,$$

where $U_i(c) = E\left[\sum_{t=0}^T u_i(t, c_t)\right]$. Denoting the solution by $\left(y^{i,\lambda}\right)_i$, we get that $\sum_{i=1}^N y^{i,(\lambda)} = e^*$ and the process $\left(\frac{1}{\lambda_i} u_i'\left(t, y_t^{i,(\lambda)}\right)\right)_t$ is independent from i. We denote this process by $p^{(\lambda)}$. Letting $M^{(\lambda)} \equiv \frac{q^*}{p^{(\lambda)}}$, we then have for all i and for all t

$$M_t^{(\lambda)} u_i'\left(t, y_t^{i,(\lambda)}\right) = M_t^i u_i'\left(t, y_t^{*^i}\right).$$

The process $M^{(\lambda)}$ is adapted and positive. Moreover, at date $t = 0$, we have for all i, $M_0^i = 1$, and $\sum_{i=1}^{N} y_0^{i,(\lambda)i} = \sum_{i=1}^N y_0^{*^i} = e_0^*$, so that $M_0^{(\lambda)} = 1$. Then, it suffices to take $M = M^{(\lambda)}$ and $\bar{y}^i = y^{i,(\lambda)}$.

As far as uniqueness is concerned, notice that any process y^i such that $\sum_{i=1}^N y^i = e^*$ and

$$M_t u_i'\left(t, y_t^i\right) = M_t^i u_i'\left(t, y_t^{*^i}\right)$$

for some positive process M is a solution of the maximization problem $\left(\mathcal{P}^{(\lambda)}\right)$. The uniqueness follows from the strict concavity of this maximization program. ■

Proof of Proposition 2.2 Similar to the proof of the analogous result in a standard setting. ■

Proof of Example 2.3 Since the representative utility function u is given by

$$u_\lambda(t, x) = \max_{\sum_{i=1}^N x_i \leq x} \sum_{i=1}^N \frac{1}{\lambda_i} u_i(t, x_i)$$

the expression of u_λ in the specific setting of linear risk tolerance utility functions is obtained as in the standard case (see e.g. Huang-Litzenberger, 1988).

The expression of M is obtained by using $M_t u_i'\left(t, \overline{y}_t^i\right) = M_t^i u_i'\left(t, y_t^{*^i}\right)$, as well as

$$\sum_{i=1}^N y^{*^i} = \sum_{i=1}^N \overline{y}^i = e^*.$$

Indeed, in the case of exponential utility functions, we have for all i,

$$M^i \exp\left(-\frac{y^{*^i}}{\theta_i}\right) = M \exp\left(-\frac{\overline{y}^i}{\theta_i}\right)$$

hence

$$\prod_{i=1}^N \left(M^i\right)^{\theta_i} \exp\left(-\sum_{i=1}^N y^{*^i}\right) = M^{\overline{\theta}} \exp\left(-\sum_{i=1}^N \overline{y}^i\right),$$

or equivalently

$$M = \prod_{i=1}^N \left(M^i\right)^{\frac{\theta_i}{\overline{\theta}}}.$$

In the case of power utility functions, we get for all i,

$$M^i \left(\theta_i + \eta y^{*^i}\right)^{-1/\eta} = M\left(\theta_i + \eta \overline{y}^i\right)^{-1/\eta} = \lambda_i M \left(\overline{\theta} + \eta e^*\right)^{-1/\eta} b$$

hence

$$\left(M^i\right)^\eta \lambda_i^{-\eta} = M^\eta \left(\overline{\theta} + \eta e^*\right)^{-1} b^\eta \left(\theta_i + \eta y^{*^i}\right)$$

and

$$M = \left[\sum_{i=1}^N \frac{\lambda_i^{-\eta}}{\sum_{i=1}^N \lambda_i^{-\eta}} \left(M^i\right)^\eta\right]^{1/\eta}.$$

■

Proof of Proposition 2.4 Immediate defining B and $\overline{M}$ by

$$B_t = B_{t-1} \frac{E_{t-1}\left[M_t\right]}{M_{t-1}}, \qquad B_0 = 1$$

and

$$\overline{M}_t = \frac{M_t}{B_t}.$$

■

Proof of Example 2.5 By Example (2.3) and Proposition (2.4), we immediately get the expressions for B_t. Hölder 's inequality permits to conclude in the exponential case. In the case of power utility functions, for $0 < \eta < 1$, we get by Minkovski's inequality that $E_{t-1}\left[\left[\sum_{i=1}^{N}\gamma_i\left(M_t^i\right)^{\eta}\right]^{1/\eta}\right] \leq \left[\sum_{i=1}^{N}\gamma_i\left(M_{t-1}^i\right)^{\eta}\right]^{1/\eta}$ so that B is nonincreasing. The case $\eta > 1$ can be treated similarly. ■

