Stability in shape optimization with second variation

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Abstract

We are interested in the question of stability in the field of shape optimization. Precisely, we prove that under structural assumptions on the hessian of the considered shape functions, the necessary second order minimality conditions imply that the shape hessian is coercive for a given norm. Moreover, under an additional continuity condition for the second order derivatives, we derive precise optimality results in the class of regular perturbations of a domain. These conditions are quite general and are satisfied for the volume, the perimeter, the torsional rigidity and the first Dirichlet eigenvalue of the Laplace operator. As an application, we provide non trivial examples of inequalities obtained in this way.

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1 Introduction

In this paper, we are interested in the question of stability in the field of shape optimization. More precisely, given $J : \mathcal{A} \to \mathbb{R}$ defined on $\mathcal{A} \subset \{\Omega \text{ smooth enough open sets in } \mathbb{R}^d\}$, we consider the optimization problem

$$\min \{J(\Omega), \;\; \Omega \in \mathcal{A}\}, \tag{1.1}$$

and we ask the following question:

if $\Omega^* \in \mathcal{A}$ is a critical domain satisfying a stability condition (that is to say a strict second order optimality condition), can we conclude that Ω^* is a strict local minimum for (1.1) *in the sense that*

$$J(\Omega) - J(\Omega^*) \geq cd(\Omega, \Omega^*)^2, \qquad \textit{for every } \Omega \in \mathcal{V}(\Omega^*) \tag{1.2}$$

where $c \in (0, \infty)$, d is a distance among sets, and $\mathcal{V}(\Omega^*)$ is a neighborhood of Ω^*, also relying on a suitable distance ?

Notations and shape calculus:

We start by introducing our notations: we shall use the usual shape calculus first introduced by Hadamard, then developed by Murat-Simon and Delfour-Zolesio. The main idea is to consider diffeomorphisms to encode variations of the domain that is to say one defines the function $\mathcal{J}$ on a neighborhood of 0 in the Banach space $\Theta = C^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d) = \{\theta : \mathbb{R}^d \to \mathbb{R}^d, \; \theta \text{ is of class } C^3 \text{ and } \forall k \in [\![0, 3]\!], D^k\theta \text{ is bounded}\}$ (this space is chosen for convenience as the space of shape differentiability in the whole paper, though most of the results can be adapted to other spaces) by

$$\forall \theta \in \Theta, \quad \mathcal{J}(\theta) = J[(I + \theta)(\Omega)].$$

One then uses (in the whole paper) the usual notion of Fréchet-differentiability: shape derivatives are the successive derivatives of $\mathcal{J}$ at 0, when they exist. In particular, the first shape derivative is $J'(\Omega) := \mathcal{J}'(0) \in \Theta'$ and the second order shape derivative is $J''(\Omega) := \mathcal{J}''(0)$, a continuous symmetric bilinear form on Θ.

Distance between domains:

We use the usual distance introduced by Michelletti:

$$d_\Theta(\Omega_1, \Omega_2) := \inf \left\{ \|\theta\|_\Theta + \|(I+\theta)^{-1} - I\|_\Theta,\ \ \theta \in \Theta \text{ diffeomorphism such that } (I+\theta)(\Omega_1) = \Omega_2 \right\}.$$

Let us emphasize that since we consider diffeomorphims that are close to the identity, the boundaries of the perturbed domains are graphs over the boundary of the initial domain: for any domain $\Omega_2 = (I+\theta)(\Omega_1)$ with $\theta \in \Theta$ close to 0, there is a unique real-valued function $h = h_{\Omega_1,\Omega_2}$ defined on $\partial\Omega_1$ such that

$$\partial\Omega_2 = \partial[(I+\theta)(\Omega_1)] = \{x + h(x)\boldsymbol{n}(x),\ x \in \partial\Omega_1\}, \tag{1.3}$$

see Lemma 3.1 in [17].

Main difficulties and contributions of the present paper:

It is well-known since Hadamard that the shape gradient is a distribution supported on the boundary of the domain acting on the normal component of the diffeomorphism of deformation. On a critical domain Ω^* for J, that is a domain such that the shape gradient of J vanishes, the shape hessian reduces to a bilinear form ℓ_2 acting also on normal components of diffeomorphisms so that the following Taylor formula holds

$$J((I+\theta)(\Omega^*)) = J(\Omega^*) + \frac{1}{2}\,\ell_2(\theta\cdot\boldsymbol{n}, \theta\cdot\boldsymbol{n}) + o(\|\theta\|_\Theta^2). \tag{1.4}$$

see Section 2.1 for more details.

Let us make precise the formulation of (1.2): we consider a domain $\Omega^* \in \mathcal{A}$, and J a shape functional such that

(A1) the domain Ω^* is critical for J,

(A2) the shape hessian at Ω^* is nonnegative i.e.: $\ell_2(\varphi,\varphi) > 0$ for all $\varphi \neq 0$

Our main result is that conclusion (1.2) holds under these assumptions and some structure assumptions on the bilinear form ℓ_2, for suitable distance to be precised.

To prove this, we have to face two main difficulties. The first one is that Assumption (A2) is not natural: in order to dominate the reminder term in (1.4), one needs a coercive bilinear form. We first provide a result (Lemma 3.3) stating that under a structure assumption a nonnegative bilinear form defined on Sobolev spaces is coercive. Notice that our statement holds for general bilinear forms on Sobolev spaces and is not connected to shape hessian. Similar statements exists for particular functionals in the literature: for example, one can mention the works of Grosse-Brauckmann [12] in the context of stable minimal surfaces and of Acerbi, Fusco and Morini [2] in the context of non local isoperimetric problems. Our main contribution about this step is to identify general assumptions: indeed, the above-mentioned structure assumptions (see $(H_1)-(H_2)$ in Lemma 3.3) are satisfied for a large class of functionals including the volume of Ω, its perimeter, the first eigenvalue of the Dirichlet Laplace operator and the Dirichlet energy, examples we will treat in this paper.

The second difficulty comes from the two norms discrepancy that appears in Taylor-Formula (1.4): the bilinear form ℓ_2 is usually coercive in a Sobolev norm which is strictly weaker than the $C^{3,\infty}$ norm of differentiability that appears in the reminder, so it is a priori not possible to control the sign of the term $\frac{1}{2}\ \ell_2(\theta\cdot\boldsymbol{n},\theta\cdot\boldsymbol{n})+o(\|\theta\|_\Theta^2)$. We may think that we should rather change the space of differentiability for the functional (choosing for example the space for which there is coercivity), unfortunately for functionals relying on PDE, as we are interested in this work, it is well-known that it is usually not possible, the functionals are no longer differentiable in those spaces (see [13] for example). This difficulty is well-known in the literature: it appeared in the work of Descloux [9], was overcome in the works of [8] of Dambrine and Pierre and of Dambrine [7], then a very similar approach can be found in the work of Acerbi, Fusco and Morini [2]. The idea is, given Ω^* a critical and stable domain and Ω a domain sufficiently close for d_Θ, to consider the path $(\Omega_t)_{t\in[0,1]}$ defined through its boundary

$$\partial\Omega_t=\{x+\ t\ h(x)\ \boldsymbol{n}(x),\ x\in\partial\Omega\}. \tag{1.5}$$

connecting Ω^* to Ω, where $h=h_{\Omega^*,\Omega}$ is defined in (1.3) and to write a Taylor formula with integral rest for the function $j(t)=J(\Omega_t)$ defined on $[0,1]$. The stability assumption provides $j''(0)>0$ and one need to propagate this sign property for $t>0$. Therefore, we introduce the following hypothesis ($\mathrm{C}_{\mathrm{H}^s}$) which can be seen as a suitable continuity of the second order shape derivative:

($\mathbf{C}_{\mathrm{H}^s}$) there exist $\eta>0$ and a modulus of continuity ω such that for every domain $\Omega=(I+\theta)(\Omega^*)$ with $\|\theta\|_\Theta\le\eta$, and all $t\in[0,1]$:

$$\left|j''(t)-j''(0)\right|\le\omega(\|\theta\|_\Theta)\|\theta\cdot n_{\partial\Omega^*}\|_{\mathrm{H}^s}^2.$$

This technical assumption is essential to provide uniform (with respect to the deformations direction) results. In the literature, this assumption is established for a lot of examples (see Section 3.2). Our contribution on this point is to generalize the strategy, and also to simplify the proofs from [7, 2], especially in the case of volume constraint and translation invariance of the functional, which is the case of most interesting examples, see the comments after Theorem 1.1 in this introduction.

