Pessimism, riskiness, risk aversion and the market price of risk

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March 24, 2004

Abstract

In this paper, we present a criterion related to the market price of risk in an equilibrium model, which can be used to order 1) the degree of pessimism of subjective probabilities, 2) the level of risk aversion of economic agents and 3) the level of risk of given prospects. This criterion is quite natural and permits to enlighten the risk premium puzzle. Moreover, it shreds light on the existing links between the different notions of pessimism, risk aversion and level of risk.

Keywords : pessimism, optimism, doubt, stochastic dominance, risk premium, riskiness, risk aversion, market price of risk

JEL numbers : D81, G11

1. Introduction

We start from an equilibrium model in which the (representative) agent is endowed with a subjective probability Q different from the initial probability P. We want to measure the impact of this subjective probability on the equilibrium market price of risk. In particular, in relation with the Equity premium puzzle[1], we are interested in the characterization of the set of subjective probabilities leading to an increase of the market price of risk for all utility functions in a given class.

Starting from the empirical findings[2] of [2], [1] has considered a closely related problem. He defines a pessimistic agent by the fact that his belief, described by a probability measure Q, is dominated by the initial probability P in the sense of the first stochastic dominance[3]. In order to analyze the impact of pessimism on the equilibrium risk premium, [1] introduces the stronger notion of uniform pessimism[4] and proves then that uniform pessimism leads to an increase of the equilibrium risk premium for power utility functions.

It is intuitively appealing to obtain that pessimism raises the risk premium. However, the concept of first stochastic dominance-pessimism introduced by [1] does not seem to be the right concept to reach such a conclusion in a sufficiently general framework. Indeed, we show that in some cases, pessimism as defined by [1] leads to a decrease of the risk premium (even in the class of power utility functions).

We then propose to adopt the converse approach and to say that an agent, described by his subjective probability Q_1 is more pessimistic than an agent Q_2 (with respect to a given aggregate wealth e) if the equilibrium market price of risk is higher under Q_1 than under Q_2 for all utility functions in a given class. We define, according to which class of utility functions we consider, the pessimism or market price of risk-dominance of the first order (for all nonincreasing utility functions) and of the second order (for all nonincreasing and concave utility functions). We show that a necessary and sufficient condition for a probability measure Q_1 to be more first order-pessimistic than another probability Q_2 (with respect to a given aggregate wealth e) is that the "density" of Q_1 with respect to Q_2 decreases with the aggregate wealth e. This is a stronger notion of pessimism than the first

[1] See [14] or [12] for a survey.

[2] The authors prove that a model in which consumers exhibit pessimistic beliefs can better match sample moments of asset returns than can a rational expectations model. See also the empirical study in [6], where the authors provide evidence of pessimism in investors forecasts.

[3] i.e., for all t, $Q(X \leq t) \geq P(X \leq t)$.

[4] defined by $Q(X \leq t) = P\left(X \leq e^{\Delta} t\right)$ for some $\Delta > 0$.

stochastic dominance. We also provide a necessary and sufficient condition for a probability measure Q_1 to be more second order-pessimistic than another probability Q_2 (with respect to a given aggregate wealth e) in terms of copositivity of given matrices. In particular, we show that second order pessimism implies lower conditional expected values for the total wealth and a higher variance when the expected wealth is the same under the two probabilities.

We then apply the same "market price of risk" approach to the study of the risk aversion of a given utility function. We study which (representative) agents described by their utility functions lead to an increase of the equilibrium market price of risk. More precisely, we wish to analyze the conditions on the utility functions u and v which characterize the fact that for all aggregate wealth, the market price of risk under u is higher than under v. We prove that a necessary and sufficient condition is that u is more risk averse than v in the sense of Arrow-Pratt ([17]).

We finally apply the same approach to the study of the level of risk of a given prospect. We shall say that a prospect X is more risky (or less desirable) than another prospect Y if the relative equilibrium price for X (i.e. its price in terms of units of riskless asset) is lower than the relative equilibrium price for Y for all representative agents in a given class. As in the analysis of pessimism, we introduce the concepts of first- and second-order relative equilibrium price dominance. We obtain that the order induced by the first-order equilibrium relative price dominance is equivalent to the well-known monotone likelihood ratio (MLR) order (see [13]) and that the order induced by the second-order equilibrium relative price dominance is equivalent to the central-riskiness order (see [7, 8]) for all the translations of X and Y. More generally, we show that the first- and second-order relative equilibrium price dominance for given prospects is equivalent to the first- and second-order market price of risk dominance for their distributions[5].

The paper is organized as follows. Section 2 is devoted to the study of the market price of risk-dominance for given subjective probabilities in relation with the level of pessimism. Section 3 is devoted to the study of the market price of risk-dominance for given utility functions in relation with the level of risk aversion. Section 4 is devoted to the study of the equilibrium relative price-dominance in relation with the level of risk of given prospects. Section 5 concludes.

[5]The distribution of a random variable X is the probability measure P_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $P_X(A) = P(X \in A)$ for all Borel set A.

2. Pessimism

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. We consider a standard 2 dates Lucas-fruit tree economy, except that we allow the representative agent to have a subjective belief/opinion. The representative agent solves a standard utility maximization problem. He has a current income at date $t = 0, 1$ denoted by (e_0, e) and a von Neumann-Morgenstern utility function for consumption of the form $u_0(c_0) + E^Q[u(c)]$, where Q is a probability measure absolutely continuous with respect to P which corresponds to the subjective belief of the agent. Throughout the paper e is assumed to be nonnegative. If we denote by M the nonnegative density of Q with respect to P, then the utility function can be rewritten as $u_0(c_0) + E^P[Mu(c)]$. We suppose that the economy is in equilibrium, i.e. that there exists a nonnegative, nonzero and uniformly bounded price process [6] q such that the optimal consumption plan for the representative agent (under his budget constraint) coincides with the aggregate wealth of the economy, i.e. $e = \arg\max_{E^P[(c_0-e_0)+q(c-e)]\leq 0} \left\{u_0(c_0) + E^P[Mu(c)]\right\}$. Such an equilibrium, when it exists, can be characterized by the first order necessary conditions for individual optimality. When the equilibrium is interior, these conditions are given by $Mu'(e) = u_0'(e_0)q$.

It is easy to obtain, as in the classical setting, the CCAPM formula. We suppose the existence of a riskless asset with price process S^0 such that $S_0^0 = 1$ and $S_1^0 = \left(1 + r^f\right)$ for some risk free rate r^f. We consider a risky asset with positive price process S and associated rate of return between date 0 and 1 denoted by $R \equiv \frac{S_1}{S_0} - 1$. In such a context, since qS is a $P-$martingale, we obtain

$$E^P[R] - r^f = -cov^P\left[\frac{q}{E^P[q]}, R\right] \tag{2.1}$$

or equivalently

$$E^P[R] - r^f = -cov^P\left[\frac{Mu'(e)}{E^P[Mu'(e)]}, R\right]. \tag{2.2}$$

The formula for the risk premium in the standard setting is given by

$$E^P[R] - r^f = -cov^P\left[\frac{u'(e)}{E^P[u'(e)]}, R\right].$$

[6] Prices are in terms of date-0 consumption units.

In order to analyze the impact of the subjective belief on the value of the risk premium, we need to be aware of the fact that the possible returns R obtained in equilibrium are not the same in both settings. What exactly is to be compared ?

We consider an asset, whose terminal payoff at date 1 is given by the aggregate wealth e. Notice that such an asset exists by market completeness, but obviously does not have the same price and return in both settings. However, the "risk level" of such an asset[7] is modified in the same proportions as the returns. We propose to compare the market price of risk (i.e., the ratio between the risk premium and the "level of risk") in both settings, which leads us to compare $-cov^P\left[\frac{u'(e)}{E^P[u'(e)]}, e\right]$ with $-cov^P\left[\frac{Mu'(e)}{E^P[Mu'(e)]}, e\right]$. The subjective belief setting leads to a higher market price of risk if and only if

$$\frac{E^Q\left[u'\left(e\right)e\right]}{E^Q\left[u'\left(e\right)\right]} \leq \frac{E^P\left[u'\left(e\right)e\right]}{E^P\left[u'\left(e\right)\right]}. \tag{2.3}$$

Notice that it is equivalent to characterize the probability measures leading to a higher equilibrium market price of risk (for an asset whose payoff is the aggregate wealth e) or to characterize the probability measures leading to a lower equilibrium relative price for the aggregate wealth e, where the relative price for e means the price for e in terms of units of riskless asset. Indeed, in the standard model, the equilibrium price $p^P\left(e\right)$ for e is given by $E^P\left[qe\right] = E^P\left[u'\left(e\right)e\right]/u_0'(e_0)$ and the price $p^P\left(1\right)$ for a riskless asset whose payoff is always 1 is given by $E^P\left[q\right] = E^P\left[u'\left(e\right)\right]/u_0'(e_0)$, hence the equilibrium relative price $rp^P\left(e\right)$ is given by $\frac{E^P[u'(e)e]}{E^P[u'(e)]}$. We find analogously that the equilibrium relative price in the model under Q denoted by $rp^Q\left(e\right)$ is given by $\frac{E^Q[u'(e)e]}{E^Q[u'(e)]}$. According to Inequality (2.3), the probability measures for which the market price of risk increases are those for which the equilibrium relative price decreases.

