Stability in shape optimization with second variation

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Abstract

We are interested in the question of stability in the field of shape optimization. We focus on the strategy using second order shape derivative. More precisely, we identify structural hypotheses on the hessian of the considered shape functions, so that critical stable domains (i.e. such that the first order derivative vanishes and the second order one is positive) are local minima for smooth perturbations. These conditions are quite general and are satisfied by a lot of classical functionals, involving the perimeter, the Dirichlet energy or the first Laplace-Dirichlet eigenvalue. We also explain how we can easily deal with volume constraint and translation invariance of the functionals. As an application, we retrieve or improve previous results from the existing literature, and provide new local isoperimetric inequalities. We finally test the sharpness of our hypotheses by giving counterexamples of critical stable domains that are not local minima.

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1 Introduction

In this paper, we are interested in the question of stability in the field of shape optimization. More precisely, given $J : \mathcal{A} \to \mathbb{R}$ defined on $\mathcal{A} \subset \{\Omega$ smooth enough open sets in $\mathbb{R}^d\}$, we consider the optimization problem

$$\min \{J(\Omega), \;\; \Omega \in \mathcal{A}\}, \tag{1.1}$$

and we ask the following question:

if $\Omega^* \in \mathcal{A}$ is a critical domain satisfying a stability condition (that is to say a strict second order optimality condition), can we conclude that Ω^* is a strict local minimum for (1.1) *in the sense that*

$$J(\Omega) - J(\Omega^*) \geq c d_1(\Omega, \Omega^*)^2, \qquad \textit{for every } \Omega \in \mathcal{V}(\Omega^*) \tag{1.2}$$

where $c \in (0, \infty)$, d_1 is a distance among sets, and $\mathcal{V}(\Omega^*) = \{\Omega, d_2(\Omega, \Omega^*) < \eta\}$ is a neighborhood of Ω^*, relying on a (possibly different) distance d_2?

(Note that the word distance is used here and in the rest of the paper as an intuitive notion here, asserting that Ω is far or close from the fixed shape Ω^*, and do not refer in general to the formal mathematical notion of distance).

Origin of the question:

For example in [38], the following terminology is used: the property that a critical point x has a positive second order derivative is called *linear*-stability, and implies that $t \mapsto f(x + ty)$ has a minimum at $t = 0$ for every y, while *nonlinear*-stability requires that $f(x)$ is less that $f(z)$ for any z close to x. It is classical that, when dealing with infinitely dimensional parameters, these two notions do not coincide in general.

In the framework of shapes, this question has been raised in different settings, and its answer has sometimes been mistakenly considered as easily valid: for example, in the context of stable constant mean curvature surfaces, literature has focused for a while on giving sufficient conditions so that (linear)-stability would occur, without proving that it actually implied local minimality. This point was raised by Finn in [18], and some answers followed quickly, see [25, 38, 39], though in the particular case of the ball and the isoperimetric problem, the difficulty was already handled by Fuglede in [19]. In the context of shape functionals involving PDE, the issue was raised by Descloux in [15] and a first solution was given in [12, 11].

Quantitative isoperimetric inequalities: different strategies

During the last decade, starting with [20], this type of question gained interest in the community of isoperimetric inequalities and shape optimization, in particular three main methods were developed in a quite extensive literature, in order to get a stability result of the form (1.2) for the most classical problems (1.1):

- Symmetrization technique,
- Mass transportation approach,
- Second order shape derivative approach.

As an example, we quote the L^1-stability result for the perimeter: for every $V_0 \in (0,\infty)$, there exists $c \in (0,\infty)$ such that

$$P(\Omega) - P(B) \geq c d_{\mathrm{L}^1}(\Omega, B)^2, \qquad \text{for every (measurable set) } \Omega \text{ such that } |\Omega| = V_0, \tag{1.3}$$

where P denotes the perimeter (in the sense of geometric measure theory), $|\cdot|$ is the volume, B is any ball of volume V_0, and

$$d_{\mathrm{L}^1}(\Omega, B) = \inf_{\tau \in \mathbb{R}^d} \frac{|(\Omega - \tau)\Delta B|}{|B|}$$

is also known as the Fraenkel asymmetry (which can be seen as the L^1-distance to the ball, up to translations). For this specific example, all of these three strategies have been successfully applied, see [20, 17, 10].

Note in particular that the result is global (in other words $\eta = \infty$), but in that case a local result implies a non-local one as it is shown in [20, Lemma 5.1 and Lemma 2.3].