Main result:

Here is the main result of this paper, stated in a simplified way for an unconstraint problem (see Section 3 for a similar statement for problems with volume constraint and translation invariant functionals):

Theorem 1.1 *Let Ω^* be a domain of class C^5, and J be a shape functional, twice Fréchet differentiable on a neighborhood of Ω^* for d_Θ, such that*

- **Structural hypotheses:** *there exists $0\le s_1<s_2\le 1$ such that*
  - *the hessian ℓ_2 of J at Ω^* can be written $\ell_2=\ell_m+\ell_r$ with*

$$\begin{cases}\ell_m \text{ is lower semi-continuous in } H^{s_2}(\partial\Omega^*) \text{ and } \ell_m(\varphi,\varphi)\ge c_1|\varphi|_{\mathrm{H}^{s_2}}^2,\ \forall\varphi\in C^\infty(\partial\Omega^*),\\ \ell_r \text{ continuous in } H^{s_1}.\end{cases} \tag{1.6}$$

    where $c_1>0$.
  - *J satisfies $C_{\mathrm{H}^{s_2}}$.*
- **Necessary optimality conditions:**

– Ω^* *is a critical domain for* J*,*

– Ω^* *is a stable shape for* J*:*

$$\ell_2(\varphi,\varphi) > 0 \text{ for all } \varphi \in H^{s_2}(\partial\Omega,\mathbb{R}) \setminus \{0\}. \tag{1.7}$$

Then Ω^* *is an* H^{s_2}**-stable local minimum of** J **in a** Θ**-neighborhood**, *that is to say there exists* $\eta > 0$ *and* $c = c(\eta) > 0$ *such that*

$$\forall\, \Omega \text{ such that } d_\Theta(\Omega^*,\Omega) \leq \eta, \quad J(\Omega) \geq J(\Omega^*) + c\|h\|^2_{\mathrm{H}^{s_2}} \tag{1.8}$$

where the function $h = h_{\Omega^*,\Omega}$ *is defined in* (1.3)*.*

Of course, we will provide several examples of functionals satisfying these conditions, but we here notice that the reader can have in mind the following generic example: if Ω^* is a ball of volume $V_0 \in (0,\infty)$, $P(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$ denotes the perimeter of Ω, and E is the Dirichlet energy:

$$E(\Omega) = \min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u, \;\; u \in H^1_0(\Omega)\right\}, \tag{1.9}$$

then the conditions of Theorem 1.1 (more precisely the conditions of Theorem 3.2 which are of the same nature, but take into account the constraint and invariance of the problem, see also below in the introduction) are fulfilled if Ω^* is the ball of volume V_0 and we can conclude from our strategy that the ball is a local minimizer of the following optimization problem

$$\min\left\{P(\Omega) + \gamma E(\Omega), \;\; |\Omega| = V_0\right\}, \tag{1.10}$$

where $|\cdot|$ denotes the volume, $V_0 \in (0,\infty)$, and $\gamma \geq \gamma_0$ where $\gamma_0 \in (-\infty, 0)$. For $\gamma \geq 0$ this result is not surprising, since the ball minimizes both the perimeter and the Dirichlet energy. Though we obtain a quantitative version of the inequality for small smooth deformations, see also [2, 5] for related statements.

However, this result is surprising when γ is nonpositive, in which case there is a competition between minimizing the perimeter and maximizing the Dirichlet energy. In that case, if γ is close enough to 0, we conclude that Ω^* is a local minimizer, again in a neighborhood subjected to a strong norm, namely (1.2) is valid for

$$\mathcal{V}(\Omega^*) = \{\Omega, d_\Theta(\Omega^*,\Omega) \leq \eta\} \quad \text{with } \eta > 0. \tag{1.11}$$

This time, this is no longer valid when one consider larger neighborhood of Ω^*, for example in the L^1-norm. A counterexample is given in Section 3. This last question of stability in an L^1-neighborhood has been investigated for the isoperimetric problem in [21, 15] and recently received again some attention in the papers [2, 5]. For a problem related to (1.10) when $\gamma < 0$, see also [19].

In the previous example as in all of our applications, $\mathcal{A}$ is the class of domains with fixed volume, and the shape functionals under consideration are translation invariant. In that case the natural second order necessary condition is, instead of **(A2)** above, that the hessian is coercive on the subspace tangent to the "manifold" of domains satisfying the constraint and the invariance, therefore it is to be expected that we only assume the positivity assumption (1.7) valid for the Lagrangian $J - \mu\mathrm{Vol}$ in the space $T(\partial\Omega^*) := \left\{\varphi : \mathbb{R}^d \to \mathbb{R}, \;\; \int_{\partial\Omega}\varphi = 0, \;\; \int_{\partial\Omega}\varphi\overrightarrow{x} = \overrightarrow{0}\right\}$ (which, in other words, is the space where the first derivative of the volume and the barycenter vanish). In the previous works [7, 2], concerned with similar invariance and constraints, the authors construct a path satisfying the volume constraint and fixing the barycenter, which is not the case of the one defined in (1.5). This leads to rather technical difficulties. We drastically simplify this strategy and show that the hypothesis ($\mathrm{C}_{\mathrm{H}^s}$) suffices to conclude that a precise stability statement as (1.2) holds, thanks to an exact penalization procedure. We state the corresponding Theorem in the beginning of Section 3. It is clear, looking at the proofs of our results, that the situation we aim to describe in this paper is quite general and can be applied to many other situations. Thanks to this degree of generality, we obtain several local isoperimetric inequalities (see Proposition 3.7, in Section 3.5).

Going deeper in the computations, it is possible, as it is done in [16] for the second inequality involving (P,λ_1), to compute the optimal value of the constant γ_0 such that these inequalities are valid in a neighborhood of the ball. We insist on the fact that even in this particular case, we improve the result of [16], since our analysis provides a uniform neighborhood where we have an isoperimetric inequality, while this author make an asymptotic analysis on each path. This is done thanks to the ($\mathrm{C}_{\mathrm{H}^s}$) assumption.

In Section 2, we remind the classical results about second order shape derivatives, in particular in which norms they are continuous, and focus on the case of the ball for which we diagonalize the shape hessians and recall (also classical) stability properties. Section 3 contains the main results of this work: we state the version of Theorem 1.1 adapted to the constrained/invariant case, we discuss coercivity assumptions, and precise the known results about the ($\mathrm{C}_{\mathrm{H}^s}$) assumption, before proving our main theorem. We then prove some local isoperimetric inequalities, some are known, some are new, see Proposition 3.7. These inequalities are simple corollaries of our main result, combined with the computations reminded in Section 2. In the last Section, we show how to compute some explicit (sometimes optimal) constants for all the inequalities of Proposition 3.7.

2 On Second order shape derivatives.

2.1 Definitions and Structure theorem

It is well-known since Hadamard's work that the shape gradient is a distribution supported on the moving boundary and acting on the normal component of the deformation field. The second order shape derivative also has a specific structure as stated by A. Novruzi and M. Pierre in [17]. We quote their result.

Theorem 2.1 (Structure theorem of first and second shape derivatives) *Let $k\geq 1$ be an integer and J a real-valued shape function defined $\mathcal{O}_k$ the set of open bounded domains of $\mathbb{R}^d$ with a C^k boundary. Let us define the function $\mathcal{J}$ on $C^{k,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ by*

$$\mathcal{J}(\theta)=J[(I+\theta)(\Omega)].$$

(i) *If $\Omega\in\mathcal{O}_{k+1}$ and $\mathcal{J}$ is differentiable at 0, then there exists a continuous linear form ℓ_1 on $C^k(\partial\Omega)$ such that $\mathcal{J}'(0)\xi=\ell_1(\xi\cdot\boldsymbol{n})$ for all $\xi\in C^{k,\infty}(\mathbb{R}^d,\mathbb{R}^d)$.*

(ii) *If $\Omega\in\mathcal{O}_{k+2}$ and $\mathcal{J}$ is twice differentiable at 0, then there exists a continuous symmetric bilinear form ℓ_2 on $C^k(\partial\Omega)\times C^k(\partial\Omega)$ such that for all $(\xi,\zeta)\in C^{k,\infty}(\mathbb{R}^d,\mathbb{R}^d)^2$*

$$\mathcal{J}''(0)(\xi,\zeta)=\ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n})+\ell_1((D_\tau\boldsymbol{n}\zeta_\tau)\cdot\xi_\tau-\nabla_\tau\,(\zeta\cdot\boldsymbol{n})\cdot\xi_\tau-\nabla_\tau\,(\xi\cdot\boldsymbol{n})\cdot\zeta_\tau),$$

where ∇_τ is the tangential gradient and ξ_τ and ζ_τ stands for the tangential components of ξ and ζ.

The so-called shape derivative are then the shape gradient usually denoted $J'(\Omega):=\mathcal{J}'(0)$ and the shape hessian usually denoted $J''(\Omega):=\mathcal{J}''(0)$. With respect to this work, it is important to notice that at a critical domain for J, the shape hessian is reduced to ℓ_2 and hence does not see the tangential components of the deformations fields.

2.2 Examples of shapes derivatives on general domains.

We need to precise some geometrical definitions. The mean curvature (understood as the sum of the principal curvatures of $\partial\Omega$) is denoted by H. We recall that $D_\tau \boldsymbol{n}$ is the second fundamental form of $\partial\Omega$ and that ${}^T D_\tau \boldsymbol{n} D_\tau \boldsymbol{n}$ is the sum of the squares of the principal curvatures of $\partial\Omega$, hence $H^2 - Tr(\,{}^T D_\tau \boldsymbol{n} D_\tau \boldsymbol{n})$ is the sum of the products of each pair of principal curvatures, it is nonnegative when Ω is convex.