Proof of the CCAPM formula and the risk free rate expression

We know that for any asset with associated price process $(S_t)_{t=0}^{T}$, we must have

$$q_t^* S_t = E_t^P\left[q_{t+1}^* S_{t+1}\right] \text{ for all } t = 0, ..., T-1. \tag{5.1}$$

or equivalently, since $\widehat{M_t} B_t u'(t, e_t^*) = q_t^*$,

$$S_t = E_t^p\left[\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)} S_{t+1}\right]$$

or by definition of R_{t+1}

$$1 = E_t^p\left[\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}(1 + R_{t+1})\right].$$

This leads to

$$1 = E_t^p\left[\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}\right]\left(1 + E_t^P\left[R_{t+1}\right]\right) + cov_t^P\left(\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}, R_{t+1}\right).$$

Now, applying equation (5.1) to the riskless asset yields $E_t^p\left[\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}\right] = \frac{1}{1+r_{t+1}^f}$, hence

$$1 + r_{t+1}^f = \left(1 + E_t^P\left[R_{t+1}\right]\right) + \left(1 + r_{t+1}^f\right) cov_t^P\left(\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}, R_{t+1}\right)$$

and

$$E_t^P\left[R_{t+1}\right] - r_{t+1}^f = -\frac{cov_t^P\left(\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}, R_{t+1}\right)}{E_t^p\left[\frac{\widehat{M_{t+1}} B_{t+1} u'\left(t+1, e_{t+1}^*\right)}{\widehat{M_t} B_t u'(t, e_t^*)}\right]}$$

or equivalently

$$\begin{aligned} E_t^P\left[R_{t+1}\right] - r_{t+1}^f &= -cov_t^P\left[\frac{\overline{M}_{t+1}B_{t+1}u'\left(t+1, e_{t+1}^*\right)}{E_t^P\left[\overline{M}_{t+1}B_{t+1}u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \\ &= -cov_t^P\left[\frac{\widehat{M}_{t+1}u'\left(t+1, e_{t+1}^*\right)}{E_t^P\left[\widehat{M}_{t+1}u'\left(t+1, e_{t+1}^*\right)\right]}, R_{t+1}\right] \end{aligned}$$

■

Proof of Proposition 3.3 1. Consider first the case of a pessimistic probability measure $\widehat{P}$. We have seen (Inequality 3.2) that the probability measures leading to an increase of the market price of risk are characterized by $cov_t^{P_u}\left(\widehat{M}_{t+1}, e_{t+1}^*\right) \leq 0$ where $\frac{dP_u}{dP}$ is given (up to a constant) by $u'\left(t+1, e_{t+1}^*\right)$. By definition, since $\widehat{P}$ is pessimistic with respect to e^*, the random variables $\widehat{M}_{t+1}$ and $-e_{t+1}^*$ are comonotonic conditionally to F_t. From Proposition 1 in Jouini-Napp (2004), we know that for positive random variables X and Z such that X and Z are comonotonic conditionally to F_t, we have $cov_t^Q\left(X, Z\right) \geq 0$ for any probability measure Q absolutely continuous with respect to P. We easily deduce from this lemma that $cov_t^{P_u}\left(\widehat{M}_{t+1}, e_{t+1}^*\right) \leq 0$.

2. Consider now the case of a probability measure $\widehat{P}$ which exhibits doubt. We have obtained (Inequality 3.1) that $\widehat{P}$ leads to a higher market price of risk if and only if

$$\frac{E_t^{\widehat{P}}\left[u'\left(e_{t+1}^*\right)e_{t+1}^*\right]}{E_t^{\widehat{P}}\left[u'\left(e_{t+1}^*\right)\right]} \leq \frac{E_t^P\left[u'\left(e_{t+1}^*\right)e_{t+1}^*\right]}{E_t^P\left[u'\left(e_{t+1}^*\right)\right]}.$$

We have

$$\frac{E_t^{\widehat{P}}\left(e_{t+1}^*u'\left(e_{t+1}^*\right)\right)}{E_t^{\widehat{P}}\left(u'\left(e_{t+1}^*\right)\right)} = \frac{E_t^P\left[\left(e_{t+1}^* - m_t\right)\widehat{M}_{t+1}u'\left(e_{t+1}^*\right)\right]}{E_t^P\left[\widehat{M}_{t+1}u'\left(e_{t+1}^*\right)\right]} + m_t$$

where $m_t = E_t\left[e_{t+1}^*\right]$. Since the random vector $\left(e_{t+1}^* - m_t, \widehat{M}_{t+1}\right)$ is distributed like $\left(m_t - e_{t+1}^*, \widehat{M}_{t+1}\right)$ conditionally to F_t, we have