In this paper, **we focus on the third strategy**, which recently receveid even more attention as in some examples, the other techniques could not be applied, or provided non-optimal results: as an example we quote the L^1-stability for the Faber-Krahn inequality, which was solved with symmetrization technique in [23], but provided a higher (and less strong) exponent in (1.2), and has been improved to an optimal exponent recently in [7] using the third strategy (see also [21]).

One specific difficulty for this strategy is to define a framework of differential calculus within shapes. This can be done for example with the notion of shape derivatives, but one main drawback is that this is available only for reasonably smooth deformations of the initial shape, or in other words, for a rather strong distance d_2 (otherwise it is clear that classical functionals are not differentiable for non-smooth perturbations). However, as it is shown for example in [2], the strategy can also provide results for very weak distances (as the Fraenkel asymmetry), and can be decomposed in two main steps:

- first, with the help of the differential setting and the fact that Ω^* satisfies a strict second order optimality condition, prove a stability result for small and smooth perturbation of Ω^*; in other words, prove that (1.2) is valid where d_2 is a strong distance (and d_1 is limited by the properties of J, and is in general different from d_2, see below),
- second, deduce from this first step that (1.2) is valid where $d_1 = d_2$ is a weak distance (for example the Fraenkel asymmetry).

For the perimeter functional, the first step goes back to [19], and the second step is inspired by results in [40, 30], though the complete result was achieved in [10]. These two steps rely on very different arguments: in particular, the second step usually requires to adapt the regularity theory related to the optimization problem (1.1), namely the notion of quasi-minimizer of the perimeter when the functional J contains a perimeter term, or the regularity of free boundaries when J involves an energy related to a PDE functional (see [2, 7] respectively), so it strongly relies on specific properties of the functional J under study. However, as we aim to show in this paper, the first step has a very large range of applications, and is valid under rather weak assumptions on the functionals.

The aim of this paper is to describe a general framework so that the first step of the above strategy applies: while this has been done in a few places in the literature, every time specifically for the functional that was under study, we aim at giving some general statements, and then show that these statements both applies to the examples already handled in the literature, and also to new examples. Despite getting a wider degree of generality, we also simplify many proofs and strategies found in the previous literature, as we describe below.

Neighborhood of shapes

In order to describe the details of the strategy, we briefly introduce two classical ways to parametrize shapes in a neighborhood of a fixed one:

- **Diffeomorphism and shape derivatives:** we consider a shape to be a neighbor of Ω if it is a deformation of Ω by a diffeomorphism which is close to the identity. More precisely, Θ being a Banach space such that $\mathrm{C}^\infty(\mathbb{R}^d,\mathbb{R}^d)\subset\Theta\subset W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, we consider shapes of the form $\Omega_\theta := (Id+\theta)(\Omega)$ where $\|\theta\|_\Theta$ is small.

  In this framework, we can consider the distance introduced by Micheletti :

  $$d_\Theta(\Omega_1,\Omega_2) := \inf\left\{\|\theta\|_\Theta + \|(Id+\theta)^{-1}-I\|_\Theta,\ \ \theta\in\Theta \text{ diffeomorphism such that } (Id+\theta)(\Omega_1)=\Omega_2\right\}.$$

  Morover, this leads to the notion of shape derivatives, first introduced by Hadamard, then developed by Murat-Simon and Delfour-Zolesio. One defines the function $\mathcal{J}_\Omega$ on a neighborhood of 0 in Θ by

  $$\forall\theta\in\Theta,\quad \mathcal{J}_\Omega(\theta) = J[(Id+\theta)(\Omega)].$$

  One then uses (in the whole paper) the usual notion of Fréchet-differentiability: shape derivatives of J at Ω are the successive derivatives of $\mathcal{J}_\Omega$ at 0, when they exist. In particular, the first shape derivative is $J'(\Omega) := \mathcal{J}'_\Omega(0)$, a continuous linear form on Θ (the shape gradient), and the second order shape derivative is $J''(\Omega) := \mathcal{J}''_\Omega(0)$, a continuous symmetric bilinear form on Θ (the shape hessian).

- **Normal graphs:**
  On the other hand, assuming that Ω is C^1 (and $\boldsymbol{n}=\boldsymbol{n}_{\partial\Omega}$ is its outer unit normal vector) we can consider "normal graph" on $\partial\Omega$, that is Ω_h such that

  $$\partial\Omega_h = \{x+h(x)\boldsymbol{n}(x)\}, \tag{1.4}$$

  where $h\in H$ is small and H is a Banach space of (scalar) functions defined on $\partial\Omega$ (with $\mathrm{C}^\infty(\mathbb{R}^d)\subset H\subset W^{1,\infty}(\mathbb{R}^d)$).