For a domain $\Omega \subset \mathbb{R}^d$, we consider its volume $|\Omega|$, its perimeter $P(\partial\Omega)$ and its Dirichlet energy $E(\Omega)$ defined as

$$E(\Omega) = -\frac{1}{2}\int_{\partial\Omega} |\nabla u_\Omega|^2,$$

where u_Ω is the solution of $-\Delta u = 1$ in $\mathrm{H}^1_0(\Omega)$ and λ_1 the first eigenvalue of the Dirichlet Laplace operator. The shape derivatives of these functionals are well known (see [13, Section 5.9.6]), and we notice briefly that we assume enough regularity on Ω so that those derivatives exist and are well-defined, therefore we do not discuss the shape differentiability of the considered functional which is also well-known in this smooth setting.

Lemma 2.2 (Expression of shape derivatives) *If Ω is C^2, one has, for any $\varphi \in C^\infty(\partial\Omega)$,*

$$\ell_1[\mathrm{Vol}](\Omega).\varphi = \int_{\partial\Omega} \varphi; \tag{2.1a}$$

$$\ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} H\varphi^2. \tag{2.1b}$$

$$\ell_1[P](\Omega).\varphi = \int_{\partial\Omega} H\varphi; \tag{2.1c}$$

$$\ell_2[P](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} |\nabla_\tau\, \varphi|^2 + \int_{\partial\Omega} \left[H^2 - Tr(\,{}^T D_\tau \boldsymbol{n} D_\tau \boldsymbol{n})\right]\varphi^2; \tag{2.1d}$$

$$\ell_1[E](\Omega).\varphi = -\frac{1}{2}\int_{\partial\Omega} (\partial_n u)^2 \varphi; \tag{2.1e}$$

$$\ell_2[E](\Omega).(\varphi,\varphi) = \langle -\partial_n u\ \varphi, \Lambda(-\partial_n u\ \varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} + \int_{\partial\Omega}\left[\partial_n u + \frac{1}{2}H(\partial_n u)^2\right]\varphi^2; \tag{2.1f}$$

$$\ell_1[\lambda_1](\Omega).\varphi = -\int_{\partial\Omega} (\partial_n v)^2 \varphi; \tag{2.1g}$$

$$\ell_2[\lambda_1](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} 2w(\varphi)\ \partial_n w(\varphi) + H(\partial_n v)^2\varphi^2; \tag{2.1h}$$

where $\Lambda:\ \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ is the Dirichlet-to-Neumann map defined as $\Lambda(\varphi) = -\partial_n V(\varphi)$ with $V(\varphi)$ is the solution of

$$-\Delta V(\varphi) = 0 \text{ in } \Omega, \quad V(\varphi) = -\varphi \text{ on } \partial\Omega, \tag{2.2}$$

and v is the associated eigenfunction solution in $\mathrm{H}^1_0(\Omega)$ of $-\Delta v = \lambda_1 v$ with $v > 0$ in Ω and $\|v\|_{\mathrm{L}^2(\Omega)} = 1$ and $w(\varphi)$ is the solution of

$$\left\{\begin{array}{rcl} -\Delta w(\varphi) & = & \lambda_1 w(\varphi) - v\displaystyle\int_{\partial\Omega} (\partial_n v)^2\varphi \text{ in } \Omega, \\ w(\varphi) & = & -\varphi\partial_n v \text{ on } \partial\Omega, \\ \displaystyle\int_\Omega v\ w(\varphi) & = & 0. \end{array}\right. \tag{2.3}$$

A fundamental fact for this work appears here in the expression of the shape hessian. Even if they are defined and derived for regular perturbations, they are naturally defined and continuous on different Sobolev spaces on $\partial\Omega$. The hessian of the perimeter is defined on $\mathrm{H}^1(\partial\Omega)$, the hessian of Dirichlet energy on $\mathrm{H}^{1/2}(\partial\Omega)$ while the hessian of the volume is defined on $\mathrm{L}^2(\partial\Omega)$ as expressed in the following continuity properties:

Lemma 2.3 (Continuity of shape Hessian) *If Ω is C^2, there is a constant $C > 0$ such that*

$$|\ell_2[P](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^1(\partial\Omega)}, \text{ and } |\ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{L}^2(\partial\Omega)},$$

$$|\ell_2[E](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)} \text{ and } |\ell_2[\lambda_1](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

Therefore, from this Lemma, it is natural to consider the extension of these bilinear forms to their space of continuity.

2.3 The case of balls

For the sequel of this work, let us explicit the shape derivatives of these functionals on the balls B_R. To explicit the derivatives of the Dirichlet energy E, we need to remark that $u(x) = (R^2 - |x|^2)/2d$ solves $-\Delta u = 1$ in $\mathrm{H}^1_0(B_R)$ and satisfies $\partial_n u = -R/d$ on ∂B_R. For λ_1, we recall that the eigenvalue and eigenfunction are

$$\lambda_1(B_R) = \frac{j^2_{d/2-1}}{R^2} \text{ associated to } v(x) \;=\; \alpha_d \; |x|^{1-d/2} \; J_{d/2-1}\left(\frac{j_{d/2-1}}{R} \;\; |x|\right),$$

where the normalization constant is defined as

$$\alpha_d = \left[|\partial B_1| \int_0^R r J^2_{d/2-1}\left(\frac{j_{d/2-1}}{R}\, r\right)\, dr\right]^{-1/2},$$

and where $j_{d/2-1}$ is the first zero of Bessel's function $J_{d/2-1}$. On the unit ball, the eigenfunction satisfies

$$\partial_n v = \sqrt{\frac{2}{P(B_1)}}\, j_{d/2-1} := \gamma_d, \text{ so that } \gamma_d^2 = \frac{2\lambda_1}{P(B_1)}; \tag{2.4}$$

from [14, p. 35]. We obtain the shape gradients:

$$\ell_1[\mathrm{Vol}](B_R).\varphi = \int_{\partial B_R} \varphi; \qquad \ell_1[P](B_R).\varphi = \frac{d-1}{R}\int_{\partial B_R} \varphi;$$
$$\ell_1[E](B_R).\varphi = -\frac{R^2}{2d^2}\int_{\partial B_R} \varphi; \qquad \ell_1[\lambda_1](B_R).\varphi = \;-\;\gamma_d^2 \int_{\partial B_R} \varphi.$$

Let us notice that these four shape gradients at balls are colinear. As a consequence, the balls are critical domains for the perimeter, λ_1 and Dirichlet energy under a fixed volume constraint since Euler-Lagrange equations are trivially satisfied, and also of λ_1 and Dirichlet energy under fixed volume constraint.

Let us turn our attention to the hessians. The value of $\ell_2[\lambda_1]$ is a bit more involved, so we deal with it in the next Lemma. For the other functionals, it is known that:

$$\ell_2[\mathrm{Vol}](B_R).(\varphi,\varphi) = \frac{d-1}{R}\int_{\partial B_R}\varphi^2; \tag{2.5a}$$

$$\ell_2[P](B_R).(\varphi,\varphi) = \int_{\partial B_R}|\nabla_\tau\,\varphi|^2 + \frac{(d-1)(d-2)}{R^2}\int_{\partial B_R}\varphi^2; \tag{2.5b}$$

$$\ell_2[E](B_R).(\varphi,\varphi) = \frac{R^2}{d^2}\langle\varphi,\Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} - \frac{d+1}{2d^2}R\int_{\partial B_R}\varphi^2. \tag{2.5c}$$

It is well known in the literature (see for example [7] for Vol, P and E) that on balls these quadratic forms are coercive on their natural space as stated in next lemma. Let us make this point precise by diagonalizing the Hessian. The useful tool to explicit the shape hessian under consideration is spherical harmonics defined as the restriction to the unit sphere of harmonic polynomials.

We recall here facts from [20, pages 139-141]. We let $\mathcal{H}_k$ denote the space of spherical harmonics of degree k. It is also the eigenspace of the Laplace-Beltrami operator on the unit sphere associated with the eigenvalue $-k(k+d-2)$. Its dimension is

$$d_k = \begin{pmatrix} d+k-1 \\ k \end{pmatrix} - \begin{pmatrix} d+k-3 \\ k-2 \end{pmatrix}.$$

Let $(Y^{k,l})_{1\le l\le d_k}$ be an orthonormal basis of $\mathcal{H}_k$ with respect to the $\mathrm{L}^2(\partial B_1)$ scalar product. The $(\mathcal{H}_k)_{k\in\mathbb{N}}$ spans a vector space dense in $\mathrm{L}^2(\partial B_1)$ and the family $(Y^{k,l})_{k\in\mathbb{N},1\le l\le d_k}$ is a Hilbert basis of $\mathrm{L}^2(\partial B_1)$. Hence, any function φ in $L^2(\partial B_1)$ can be decomposed as the Fourier series:

$$\varphi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)Y^{k,l}(x), \quad \text{for } |x|=1. \tag{2.6}$$

Then, by construction, the function u defined by

$$u(x) = \sum_{k=0}^{\infty}|x|^k\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)Y^{k,l}\left(\frac{x}{|x|}\right), \quad \text{for } |x|\le 1,$$

is harmonic in B_1 and satisfies $u=\varphi$ on ∂B_1. Moreover, the sequence of coefficients $\alpha_{k,l}$ characterizes the Sobolev regularity of φ: indeed $\varphi\in\mathrm{H}^s(\partial B_1)$ if and only if the sum $\sum_k(1+k^2)^s\sum_l|\alpha_{k,l}|^2$ converges. Let us now prove the following Lemma expressing the fact that the shape hessian of the volume, the perimeter, the Dirichlet energy and the first eigenvalue are diagonal on the basis of spherical harmonics.

