New results about shape derivatives and convexity constraint

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Geometric aspects of PDE's and functional inequalities

Main motivation

Convexity constraint in shape optimization

We are interested in problems of the form :

$$\min\left\{J(\Omega), \quad \Omega \;\; is \;\; convex, \;\; \Omega \in \mathcal{F}_{ad}\right\}$$

where J is a shape functional.

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Examples of constraints :

- $\mathcal{F}_{ad}=\left\{\Omega,\ B(0,a)\subset\Omega\subset B(0,b)\right\}$,
- $\mathcal{F}_{ad}=\left\{|\Omega|=V_0,\text{ and }\Omega\subset B(0,b)\right\}$.

Outline

1. Convexity constraint
   - Old, new, open problems
   - Case of concave functionals
   - Applications
2. Shape derivatives
   - Previous results (2-dim)
   - New results (N-dim)

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Example I

Newton's problem of the body of minimal resistance [1685]

$$\min\left\{\int_D \frac{1}{1+|\nabla f|^2}\ \ /\ \ f:D\to[0,M], f\ \ \textit{concave}\ \right\}$$
$$D=\{x\in\mathbb{R}^2,\ |x|\leq 1\}$$

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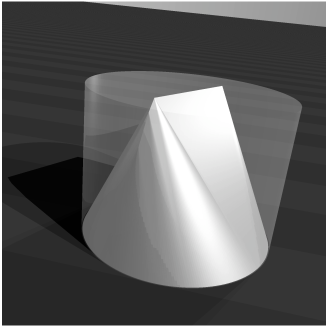
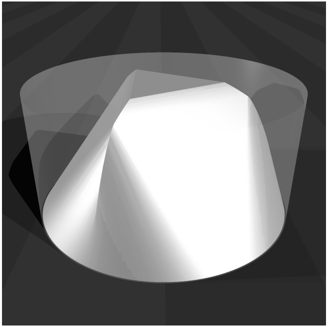
Numerical computations : Lachand-Robert, Oudet, 2004 :





Example II

Mahler's conjecture [1939]

Conjecture : is the cube $Q_N := [-1,1]^N$ solution of

$$\min\left\{M(K) := |K||K^\circ|,\ \ K \text{ convex of } \mathbb{R}^N,\ \ -K = K\right\} ?$$

where

$$K^\circ := \left\{\xi \in \mathbb{R}^N,\ \ \langle \xi, x\rangle \leq 1,\ \ \forall x \in K\right\}.$$

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Theorem (Mahler, 1939)

If $N = 2$, ok.

Example III

Pólya-Szegö's conjecture [1951]

The electrostatic capacity of a set $\Omega \subset \mathbb{R}^3$ is defined by

$$Cap(K) := \int_{\mathbb{R}^3 \setminus K} |\nabla u_K|^2 \quad \text{where} \quad \begin{cases} \Delta u_K & = & 0 & in & \mathbb{R}^3 \setminus K \\ u_K & = & 1 & on & \partial K \\ \lim\limits_{|x| \to +\infty} u_K & = & 0 & & \end{cases}$$

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or equivalently solution of

$$\min\left\{\frac{Cap(K)^2}{P(K)},\ K \text{ convex}\right\} \Leftrightarrow \min\left\{Cap(K)^2 - \mu P(K),\ K \text{ convex}\right\}$$

Example IV

Variations on the charged liquid drop problem [Goldman, Novaga, Ruffini, 2016]

$$\min\{P(\Omega) + Q^2 I_0(\Omega),\ \ \Omega \textit{ convex } \subset \mathbb{R}^N,\ \ |\Omega| = V_0\}$$

$$I_0(\Omega) = \inf_{\mu \in \mathcal{P}(\Omega)} \int_{\Omega\times\Omega} \log\left(\frac{1}{|x-y|}\right) d\mu(x)d\mu(y) = \frac{1}{Lcap(\Omega)}$$

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- Existence ok
- If $N = 2$, $C^{1,1}$-regularity of solutions
- Ball if Q is small enough
- Convergence to a segment if $Q \to \infty$ (after rescaling).

Example V : Reverse isoperimetry

[Bianchini-Henrot 2012]

$$\min\left\{\mu|\Omega| - P(\Omega),\ \ \Omega \textit{ convex } \subset \mathbb{R}^N,\ \ B(0,a)\subset\Omega\subset B(0,b)\right\}$$

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- Complete description available [Bianchini-Henrot] : 'true' polygons

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- understanding $\partial\Omega^*\cap\partial C$? Can it be strictly convex ?