  Then once can see $\widetilde{d}_H(\Omega,\Omega_h) := \|h\|_H$ as a measure of the distance between Ω_h and Ω (though it is not formally a distance), and one can also define derivatives in this framework, as $j_\Omega(h) := J(\Omega_h)$ is defined on a Banach space. Notice that given Ω_1 and Ω_2, there is at most one h defined on $\partial\Omega_1$ such that $\partial\Omega_2 = (\partial\Omega_1)_h$ as defined in (1.4), so we can denote $h = h_{\Omega_1,\Omega_2}$ this function if it exists.

Let us emphasize that even if the second method seems more restrictive, the two methods are equivalent in a neighborhood of Ω (if Ω is smooth enough) in the sense that one descibes as many shapes with each methods (for suitable Θ and H): first, a normal graph Ω_h is a deformation of Ω for any θ_h which is an extension to $\mathbb{R}^d$ of $h\boldsymbol{n}$ (and then $j_\Omega(h) = \mathcal{J}_\Omega(\theta_h)$). Second, if we consider diffeomorphims that are close to the identity, the boundaries of the perturbed domains are graphs over the boundary of the initial domain: in other words, for any domain $\Omega_\theta = (Id+\theta)(\Omega)$ with $\theta\in\Theta$ close to 0, there is a unique real-valued function $h = h_{\Omega,\Omega_\theta}$ defined on $\partial\Omega$ such that (1.4), see Lemma 3.1 in [32].

However, it is not clear a priori that computing derivatives for normal graphs (derivatives of $j_\Omega : h \mapsto J(\Omega_h)$) is enough to describe shape derivatives (derivatives of $\mathcal{J}_\Omega : \theta \mapsto J(\Omega_\theta)$): this issue is handled in the first point below. Also, it may be of interest to focus on paths $t \in \mathbb{R} \mapsto \Omega_t$ (as we will need below), and in that case Ω_{t+s} is not in general a normal graph over Ω_t; it is therefore important to understand the framework of diffeomorphisms, though in some case it is enough to remain in the framework of normal graphs.

We may also be interested in having distances taking into account some invariance with translation. For example, we define

$$\bar{d}_H(\Omega_1, \Omega_2) = \inf_{\tau \in \mathbb{R}^d} \|h_{\Omega_1, \Omega_2 + \tau}\|_H \tag{1.5}$$

where the infimum is taken over τ such that $\partial\Omega_2 + \tau$ is a normal graph on $\partial\Omega_1$.

Main contributions of the paper:

We are now in position to describe the main steps of the strategy to obtain a stability inequality of the form (1.2) *for a strong distances d_2: we describe here these steps, and insist on the contributions of the present paper in each step.*

- **Structure of derivatives:** The way of differentiating shape functionals with diffeomorphism described just above is very convenient as most shape functionals are easily proven to be smooth in this setting (usually not using any regularity on the initial shape Ω^*, see more details in [26] for example), and as noticed before, it is clear that computing derivatives in the sense of normal graphs is just a particular case (while the opposite seems not clear). Nevertheless, one main drawback for our purpose is that as we are dealing with shapes, there is a lot of invariance for $\mathcal{J}_\Omega$ (any non-trivial diffeomorphisms that leaves Ω invariant (at first, second order) must lead to vanishing derivatives). It is therefore unreasonable to expect that the stability condition for optimal shapes writes $J''(\Omega^*) \cdot (\xi, \xi) > 0$ for $\xi \in \Theta \setminus \{0\}$. This difficulty is well-known since Hadamard, who observed (in particular examples) that the shape gradient is a distribution supported on the boundary of the domain, acting only on the normal component of the deformation. In other words, for any C^1 domain Ω, there is a linear form $\ell_1 = \ell_1[J](\Omega)$ acting on scalar functions defined on $\partial\Omega$, such that

  $$\forall \xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega) \cdot \xi = \ell_1(\xi_{|\partial\Omega} \cdot \boldsymbol{n}),$$