Lemma 2.4 *Assume that φ is decomposed on the basis of spherical harmonics as in* (2.6)*, then*

$$\ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)\ \alpha_{k,l}(\varphi)^2, \tag{2.7}$$

$$\ell_2[E](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{1}{d^2}\ k\ -\frac{d+1}{2d^2}\right]\alpha_{k,l}(\varphi)^2, \tag{2.8}$$

$$\ell_2[P](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[k^2+(d-2)k+(d-1)(d-2)\right]\ \alpha_{k,l}(\varphi)^2, \tag{2.9}$$

$$\ell_2[\lambda_1](B_1)(\varphi,\varphi) = \gamma_d^2\left(3P(B_1)^2\alpha_{0,1}^2(\varphi)+\sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\left[k-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)\right). \tag{2.10}$$

where γ_d is the constant defined in (2.4)*.*

Proof. We decompose $\varphi \in \mathrm{L}^2(\partial B_1)$ on the spherical harmonics basis as

$$\varphi(x) = \sum_{k=0}^{\infty}\left(\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)Y^{k,l}(x)\right), \quad \text{for } |x|=1. \tag{2.11}$$

and let us express the various integrals arising in the shape hessian in terms of the spherical harmonics decomposition. First we check that

$$\int_{\partial B_1}\varphi^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

$$\int_{\partial B_1}|\nabla_\tau\ \varphi|^2 = -\int_{\partial B_1}\varphi\ \Delta_\tau\ \varphi = \sum_{k=0}^{\infty}k(k+d-2)\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

Then, we precise the term involving the Dirichlet-to-Neumann map that appears in the shape hessian of the Dirichlet energy. The series defining u is normally convergent inside B_1, we cannot directly differentiate with respect to r up to the boundary. Though, by Green formula, we have:

$$\begin{aligned}
\langle\varphi,\Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} &= \int_{\partial B_1}\varphi\partial_n u = \int_{B_1}|\nabla u|^2\\
&= \int_0^1\left(\int_{\partial B_r}\left((\partial_n u^2+|\nabla_\tau\ u|^2\right)d\sigma\right)dr = \int_0^1\left(\int_{\partial B_r}\left((\partial_n u^2-u\Delta_\tau\ u\right)d\sigma\right)dr\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\int_0^1 r^{d-1}\left[k^2r^{2(k-1)}+\frac{k(k+d-2)}{r^2}\ r^{2k}\right]dr\ \alpha_{k,l}(\varphi)^2\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{k^2}{2k+d-2}+\frac{k(k+d-2)}{2k+d-2}\right]\alpha_{k,l}(\varphi)^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}k\ \alpha_{k,l}(\varphi)^2.
\end{aligned}$$

We obtain $\ell_2[\mathrm{Vol}]$, $\ell_2[P]$ and $\ell_2[E]$ by gathering these elementary terms.

Let us now consider the case of the first eigenvalue. Again we decompose φ on the basis of spherical harmonics according to (2.11). The computation is given by D. Henry in [14, p. 35]. He uses a volume preserving deformation field V generating a family of diffeomorphisms T_t and he gets

$$\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{t=0} = \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\gamma_d^2\ \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(V_n)$$

where we have used the recurrence formula for Bessel function $J'_\nu(z)=(\nu/z)J_\nu(z)-J_{\nu+1}(z)$ to adapt his expression to our notations ([1, 9.1.27, p 361]). To deduce $\ell_2[\lambda_1]$ from his computation, we note that the deformations path T_t preserves the volume so its second order Taylor expansion $I+t\varphi+t^2/2\ \psi$ satisfies

$$0=\left(\frac{d^2}{dt^2}\mathrm{Vol}''(T_t(B_1)\right)_{t=0} = \ell_2[\mathrm{Vol}](\varphi,\varphi)+\ell_1[\mathrm{Vol}](\psi) \text{ hence } (d-1)a_{k,l}(\varphi)^2+a_{k,l}(\psi)=0.$$

Then, we get:

$$
\begin{aligned}
\ell_2[\lambda_1](\varphi,\varphi) &= \left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1)\right)_{t=0} - \ell_1[\lambda_1](\psi),\\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\gamma_d^2\ \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)+\gamma_d^2\sum_{k=1}^{\infty}\sum_{l=1}^{d_k}a_{k,l}(\psi),\\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\gamma_d^2\ \left[k-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi).
\end{aligned}
$$

It remains to compute the coefficient associated to the mode $k=0$. It suffices to consider the deformations as $T_t(x)=x+t\|Y^{0,1}\|x$ mapping the ball B_1 onto the ball of radius $1+tP(B_1)^{1/2}$. Since λ_1 is homogeneous of degree -2, we get $\lambda(t)=(1+tP(B_1)^{1/2})^{-2}\lambda_1(B_1)$ so that $\lambda''(0)=6P(B_1)\lambda_1(B_1)$.

□

3 Main Theorem

Let us precise here the suitable definitions of critical and stable domains for problems with volume constraint and translation invariant functionals:

Definition 3.1 *Let Ω be a shape and J a shape functional defined and twice Fréchet differentiable on a neighborhood of Ω^* for d_Θ.*

- *We say that Ω is a critical domain for J under volume constraint if*

$$
\forall\varphi\in C^\infty(\partial\Omega)\ \text{such that}\ \ell_1[\mathrm{Vol}](\Omega).\varphi=\int_{\partial\Omega}\varphi=0,\ \ \ell_1[J](\Omega).(\varphi)=0. \tag{3.1}
$$

  It is well-known that it is equivalent to the existence of $\mu\in\mathbb{R}$ such that $(\ell_1[J]-\mu\ell_1[\mathrm{Vol}])(\Omega)=0$ on $C^\infty(\partial\Omega)$; in that case, μ is called a Lagrange multiplier associated to J.

- *When Ω is a critical domain for J under volume constraint, we say that Ω is a stable shape for J under volume constraint and up to translations if*

$$
\forall\varphi\in T(\partial\Omega)\setminus\{0\},\ (\ell_2[J]-\mu\ell_2[\mathrm{Vol}])(\Omega).(\varphi,\varphi)>0 \tag{3.2}
$$

  where

$$
T(\partial\Omega):=\left\{\varphi\in H^s(\partial\Omega),\ \int_{\partial\Omega}\varphi=0,\ \int_{\partial\Omega}\varphi\,\overrightarrow{x}=\overrightarrow{0}\right\},
$$

  μ is the Lagrange multiplier associated to J and $s\geq 0$ is the lowest index so that $\ell_2(J)$ is continuous on $H^s(\partial\Omega)$.

Here is the main result of this paper:

Theorem 3.2 *Let Ω^* of class C^5 and J a shape functional, translation invariant, twice Fréchet differentiable on a neighborhood of Ω^* for d_Θ, such that*

- **Structural hypotheses:** *there exists $0\leq s_1<s_2\leq 1$ such that*

- $\ell_2[J](\Omega^*)$ *can be written* $\ell_2[J](\Omega^*) = \ell_m + \ell_r$ *satisfying* (1.6)
- J *satisfies* $C_{H^{s_2}}$.

- **Necessary optimality conditions:**
  - Ω^* *is a critical shape under volume constraint for* J,
  - Ω^* *is a stable shape for* J *under volume constraint and up to translations:*

Then Ω^* *is an* H^{s_2}**-stable local minimum of** J **in a** Θ**-neighborhood under volume constraint**, *that is to say there exists* $\eta > 0$ *and* $c = c(\eta)$ *such that* (1.8) *holds:*

$$\forall\, \Omega \text{ such that } d_\Theta(\Omega, \Omega^*) \leq \eta \text{ and } |\Omega| = |\Omega^*|, \quad J(\Omega) \geq J(\Omega^*) + c \inf \|h\|^2_{H^{s2}}.$$

where $h = h_{\Omega^*, \Omega+\tau}$ *is defined in* (1.3) *and the infimum is taken over* $\tau \in \mathbb{R}$.

Note that the minimization in $\tau \in \mathbb{R}^d$ is here to take into account the translation invariance of the functional.

3.1 About coercivity

Usually the coercivity property for the augmented Lagrangian has to be proved by hand on each specific example by studying the lower bound of the spectrum of the bilinear form ℓ_2 defined in Theorem 2.1 typically thanks to Lemma 2.4. Nevertheless, when ℓ_2 enjoys some structural property, coercivity can be more easily checked as a consequence of the following general lemma.