A similar problem is studied in [1] (in a dynamic setting). The definition of the risk premium is slightly different in [1], since it is equal to the ratio between the expected return of the considered asset and the riskfree return, instead of the difference between these two quantities. With this definition (and our notations), the risk premium in the standard setting (for an asset whose payoff is e) is given by

$$\frac{E^P\left[R+1\right]}{1+r^f} = \frac{E^P\left[\frac{e}{p^P(e)}\right]}{1+r^f} = E^P\left[e\right]\frac{1/\left(1+r^f\right)}{p^P\left(e\right)} = \frac{E^P\left[e\right]}{rp^P\left(e\right)}.$$

[7] measured either by the standard deviation or by a market, (resp. consumption) beta

Analogously, we obtain that the risk premium in the sense of [1] in the model with the subjective probability Q is given by $\frac{E^P[e]}{rp^Q(e)}$. The introduction of a subjective belief leads then to an increase of the equilibrium risk premium in the sense of [1] if and only if it leads to a decrease of the relative equilibrium price (for e).

To summarize what we have just seen, the probability measures for which the market price of risk increases are those for which the equilibrium relative price decreases and they are also those for which the equilibrium risk premium in the sense of [1] increases. We wish to characterize the set of such probabilities.

The concept of pessimism introduced by [1] is directly related to the first order stochastic dominance. More precisely, a consumer with a belief described by a probability measure Q is said to be pessimistic in the sense of [1] (with respect to a random variable e) if for all t, $Q(e \leq t) \geq P(e \leq t)$. It is then proved in [1] that, in a standard Lucas economy, pessimism (with respect to the aggregate wealth) reduces the equilibrium riskfree rate for power utility functions. However, in order to analyze the impact of pessimism on the equilibrium risk premium (with his definition), he introduces the stronger notion of uniform pessimism defined by

$$Q(e \leq t) = P(e \leq t \exp \Delta)$$

for some $\Delta > 0$. He proves then that uniform pessimism leads to an increase of the equilibrium risk premium (for power utility functions).

It is intuitively appealing to obtain that pessimism raises the risk premium. However, the concept of pessimism introduced by [1] does not seem to be the right concept to reach such a conclusion in a sufficiently general framework. Indeed, we show that pessimism as defined by [1] can lead to a decrease of the risk premium (or equivalently to a decrease of the market price of risk or to an increase of the relative price) even in the class of power utility functions.

Suppose that $e \sim \mathcal{U}_{[1;3]}$ and that the density of Q with respect to P is given by $f(e)$ where

$$f(x) = \begin{cases} x & \text{on } [1;2[\\ x-2 & \text{on } [2;3] \end{cases}.$$

It is easy to see on Figure 1 that P dominates Q in the sense of the first stochastic dominance. The probability measure Q is then pessimistic in the sense of [1].

If the representative agent utility function is such that $u'(x) = 1_{x\leq 2}$, we can easily check that $cov^{P_u}(e, f(e)) > 0$ where P_u has a density $\frac{u'(e)}{E^P[u'(e)]}$ with respect to P. Indeed, P_u does not charge $]2;3]$ and $f(e)$ increases with e under P_u. Now, since $\left(\frac{E^Q[u'(e)e]}{E^Q[u'(e)]} - \frac{E^P[u'(e)e]}{E^P[u'(e)]}\right) = \frac{E^P[u'(e)]}{E^Q[u'(e)]} cov^{P_u}(e, f(e))$, this leads to a decrease of the equilibrium risk premium.

As it can be seen in Figure 2., Q is pessimistic with respect to P but optimistic with respect to P_u (since it is almost surely increasing under P_u) and this is the reason why we obtain a decrease of the equilibrium risk premium.

One could argue that this representative agent utility function is too specific since it is not strictly increasing nor strictly concave and that P_u is not equivalent to P. However, small perturbations of this function would lead to the same result. Furthermore, we can check that we would obtain the same result for well chosen power utility functions (e.g. $u'(x) = x^{-6}$). Hence, even in the class of utility functions studied in [1], the concept of pessimism introduced therein is not sufficient to guarantee an increase of the equilibrium risk premium.

We propose to adopt the converse approach and to define a pessimistic agent by the fact that the equilibrium market price of risk is higher under his subjective probability Q than under P for all utility functions in a given class. Analogously, we shall say that an agent, described by his subjective belief Q_1 is more pessimistic than an agent Q_2 if the equilibrium market price of risk is higher under Q_1 than under Q_2 for all utility functions in a given class. We recall that it would be equivalent to require an increase of the risk premium in the sense of [1] or a decrease of the equilibrium relative price. We shall consider the sets $\mathcal{U}_1$ of all continuous nondecreasing functions u on $\mathbb{R}_+^*$ such that $\limsup_{x\to\infty} u'(x) < \infty$ and $\mathcal{U}_2$ of all continuous nondecreasing concave[8] functions u on $\mathbb{R}_+^*$.

Definition 2.1. *Let e be a nonnegative random variable on Ω and let F be the sigma-algebra generated by e. Let Q_1 and Q_2 denote two probability measures on (Ω, F) such that $E^{Q_i}[e] < \infty$, $i = 1, 2$. We say that Q_1 is more pessimistic than Q_2 - with respect to e - in the sense of the market price of risk ($Q_1 \succcurlyeq_{MPR_i} Q_2$) when for all utility functions u in $\mathcal{U}_i$, $E^{Q_1}[u'(e)\,e]\,E^{Q_2}[u'(e)] \leq E^{Q_2}[u'(e)\,e]\,E^{Q_1}[u'(e)]$.*

Note that we do not refer anymore to an objective probability in this definition. However it suffices to consider any probability measure P such that Q_1 and Q_2 are absolutely continuous with respect to P to recover the interpretation of $\frac{E^{Q_1}[u'(e)e]}{E^{Q_1}[u'(e)]}$ (resp. $\frac{E^{Q_2}[u'(e)e]}{E^{Q_2}[u'(e)]}$) as the equilibrium relative price of e in a market where the representative agent subjective belief is given by Q_1 (resp. by Q_2). Such a probability P can be, for instance, any positive combination of Q_1 and Q_2.

[8] Note that all functions in $\mathcal{U}_1$ are almost everywhere differentiable and all functions in $\mathcal{U}_2$ are almost everywhere twice differentiable.

Remark also that for u in $\mathcal{U}_i$ such that $Q_i(u'(e) \neq 0) \neq 0$, $i = 1, 2$, $\limsup_{x\to\infty} u'(x) < \infty$ and since e is integrable, $\frac{E^{Q_i}\left[u'(e)e1_{e\geq x}\right]}{E^{Q_i}\left[u'(e)1_{e\geq x}\right]}$ is well defined for x sufficiently large. Furthermore this quantity decreases when x goes to zero. Hence $\frac{E^{Q_i}[u'(e)e]}{E^{Q_i}[u'(e)]}$ is well defined at least as the limit of $\frac{E^{Q_i}\left[u'(e)e1_{e\geq x}\right]}{E^{Q_i}\left[u'(e)1_{e\geq x}\right]}$ when x goes to zero.

In the next, we shall also equivalently say that Q_1 dominates Q_2 in the sense of the market price of risk (of the first or of the second order). We shall see that this dominance concept indeed corresponds to pessimism.

The next Proposition characterizes the first order market price of risk dominance (or first order pessimism). If A is a given subset of Ω, we denote by $e|_A$ (resp. $Q\,|_A$) the restriction of e (resp. Q) to A.

Proposition 2.2. *Let Q_1 and Q_2 denote two probability measures on (Ω, F).*

1. *If Q_1 and Q_2 are equivalent then $Q_1 \succcurlyeq_{MPR_1} Q_2$ if and only if there exists a nondecreasing function $h : R \to [0, \infty]$ such that $\frac{dQ_2}{dQ_1} = h(e)$.*

2. *In the general case, $Q_1 \succcurlyeq_{MPR_1} Q_2$ if and only if there exist numbers $-\infty \leq x_2 \leq x_1 \leq \infty$ and a nondecreasing function $h : [x_1, x_2] \to [0, \infty]$ such that $Q_2(e < x_2) = 0$, $Q_1(e > x_1) = 0$, $Q_2|_{\{x_2\leq e\leq x_1\}}$ is absolutely continuous with respect to $Q_1|_{\{x_2\leq e\leq x_1\}}$ and $\frac{dQ_2|_{\{x_2\leq e\leq x_1\}}}{dQ_1|_{\{x_2\leq e\leq x_1\}}} = h(e)$.*

Proof. 1. Let us assume that Q_1 and Q_2 are equivalent and that $Q_1 \succcurlyeq_{MPR_1} Q_2$. We have then $\frac{E^{Q_1}[u'(e)e]}{E^{Q_1}[u'(e)]} \leq \frac{E^{Q_2}[u'(e)e]}{E^{Q_2}[u'(e)]}$ for any u in $\mathcal{U}_1$. Since F is generated by e, $\frac{dQ_2}{dQ_1}$ can be written in the form $\frac{dQ_2}{dQ_1} = h\,(e)$. Let us prove that the function h is nondecreasing, or more precisely that the random variable φ defined on $(\mathbb{R}^2, \mathcal{B}\,(\mathbb{R}^2))$ by $\varphi\,(x_1, x_2) = h\,(x_2) - h\,(x_1)$ is $P_e \otimes P_e$ almost surely nonnegative on the open half plane $\{(x_1, x_2)\,, x_1 < x_2\}$.

Let us assume on the contrary that there exist two Borel real sets A_1 and A_2 such that [9] $A_1 < A_2$, $P_e(A_i) \neq 0$, $i = 1, 2$ and φ is $P_e \otimes P_e$ almost surely negative on $A_1 \times A_2$. We can choose A_1 and A_2 such that $h\,(A_1) \subset [a - \varepsilon, a]$ and $h\,(A_2) \subset [b, b + \varepsilon]$ for some $a < b$ and $0 < \varepsilon < b - a$.