Few formal remarks/results

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Use second order optimality condition !

Outline

The gauge function

If $N \geq 2$, and $u : \mathbb{S}^{N-1} \to (0, \infty)$ is given, we define

$$\Omega_u := \left\{ (r, \theta) \in [0, \infty) \times \mathbb{S}^{N-1}, \quad r < \frac{1}{u(\theta)} \right\}.$$

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$$\Omega_u := \left\{ (r, \theta) \in [0, \infty) \times \mathbb{S}^{N-1}, \quad r < \frac{1}{u(\theta)} \right\}.$$

Then

$$\Omega_u \text{ is convex if and only if } u \text{ is convex on } \mathbb{S}^{N-1}.$$

Reformulation of the problem

$$\min\left\{J(\Omega),\ \Omega\ convex,\ \Omega\in\mathcal{F}_{ad}\right\}$$

$$\iff \min\left\{j(u):=J(\Omega_u),\ u\ convex,\ \ u\in S_{ad}\right\}$$

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For example,

$$S_{ad}=\left\{u:\mathbb{S}^{N-1}\to\mathbb{R},\ \ a\leq 1/u\leq b\right\}$$

Minimization of locally concave functionals

Let u_0 such that :

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Theorem (L.,Novruzi,Pierre 2015)

Assume, for any $v \in W^{1,\infty}(\mathbb{S}^{N-1})$,

$$j''(u_0)(v,v) \leq -\alpha |v|^2_{H^1(\mathbb{S}^{N-1})} + \beta \|v\|^2_{H^s(\mathbb{S}^{N-1})}. \tag{H}$$

for some $\alpha > 0, s \in [0,1)$.

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for some $\alpha > 0, s \in [0,1)$. *Then the set*

$$T_{u_0} = \left\{ v / \exists \varepsilon > 0, \forall |t| < \varepsilon,\ u_0 + tv \text{ is convex and } \in S_{ad} \right\},$$

is a linear vector space of finite dimension.

Consequences

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Problem : Ω_{u_0} is not necessarily a polyhedra.

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   - Previous results (2-dim)
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Results

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- Example VI (Enea's problem) ?

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2-dim result

The perimeter, the volume

$$p(u) = P(\Omega_u) = \int_{\mathbb{S}^1} \frac{\sqrt{u^2 + u'^2}}{u^2} d\theta, \quad a(u) = |\Omega_u| = \frac{1}{2} \int_{\mathbb{S}^1} \frac{1}{u^2} d\theta$$

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For any u_0 such that $a \leq 1/u_0 \leq b$,

- $p''(u_0)(v, v) \geq \alpha |v|^2_{H^1} - \beta \|v\|^2_{L^2}$
- $|a''(u_0)(v, v)| \leq \beta \|v\|^2_{L^2}$

2-dim result

PDE functional

$$\ell(u) := \lambda_1(\Omega_u)$$

Theorem (L., Novruzi, Pierre, 2011)

Let Ω_u be convex in $\mathbb{R}^2$. Then for any $\varepsilon > 0$, there exist β such that

$$|\ell''(u)(v,v)| \le \beta \|v\|^2_{H^{\frac{1}{2}+\varepsilon}}.$$

Applications :

- Example V bis in $\mathbb{R}^2$.

Outline

N-dim result

The perimeter, the volume

$$p(u) = \int_{\mathbb{S}^{N-1}} \frac{\sqrt{u^2 + |\nabla_\tau u|^2}}{u^N} d\theta, \quad m(u) = \frac{1}{N} \int_{\mathbb{S}^{N-1}} \frac{1}{u^N} d\theta,$$

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PDE functional, convex domains

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Theorem (L., Novruzi, Pierre, 2015)

Let Ω_u be semi-convex in $\mathbb{R}^N$ (exterior ball condition). Then there exist β such that

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Main tools for the proof :

- New way to estimate shape derivative
- $W^{1,p}$ regularity theory for elliptic PDE in semi-convex domains (valid for any $p \in (1, \infty)$).

N-dim result

PDE functional, lipschitz domains

Theorem (L., Novruzi, Pierre, 2015)

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- Allows to deal with Exterior PDE problems
- Estimate available in $\mathbb{R}^N$, but too weak to handle Example III (Pólya-Szëgo)

Perspectives - More open problems

- Reverse Faber-Krahn inequality

$$\max\{\lambda_1(K) \;/\; K \text{ convex} \subset B(0,b),\ |K| = V_0\}$$

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- Polyhedra in dimension ≥ 3 ?