Lemma 3.3 *If the bilinear form* ℓ *can be written as* $\ell = \ell_m + \ell_r$ *where*

(H_1) *the main part* ℓ_m *is lower semicontinuous on* $\mathrm{H}^{s_2}(\partial\Omega)$ *and there exists a constant* $C > 0$ *such that* $\ell_m(\varphi, \varphi) \geq C|\varphi|^2_{\mathrm{H}^{s_2}}$ *when* $\varphi \in V$.

(H_2) *the remainder part* ℓ_r *is continuous on* $\mathrm{H}^{s_1}(\partial\Omega)$.

where $0 \leq s_1 < s_2 \leq 1$ *and* V *a vectorial subspace of* $\mathrm{H}^{s_2}(\partial\Omega)$, *closed for the weak convergence in* $\mathrm{H}^{s_2}(\partial\Omega)$. *Then the following propositions are equivalent:*

(i) $\ell(\varphi, \varphi) > 0$ *for any* $\varphi \in V \setminus \{0\}$.

(ii) $\exists \lambda > 0, \quad \ell(\varphi, \varphi) \geq \lambda \|\varphi\|^2_{\mathrm{H}^{s_1}}$ *for any* $\varphi \in V$.

(iii) $\exists \lambda > 0, \quad \ell(\varphi, \varphi) \geq \lambda \|\varphi\|^2_{\mathrm{H}^{s_2}}$ *for any* $\varphi \in V$.

Proof. Since the implications $(iii) \implies (ii)$ and $(ii) \implies (ii)$ are trivial, it suffices to prove $(i) \implies (iii)$. To that end, let $(\varphi_k)_k$ a minimizing sequence for the problem

$$\inf \left\{ \ell(\varphi, \varphi),\ \varphi \in V, \|\varphi\|_{\mathrm{H}^{s_2}(\partial\Omega)} = 1 \right\}.$$

Up to a subsequence, φ_k weakly converges in $\mathrm{H}^{s_2}(\partial\Omega)$ to some $\varphi_\infty \in V$. By the compactness of the embedding of $\mathrm{H}^{s_2}(\partial\Omega)$ into $\mathrm{H}^{s_1}(\partial\Omega)$, $\varphi_k \to \varphi_\infty$ in $\mathrm{H}^{s_1}(\partial\Omega)$ so that $\ell_r(\varphi_k, \varphi_k) \to \ell_r(\varphi_\infty, \varphi_\infty)$. We distinguish the cases $\varphi_\infty = 0$ or not. If $\varphi_\infty \neq 0$, $\lim \ell_m(\varphi_k, \varphi_k) \geq \ell_m(\varphi_\infty, \varphi_\infty)$ by the lower semi continuity of ℓ_m, so that $\lim \ell(\varphi_k, \varphi_k) \geq \ell(\varphi_\infty, \varphi_\infty) > 0$ by assumption (i). If $\varphi_\infty = 0$, then $\lim |\varphi_k|_{\mathrm{H}^1} = \lim \|\varphi_k\|_{\mathrm{H}^1} = 1$ and $\lim \ell(\varphi_k, \varphi_k) \geq C \lim |\varphi_k|_{\mathrm{H}^1} = C > 0$. $\square$

Remark 3.4 The equivalence between coercivity in $\mathrm{L}^2(\partial\Omega)$ and $\mathrm{H}^1(\partial\Omega)$ is already known in the context of stable minimal surface it appears in the work [12] of Grosse-Brauckmann.

Remark 3.5 When, one applies this lemma to a shape hessian, assumption (i) is not natural. Indeed, shape derivatives are defined for regular perturbations that are dense subsets of $\mathrm{H}^s(\partial\Omega)$ and one would have expect: $\ell(\varphi,\varphi) > 0$ in $\varphi \in C^2 \cap V \setminus \{0\}$. But, in that case, our proof is not valid since φ_∞ may not be smooth and therefore not admissible to test the positivity property. Therefore, the bilinear form ℓ has to be extended by continuity to the whole $\mathrm{H}^s(\partial\Omega)$ (see assumption (1.7) in Theorem 1.1 and (3.2) for Theorem 3.2). Notice that this extension is for free once the expression of the shape derivative has been computed as illustrated by Lemma 2.2.

The shape hessians of the model functionals admit such a splitting. The shape hessian of the perimeter can be written as $\ell_2[P] = \ell_m[P] + \ell_r[P]$ where

$$\ell_m[P](\varphi,\varphi) = \int_{\partial\Omega} |\nabla_\tau\, \varphi|^2 \text{ and } \ell_r[P](\varphi,\varphi) = \int_{\partial\Omega} \left[H^2 - Tr(\,{}^T D_\tau \boldsymbol{n} D_\tau \boldsymbol{n})\right]\varphi^2$$

satisfy $(H_1)-(H_2)$ with $s_1 = 0$ and $s_2 = 1$. The same holds for the Dirichlet energy and λ_1 with

$$\ell_m[E](\varphi,\varphi) = \langle -\partial_n u\varphi, \Lambda(-\partial_n u\varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} \text{ and } \ell_r[E](\varphi,\varphi) = \int_{\partial\Omega}\left[\partial_n u + \frac{1}{2}H(\partial_n u)^2\right]\varphi^2;$$

$$\ell_m[\lambda_1](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} 2w(\varphi)\ \partial_n w(\varphi) \text{ and } \ell_r[\lambda_1](\varphi,\varphi) = \int_{\partial\Omega} H(\partial_n v)^2\varphi^2;$$

satisfy $(H_1)-(H_2)$ with $s_1 = 0$ and $s_2 = 1/2$.

However, this splitting property is not universal: shape functionals used for domain reconstruction from boundary measurements are such that (i) holds while (ii) and (iii) are false (see [3], [4]). The general situation in the general class of such inverse problems is then: for a reconstruction function J (for example the least square fitting to data), the Riesz operator corresponding to the shape Hessian $\ell_2[J]$ at a critical domain is compact. This means, roughly speaking, that, in a neighborhood of the critical domain (i.e. for t small), J behaves as its second order approximation and one cannot expect an estimate of the kind $J(\Omega_t) - J(\Omega_0) \geq ct^2$ with a constant c uniform in the deformation direction. This explains also why regularization is required in the numerical treatment of this type of problem.

3.2 About Condition C_{H^s}

This condition is the main ingredient to overcome the two norms discrepancy problem that appears when one wants to use a Taylor formula to use second order information: the norm where coercivity holds is strictly weaker than the norm of differentiability. As a consequence, Taylor formula (1.4) is not sufficient since the second order reminder small in the norm of differentiability can be larger than the positive second order term. One has to use the integral form of the remainder and to that end build a path connecting domains and estimate the second order derivative of the shape function along that path.

Condition C_{H^s} expresses the continuity property of that second order derivative so that its sign at the original shape is preserved along the path. This continuity is a keystone in proving stability with second order based methods, it has be proven on a lot of examples: first in dimension two for Dirichlet energy in [8], then for general functions in any dimension in [7], and then in [2]. For the completeness of the presentation, we recall now the leading steps of the approach of [8, 7] where Condition C_{H^0} is established for the volume, C_{H^1} for the perimeter and $C_{\mathrm{H}^{1/2}}$ for Dirichlet energy. Note that Condition $C_{\mathrm{H}^{1/2}}$ is also established for the drag in a Stokes flow in [6] with the same strategy.

The first step is to build the path connecting a domain Ω to some perturbed domain $(I+\theta)(\Omega)$ with normal deformations. The idea is that the boundary of a perturbed domain $(I+\theta)(\Omega)$ close Ω for d_Θ is in fact a graph over $\partial\Omega$: the boundary of $(I+\theta)(\Omega)$ is parametrized as $\{x+h(x)\boldsymbol{n}(x), x\in\partial\Omega\}$ where $h=h_{\Omega,(I+\theta)(\Omega)}$ is a real-valued function defined on $\partial\Omega$. Then, we consider the path (Ω_t) defined by the boundary $\partial\Omega_t=\{x+th(x)\boldsymbol{n}(x); x\in\partial\Omega\}$. This corresponds to the deformation of Ω under the flow T_t of the vector field $V_\theta(x) = h(p_{\partial\Omega}(x))\boldsymbol{n}(x)$ in the neighborhood of $\partial\Omega$ and extended in the complement.

The second step is to compute the derivatives of the function along the path. The use of the speed method simplifies the computation of that second order derivative for all $t\in[0,1]$.

