Jump processes
Session 1

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Exercise 1

Let N denote a Poisson process with intensity $\lambda > 0$ and

$$X_t = \sum_{k=1}^{N_t} Z_k$$

where $(Z_k)_{k\in\mathbb{N}}$ are i.i.d. with distribution ν and independent of N such that $m := \mathbb{E}[Z_1] < \infty$.

1. Prove that

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda, \text{ a.s.} \quad \text{and} \quad \frac{N_t - \lambda t}{\sqrt{\lambda t}} \to \mathcal{N}(0,1) \text{ in distribution.}$$

2. Find the compensated compound Poisson process of X, i.e. find a deterministic $(A_t)_{t\geq 0}$ such that $X - A$ is a martingale.

3. Sketch a path of a Poisson process and a compensated Poisson process.

4. Compute the characteristic function of the compound Poisson process X.

5. Let N' denote another Poisson process independent of N with intensity $\lambda' > 0$. What can you say of the following processes

   (i) $N + N'$,

   (ii) $N - N'$.

Exercise 2

Part 1: Cauchy distribution.

1. Compute the characteristic function of the Laplace distribution with density $f(x) = \frac{\lambda}{2}\exp(-\lambda|x|)$, where $\lambda > 0$.

2. Using the inverse Fourrier transform, compute the characteristic function of the Cauchy distribution with parameter $c > 0$ and with density $g(x) = \frac{c}{\pi(c^2+x^2)}$.

3. Deduce that the Cauchy distribution is infinitely divisible.

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Part 2: One sided $1/2$-stable distribution.
Fix $c > 0$ and consider the density function

$$f(x) = \frac{c}{\sqrt{2\pi}} e^{-c^2/(2x)} x^{-3/2} \mathbb{1}_{x>0}.$$

4. Prove that $\mu(dx) := f(x)dx$ is a probability measure.

5. Compute the Laplace transform $\mathcal{L}_\mu(u) = \int_{\mathbb{R}} e^{-ux}\mu(dx)$ for $u \geq 0$. (*Hint:* Use the change variable $ux = c^2/(2y)$ and find the ordinary differential equation satisfied by $\mathcal{L}_\mu$.)

6. Deduce that the characteristic function is

$$\hat{\mu}(z) = \exp\left(-c|z|^{1/2}(1 - \mathrm{isgn}(z))\right), \quad \text{for all } z \in \mathbb{R}.$$

(*Hint:* It is enough to prove that this expression matches with the Laplace transform for $u \geq 0$.)

7. Is μ infinitely divisible?

8. Prove that

$$\int_0^\infty \left(e^{-ux} - 1\right) x^{-3/2} dx = -2\sqrt{\pi u}, \quad \text{for all } u \geq 0.$$

(Recall the identities $\Gamma(1/2) = \int_0^\infty s^{-1/2}e^{-s}ds = \sqrt{\pi}$ and $e^{-ux} - 1 = -u\int_0^x e^{-uy}dy$.)

9. Is it the symbol of a compound Poisson process? What is the Lévy triplet of μ?

Exercise 3

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (b, σ^2, ν), where $\nu(dz)$ is the Lévy measure given by

$$\nu(dz) = f(z)dz = Kz^{\alpha-1}e^{-\beta z^\alpha}\mathbf{1}_{(0,\infty)}(z)dz,$$

for some positive constants $K > 0$, $\alpha > 0$, $\beta > 0$.

1. Does one need additional conditions on (K, α, β) to make ν a Lévy measure ?

   If yes, provide the conditions and we assume them in the following.

   If no, provide the reason.

2. Compute and prove that

$$\lambda \ := \ \nu(\mathbb{R}) \ = \ \int_{\mathbb{R}} f(z)dz \ < \ \infty.$$

3. Using the Lévy-Itô decomposition, X can be decomposed as

$$X_t = \mu t + \sigma W_t + Y_t,$$

where W is a Brownian motion, and Y is a compound Poisson process:

$$Y_t = \sum_{k=1}^{N_t} Z_k.$$

Identify the constant $\mu \in \mathbb{R}$ and the intensity of the Poisson process N and the distribution density of random variables $(Z_k)_{k\geq 1}$.

4. Prove that $\mathbb{E}[|Z_1|] < \infty$ and then that $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$.