[9] For two real subsets A_1 and A_2, $A_1 < A_2$ stands for $\forall (a_1, a_2) \in A_1 \times A_2$, $a_1 < a_2$.

We have then $\frac{Q_2(e\in A_1)}{Q_1(e\in A_1)}=\frac{E^{Q_1}\left[\frac{dQ_2}{dQ_1}1_{e\in A_1}\right]}{Q_1(e\in A_1)}\in[a-\varepsilon,a]$ and $\frac{Q_2(e\in A_2)}{Q_1(e\in A_2)}\in[b,b+\varepsilon]$. Let $\alpha>0$ be given and let $\theta=\frac{Q_1(A_1)}{Q_1(A_1)+\alpha Q_1(A_2)}$. We have

$$\theta(a-\varepsilon)+(1-\theta)b\leq\theta\frac{Q_2(A_1)}{Q_1(A_1)}+(1-\theta)\frac{Q_2(A_2)}{Q_1(A_2)}\leq\theta a+(1-\theta)(b+\varepsilon)$$

and for $\alpha\in\left(\frac{Q_1(A_1)}{Q_1(A_2)}\frac{\varepsilon}{b-a},\frac{Q_1(A_1)}{Q_1(A_2)}\frac{b-a}{\varepsilon}\right)$, we obtain $\theta\frac{Q_2(A_1)}{Q_1(A_1)}+(1-\theta)\frac{Q_2(A_2)}{Q_1(A_2)}\in(a,b)$. It follows then that for such α, $\frac{dQ_2}{dQ_1}$ is higher (resp. lower) than $\frac{Q_2(A_1)+\alpha Q_2(A_2)}{Q_1(A_1)+\alpha Q_1(A_2)}$ on A_1 (resp. A_2).

Let γ denote some real number in $[\sup A_1,\inf A_2]$ and let the utility function u be defined by $u'(x)=1_{A_1}+\alpha 1_{A_2}$. We have

$$\begin{aligned}\frac{E^{Q_2}\left[eu'(e)\right]}{E^{Q_2}\left[u'(e)\right]}&=\frac{E^{Q_2}\left[(e-\gamma)1_{A_1}\right]+\alpha E^{Q_2}\left[(e-\gamma)1_{A_2}\right]}{E^{Q_2}\left[1_{A_1}\right]+\alpha E^{Q_2}\left[1_{A_2}\right]}+\gamma\\&=\frac{E^{Q_1}\left[(e-\gamma)\frac{dQ_2}{dQ_1}1_{A_1}\right]+\alpha E^{Q_1}\left[(e-\gamma)\frac{dQ_2}{dQ_1}1_{A_2}\right]}{E^{Q_2}\left[1_{A_1}\right]+\alpha E^{Q_2}\left[1_{A_2}\right]}+\gamma.\end{aligned}$$

Now, since $(e-\gamma)\leq 0$ and $\frac{dQ_2}{dQ_1}>\frac{Q_2(A_1)+\alpha Q_2(A_2)}{Q_1(A_1)+\alpha Q_1(A_2)}$ on A_1, we have $(e-\gamma)\frac{dQ_2}{dQ_1}\leq(e-\gamma)\frac{Q_2(A_1)+\alpha Q_2(A_2)}{Q_1(A_1)+\alpha Q_1(A_2)}$ on A_1. Analogously we get that $(e-\gamma)\frac{dQ_2}{dQ_1}\leq\frac{Q_2(A_1)+\alpha Q_2(A_2)}{Q_1(A_1)+\alpha Q_1(A_2)}$ on A_2 and one of these inequalities is strict. Hence

$$\begin{aligned}\frac{E^{Q_2}\left[eu'(e)\right]}{E^{Q_2}\left[u'(e)\right]}&<\frac{Q_2(A_1)+\alpha Q_2(A_2)}{Q_1(A_1)+\alpha Q_1(A_2)}\frac{E^{Q_1}\left[(e-\gamma)1_{A_1}\right]+\alpha E^{Q_1}\left[(e-\gamma)1_{A_2}\right]}{E^{Q_2}\left[1_{A_1}\right]+\alpha E^{Q_2}\left[1_{A_2}\right]}+\gamma\\&<\frac{E^{Q_1}\left[(e-\gamma)1_{A_1}\right]+\alpha E^{Q_1}\left[(e-\gamma)1_{A_2}\right]}{E^{Q_1}\left[1_{A_1}\right]+\alpha E^{Q_1}\left[1_{A_2}\right]}+\gamma\\&<\frac{E^{Q_1}\left[eu'(e)\right]}{E^{Q_1}\left[u'(e)\right]}\end{aligned}$$

which contradicts the fact that $Q_1\succcurlyeq_{MPR_1}Q_2$.

Conversely, $\frac{E^{Q_2}[u'(e)e]}{E^{Q_2}[u'(e)]}=\frac{E^{Q_1}\left[\frac{dQ_2}{dQ_1}u'(e)e\right]}{E^{Q_1}\left[\frac{dQ_2}{dQ_1}u'(e)\right]}=\frac{E^{Q_u}\left[\frac{dQ_2}{dQ_1}e\right]E^{Q_1}[u'(e)]}{E^{Q_1}\left[\frac{dQ_2}{dQ_1}u'(e)\right]}$ where Q_u is defined by a density with respect to Q_1 equal (up to a constant) to $u'(e)$. Since $\frac{dQ_2}{dQ_1}$ is nondecreasing in e, we have

$$E^{Q_u}\left[\frac{dQ_2}{dQ_1}e\right]\geq E^{Q_u}\left[\frac{dQ_2}{dQ_1}\right]E^{Q_u}\left[e\right]$$

hence

$$\frac{E^{Q_2}\left[u'(e)\,e\right]}{E^{Q_2}\left[u'(e)\right]} \geq \frac{E^{Q_u}\left[\frac{dQ_2}{dQ_1}\right]E^{Q_u}\left[e\right]E^{Q_1}\left[u'(e)\right]}{E^{Q_1}\left[\frac{dQ_2}{dQ_1}u'(e)\right]}$$
$$\geq \frac{E^{Q_1}\left[u'(e)\,e\right]}{E^{Q_1}\left[u'(e)\right]}.$$

2. Let us now consider the general case. We first prove that if $Q_1 \succcurlyeq_{MPR_1} Q_2$, subsets on which Q_1 is zero correspond to values of e at the right-end of the real line. Let $(A_i)_{i=1,2}$ be disjoint real subsets such that[10] $co(A_1) \cap co(A_2) = \emptyset$ and such that $Q_1(e \in A_1) = 0$. Let us define u such that $u' = \alpha 1_{A_1} + 1_{A_2}$ with $\alpha \geq 0$. We have $E^{Q_2}\left[u'(e)\,e\right]E^{Q_1}\left[u'(e)\right] - E^{Q_2}\left[u'(e)\right]E^{Q_1}\left[eu'(e)\right] \geq 0$ or equivalently

$$\alpha\left(E^{Q_2}\left[e1_{e\in A_1}\right]Q_1\left[A_2\right] - Q_2\left[A_1\right]E^{Q_1}\left[e1_{e\in A_2}\right]\right) + E^{Q_2}\left[e1_{e\in A_2}\right]Q_1\left[A_2\right] - Q_2\left[A_2\right]E^{Q_1}\left[e1_{e\in A_2}\right] \geq 0$$

for all $\alpha \geq 0$. If we assume $Q_i(e \in A_j) \neq 0$ for $i \neq j$, we have $\frac{E^{Q_2}\left[e1_{e\in A_1}\right]}{Q_2[e\in A_1]} \geq \frac{E^{Q_1}\left[e1_{e\in A_2}\right]}{Q_1[e\in A_2]}$ which leads to $A_2 < A_1$.

Let us now define $x_1 \equiv \inf\{x : Q_1(e > x) = 0\}$ and let $A \subset (-\infty, x_1 - \varepsilon)$ for some $\varepsilon > 0$ such that $Q_1(e \in A) = 0$. By construction, $Q_1(e \in [x_1 - \varepsilon, x_1]) \neq 0$. If we assume $Q_2(e \in A) \neq 0$, we have $[x_1 - \varepsilon, x_1] < A$ and this is impossible. We have then $Q_2(e \in A) = 0$. Consequently, for all $\varepsilon > 0$, $Q_2|_{\{e \leq x_1 - \varepsilon\}}$ is absolutely continuous with respect to $Q_1|_{\{e \leq x_1 - \varepsilon\}}$, hence $Q_2|_{\{e < x_1\}}$ is absolutely continuous with respect to $Q_1|_{\{e < x_1\}}$.

Symmetrically, if we define $x_2' = \sup\{x : Q_2(e < x) = 0\}$, we obtain that $Q_1|_{\{x_2' < e\}}$ is absolutely continuous with respect to $Q_2|_{\{x_2' < e\}}$.

If $x_2' > x_1$, we take $x_2 = x_1$ and $h = 0$ and we have $-\infty \leq x_2 \leq x_1 \leq \infty$, $Q_2(e < x_2) = 0$, $Q_1(e > x_1) = 0$, $Q_2|_{\{x_2 \leq e \leq x_1\}}$ is absolutely continuous with respect to $Q_1|_{\{x_2 \leq e \leq x_1\}}$ and $\frac{dQ_2|_{\{x_2 \leq e \leq x_1\}}}{dQ_1|_{\{x_2 \leq e \leq x_1\}}} = h(e)$.