The third step is to obtain control on the variations of geometrical quantities along the previously defined path given in next Lemma proven in [7]:

Lemma 3.6 *There is a constant $C>0$ depending on Ω such that*

- *the surface jacobian $J(t):=\det DT_t/\|({}^tDT_t^{-1})\boldsymbol{n}\|$ satisfies*
$$\|J(t)-1\|_{C^1(\partial\Omega)} \leq C\|T_t-I\|_\Theta,\ \forall t\in[0,1]; \tag{3.3}$$
- *the normal field $\boldsymbol{n}_t$ to $\partial\Omega_t$ satisfies*
$$\|\boldsymbol{n}_t\circ T_t - \boldsymbol{n}\|_{\mathcal{C}^1(\partial\Omega)} \leq C\|T_t-I\|_\Theta,\ \forall t\in[0,1]. \tag{3.4}$$
- *Set $\varphi_\theta := V_\theta\cdot\tilde{\boldsymbol{n}}$ where $\tilde{\boldsymbol{n}}$ is a unitary extension of $\boldsymbol{n}$ to a neighborhood of Ω, then for all $t\in[0,1]$:*
$$\|\varphi_\theta\circ T_t - \varphi_\theta\|_{\mathrm{L}^2(\partial\Omega)} \leq C\,\|\varphi_\theta\|_{\mathrm{L}^2(\partial\Omega)}\,\|T_t-I\|_\Theta, \tag{3.5}$$
$$\|\varphi_\theta\circ T_t - \varphi_\theta\|_{\mathrm{H}^{1/2}(\partial\Omega)} \leq C\,\|\varphi_\theta\|_{\mathrm{H}^{1/2}(\partial\Omega)}\,\|T_t-I\|_\Theta. \tag{3.6}$$

The fourth step is to obtain control on the variations of derivatives of the state functions along the previously defined path. Obviously, this step is problem dependent. The general idea is to transport the boundary value problem defined on $T_t(\Omega)$ on the original domain Ω, this provides a new boundary value problem of the type $L(t)v=f\circ T_t^{-1}$ on Ω. Using estimates, uniform in θ and t, on the coefficients of the variable coefficient operator $L(t)$, one can apply uniform *a priori* estimates up to the boundary in order to obtain uniform control of the variations of the derivatives of the state.

3.3 Proof of Theorem 1.1

We are now in position to prove Theorem 1.1 corresponding to the unconstrained case. Let Ω^* be a domain satisfying the assumption of Theorem 1.1. Let $\eta>0$ and let Ω be a domain in a ball centered in Ω^* of radius η for d_Θ. Then, there exists h such that the boundary of Ω is the set $\{x+h(x)\boldsymbol{n}(x),\ x\in\partial\Omega^*\}$. Consider the path $(\Omega_t)_{t\in[0,1]}$ defined in (1.5), j the restriction of J to the path Ω_t. We write Taylor formula along this path:

$$\begin{aligned}J(\Omega)-J(\Omega^*) &= \int_0^1 j''(t)(1-t)dt\\ &= \frac{1}{2}j''(0)+\int_0^1[j''(t)-j''(0)](1-t)dt \geq \frac{1}{2}j''(0)-\int_0^1|j''(t)-j''(0)|(1-t)dt.\end{aligned}$$

By Lemma 3.3, there is a constant $\lambda > 0$ such that

$$\ell_2[J](\Omega^*).(h,h) \geq \lambda \|h\|^2_{\mathrm{H}^{s_2}}.$$

Applying the $C_{\mathrm{H}^{s_2}}$ assumption, we obtain that for η small enough,

$$\left|j''(t) - j''(0)\right| \leq \frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}, \;\; \forall t \in [0,1], \text{ then } J(\Omega) - J(\Omega^*) \geq \frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}.$$

3.4 Proof of Theorem 3.2

We denote μ the Lagrange multiplier associated to J. Therefore we consider $J_\mu = J - \mu\mathrm{Vol}$ and Ω^* satisfies $J'_\mu(\Omega^*) = 0$.

Step 1: Stability under volume and barycenter constraint: The bilinear form ℓ_2 associated to the second order derivative of the Lagragian J_μ is $\ell_2[J] - \mu\ell_2[\mathrm{Vol}]$. Under the structural hypotheses on $\ell_2[J]$, we can applied Lemma 3.3 to $\ell[J_\mu]$. We introduce the constants c_1, c_2, c_3 and $c_4 > 0$ such that

$$\forall \varphi \in C^\infty(\partial\Omega^*), \;\; |\ell_m[J].(\varphi,\varphi)| \geq c_1|\varphi|^2_{\mathrm{H}^{s_1}} \quad |\ell_r[J].(\varphi,\varphi)| \leq c_2\|\varphi\|^2_{\mathrm{H}^{s_1}}, \quad |\ell_2[\mathrm{Vol}].(\varphi,\varphi)| \leq c_3\|\varphi\|^2_{\mathrm{L}^2},$$

$$\forall \varphi \in T(\partial\Omega^*) \cap C^\infty(\partial\Omega^*),, \quad \ell_2[J - \mu\mathrm{Vol}](\Omega^*).(\varphi,\varphi) \geq c_4\|\varphi\|^2_{\mathrm{H}^{s_2}}, \tag{3.7}$$

where we have used that $\ell_2[\mathrm{Vol}](\Omega^*)$ is continuous in the L^2-norm, and therefore in the H^{s_2}-norm as well.

Step 2: Stability without constraint: In order to deal with the volume constraint and the invariance with respect to translations, we use an idea of [21, 12] by considering

$$J_{\mu,C} = J - \mu\mathrm{Vol} + C\,(\mathrm{Vol} - V_0)^2 + C\,\|\mathrm{Bar}(\Omega) - \mathrm{Bar}(\Omega^*)\|^2,$$

where $\mathrm{Bar}(\Omega) := \int_\Omega x$ and $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^d$. The shape Ω^* still satisfies $J'_{\mu,C}(\Omega^*) = 0$. We claim that Ω^* is an H^{s_2}-strictly stable shape for $J_{\mu,C}$ on the entire space $C^\infty(\partial\Omega^*)$ when C is big enough: there is a constant $\lambda > 0$ such that for all φ in H^{s_2},

$$\ell_2[J_{\mu,C}](\Omega^*).(\varphi,\varphi) \geq \lambda\|\varphi\|^2_{\mathrm{H}^{s_2}}.$$

Indeed, if it was not the case, we would have the existence of $\varphi_n \in C^\infty(\partial\Omega^*)$ such that

$$\ell_2[J_{\mu,n}](\Omega^*).(\varphi_n,\varphi_n) \leq 0. \tag{3.8}$$

According to the structure of the shape hessian of J_γ, this leads to

$$c_1|\varphi_n|^2_{\mathrm{H}^{s_2}} - c_2\|\varphi_n\|^2_{\mathrm{H}^{s_1}} - |\mu|c_3\|\varphi_n\|^2_{\mathrm{H}^{s_2}} + 2n\left(\int \varphi_n\right)^2 + 2n\left\|\int_{\partial\Omega^*} \varphi_n x\right\|^2 \leq 0. \tag{3.9}$$

Assuming by homogeneity that $\|\varphi_n\|_{\mathrm{H}^{s_2}} = 1$ for every n, and using the compactness of $\mathrm{H}^{s_2}(\partial\Omega^*)$ in $\mathrm{H}^{s_1}(\partial\Omega^*)$, we have, up to a subsequence, that φ_n converges weakly in H^{s_2} and strongly in H^{s_1} and L^2. Therefore, (3.9) implies first that $2n[\mathrm{Vol}'(\varphi_n)^2 + \mathrm{Bar}'(\varphi_n)^2]$ is bounded, then that $\varphi \in T(\partial\Omega^*)$ that is

$$\int_{\partial\Omega^*} \varphi = 0 \text{ and } \int_{\partial\Omega^*} \varphi x = 0,$$

where φ is the limit of (φ_n) and then the semi-lower continuity assumption (1.6) implies

$$\ell_2[J_\mu](\Omega^*).(\varphi,\varphi) \leq 0, \quad \text{ with } \|\varphi\|_{\mathrm{H}^{s_1}} = 1$$

which contradicts (3.7), since $\varphi \neq 0$.

Step 3: Taylor expansion: We consider now Ω close to Ω^* for d_Θ and introduce $h = h_{\Omega^*,\Omega}$ as in (1.3), and proceed as in the unconstrained case using Lemma 3.3 and hypothesis $C_{\mathrm{H}^{s_2}}$ to prove

$$\begin{aligned} J_{\mu,C}(\Omega) - J_{\mu,C}(\Omega^*) &= \frac{1}{2}\frac{d^2}{ds^2}J_{\mu,C}(\Omega_s)_{|s=0} + \int_0^1 (1-t)\left[\frac{d^2}{ds^2}J_{\mu,C}(\Omega_s)_{|s=t} - \frac{d^2}{ds^2}J_{\mu,C}(\Omega_s)_{|s=0}\right] dt \\ &\geq \frac{\lambda}{2}\|h\|^2_{\mathrm{H}^{s_2}} - c\omega(\eta)\|h\|^2_{\mathrm{H}^{s_2}} \geq \frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}, \end{aligned} \tag{3.10}$$

the last inequality holds if we assume that η is small enough. Writing this inequality in particular for shapes Ω of volume V_0 and having the same barycenter as Ω^*,

$$J_\mu(\Omega) - J_\mu(\Omega^*) \geq \frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}.$$

We conclude using the invariance of J_μ with translations. $\square$

3.5 Applications

Combining the general Theorem 3.2 to the computations of shape derivatives from Section 2.1, we easily obtain the following:

Proposition 3.7 *Let $V_0 \in (0,\infty)$, and B a ball of volume V_0. Then there exists $\gamma_0 \in (0,\infty)$ such that for every $\gamma \in [-\gamma_0,\infty)$, and every φ in $C^\infty(\partial\Omega,\mathbb{R}) \cap T(\partial\Omega)$:*

$$\ell_2[P+\gamma E](B)(\varphi,\varphi) > 0, \quad \ell_2[P+\gamma\lambda_1](B)(\varphi,\varphi) > 0,$$

$$\ell_2[E+\gamma\lambda_1](B)(\varphi,\varphi) > 0, \quad \ell_2[\lambda_1+\gamma E](B)(\varphi,\varphi) > 0.$$

Therefore, there exists $\eta = \eta(\gamma) > 0$ and $c = c(\gamma) > 0$ such that for every $\Omega \in \mathcal{V}_\eta := \{\Omega', d_\Theta(\Omega', B) < \eta\}$ having the same barycenter as B,

$$(P+\gamma E)(\Omega) \;\geq\; (P+\gamma E)(B) + c\|h\|^2_{H^1}, \quad (P+\gamma\lambda_1)(\Omega) \;\geq\; (P+\gamma\lambda_1)(B) + c\|h\|^2_{H^1}$$

$$(E+\gamma\lambda_1)(\Omega) \;\geq\; (E+\gamma\lambda_1)(B) + c\|h\|^2_{H^{1/2}}, \quad (\lambda_1+\gamma E)(\Omega) \;\geq\; (\lambda_1+\gamma E)(B) + c\|h\|^2_{H^{1/2}},$$

where $h = h_{B,\Omega}$ is such that $\partial\Omega = \{x + h(x)\boldsymbol{n}(x), x \in \partial B\}$.

Proof of Proposition 3.7: It suffices to prove that Theorem 3.2 can be applied to $\Omega^* = B$ and

$$(F_1,F_2) \in \{(P,E),(P,\lambda_1),(E,\lambda_1),(\lambda_1,E)\}.$$

It is explained in Section 3.1 that (P,E,λ_1) satisfies the structural hypotheses, and in Lemma 2.4 that the ball is stable in $T(\partial B)$. $\square$

Corollary 3.8 *With the same notations as in Proposition 3.7, we have, with $\eta_0 = \eta(\gamma_0)$:*

$$\forall \Omega \in \mathcal{V}_{\eta_0}, \quad \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \geq \gamma_0, \qquad \frac{P(\Omega) - P(B)}{\lambda_1(\Omega) - \lambda_1(B)} \geq \gamma_0$$

$$\gamma_0 \leq \frac{\lambda_1(\Omega) - \lambda_1(B)}{E(\Omega) - E(B)} \leq \gamma_0^{-1}.$$

Remark 3.9 In [16], the second inequality in Corollary 3.8 is also investigated, and the author computes the optimal value γ_0 when the size of the neighborhood $\mathcal{V}_{\eta_0}$ goes to 0. We also refer to [18] for some result of this kind.
To the contrary to the last two-sided inequality, it is not possible to bound the first two ratio from above. Indeed, for every $\gamma \in (0, \infty)$, there exists $\Omega_\gamma = (Id + \theta_\gamma)(B)$ of class C^∞ such that

$$|\Omega_\gamma| = |B|, \ \ \|\theta_\gamma\|_\Theta \leq \gamma^{-1} \text{ and } \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} > \gamma.$$

3.6 Counterexample for non smooth perturbations

Let us consider P the perimeter and E the Dirichlet energy with second right hand side 1 (defined in (1.9)), and $\Omega^* = B$ a ball of volume V_0. We have seen in Proposition 3.7 that there is a real number $\gamma_0 \in (0, \infty)$ such that for every $\gamma \in (-\gamma_0, \infty)$, B is a stable local minimum for $P + \gamma E$.

For $\gamma \geq 0$ this is not very surprising: since the ball minimizes E among sets of given volume, it is enough to prove that the ball is a stable minimizer for the perimeter, which goes back to Fuglede [10]. Moreover, it has been proven that B is an L^1-stable minimizer of the perimeter in a L^1-neighborhood of the ball, that is to say there exists $\eta > 0$ such that

$$\forall \ \Omega \text{ such that } |\Omega \Delta B| \leq \eta, |\Omega| = |B|, \quad P(\Omega) - P(B) \geq c|\Omega \Delta B|^2 \tag{3.11}$$

where we assume the barycenter of Ω to be the same as the one of B (actually this is no longer local, this inequality can be stated for every set Ω of finite perimeter, see [11]). Therefore a similar inequality is valid for $P + \gamma E$ if $\gamma \geq 0$.

However, *for $\gamma < 0$, the fact that the ball is a local minimizer is no longer trivial*, there is a competition between the minimization of the perimeter and maximization the Dirichlet energy. Though if the coefficient in E is small enough, our result state that B is still a local minimizer in a Θ-neighborhood. Nevertheless, in that case B is no longer a local minimizer in a L^1-neighborhood. In other words, for every $\gamma < 0$ and any $\varepsilon > 0$ one can find Ω_ε such that

$$d_{L^1}(\Omega_\varepsilon, B) < \varepsilon, \quad |\Omega_\varepsilon| = |B|, \quad \text{ and } \quad (P + \gamma E)(\Omega_\varepsilon) < (P + \gamma E)(B).$$

To prove this, we use the idea of topological derivative, it is well known that if one consider a small hole of size ε in the interior of a fixed shape. The energy will change at order ε^{d-2} if $d \geq 3$ and $1/\log(\varepsilon)$ if $d = 2$, which is strictly bigger than the change of perimeter which is of order ε^{d-1}, and therefore will strictly decrease the energy $P + \gamma E$ when $\gamma < 0$.

We compute here explicitly these estimates when the hole is at the center of the ball: let us consider a fixed ball $B_1 = B(0, 1)$ of radius 1 (to simplify the computations) and define $\Omega_\varepsilon = B_1 \setminus B(0, \varepsilon)$ an annulus. Using that $\Delta u = \partial_{rr} u + \frac{d-1}{r} \partial_r u$ when u is radial, the state function is:

$$u_{\Omega_\varepsilon}(r) = \frac{(\varepsilon^{d-2} - \varepsilon^d) r^{2-d} + \varepsilon^d - 1}{2d(\varepsilon^{d-2} - 1)} - \frac{r^2}{2d}, \ \text{ if } d \geq 3$$

$$u_{\Omega_\varepsilon}(r) = \frac{1-\varepsilon^2}{-4\log(\varepsilon)}\log(r) + \frac{1-r^2}{4}, \quad \text{if } d=2$$

and therefore

$$\begin{aligned}
\text{if } d\geq 3,\ \ E(\Omega_\varepsilon) &= -\frac{1}{2}\int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{d(1-\varepsilon^2)^2\varepsilon^{d-2} - 2(1-\varepsilon^d)^2}{8d^2(1-\varepsilon^{d-2})} + \frac{1-\varepsilon^{d+2}}{4d(d+2)}\right] P(B_1)\\
&= \left[-\frac{1}{2d^2(d+2)} + \frac{d-2}{8d^2}\varepsilon^{d-2} + o(\varepsilon^{d-2})\right] P(B_1),
\end{aligned}$$

$$\begin{aligned}
\text{if } d=2,\ \ E(\Omega_\varepsilon) &= -\frac{1}{2}\int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{(1-\varepsilon^2)}{-8\log(\varepsilon)}(1-\varepsilon^2(1-2\log(\varepsilon))) - \frac{1}{16}(1-\varepsilon^2+\frac{\varepsilon^4}{2})\right] P(B_{1B_1})\\
&= \left[-\frac{1}{16} - \frac{1}{8\log(\varepsilon)} + o\left(\frac{1}{\log(\varepsilon)}\right)\right] P(B_1).
\end{aligned}$$

We now define $\widetilde{\Omega_\varepsilon} = \mu_\varepsilon \Omega_\varepsilon$ where $\mu_\varepsilon = (1-\varepsilon^d)^{-1/d}$ so that

$$|\widetilde{\Omega_\varepsilon}| = |B_1|, \quad P(\widetilde{\Omega_\varepsilon}) - P(B_1) = \left[\mu_\varepsilon^{d-1}(1+\varepsilon^{d-1}) - 1\right] P(B_1) \sim_{\varepsilon\to 0} \varepsilon^{d-1}P(B_1)$$

$$E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon\to 0} \frac{(d-2)P(B_1)}{8d^2}\varepsilon^{d-2} > 0, \quad \text{if } d\geq 3, \quad E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon\to 0} \frac{P(B_1)}{-8\log(\varepsilon)} > 0, \quad \text{if } d=2$$

so that in both cases, for any nonpositive γ, $(P+\gamma E)(\Omega_\varepsilon) - (P+\gamma E)(B_1) < 0$ for small ε.

4 Explicit constants

In this section, we are interested in computing explicit numbers γ such that the inequalities of Proposition 3.7 holds. To simplify the expressions, we restrict ourselves to the case of the unit ball.