  see classical monographs, or [29] for a statement and a proof in a possibly non-smooth setting. A similar observation can be made about second order shape derivatives, though the situation is in general more involved. Let us describe a simple case, which happens to be particularly relevant for our purpose here: if Ω^* is a critical domain for J, that is a domain such that the shape gradient of J vanishes, then the shape hessian reduces to a symmetric bilinear form $\ell_2 = \ell_2[J](\Omega^*)$, also acting only on normal components of diffeomorphisms. When we do not assume that Ω^* is a critical domain, contrary to the first order one, the second order shape derivative may involve the tangential component of the deformation. However, as it has been proven in [32], there is a general structure, involving a quadratic form acting only on normal components (as in the critical case) and another term involving the first order derivative: more precisely, for any C^2 domain Ω, there exists $\ell_2 = \ell_2[J](\Omega)$ acting on normal components such that

  $$\forall \xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d), \quad J''(\Omega) \cdot (\xi, \xi) = \ell_2(\xi \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_1(Z_\xi) \qquad \text{where } Z_\xi = \boldsymbol{B}(\xi_\tau, \xi_\tau) - 2\nabla_\tau(\xi \cdot \boldsymbol{n}) \cdot \xi_\tau,$$

  where ξ_τ is the tangential component of ξ, and $B = D_\tau \boldsymbol{n}$ is the tangential differential of $\boldsymbol{n}$, or in other words, the second fondamental form of $\partial\Omega$. This fact is often observed in the literature on specific examples and after lengthy computations, while it can be used a priori to simplify the computations: indeed, once we know the shape functional is smooth (in the Fréchet sense), this result implies that the computation of shape derivatives for purely normal deformations, is enough to describe the result for any deformation, with the use of this structure result, and if needed, the chain-rule formula (see also Remark 2.6). In particular, using the framework of normal graphs, we get $j''_\Omega(0)(v, v) = \ell_2[J](\Omega)(v, v)$.

The first contribution of this paper is to give a new proof of this result, see **Theorem 2.1**. *Though the strategy in [32] is quite natural as it shows that any small deformation of a shape can be seen (in a smooth way) as a normal deformation defined on the boundary, up to a change of parametrization of the boundary, we believe this new proof is less technical, and also quite natural as it only relies on the invariance properties mentioned before.*

Even if it may seem that for our purpose this result in only helpful in the particular case where Ω^* is a critical shape, as we will notice in the following items, we will actually need to deal with second order shape derivatives at non-critical shapes as well, when proving the stability result, as we will use of the Taylor formula with an integral form of the remainder.

- **Coercivity assumption:** As noticed earlier, we are dealing with infinite dimensional differential calculus, and it is well-known that usual sufficient conditions for getting optimality is that the second order derivative is coercive rather than just assuming that it is positive. In particular, we need to wonder for which norm the coercivity might be valid, with view of applications. We will see that in most examples we are dealing with, the quadratic form $\ell_2[J](\Omega)$ (normal part of the hessian of J at Ω) satisfies a structural property emphasizing a particular norm, that will restrict the choice of d_1 for applications (actually, there are examples that do not satisfy the following assumption, and for which stability do not imply local minimality, see Section 5.2): for some $s_2 \in (0,1]$, the bilinear form ℓ on $\mathrm{C}^\infty(\partial\Omega)$ satisfies condition $(\mathbf{C}_{\mathrm{H}^{s_2}})$ if (and by extension we say that J satisfies the condition at Ω^* if $\ell_2[J](\Omega^*)$ does):

$(\mathbf{C}_{\mathrm{H}^{s_2}})$ there exists $0 \leq s_1 < s_2$ and $c_1 > 0$ such that $\ell = \ell_m + \ell_r$ with

$$\begin{cases} \ell_m \text{ is lower semi-continuous in } \mathrm{H}^{s_2}(\partial\Omega) \text{ and } \ell_m(\varphi,\varphi) \geq c_1 |\varphi|^2_{\mathrm{H}^{s_2}(\partial\Omega)}, \ \forall \varphi \in \mathrm{C}^\infty(\partial\Omega), \\ \ell_r \text{ continuous in } \mathrm{H}^{s_1}(\partial\Omega). \end{cases}$$

where $|\cdot|_{\mathrm{H}^{s_2}(\partial\Omega)}$ denote the $\mathrm{H}^{s_2}(\partial\Omega)$ semi-norm. In that case, ℓ is naturally extended (by a density argument) to the space $\mathrm{H}^{s_2}(\partial\Omega)$, and we prove that under this assumption

$$\ell > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega) \setminus \{0\} \Leftrightarrow \exists \lambda > 0, \quad \forall \varphi \in \mathrm{H}^{s_2}(\partial\Omega), \quad \ell(\varphi,\varphi) \geq \lambda \|\varphi\|^2_{\mathrm{H}^{s_2}(\partial\Omega)}, \tag{1.6}$$

(here ℓ is a quadratic form, so $\ell > 0$ on X means $\ell(\varphi,\varphi) > 0$ for any $\varphi \in X$). Notice also that this statement holds for general bilinear forms on Sobolev spaces and is not connected to shape hessians.