If $x_2' \leq x_1$,we take $x_2 = x_2'$ and $Q_2|_{\{x_2 \leq e \leq x_1\}}$ and $Q_1|_{\{x_2 \leq e \leq x_1\}}$ are equivalent. The same argument as in 1. gives us then the existence of a nondecreasing nonnegative function h such that $\frac{dQ_2|_{\{x_2 < e < x_1\}}}{dQ_1|_{\{x_2 < e < x_1\}}} = h(e)$.

If $Q_1(e = x_1) = 0$ (resp. $\neq 0$) and $Q_2(e = x_1) = 0$ (resp. $\neq 0$) then the previous reasoning can be extended to $\{x_2 < e \leq x_1\}$. If $Q_1(e = x_1) = 0$ and

[10] If A is a given real subset, we denote by $co(A)$ the convex hull of A.

$Q_2(e = x_1) \neq 0$ then it suffices to take $h(x_1) = \infty$. Finally, it is easy to check that we can not have $Q_1(e = x_1) \neq 0$ and $Q_2(e = x_1) = 0$. The point x_2 is treated analogously.

In order to establish the converse implication, it suffices to prove, for $A = \{x_2 \leq e \leq x_1\}$, that $E^{Q_2}[u'(e) e 1_A] E^{Q_1}[u'(e) 1_A] - E^{Q_2}[u'(e) 1_A] E^{Q_1}[e u'(e) 1_A] \geq 0$ for all u in $\mathcal{U}_1$. Remark that this quantity is not modified if we replace Q_1, Q_2 and $u'(e)$ by $Q_1|_{\{x_2 \leq e \leq x_1\}}$, $Q_2|_{\{x_2 \leq e \leq x_1\}}$ and $u'(e) 1_{\{x_2 \leq e \leq x_1\}}$. The result is then a direct consequence of 1. ■

It is now easy to provide examples of first order market price of risk dominance. If e has a normal distribution $\mathcal{N}(\mu_1, \sigma_1)$ (resp. $\mathcal{N}(\mu_2, \sigma_2)$) under Q_1 (resp. under Q_2) then $Q_1 \succcurlyeq_{MPR_1} Q_2$ if and only if $\sigma_1 = \sigma_2$ and $\mu_1 \leq \mu_2$.

Notice that the comonotonicity condition between $\frac{dQ_2}{dQ_1}$ and e (hence our definition of pessimism) indeed corresponds to some concept of pessimism and this is, in particular, reflected in the previous examples where pessimism is equivalent to a lower expected value for e without modification of the risk level. It corresponds to a stronger notion of pessimism than the first stochastic dominance-pessimism introduced in [1], since it is immediate to see that if $\frac{dQ_2}{dQ_1}$ and e are comonotone then $Q_1(e \leq t) \geq Q_2(e \leq t)$ for all t.

The second order market price of risk is harder to characterize. Let us first introduce the following notations. Let x_1 and x_2 be given nonnegative real numbers, we define $a(x_1)$ and $b(x_1, x_2)$ respectively by

$$\begin{aligned}
a(x_1) &= E^{Q_1}[e 1_{e \leq x_1}] E^{Q_2}[1_{e \leq x_1}] - E^{Q_2}[e 1_{e \leq x_1}] E^{Q_1}[1_{e \leq x_1}] \\
b(x_1, x_2) &= E^{Q_1}[e 1_{e \leq x_1}] E^{Q_2}[1_{e \leq x_2}] + E^{Q_1}[e 1_{e \leq x_2}] E^{Q_2}[1_{e \leq x_1}] \\
&\quad - E^{Q_2}[e 1_{e \leq x_1}] E^{Q_1}[1_{e \leq x_2}] - E^{Q_2}[e 1_{e \leq x_2}] E^{Q_1}[1_{e \leq x_1}]
\end{aligned}$$

and let us define the polynomial R_{x_1,x_2} by $R_{x_1,x_2}(X) = a(x_1)X^2 + b(x_1, x_2)X + a(x_2)$. We are now able to provide the following characterization.

Theorem 2.3. *Let Q_1 and Q_2 denote two probability measures on (Ω, F). The following conditions are equivalent:*

1. *$Q_1 \succcurlyeq_{MPR_2} Q_2$,*
2. *for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ and for all $X \geq 0$ we have $R_{x_1,x_2}(X) \leq 0$*
3. *for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ we have $a(x_1) \leq 0$ and $\frac{1}{2} b(x_1, x_2) \leq \sqrt{a(x_1) a(x_2)}$*

Proof. We want to have

$$\frac{E^{Q_1}\left[u'\left(e\right)e\right]}{E^{Q_1}\left[u'\left(e\right)\right]} \leq \frac{E^{Q_2}\left[u'\left(e\right)e\right]}{E^{Q_2}\left[u'\left(e\right)\right]}$$

for all u in $\mathcal{U}_2$. Since any nonincreasing function is an increasing limit of positive combinations of step functions on the form $u'(z) = \sum_{i=1}^n y_i 1_{z\leq x_i}$ for some n, and some $(x, y) \in \mathbb{R}_+^n$, it suffices to restrict our attention to such functions. We want then to characterize the following property

$$\forall (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n, \qquad \frac{\sum_{i=1}^n y_i E^{Q_1}\left[e 1_{e\leq x_i}\right]}{\sum_{i=1}^n y_i E^{Q_1}\left[1_{e\leq x_i}\right]} \leq \frac{\sum_{i=1}^n y_i E^{Q_2}\left[e 1_{e\leq x_i}\right]}{\sum_{i=1}^n y_i E^{Q_2}\left[1_{e\leq x_i}\right]}$$

or equivalently for all real number λ, and all (x, y) in $\mathbb{R}_+^n \times \mathbb{R}_+^n$,

$$\sum_{i=1}^n y_i E^{Q_2}\left[e 1_{e\leq x_i}\right] - \lambda \sum_{i=1}^n y_i E^{Q_2}\left[1_{e\leq x_i}\right] \leq 0 \Rightarrow \sum_{i=1}^n y_i E^{Q_1}\left[e 1_{e\leq x_i}\right] - \lambda \sum_{i=1}^n y_i E^{Q_1}\left[1_{e\leq x_i}\right] \leq 0$$

or in other words for all real number λ, and all x in $\mathbb{R}_+^n$,

$$\sup_{\sum_{i=1}^n y_i E^{Q_2}\left[e 1_{e\leq x_i}\right] - \lambda \sum_{i=1}^n y_i E^{Q_2}\left[1_{e\leq x_i}\right] \leq 0} \sum_{i=1}^n y_i E^{Q_1}\left[e 1_{e\leq x_i}\right] - \lambda \sum_{i=1}^n y_i E^{Q_1}\left[1_{e\leq x_i}\right] \leq 0.$$

Without loss of generality, we can restrict the last maximization program to the unit simplex. We have then for any λ a linear programming problem on the unit simplex and the maximum value of the objective must be achieved by a function on the form $u'(z) = \sum_{i=1}^n y_i 1_{z\leq x_i}$ with $n \leq 2$. Going backward[11], it is easy to check then that the MPR_2 dominance is now characterized by the following property

$$\forall (x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2, \qquad \frac{\sum_{i=1}^2 y_i E^{Q_1}\left[e 1_{e\leq x_i}\right]}{\sum_{i=1}^2 y_i E^{Q_1}\left[1_{e\leq x_i}\right]} \leq \frac{\sum_{i=1}^2 y_i E^{Q_2}\left[e 1_{e\leq x_i}\right]}{\sum_{i=1}^2 y_i E^{Q_2}\left[1_{e\leq x_i}\right]}.$$

The last condition is satisfied if and only if for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ and for all $X \geq 0$ we have $R_{x_1,x_2}(X) \leq 0$.

A necessary and sufficient condition for R_{x_1,x_2} to be nonpositive on $\mathbb{R}_+$ is $a(x_1) \leq 0$, $a(x_2) \leq 0$ and $\frac{1}{2} b(x_1, x_2) \leq \sqrt{a(x_1) a(x_2)}$. This ends the proof. ■

[11] This argument that permits to restrict our attention to the case $n = 2$ is adapted from "New methods in the classical economics of uncertainty : comparing risks" by Gollier and Kimball (1997), unpublished.

Even if the previous proposition proposes a characterization of the second order market price of risk dominance, it is clear that it is still difficult to identify whether a given probability measure Q_1 is more pessimistic than a probability measure Q_2 in the sense of the second order market price of risk dominance. The following proposition gives some necessary and/or sufficient conditions for this dominance.

Proposition 2.4. *Let Q_1 and Q_2 denote two probability measures on (Ω, F).*

1. *If $Q_1 \succcurlyeq_{MPR_2} Q_2$, then for all x, $E^{Q_1}[e \mid e \leq x] \leq E^{Q_2}[e \mid e \leq x]$*
2. *If $Q_1 \succcurlyeq_{MPR_2} Q_2$, if $E^{Q_i}[e^2] < \infty$, $i = 1,2$ and if $E^{Q_1}[e] = E^{Q_2}[e]$ then $Var^{Q_1}[e] \geq Var^{Q_2}[e]$*
3. *If there exists λ such that, for all x, $E^{Q_1}[e1_{e\leq x}] \leq \lambda E^{Q_2}[e1_{e\leq x}]$ and $Q_1\{e \leq x\} \geq \lambda Q_2\{e \leq x\}$, then $Q_1 \succcurlyeq_{MPR_2} Q_2$.*
4. *In particular, if for all x, $E^{Q_1}[e1_{e\leq x}] \leq E^{Q_2}[e1_{e\leq x}]$ and $Q_1\{e \leq x\} \geq Q_2\{e \leq x\}$, then $Q_1 \succcurlyeq_{MPR_2} Q_2$*

Proof. 1. This is an immediate consequence of the nonpositivity of $a(x)$.

