Jump processes
Session 2

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Exercise 1

Let X be a Lévy process of dimension 1. Let Ψ be the characteristic function of X:

$$\Psi(u) = \mathbb{E}[e^{iuX_1}] , \quad u \in \mathbb{R} .$$

1. Recall the generic expression of Ψ.
2. Let $c > 0$ be fixed. We set $Y_t := X_{ct}$ for all $t \geq 0$. Show that Y is a Lévy process and determine its characteristic function.
3. We assume that X is a compound Poisson process. Let $T > 0$ be fixed. Are the laws of $(X_t, t \in [0, T])$ and of $(X_{ct}, t \in [0, T])$ equivalent ?
4. Same question when X is a Brownian motion.

Exercise 2

Let B be a standard Brownian motion of dimension 1 and let N be a Poisson process of intensity $\lambda > 0$ independent of B. We set

$$X_t := B_{N_t} , \quad t \geq 0 .$$

1. Compute the characteristic function of X_t for $t \geq 0$.
2. Show that X is a Lévy process.
3. Identify the law of the process X.
4. If B is replaced by an arbitrary Lévy process L, what are the answers to the previous three questions ?

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Exercise 3

Let X be a Lévy process of dimension 1 of characteristic exponent

$$\eta(u) = \int_{x\in[-1,1]} (e^{iux} - 1 - iux)\nu(dx) \ , \quad u \in \mathbb{R} \ ,$$

where ν is a Lévy measure. For any integer $N \geq 1$ and for some parameter $\alpha > 0$, we set

$$X_t^{(N)} := \frac{1}{N^\alpha} X_{tN} \ , \quad t \geq 0 \ .$$

1. Assume that $\nu(dx) = \delta_1(dx) + \delta_{-1}(dx)$. Show that X is a mean zero compound Poisson process.
2. Show that $X^{(N)}$ is a Lévy process and identify its characteristic exponent.
3. Show that, for some well-chosen parameter α, the finite dimensional marginals of $X^{(N)}$ converge to those of a Brownian motion and determine the variance of this Brownian motion.
4. We now assume that $\nu(dx) = f(x)dx$ where $f : [-1,1] \to \mathbb{R}_+$. Show that the finite dimensional marginals of $X^{(N)}$ converge to those of a Brownian motion and determine the variance of this Brownian motion.

Exercise 4

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$ denote a filtered probability space and X be a Lévy process on $\mathbb{R}$ with characteristic triplet (γ, A, ν). Assume that $\int_{|z|>1} z^2 \nu(dz) < \infty$. Denote by J_X the jump-measure of X. The aim is to prove that the following assertions are equivalent:

(i) $X_t \geq 0$ for every $t \geq 0$ $\mathbb{P}-$almost surely,

(ii) Sample paths of X are almost surely non-decreasing: $t \geq s \Rightarrow X_t \geq X_s$ $\mathbb{P}-$almost surely,

(iii)

$$b := \gamma - \int_0^1 z\nu(dz) \geq 0, \quad A = 0, \quad \nu((-\infty, 0]) = 0, \quad \int_0^\infty (z \wedge 1)\nu(dz) < \infty.$$

1. Prove $(i) \Leftrightarrow (ii)$.
2. Prove $(iii) \Rightarrow (i)$.
3. Assume (i).
   (a) Explain why necessarily $\nu((-\infty, 0]) = 0$.
   (b) Fix $\eta < 0$. Prove that $\int_0^\cdot \int_{\mathbb{R}} (e^{\eta z} - 1)(J_X(ds, dz) - \nu(dz)ds)$ is a square integrable $\mathbb{P}-$martingale.
   (c) Define $U^\eta := \eta\sqrt{A}W + \int_0^\cdot \int_{\mathbb{R}} (e^{\eta z} - 1)(J_X(ds, dz) - \nu(dz)ds)$ where W is a $\mathbb{P}-$Brownian motion. Prove that the following equation admits a unique solution

   $$dZ_t = Z_{t-} dU_t^\eta, \quad Z_0 = 1.$$

   Explicit this solution which we denote by Z^η.

(d) Prove that Z^η is a positive martingale.

(e) Define $\mathbb{P}^\eta$ by $\frac{d\mathbb{P}^\eta}{d\mathbb{P}}|_{\mathcal{F}_t} = Z_t^\eta$. Recall that X is still a Lévy process under $\mathbb{P}^\eta$ and explicit the corresponding triplet $(\gamma^\eta, A^\eta, \nu^\eta)$ in terms of (γ, A, ν, η).

(f) Justify that $X_t \geq 0$ for every $t \geq 0$ $\mathbb{P}^\eta-$almost surely.

(g) Prove that $\int_{|z|>1} z^2 \nu^\eta(dz)$ and deduce (iii).

4. Give a geometric interpretation of (iii).

5. Provide a qualitative explanation for the following statement: *There exist Lévy processes without diffusion components having no negative jumps that can become negative.*