$$E_t^P\left[\left(e_{t+1}^* - m_t\right)\widehat{M}_{t+1}u'\left(e_{t+1}^*\right)1_{e_{t+1}^* \leq m_t}\right] = E_t^P\left[\left(m_t - e_{t+1}^*\right)\widehat{M}_{t+1}u'\left(2m_t - e_{t+1}^*\right)1_{e_{t+1}^* \geq m_t}\right]$$

hence

$$E_t^P\left[\left(e_{t+1}^*-m_t\right)\widehat{M}_{t+1}u'\left(e_{t+1}^*\right)\right]=E_t^P\left[\left(m_t-e_{t+1}^*\right)\widehat{M}_{t+1}\left[u'\left(2m_t-e_{t+1}^*\right)-u'\left(e_{t+1}^*\right)\right]1_{e_{t+1}^*\geq m_t}\right].$$

Then

$$\frac{E_t^{\widehat{P}}\left(e_{t+1}^*u'\left(e_{t+1}^*\right)\right)}{E_t^{\widehat{P}}\left(u'\left(e_{t+1}^*\right)\right)}=\frac{E_t^P\left[\left(m_t-e_{t+1}^*\right)\widehat{M}_{t+1}\left[u'\left(2m_t-e_{t+1}^*\right)-u'\left(e_{t+1}^*\right)\right]1_{e_{t+1}^*\geq m_t}\right]}{E_t^P\left[\widehat{M}_{t+1}\left[u'\left(2m_t-e_{t+1}^*\right)+u'\left(e_{t+1}^*\right)\right]1_{e_{t+1}^*\geq m_t}\right]}+m_t.$$

We want to compare this quantity with

$$\frac{E_t^P\left[\left(m_t-e_{t+1}^*\right)\left[u'\left(2m_t-e_{t+1}^*\right)-u'\left(e_{t+1}^*\right)\right]1_{e_{t+1}^*\geq m_t}\right]}{E_t^P\left[\left[u'\left(2m_t-e_{t+1}^*\right)+u'\left(e_{t+1}^*\right)\right]1_{e_{t+1}^*\geq m_t}\right]}+m_t.$$

Letting $g\left(m_t,e_{t+1}^*\right)\equiv\left(m_t-e_{t+1}^*\right)\left[u'\left(2m_t-e_{t+1}^*\right)-u'\left(e_{t+1}^*\right)\right]$ and $h\left(m_t,e_{t+1}^*\right)\equiv u'\left(2m_t-e_{t+1}^*\right)+u'\left(e_{t+1}^*\right)$, we are led to compare $\frac{E_t^P\left[\widehat{M}_{t+1}g\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}\right]}{E_t^P\left[\widehat{M}_{t+1}h\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}\right]}$ with $\frac{E_t^P\left[g\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}\right]}{E_t^P\left[h\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}\right]}$. Let us now define the probability measure P^g by $\frac{dP^g}{dP}=\frac{g\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}}{E^P\left[g\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}\right]}$. We are led to compare $\frac{E_t^{P^g}\left[\widehat{M}_{t+1}\right]}{E_t^{P^g}\left[\widehat{M}_{t+1}\frac{h}{g}\left(m_t,e_{t+1}^*\right)\right]}$ with $\frac{1}{E_t^{P^g}\left[\frac{h}{g}\left(m_t,e_{t+1}^*\right)\right]}$.

It is easy to check that the function $\frac{h}{g}\left(x,\cdot\right):y\mapsto\frac{(u'(x+y)+u'(x-y))}{y(u'(x-y)-u'(x+y))}$ is decreasing on $\mathbb{R}_+$, so that $\widehat{M}_{t+1}1_{e_{t+1}^*\geq m_t}$ and $-\frac{h}{g}\left(m_t,e_{t+1}^*\right)1_{e_{t+1}^*\geq m_t}$ are comonotonic conditionally to F_t. We have then (Jouini and Napp 2004, Proposition 1), $cov_t^{P^g}\left(\frac{h}{g}\left(m_t,e_{t+1}^*\right),\widehat{M}_{t+1}\right)\leq 0$ or equivalently

$$E_t^{P^g}\left[\widehat{M}_{t+1}\frac{h}{g}\left(m_t,e_{t+1}^*\right)\right]\leq E_t^{P^g}\left[\widehat{M}_{t+1}\right]E_t^{P^g}\left[\frac{h}{g}\left(m_t,e_{t+1}^*\right)\right]$$

which concludes the proof. ■

Proof of Proposition 3.4 1. If $\widehat{P}$ exhibits pessimism with respect to e^*, then by definition of pessimism and since u is concave, we easily obtain that for all t, $\overline{M}_{t+1}$ and $u'\left(e_{t+1}^*\right)$ are comonotonic conditionally to F_t. By Proposition 1 in Jouini-Napp (2004), we deduce that for all probability measure $\widehat{P}$ absolutely continuous with respect to P, we have for all t, $cov_t^{\widehat{P}}\left(\overline{M}_{t+1},u'\left(e_{t+1}^*\right)\right)\geq 0$. In particular, we have for all t, $cov_t^P\left(\overline{M}_{t+1}, u'\left(e_{t+1}^*\right)\right) \geq 0$ and $E_t^{\widehat{P}}\left[u'\left(e_{t+1}^*\right)\right] \geq E_t^P\left[u'\left(e_{t+1}^*\right)\right]$.