2. Let us first assume that e has a finite support and let $\bar{e}$ be such that $P\{e \geq \bar{e}\} = 0$. Taking $u'(x) = (\bar{e} - x)1_{x\leq \bar{e}}$, we have

$$\frac{\bar{e}E^{Q_1}[e] - E^{Q_1}[e^2]}{\bar{e} - E^{Q_1}[e]} \leq \frac{\bar{e}E^{Q_2}[e] - E^{Q_2}[e^2]}{\bar{e} - E^{Q_2}[e]}.$$

Since we assumed $E^{Q_1}[e] = E^{Q_2}[e]$, we have $E^{Q_1}[e^2] \geq E^{Q_2}[e^2]$ or equivalently $Var^{Q_1}[e] \geq Var^{Q_2}[e]$. If e does not have a finite support, we still take $u'(x) = (\bar{e} - x)1_{x\leq \bar{e}}$ for some $\bar{e}$. Since $E^{Q_1}[e^2 1_{e\leq \bar{e}}]$ (resp. $E^{Q_2}[e^2 1_{e\leq \bar{e}}]$) converges to $E^{Q_1}[e^2]$ (resp. $E^{Q_2}[e^2]$) when $\bar{e}$ goes to infinity, we have for $\varepsilon > 0$ given and for $\bar{e}$ sufficiently large

$$E^{Q_i}\left[e^k\right] - \bar{e}^{k-2}\varepsilon \leq E^{Q_i}\left[e^k 1_{e\leq \bar{e}}\right] \leq E^{Q_i}\left[e^k\right], \qquad k = 0,1,2, \quad i = 1,2$$

We have then

$$\frac{\bar{e}\left(E^{Q_1}[e] - \frac{1}{\bar{e}}\varepsilon\right) - E^{Q_1}[e^2]}{\bar{e} - \left(E^{Q_1}[e] - \frac{1}{\bar{e}}\varepsilon\right)} \leq \frac{\bar{e}E^{Q_2}[e] - \left(E^{Q_2}[e^2] - \varepsilon\right)}{\bar{e}\left(1 - \frac{1}{\bar{e}^2}\varepsilon\right) - E^{Q_2}[e]}$$

which leads to

$$\left[\bar{e}E^{Q_1}\left[e-\frac{e^2}{\bar{e}}\right]-\varepsilon\right]\left[\bar{e}-E^{Q_1}[e]-\frac{\varepsilon}{\bar{e}}\right]\leq\left[\bar{e}E^{Q_1}\left[e-\frac{e^2}{\bar{e}}\right]+\varepsilon\right]\left[\bar{e}-E^{Q_1}[e]+\frac{\varepsilon}{\bar{e}}\right].$$

Dividing by $\bar{e}$ and taking the limit when $\bar{e}$ goes to infinity and ε goes to zero we obtain $E^{Q_1}[e^2]\geq E^{Q_2}[e^2]$ or equivalently $Var^{Q_1}[e]\geq Var^{Q_2}[e]$.

3. Let us assume that these inequalities are satisfied. By positive combinations and taking the limit (using the monotone convergence theorem) we obtain that $E^{Q_1}[eu'(e)]\leq\lambda E^{Q_2}[eu'(e)]$ and $E^{Q_1}[u'(e)]\geq\lambda E^{Q_2}[u'(e)]$ for any u in $\mathcal{U}_2$. Dividing the first inequality by the second one leads to the result.

4. It suffices to take $\lambda=1$ in 3. ■

Remark that the necessary condition provided in the first assertion means that the unconditional as well as the conditional expected values of e under Q_1 are lower than the corresponding expected values under Q_2. This condition clearly corresponds to some kind of pessimism.

The sufficient condition provided in the fourth assertion corresponds to the first-order stochastic dominance (Q_1 puts more weight on $\{e\leq x\}$ than Q_2) but also imposes that on $\{e\leq x\}$, Q_1 puts more weight on the low values. This condition seems to be closely related to the comonotonicity condition that characterizes the first order market price of risk dominance but it can be shown on simple examples that these two conditions are not equivalent. Figure 3 provides an example where the conditions of the fourth assumption are satisfied but where the comonotonicity condition is not.

It is also easy to construct examples where the sufficient condition of the assertion 3 of Proposition 2.4 is satisfied but where the first-order stochastic dominance is not. Figures 4 and 5 provide such an example.

Finally, remark that we do not modify our dominance concept if we replace e by $e+\alpha$ where α is a given constant such that $e+\alpha$ remains nonnegative. However, replacing e by $e+\alpha$ in the sufficient condition of assertion 3. leads to a weaker condition. We can then replace 3. by the following weaker sufficient condition 3' when we still adopt the convention $\frac{0}{0}=0$.

3' If $\sup_x \frac{E^{Q_1}\left[(e-ess\inf e)1_{e\leq x}\right]}{E^{Q_2}\left[(e-ess\inf e)1_{e\leq x}\right]}\leq\inf_x\frac{Q_1\{e\leq x\}}{Q_2\{e\leq x\}}$, then $Q_1\succcurlyeq_{MPR_2}Q_2$

The necessary conditions provided by the first assertion of Proposition 2.4 can clearly be interpreted as a form of pessimism. Nevertheless, the identification of the "pessimistic nature" of a probability measure is not always immediate. The next proposition provides us with an example of a class of probability measures which are shown to be second order pessimistic although the pessimism is not clearly reflected in the probability distribution.

Proposition 2.5. *If the distribution of* e *under* Q_2 *is symmetric (with respect to* $E^{Q_2}[e]$*) and if the density of* Q_1 *with respect to* Q_2 *is a function of* e*, symmetric with respect to* $E^{Q_2}[e]$*, nonincreasing before* $E^{Q_2}[e]$ *and nondecreasing after* $E^{Q_2}[e]$ *then* $Q_1 \succcurlyeq_{MPR_2} Q_2$*.*

Proof. Suppose that the density of Q_1 with respect to Q_2 can be written in the form $\frac{dQ_1}{dQ_2} = f\left(e - E^{Q_2}[e]\right)$ where f is an even function, increasing on $\mathbb{R}_+$. We have

$$\frac{E^{Q_1}(eu'(e))}{E^{Q_1}(u'(e))} = \frac{E^{Q_2}\left[\alpha f(\alpha)\left(u'(m+\alpha) - u'(m-\alpha)\right)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[f(\alpha)\left(u'(m+\alpha) + u'(m-\alpha)\right)1_{\alpha\geq 0}\right]}.$$

where $m = E^{Q_2}[e]$ and $\alpha = e - m$. We want to compare this quantity with

$$\frac{E^{Q_2}(eu'(e))}{E^{Q_2}(u'(e))} = \frac{E^{Q_2}\left[\alpha\left(u'(m+\alpha) - u'(m-\alpha)\right)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[\left(u'(m+\alpha) + u'(m-\alpha)\right)1_{\alpha\geq 0}\right]}.$$

Letting $g(\alpha) = \alpha\left(u'(m-\alpha) - u'(m+\alpha)\right)$ and $h(\alpha) \equiv \left(u'(m+\alpha) + u'(m-\alpha)\right)$, we are led to compare $\frac{E^{Q_2}\left[f(\alpha)g(\alpha)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[f(\alpha)h(\alpha 1_{\alpha\geq 0}\right]}$ with $\frac{E^{Q_2}\left[g(\alpha)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[h(\alpha)1_{\alpha\geq 0}\right]}$, or equivalently $\frac{E^{Q_2}\left[f(\alpha)g(\alpha)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[g(\alpha)1_{\alpha\geq 0}\right]}$ with $\frac{E^{Q_2}\left[f(\alpha)h(\alpha)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[h(\alpha)1_{\alpha\geq 0}\right]}$. Let us now define the probability measure P^g by $\frac{dP^g}{dQ_2} = \frac{g(\alpha)1_{\alpha\geq 0}}{E^{Q_2}\left[g(\alpha)1_{\alpha\geq 0}\right]}$. We are led to compare $E^{P^g}[f(\alpha)]$ with $E^{P^g}\left[f(\alpha)\frac{h}{g}(\alpha)\right]\frac{E^{Q_2}\left[g(\alpha)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[h(\alpha)1_{\alpha\geq 0}\right]}$. The probability measure Q_1 dominates Q_2 in the sense of the market price of risk of the second order if and only if

$$E^{P^g}[f(\alpha)] \geq E^{P^g}\left[f(\alpha)\frac{h}{g}(\alpha)\right]\frac{E^{Q_2}[g(\alpha)1_{\alpha\geq 0}]}{E^{Q_2}[h(\alpha)1_{\alpha\geq 0}]}.$$

It is easy to check that the function $\frac{h}{g} : x \mapsto \frac{(u'(m+x)+u'(m-x))}{x(u'(m-x)-u'(m+x))}$ is decreasing on $\mathbb{R}_+$. Since f is increasing, then for any probability measure Q, we have $cov^Q\left(\frac{h}{g}(\alpha), f(\alpha)\right) \leq 0$, and in particular, $cov^{P^g}\left(\frac{h}{g}(\alpha), f(\alpha)\right) \leq 0$. This implies that

$$\begin{aligned} E^{P^g}\left[f(\alpha)\frac{h}{g}(\alpha)\right] &\leq E^{P^g}[f(\alpha)]\,E^{P^g}\left[\frac{h}{g}(\alpha)\right] \\ &\leq E^{P^g}[f(\alpha)]\frac{E^{Q_2}[h(\alpha)1_{\alpha\geq 0}]}{E^{Q_2}[g(\alpha)1_{\alpha\geq 0}]} \end{aligned}$$

and $E^{P^g}[f(\alpha)] \geq E^{P^g}\left[f(\alpha)\frac{h}{g}(\alpha)\right]\frac{E^{Q_2}\left[g(\alpha)1_{\alpha\geq 0}\right]}{E^{Q_2}\left[h(\alpha)1_{\alpha\geq 0}\right]}$. This concludes the proof of the Proposition. ■

If the probability measures Q_1 and Q_2 are like in Proposition 2.5, then e has the same expectation under the two probabilities and has a higher variance under Q_1. Some authors ([1], [6]) interpret this property as doubt. However it is just a specific case of pessimism and in particular, we have, as stated in our necessary conditions, for all x, $E^{Q_1}[e \mid e \leq x] \leq E^{Q_2}[e \mid e \leq x]$.