The proof of this fact is rather simple, and similar arguments can be found in [25, 2]. Our contribution principally lies in the fact that we explicitly formulate the underlying assumption so that positivity implies coercivity, see **Lemma 3.3** *in Section 3.1.*

Note in particular that the value of s_2 is determined by the shape functional J (in practice s_2 usually does not depend on Ω), and the choice of the distance d_1 in (1.2) is limited by this coercivity property; in other words, we can not expect (1.2) to be valid for any distance d_1 stronger than the $\widetilde{d}_{\mathrm{H}^{s_2}}$ (see also [19] where an upper bound of the isoperimetric deficit is given, in a smooth neighborhood). As we will notice through computations, when J contains a perimeter term, $s_2 = 1$, while for PDE functionals we are dealing with here (see below when we describe examples), $s_2 = 1/2$. For an interesting result about the choice of d_1 in a non-smooth setting, see [22] where they obtain an improved version of (1.3) where d_1 is a stronger distance than the Fraenkel asymmetry (see also [34] for the anisotropic case).

- **Stability and Norm discrepancy:** With the help of the previous items, we are able to properly state the stability question: if Ω^* is such that

$$\ell_1 = 0 \text{ and } \ell_2 > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\} \tag{1.7}$$

(where ℓ_1, ℓ_2 are associated to $J'(\Omega^*)$ and $J''(\Omega^*)$ respectively, through the structure theorem), can we conclude to a *nonlinear*-stability inequality of the form (1.2), for $\Omega = (Id + \theta)(\Omega^*)$ small and smooth deformation of the set Ω^*?

From the Taylor formula, we can write:

$$J((Id+\theta)(\Omega^*)) - J(\Omega^*) = \frac{1}{2}\ \ell_2(\theta\cdot\boldsymbol{n},\theta\cdot\boldsymbol{n}) + o(\|\theta\|_\Theta^2). \tag{1.8}$$

which leads to two issues:

- first, the remainder depends on the full norm of θ, while the second order term is only controlled with the norm of $\theta\cdot\boldsymbol{n}$,
- the norm of differentiability Θ is in most cases stronger than the norm of coercivity given in the previous item, namely H^{s_2},

so it is a priori not possible to control the sign of the term $\frac{1}{2}\ \ell_2(\theta\cdot\boldsymbol{n},\theta\cdot\boldsymbol{n}) + o(\|\theta\|_\Theta^2)$. To solve the first issue, one could only work with normal deformation (which is enough to describe every small and smooth deformation of Ω^*), but this is in some cases a restriction (as noticed before, see [11, 2]). The second issue is more serious. In the literature, this phenomenon was first observed when minimizing the perimeter as it naturally differentiable in $\mathrm{W}^{1,\infty}$ while the coercivity can only be valid for the H^1-norm. This has been observed in two places in the literature:

- Fuglede in [19] proved a local H^1-stability result for the classical isoperimetric problem whose solution is the ball (see the next step to explain how to handle the translation invariance of the functional and the volume constraint): when writing everything in radial coordinates, that is considering $B_u = \{(r,\omega)\in[0,\infty)\times\mathbb{S}^{d-1},\ r<1+u(\omega)\}$ with u preserving the barycenter and the volume (at the second order), Fuglede proved that

$$P(B_u) - P(B) = \ell(u,u) + \mathcal{O}(\|u\|_{\mathrm{W}^{1,\infty}})\|u\|^2_{\mathrm{H}^1(\mathbb{S}^{d-1})} \tag{1.9}$$

  where $u\mapsto\ell(u,u)$ is quadratic form, coercive in the H^1-norm; this allows to conclude to the strict optimality of the ball in small $\mathrm{W}^{1,\infty}$ neighborhood.
- For stable constant mean curvature surfaces, a similar differentiability statement is quoted in [25, Proof of Theorem 6], [6, Equation 3.23], [40, Equation (1)], and this result is not restricted to the ball. See also [38, 39] for similar observations with a different parametrization.