Proposition 4.1 *Using notations of Proposition 3.7 and γ_d defined in (2.4), under the constraint* $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B_1)$

(i) *if $\gamma > -(d+1)d^2$, then B_1 is a local strict minimizer of $P+\gamma E$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (P+\gamma E)(\Omega) \ \geq \ (P+\gamma E)(B).$$

Moreover, when $\gamma = -(d+1)d^2$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -(d+1)d^2$, the ball is a saddle shape for $P+\gamma E$.

(ii) *if $\gamma > -\dfrac{d(d+1)}{\gamma_d^2(d+j_{d/2-1}^2)}$, then B_1 is a local strict minimizer of $P+\gamma\lambda_1$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (P+\gamma \lambda_1)(\Omega) \ \geq \ (P+\gamma \lambda_1)(B);$$

Moreover, when $\gamma = -\dfrac{d(d+1)}{\gamma_d^2(d+j_{d/2-1}^2)}$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -\dfrac{d(d+1)}{\gamma_d^2(d+j_{d/2-1}^2)}$, the ball is a saddle shape for $P+\gamma\lambda_1$.

(iii) *if* $\gamma > -\dfrac{1}{d^2(d+1)\gamma_d^2}$*, then* B_1 *is a local strict minimizer of* $E + \gamma\lambda_1$*: there exists* $\eta = \eta(\gamma) > 0$ *such that*

$$\forall \Omega \in \mathcal{V}_\eta, (E + \gamma\lambda_1)(\Omega) \;\geq\; (E + \gamma\lambda_1)(B);$$

(iv) *if* $\gamma > -\gamma_d^2 d^2$*, then* B_1 *is a local strict minimizer of* $\lambda_1 + \gamma E$*: there exists* $\eta = \eta(\gamma) > 0$ *such that*

$$\forall \Omega \in \mathcal{V}_\eta, (\lambda_1 + \gamma E)(\Omega) \;\geq\; (\lambda_1 + \gamma E)(B).$$

Note that the additional term $\|h\|^2_{H^{s_2}}$ can be added in the former inequalities with $s_2 = 1$ for the cases (i)-(ii) and with $s_2 = 1/2$ for the cases (iii)-(iv).

Remark 4.2 In the cases (iii) and (iv), the constants we compute are not optimal, in particular we do not claim the ball is a saddle point once we go beyond the computed value. Though it is possible to compute the optimal value, one just need to compute explicitly the value of $\sup_{k\geq 2} \tau'_k$ and $\sup_{k\geq 2} \tau''_k$ (see the notations in the proof below) as it is done in the cases (i) and (ii). As it is seen in the second case handled by Nitsch in [16], this computation can be rather technical.

Proof of Proposition 4.1:

Proof of (i): We first compute the Lagrange multiplier $\mu(t)$ associated to the volume constraint at B_1: it is defined as $\ell_1[P + tE) + \mu(t)\mathrm{Vol}] = 0$ that is from the expression of the shape gradients of Vol, P and E:

$$\mu(t) = \frac{1}{2d^2}\, t \;-(d-1).$$

Let us now turn our attention to hessian of the function $P + tE + \mu\mathrm{Vol}$ on the balls B_1. As a consequence of Lemma 2.4, the shape hessian of the lagrangian $P + tE + \lambda(t)\mathrm{Vol}$ at balls is

$$\ell_2[P + tE + \mu(t)\mathrm{Vol}](B_1).(\varphi, \varphi) = \sum_{k=0}^{\infty} c_k \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = k^2 + \left[(d-2) + \frac{1}{d^2}\, t\right]\; k - \left[(d-1) + \frac{1}{d^2}\, t\right] = (k-1)\left[k + (d-1) + \frac{1}{d^2}\, t\right].$$

Therefore, the hessian of the Lagrangian $\ell_2[P + tE + \mu(t)\mathrm{Vol}](B_1)$ is coercive in $\mathrm{H}^1(\partial B_1)$ when t solves the inequalities

$$k + (d-1) + \frac{1}{d^2}\, t > 0$$

for all $k \geq 2$. Of course, it suffices to solves that inequality in the special case $k = 2$ that provides $t > -(d+1)d^2$.

Proof of (ii): Notice that the case $t \geq 0$ is well known so we consider the case where $t < 0$. We compute the Lagrange multiplier $\mu(t)$ associated to the volume at B_1 defined by $\ell_1[P + t\lambda_1) + \mu(t)\mathrm{Vol}] = 0$ that is from the expression of the shape gradient of the volume, the perimeter and λ_1:

$$\mu(t) = \gamma_d^2\, t \;-(d-1).$$

Let us now turn our attention to the hessian of the Lagragian $P+tE+\mu(t)\mathrm{Vol}$ on the balls B_1:

$$\ell_2[P+t\lambda_1+\mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi)=\sum_{k=0}^{\infty}c_k(t)\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t)=k^2+(d-2+t\gamma_d^2)k-(d-1)+t\gamma_d^2\left[d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right].$$

We introduce the sequences $a_k=J_{k-1+d/2}(j_{d/2-1})$ and $b_k=a_{k+1}/a_k$ so that:

$$c_k(t)=k^2+(d-2)k-(d-1)+t\gamma_d^2\left[k+d-1-j_{d/2-1}b_k\right].$$

One should have $c_1(t)=0$ for any t, as known for the invariance by translations of all the involved functions, we can attest this once we describe how one can compute the numbers b_k, see below. For a given integer $k\geq 2$, $c_k(t)>0$ holds when $t>\tau_k$ defined as

$$\tau_k=-\frac{(k-1)(k+d-1)}{\gamma_d^2(k+d-1-j_{d/2-1}b_k)}.$$

In order to obtain to find the optimal value of t so that these inequalities are satisfied for every $k\geq 2$, we need to compute the supremum of $\{\tau_k,k\geq 2\}$. It is proven by Nitsch in [16, p. 332, proof of Lemma 2.3] that for all $k\geq 2,\tau_k\leq\tau_2$. We describe here how one can obtain a more explicit version of τ_2: from the recurrence formula for Bessel function ([1, 9.1.27, p 361])

$$(2\nu/z)J_\nu(z)=J_{\nu-1}(z)+J_{\nu+1}(z)$$

applied to $\nu=k-1+d/2$ and $z=j_{d/2-1}$, the sequences a_k and b_k satisfy the recurrence property

$$a_{k+1}=\frac{2(k-1)+d}{j_{d/2-1}}a_k-a_{k-1}\text{ and }b_{k+1}=\frac{2(k-1)+d}{j_{d/2-1}}-\frac{1}{b_k}$$

with the initial terms $a_0=0$ and $a_1=J_{d/2}(j_{d/2-1})$ so that $b_1=a_2/a_1=d/j_{d/2-1}$. Therefore, we have:

$$b_2=\frac{d}{j_{d/2-1}}-\frac{j_{d/2-1}}{d}=\frac{d^2-j_{d/2-1}^2}{dj_{d/2-1}}$$

and as a consequence, we obtain that

$$\tau_2=-\frac{d(d+1)}{\gamma_d^2(d+j_{d/2-1}^2)}.$$

Proof of (iii): The Lagrange multiplier is $\mu(t)=(1/d^2)+t\gamma_d^2$. The Hessian of the Lagrangian is

$$\ell_2[E+t\lambda_1+\mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi)=\sum_{k=0}^{\infty}c_k(t)\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t)=\left(\frac{1}{d^2}+t\gamma_d^2\right)k-\frac{1}{d^2}+t\gamma_d^2\left[d-1-j_{d/2-1}b_k\right].$$

Again $c_1(t) = 0$ and $c_k(t) > 0$ if and only if

$$t > \tau'_k = -\frac{k-1}{d^2\gamma_d^2(k+d-1-j_{d/2-1}b_k)}.$$

Using that $b_1 \geq b_k > 0$, we obtain

$$\tau'_k < -\frac{1}{d^2\gamma_d^2}\,\frac{k-1}{k+d-1} = -\frac{1}{d^2\gamma_d^2}\,\left(1-\frac{d}{k+d-1}\right) \leq -\frac{1}{d^2(d+1)\gamma_d^2}.$$

Therefore, if $t > -\dfrac{1}{d^2(d+1)\gamma_d^2}$ then for any $k \geq 2$, $t > \tau'_k$, which leads to the result.

Proof of (iv): The Lagrange multiplier is $\mu(t) = (t/d^2) + \gamma_d^2$. The Hessian of the Lagrangian is

$$\ell_2[\lambda_1 + tE + \mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty} c_k(t) \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = \left(\frac{t}{d^2} + \gamma_d^2\right) k - \frac{t}{d^2} + \gamma_d^2\left[d-1-j_{d/2-1}b_k\right].$$

We check $c_1(t) = 0$ and $c_k(t) > 0$ if and only if

$$t > \tau''_k = -\gamma_d^2 d^2\,\left(1 + \frac{d - j_{d/2-1}b_k}{k-1}\right).$$

Using that $b_1 \geq b_k > 0$, we obtain

$$\tau_k \leq -\gamma_d^2 d^2,$$

and therefore, if $t > -\gamma_d^2 d^2$ then for any $k \geq 2$, $t > \tau''_k$, which leads to the result.

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