Example 2.6. *Let us assume that the distribution of* e *under* Q_1 *(resp.* Q_2*) is normal with parameters* (μ_1, σ_1) *and* (μ_2, σ_2).

- *If* $\mu_1 = \mu_2$ *and* $\sigma_1 \geq \sigma_2$ *then* $Q_1 \succcurlyeq_{MPR_2} Q_2$,
- *If* $\mu_1 \leq \mu_2$ *and* $\sigma_1 = \sigma_2$ *then* $Q_1 \succcurlyeq_{MPR_2} Q_2$,
- *If* $\mu_1 \leq \mu_2$ *and* $\sigma_1 \geq \sigma_2$ *then* $Q_1 \succcurlyeq_{MPR_2} Q_2$.

Proof. If $\mu_1 = \mu_2$ and $\sigma_1 \geq \sigma_2$ then the assumptions of Proposition 2.5 are satisfied and consequently $Q_1 \succcurlyeq_{MPR_2} Q_2$. If $\mu_1 \leq \mu_2$ and $\sigma_1 = \sigma_2$, we have already seen that $Q_1 \succcurlyeq_{MPR_1} Q_2$ and then $Q_1 \succcurlyeq_{MPR_2} Q_2$. If $\mu_1 \leq \mu_2$ and $\sigma_1 \geq \sigma_2$, the result is obtained by transitivity. ■.

3. Risk aversion

In the previous section, we have introduced a pessimism-based order on subjective probabilities by saying that a probability measure is more pessimistic than another when the market price of risk is higher for all possible representative agent. We shall in this section proceed analogously concerning the economic agents and their risk aversion. Intuition suggests that the effect of an increase in risk aversion (in the sense of Arrow-Pratt, [17]) on the market price of risk ought to be the same as the effect of an increase in pessimism.

We shall say that an economic agent represented by his utility function v is more risk averse than an agent u (in the sense of the market price of risk) or equivalently that v dominates u in the sense of the market price of risk when the equilibrium market price of risk is higher for agent v than for agent u and we shall characterize this condition.

We consider the same equilibrium model as in the previous section, except that we do not assume subjective probabilities. We have seen that if the representative agent's utility function is given by u, then the equilibrium risk premium for an asset, whose terminal payoff at date 1 is the aggregate wealth e, is given by $-cov^P\left[\frac{u'(e)}{E^P[u'(e)]},\frac{e}{p^P(e)}\right]$. It is immediate that the market price of risk under v is higher than under u if and only if $\frac{E[u'(e)e]}{E[u'(e)]}\leq\frac{E[v'(e)e]}{E[v'(e)]}$. It is easy to verify that it would be equivalent to impose that the risk premium in the sense of [1] be higher or that the equilibrium relative price for e, as defined in the previous section, be lower under v than under u.

Definition 3.1. *Let u, v be in $\mathcal{U}_2$. We say that v is more risk averse than u in the sense of the market price of risk ($v \succcurlyeq_{MPR} u$) when for all probability space $(\Omega,\mathcal{F},P)$ and all random variable e on $(\Omega,\mathcal{F},P)$, $\frac{E[v'(e)e]}{E[v'(e)]}\geq\frac{E[u'(e)e]}{E[u'(e)]}$.*

We obtain the following characterization, that proves that our order on the risk aversion of economic agents is equivalent to the one of Arrow-Pratt ([17]).

Proposition 3.2. *For utility functions u and v in $\mathcal{U}_2$, the following conditions are equivalent*

1. $v \succcurlyeq_{MPR} u$
2. $h\equiv\frac{v'}{u'}$ *is nonincreasing*
3. *if we further assume that u' and v' are continuous then the previous conditions are also equivalent to* $-\frac{v''}{v'}\geq-\frac{u''}{u'}$.

Proof. In order to obtain the first implication, we suppose that h is not nonincreasing, and we shall prove that the first property is then not satisfied. Suppose that there exist $x_1 < x_2$ for which $h(x_1) < h(x_2)$. Consider the discrete random variable $\overline{e}$ which takes two distinct values x_1 and x_2 with an equal probability. Then $\frac{E[u'(\overline{e})\overline{e}]}{E[u'(\overline{e})]}=\frac{\frac{h(x_2)}{h(x_1)}v'(x_1)x_1+v'(x_2)x_2}{\frac{h(x_2)}{h(x_1)}v'(x_1)+v'(x_2)}$. Since the function $\varphi : t\mapsto\frac{tax_1+bx_2}{ta+b}$ is decreasing for all $a>0$, $b>0$, $x_1<x_2$, and since $\frac{h(x_2)}{h(x_1)}>1$, we easily obtain that $\frac{\frac{h(x_2)}{h(x_1)}v'(x_1)x_1+v'(x_2)x_2}{\frac{h(x_2)}{h(x_1)}v'(x_1)+v'(x_2)}<\frac{v'(x_1)x_1+v'(x_2)x_2}{v'(x_1)+v'(x_2)}$, hence $\frac{E[u'(\overline{e})\overline{e}]}{E[u'(\overline{e})]}<\frac{E[v'(\overline{e})\overline{e}]}{E[v'(\overline{e})]}$.

For the converse implication, suppose that $h\equiv\frac{v'}{u'}$ is nonincreasing. Let $A\equiv E[v'(e)e]\,E[u'(e)]-E[u'(e)e]\,E[v'(e)]$. Then $A=E[u'(e)]^2\,cov^{P_{u'(e)}}(h(e),e)$ where $\frac{dP_{u'(e)}}{dP}$ is given (up to a constant) by $u'(e)$. We have $A\leq 0$ if and only if $cov^{P_{u'(e)}}(h(e), e) \leq 0$. Now, since h is nonincreasing, we know that for any probability measure Q , $cov^{Q}(h(e), e) \leq 0$, which proves the implication.

Now, h is nonincreasing if and only if $v''u' - v'u'' \leq 0$, or equivalently for u and v in $\mathcal{U}_2$, if and only if $-\frac{v''}{v'} \geq -\frac{u''}{u'}$. ■

Other necessary and sufficient conditions for v to dominate u in the sense of the market price of risk are that v can be written in the form $v = T \circ u$ for some concave function T or that for all random variables, the "risk premium"[12] under v, denoted by π^v is greater than π^u, the risk premium under u . Moreover, we deduce from this proposition that in the classical portfolio allocation problem, with one risky asset and one riskless asset with zero rate of return, if $v \succcurlyeq_{MPR} u$, then for the same initial wealth w, the optimal fraction of the portfolio invested in risky asset for agent v denoted by α^v, is lower than the optimal fraction of the portfolio invested in risky asset for agent u denoted by α^u. More generally, all results that are valid under Arrow-Pratt (absolute) risk aversion order, remain valid under our "market price of risk" order.

4. Level of risk

The most common orders on risky assets are the first and the second stochastic dominance relations (see [18, 19]). They provide a natural and easily verifiable partial order on assets that certain classes of utility functions preserve. However, they fail to provide nice comparative statics. In particular, an FSD shift in the distribution of random returns of an asset does not necessarily induce a risk averse decision maker to increase his holdings of that improved asset ([5]) and an SSD increase in risk in the return of an asset does not induce all risk averters to reduce their demand for that asset (see [19]). Facing such a negative result, essentially three stratregies have been adopted in the literature. First, to impose restrictions on the economic model represented by the payoff function accruing to the decision maker (see e.g. [15]). Second, to impose restrictions on the preferences of the economic agents (e.g. [9, 18, 19]). The third strategy is the one that has been the most adopted, and consists in introducing new orders on the desirability of risky assets. For instance, in [13] the authors prove that if returns on risky assets are ordered by the monotone likelihood ratio order, then dominating assets will be more desired by all investors with nondecreasing utility functions. In [16] , the authors prove that the changes in risky asset distribution that lead all agents whose

[12] The "risk premium" under u of a random variable z denotes here the quantity $\pi^u(z) = E[z] - C^u(z)$, where the certainty equivalent $C^u(z)$ is such that $u(C^u(z)) = E[u(z)]$.

choice under certainty is monotonic in the exogeneous variable to increase their choice variable under uncertainty are those satisfying the monotone likelihood ratio dominance. The monotone probability ratio order is introduced in [4] and is more general than the monotone likelihood ratio. In [7], the author shows that a necessary and sufficient condition for unambiguous comparative statics for any risk averse agent in the standard portfolio problem is the central riskiness dominance.