2. If $\widehat{P}$ exhibits doubt with respect to e^*, then we know that

$$E_t^{\widehat{P}}\left[u'\left(e_{t+1}^*\right)\right] = E_t^{\widehat{P}}\left[\left[u'\left(2m_t - e_{t+1}^*\right) + u'\left(e_{t+1}^*\right)\right] 1_{e_{t+1}^* \geq m_t}\right]$$

and

$$E_t^P\left[u'\left(e_{t+1}^*\right)\right] = E_t^P\left[\left[u'\left(2m_t - e_{t+1}^*\right) + u'\left(e_{t+1}^*\right)\right] 1_{e_{t+1}^* \geq m_t}\right].$$

Since u has a convex derivative, $u'(2m - x) + u'(x)$ is nondecreasing on $x \geq m$ and we conclude as in the previous case. ■

Proof of Proposition 3.5 1. Immediate consequence of Proposition 3.3 and the fact that the market price of risk under heterogeneous beliefs is the same as the market price of risk under the subjective belief Q.

2. Immediate consequence of Equality 2.6, Proposition 3.4 and Example 2.5. ■

Proof of Proposition 4.1 We have $\frac{\mathcal{E}_0^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)}{E_t\left[\mathcal{E}_0^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)\right]} = 1$ and $\mathcal{E}_0^{\theta_{\cdot}}\left(M_{t+1}^{\cdot}\right) = \prod_{i=1}^N \left(M_{t+1}^i\right)^{\left(\frac{\theta_i}{\bar{\theta}} - \frac{1}{N}\right)} \mathcal{E}_0^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right) = \prod_{i=1}^N \left(M_{t+1}^i\right)^{\left(\frac{\theta_i}{\bar{\theta}} - \frac{1}{N}\right)} E_t\left[\mathcal{E}_0^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)\right]$. Consequently, $\bar{M} = \prod_{i=1}^N \left(M_{t+1}^i\right)^{\left(\frac{\theta_i}{\bar{\theta}} - \frac{1}{N}\right)} N_t$ where N_t is some F_t-measurable random variable. It is easy to see that under our assumptions, the random variable $M = \prod_{i=1}^N \left(M_{t+1}^i\right)^{\left(\theta_i - \frac{\bar{\theta}}{N}\right)/\bar{\theta}}$ is comonotonic with $-e_{t+1}^*$ conditionally to F_t, hence the consensus probability is pessimistic. Proposition 3.3 concludes. ■

Proof of Proposition 4.2 For power utility functions, let us remark that $\left[\mathcal{E}_\eta^{\gamma_{\cdot}}\left(M_{t+1}^{\cdot}\right)\right]^\eta = \sum_{i=1}^N \left(\gamma_i - \frac{1}{N}\right)\left(M_{t+1}^i\right)^\eta + \left[\mathcal{E}_\eta^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)\right]^\eta$. Under our assumptions, since $\left[\mathcal{E}_\eta^{\mathbf{1}}\left(M_{t+1}^{\cdot}\right)\right]^\eta$ is F_t-measurable, this expression is comonotonic with $-e_{t+1}^*$ conditionally to F_t and again, Proposition 3.3 concludes. ■

Proof of Proposition 4.3 For $\eta = 0$, as in the proof of Proposition 4.1, we have $\bar{M} = \prod_{i=1}^N \left(M_{t+1}^i\right)^{\left(\frac{\theta_i}{\bar{\theta}} - \frac{1}{N}\right)} N_t$ where N_t is some F_t-measurable random variable. Now, if the more risk tolerant agents (i.e. those for which $\theta_i > \frac{\bar{\theta}}{N}$) exhibit doubt (i.e. have a density M_{t+1}^i that is symmetric with respect to $E_t\left[e_{t+1}^*\right]$ nondecreasing after $E_t\left[e_{t+1}^*\right]$ conditionally to F_t) and if the less risk tolerant agents (i.e. those for which $\theta_i < \frac{\bar{\theta}}{N}$) are overconfident (i.e. have a density M_{t+1}^i that is nonincreasing with e_{t+1}^* after $E_t\left[e_{t+1}^*\right]$ conditionally to F_t), we clearly obtain that $\overline{M}$ exhibit doubt.

The case $\eta \neq 0$ can be similarly treated using the arguments of the proof of Propositions 4.2. ■

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