In this section, we adopt the same "market price of risk" approach as in the previous sections in order to rank the desirability (or the level of risk) of different prospects. Intuition suggests that the effect on the market price of risk of an increase in risk ought to be the same as the effect of an increase in pessimism or in risk aversion[13]. By analogy with the previous sections, we shall say that a prospect X is more risky (or less desirable) than another prospect Y when $\frac{E[u'(X)X]}{E[u'(X)]} \leq \frac{E[u'(Y)Y]}{E[u'(Y)]}$. Since the equilibrium relative price for (the aggregate wealth) X, i.e the price for X in terms of units of riskless asset, denoted by $rp^P(X)$, is given by $\frac{E^P[u'(X)X]}{E^P[u'(X)]}$, this amounts to saying that a prospect X is more risky (or less desirable) than another prospect Y when the equilibrium relative price for (the aggregate wealth) X is lower than the equilibrium relative price for Y. We shall say equivalently that Y dominates X in the sense of the relative price. If $E[X] = E[Y]$, this is equivalent to the fact that the risk premium in the sense of [1] is higher for X than for Y. If the prospects X and Y have the same standard deviation $\sigma_X = \sigma_Y$, then the condition that $rp^P(X)$ be lower than $rp^P(Y)$ is equivalent to the condition that the market price of risk (i.e., the ratio between the risk premium and the standard deviation) for X be higher than the market price of risk for Y. As in the analysis of pessimism, we define the concepts of first- and second-order relative price dominance.

Definition 4.1. *Let X and Y denote nonnegative random variables on (Ω, F, P) such that $E[X] < \infty$ and $E[Y] < \infty$. Then X is said to be more risky than Y in the sense of the equilibrium relative price ($Y \succcurlyeq_{RP_i} X$) when for all utility functions u in $\mathcal{U}_i$, $\frac{E[u'(X)X]}{E[u'(X)]} \leq \frac{E[u'(Y)Y]}{E[u'(Y)]}$.*

The following lemma permits to establish a link between this concept of riskiness and the concept of pessimism that we developed in Section 2.

[13] See [3, 11, 16], among others on the relation between the effect of an increase in risk aversion and the effect of an increase in risk on the level of a choice variable.

Lemma 4.2. *Let X and Y denote nonnegative random variables on (Ω, F, P) such that $E[X] < \infty$ and $E[Y] < \infty$. Let P_X and P_Y denote respectively the distributions of X and Y and let e be the random variable defined on $\mathbb{R}_+$ by $e(\omega) = \omega$. Then the two following conditions are equivalent :*

1. $Y \succcurlyeq_{RP_i} X$
2. $P_X \succcurlyeq_{MPR_i} P_Y$ *(with respect to e).*

Proof. By definition, $Y \succcurlyeq_{RP_i} X$ if and only if $\frac{E[u'(X)X]}{E[u'(X)]} \leq \frac{E[u'(Y)Y]}{E[u'(Y)]}$ for all u in $\mathcal{U}_i$ or equivalently if and only if for all u in $\mathcal{U}_i$, $\frac{\int xu'(x)dP_X}{\int u'(x)dP_X} \leq \frac{\int xu'(x)dP_Y}{\int u'(x)dP_Y}$ or $\frac{E^{P_X}[eu'(e)]}{E^{P_X}[u'(e)]} \leq \frac{E^{P_Y}[eu'(e)]}{E^{P_Y}[u'(e)]}$. ■

We easily obtain that the order induced by the first-order relative price dominance (RP_1) is equivalent to the well-known monotone likelihood ratio (MLR) order. We recall[14] that Y is said to dominate X in the sense of the monotone likelihood ratio ($Y \succcurlyeq_{MLR} X$) if there exist numbers $-\infty \leq x_1 \leq x_2 \leq \infty$ and a nondecreasing function $h : [x_1, x_2] \to [0, \infty]$ such that $P(Y < x_1) = 0$, $P(X > x_2) = 0$ and $dF_Y(x) = h(x)\,dF_X(x)$ on $[x_1, x_2]$. The MLR order is widely used in the statistical literature. It has been introduced for measuring the desirability of risky assets in a portfolio setting in [13].

Proposition 4.3. *Let X and Y denote random variables on (Ω, F, P) such that $E[X] < \infty$ and $E[Y] < \infty$. The following conditions are equivalent:*

1. $Y \succcurlyeq_{RP_1} X$
2. $Y \succcurlyeq_{MLR} X$

Proof. This is a direct consequence of Lemma 4.2 and of Proposition 2.2. ■

This means that a necessary and sufficient condition for a random variable Y to have a higher equilibrium relative price than X for all nondecreasing utility functions in $\mathcal{U}_1$ is that Y dominates X in the sense of the monotone likelihood ratio. We deduce from Proposition 4.3 that all results that are known to be valid for the MLR order remain valid for our order. In particular, we deduce that the RP_1 order is stronger than the FSD order. We have in mind some classical examples of MLR dominance that become classical examples of RP_1 dominance.

[14] see [13].

First, if two distributions have disjoint convex supports, then the distribution with the "higher" support dominates in the sense of the relative price the other one. Second, if a random variable has two-point support, then increasing the probability of the good outcome is an RP_1 improvement. If the initial distribution is strongly unimodal, that is, it possesses a density function that is log concave (a class that includes the normal, uniform and exponential distributions), then adding a constant to the random variable leads to an RP_1 change[15]. In the setting of portfolio allocations, we easily deduce from the result in [13] mentioned at the beginning of this section, that if returns on risky assets are ordered by the RP_1 order, then dominating assets will be more desired by all investors with nondecreasing utility functions.

The second order dominance is harder to characterize. However, as we have seen in Lemma 4.2, Y dominates X in the sense RP_2 if and only if $P_X \succcurlyeq_{MPR_2} P_Y$ (with respect to e). Let us introduce the following notations. Let x_1 and x_2 be given nonnegative real numbers, we define $a\left(x_1\right)$ (res $b(x_1, x_2)$ respectively by

$$\begin{aligned}
\tilde{a}\left(x_1\right) &= E\left[X 1_{X \leq x_1}\right] E\left[1_{Y \leq x_1}\right] - E\left[Y 1_{Y \leq x_1}\right] E\left[1_{X \leq x_1}\right] \\
\tilde{b}\left(x_1, x_2\right) &= E\left[X 1_{X \leq x_1}\right] E\left[1_{Y \leq x_2}\right] + E\left[X 1_{X \leq x_2}\right] E\left[1_{Y \leq x_1}\right] \\
&\quad - E\left[Y 1_{Y \leq x_1}\right] E\left[1_{X \leq x_2}\right] - E\left[Y 1_{Y \leq x_2}\right] E\left[1_{X \leq x_1}\right]
\end{aligned}$$

and let us define the polynomial R_{x_1,x_2} by $\tilde{R}_{x_1,x_2}(X) = \tilde{a}(x_1)X^2 + \tilde{b}(x_1, x_2)X + \tilde{a}(x_2)$.The following results are then direct consequences of the MPR_2 properties.

Theorem 4.4. *Let X and Y denote random variables on (Ω, F, P) such that $E[X] < \infty$ and $E[Y] < \infty$. The four following conditions are equivalent:*

1. $Y \succcurlyeq_{RP_2} X$,
2. $\forall (x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$, $\quad \frac{\sum_{i=1}^2 y_i E\left[X 1_{X \leq x_i}\right]}{\sum_{i=1}^2 y_i E\left[1_{X \leq x_i}\right]} \leq \frac{\sum_{i=1}^2 y_i E\left[Y 1_{Y \leq x_i}\right]}{\sum_{i=1}^2 y_i E\left[1_{Y \leq x_i}\right]}$
3. *for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ and for all $X \geq 0$ we have $\tilde{R}_{x_1,x_2}(X) \leq 0$*
4. *for all $x = (x_1, x_2) \in \mathbb{R}_+^2$ we have $\tilde{a}(x_1) \leq 0$ and $\frac{1}{2}\tilde{b}(x_1, x_2) \leq \sqrt{\tilde{a}(x_1)\tilde{a}(x_2)}$*

Proposition 4.5. *Let X and Y denote random variables on (Ω, F, P) such that $E[X] < \infty$ and $E[Y] < \infty$.*

[15] Alternative restrictions on the initial distribution allow for different linear transformations (see [16]).

1. *If* $Y \succcurlyeq_{RP_2} X$, *then for all* x, $E[X \mid X \leq x] \leq E[Y \mid Y \leq x]$

2. *If* $Y \succcurlyeq_{RP_2} X$ *and* $E[X] = E[Y]$ *then* $Var[X] \geq Var[Y]$

3. *If* $\sup_x \frac{E[X1_{X\leq x}]}{E[Y1_{Y\leq x}]} \leq \inf_x \frac{P\{X\leq x\}}{P\{Y\leq x\}}$ *then*[16] $Y \succcurlyeq_{RP_2} X,.$

4. *In particular, if for all* x, $E[X1_{X\leq x}] \leq E[Y1_{Y\leq x}]$ *and* $P\{X \leq x\} \geq P\{Y \leq x\}$, *then* $Y \succcurlyeq_{RP_2} X$.

Example 4.6. *Let us assume that* X *and* Y *have normal (or log-normal) distributions respectively with parameters* (μ_X, σ_X) *and* (μ_Y, σ_Y). *If* $\mu_Y \geq \mu_X$ *and* $\sigma_X \geq \sigma_Y$ *then* $Y \succcurlyeq_{RP_2} X$.

The equivalence between 1, 3 and 4 in Theorem 4.4 is a direct consequences of Lemma 4.2 and of Theorem 2.3. Condition 2 appears in the proof of Theorem 2.3. In fact this condition is exactly the same as in [8] and has been introduced in this last reference in order to characterize a reduction of the demand in risky asset when there is a shift in returns from Y to X and when the riskless rate is unspecified.

More precisely, let us assume that a given agent has the possibility to invest in a given riskless asset whose return is given by $1 + r > 0$ and in an asset whose returns are given by X. The agent is endowed with a concave utility function u and an initial wealth w and maximizes his expected utility $E[u(w + \alpha(X - (1 + r))]$ by the choice of the amount α invested in the risky asset. Let us denote by α_X his optimal demand in that asset. Let us also denote by α_Y his optimal demand in the risky asset when we replace X by Y. It is proved in [8] that we have $\alpha_Y \geq \alpha_X$ if and only if the second assertion of Theorem 4.4 is satisfied. The next proposition and its proof provide a direct argument starting from our definition.

Proposition 4.7. *If* $Y \succcurlyeq_{RP_2} X$ *and if* $E[X] \geq 1 + r$ *then regardless of the initial endowment and of the (concave) utility function defined on* $\mathbb{R}$ *as a whole we have* $\alpha_Y \geq \alpha_X$.

Proof. The amount α_X satisfies $E[(X - (1 + r))u'(w + \alpha_X(X - (1 + r)))] = 0$ or equivalently $E[(Xv'(X)] = E[(v'(X)]$ where $v(x) = u(w + \alpha_X(x - (1 + r)))$. The

[16] Once again we adopt the convention $\frac{0}{0} = 0$.

function $\alpha \to E[(X-(1+r))u'(w+\alpha(X-(1+r)))]$ has a nonpositive derivative and since $E[X] \geq 1+r$ we have $\alpha_X \geq 0$. The function v is then nondecreasing and concave and by definition of the RP_2 order we have, $E[(Yv'(Y)] \geq E[(v'(Y)]$ or equivalently $E[(Y-(1+r))u'(w+\alpha_X(Y-(1+r)))] \geq 0$. Since α_Y solves $E[(Y-(1+r))u'(w+\alpha_Y(Y-(1+r)))] = 0$ and by monotonicity of $\alpha \to E[(Y-(1+r))u'(w+\alpha(Y-(1+r)))]$, we have $\alpha_Y \geq \alpha_X$. ■

The condition $E[X] \geq 1+r$ guarantees that $\alpha_X \geq 0$ and means that X is desirable. This condition is natural when we have equilibrium models in mind. The fact that u is defined on $\mathbb{R}$ as a whole eliminates boundary problems. We can instead impose interior solutions for the first order conditions.

When the riskless rate is given, then [7] characterizes the reduction of the demand in risky asset by the "central riskiness around r" ($CR(r)$)dominance. It appears then that our RP_2 dominance is equivalent to the $CR(r)$ for all r or equivalently to the $CR(0)$ dominance for $X-r$ and $Y-r$ for all r. In [8], this property is denoted by $\cap_r CR(r)$.

Proposition 2.5 permits then to construct simple examples of such central riskiness dominance.

5. Conclusion

We have introduced a new criterion directly linked to the market price of risk in an equilibrium model in order to measure the pessimism of a given probability belief. We have seen how this criterion can be applied to propose a measure of the risk aversion of a given utility function as well as a measure of the level of risk of a given prospect.

We have given necessary and sufficient conditions on the probability belief, on the utility function, on the level of risk of the aggregate wealth for an increase of the market price of risk. For given subjective probability beliefs Q_1 and Q_2, the market price of risk is higher under Q_1 than under Q_2 for any nondecreasing utility function if and only if Q_1 is more pessimistic in the sense that the density of Q_1 with respect to Q_2 is anticomonotone with aggregate wealth. We also propose a concept of pessimism associated to a second order dominance (when we deal with the class of nondecreasing concave utility functions). For given utility functions u and v, the market price of risk is higher under v than under u if and only if v is more risk averse in the sense of Arrow-Pratt. For given prospects X and Y, the market price of risk is higher for Y than for X for any nondecreasing utility function if and only if X dominates Y in the sense of the monotone likelihood ratio. Furthermore, the market price of risk is higher for Y than for X for any nondecreasing concave utility function if and only if, for all r, $X - r$ dominates $Y - r$ in the sense of the central riskiness.

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FIGURES

Figure 1

This figure represents the distribution functions of Q (dashed line) and P (solid line).

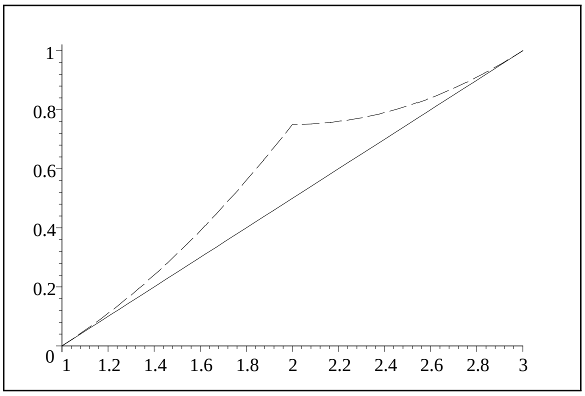

Figure 2

This figure represents the density of P_u (solid line) as well as the density of Q with respect to P.

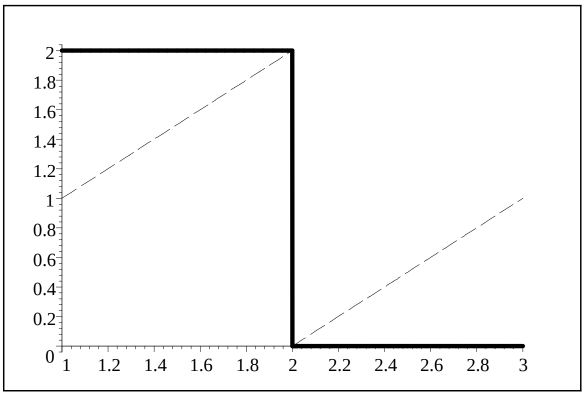

Figure 3

This figure represents a situation where the conditions of the fourth assertion of Proposition 2.4 are satisfied but where the comonotonicity condition is not satisfied. More precisely, we have $e = (0, 1, 2)$, $Q_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $Q_2 = (\frac{1}{8}, \frac{4}{8}, \frac{3}{8})$. The solid (resp. dashed) thin line represents the distribution function of Q_2 (resp. Q_1). The solid (resp. dashed) thick line represents the function $E^{Q_1}[e1_{e\leq x}]$ (resp. $E^{Q_2}[e1_{e\leq x}]$). The thin (resp. thick) dashed line is clearly above (resp. below) the solid one. The sufficient condition of assertion 4. is then satisfied. The dot-dash line represents $\frac{dQ_1}{dQ_2}$ and is clearly nonmonotone.

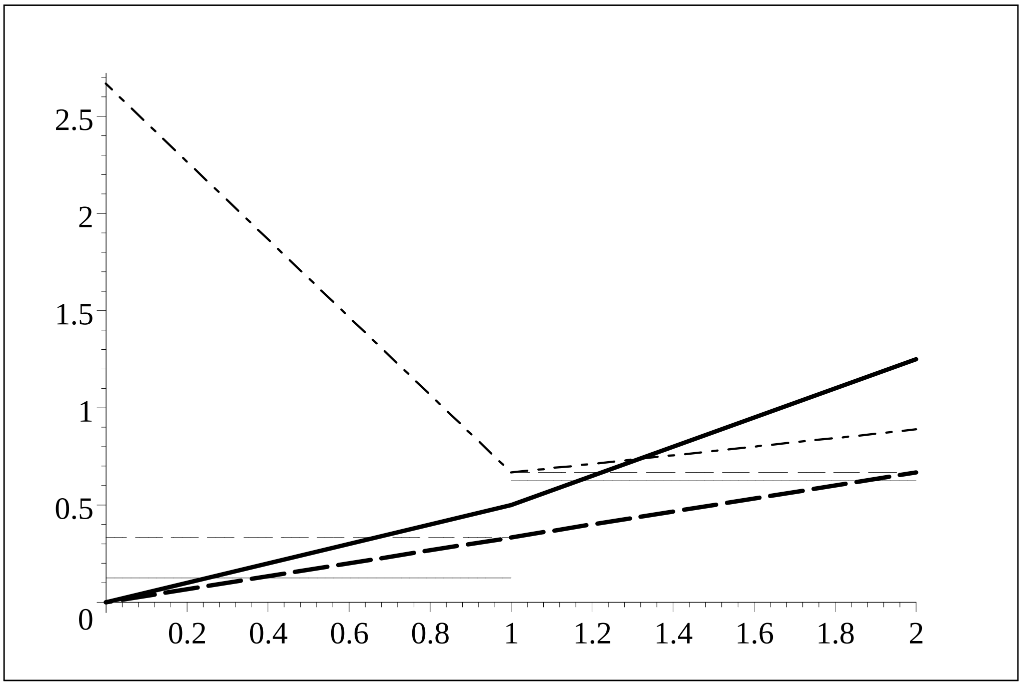

Figure 4 & 5

These figures represent a situation where the conditions of the third assertion of Proposition 2.4 are satisfied but without the first order stochastic dominace. More precisely, we have $e = (0, 1, 2)$, $Q_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $Q_2 = \left(\frac{1}{8}, \frac{5}{8}, \frac{2}{8}\right)$. In the first figure, the solid (resp. dashed) thin line represents the distribution function of Q_2 (resp. Q_1). The solid (resp. dashed) thick line represents the function $E^{Q_1}\left[e1_{e\leq x}\right]$ (resp. $E^{Q_2}\left[e1_{e\leq x}\right]$). We clearly do not have first stochastic dominance. In the second figure, the dashed lines are the same as in the previous one and the solid lines correspond to $\lambda Q_1 \left[e \leq x\right]$ and $\lambda E^{Q_1}\left[e1_{e\leq x}\right]$ with $\lambda = \frac{16}{18}$ and the conditions of the third assertion are clearly satisfied.