This difficulty is well-known in the literature on second order optimality conditions in infinite dimension. We may think that we could change the space of differentiability for the functional (choosing for example the space for which there is coercivity), unfortunately in most examples, the shape functional is not differentiable in the space for which there is coercivity, which is weaker than $\mathrm{W}^{1,\infty}$. Various geometric examples have been handled in the literature since these first examples, see [13, 16, 5, 34].

In the specific context of shape optimization involving PDE functionals, the question was raised in the work of Descloux [15], and was overcome in [12, 11], and more recently a very similar approach can be found in [2], see also Section 4.1. The situation is much more involved than for geometric functionals, as it is much harder to write the remainder term in order to obtain an estimate like (1.9).

Therefore, the idea is, given Ω^* a critical and stable domain and Ω a domain sufficiently close for d_Θ, to consider the path $(\Omega_t)_{t\in[0,1]}$ defined through its boundary

$$\partial\Omega_t = \{x+\ t\ h(x)\ \boldsymbol{n}(x),\ x\in\partial\Omega^*\}. \tag{1.10}$$

connecting Ω^* to Ω, where $h = h_{\Omega^*,\Omega}$ is defined in (1.4), and to assume an improved continuity property for the second order shape derivative, namely, given $s\in[0,1]$ (chosen as being s_2 from the coercivity property) and $\Theta\subset W^{1,\infty}$ a Banach space (which has to be chosen wisely, see below) so that $\mathcal{J}_\Omega$ is C^2 around 0, condition $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$ is:

$(\mathbf{EC}_{\mathrm{H}^s,\Theta})$ there exist $\eta>0$ and a modulus of continuity ω such that for every domain $\Omega=(Id+\theta)(\Omega^*)$ with $\|\theta\|_\Theta\le\eta$, and all $t\in[0,1]$:

$$\left|j''(t)-j''(0)\right| \le \omega(\|\theta\|_\Theta)\|h\|^2_{\mathrm{H}^s},$$

where $j : t \in [0,1] \mapsto J(\Omega_t)$ and $(\Omega_t)_{t\in[0,1]}$ is given by (1.10). Then, the Taylor formula with integral remainder gives

$$J((Id+\theta)(\Omega^*)) - J(\Omega^*) = \frac{1}{2}\ell_2(h,h) + \int_0^1 [j''(t) - j''(0)](1-t)dt \geq \lambda \|h\|^2_{\mathrm{H}^s} + \omega(\|\theta\|_\Theta)\|h\|^2_{\mathrm{H}^s},$$

which easily leads to the stability property, see the proof of Theorem 1.1 just stated below.

Our main contribution about this step is to insist on the list of examples of functionals satisfying this condition, and also on possible choices of Θ*, depending on the functionals.*

In [12, 11] they choose Hölder-spaces of the form $\mathrm{C}^{2,\alpha}$, while in [2] they prefer a Sobolev type of spaces, namely $\mathrm{W}^{2,p}$ for p large enough, which is a better result as it leads to a larger neighborhood in the stability equation (1.2). Note however that Θ cannot be chosen, in general, as the best space for differentiability (which for our purpose would be the bigger one, namely $\mathrm{W}^{1,\infty}$).

We prove that for the PDE functionals from [12, 11], which seems less smooth than the functional from [2] (see also Section 4.1), the previous condition is also valid for $\mathrm{W}^{2,p}$*, for* p *large enough, therefore improving the stability properties proven in [12, 11]. Another important contribution about this step is that thanks to the next item, it is enough to focus on the path* (1.10)*.*

Note that this improvement about spaces is not just a technical issue, as in [2] the choice of $\mathrm{W}^{2,p}$ rather than $\mathrm{C}^{2,\alpha}$ is relevant for the second step of the strategy when proving stability in an L^1-neighborhood ([2, Section 4]): indeed their regularization procedure needs to allow discontinuities of the mean curvature, see equation (4.9) in the proof of [2, Theorem 4.3]. This difficulty is handled in another way in [7] for the quantitative Faber-Krahn inequality as a stability in $\mathrm{C}^{2,\alpha}$ is enough to make the regularization procedure work.

- **Stability result:** We are now in position to state the main stability result in the framework of shape optimization. For a similar statement for problems with volume constraint and translation invariant functionals, which is very useful in practice, see the next item and **Theorem 3.2**:

  Theorem 1.1 *Let* Ω^* *be a domain of class* C^3*, and* J *a shape functional, twice Fréchet differentiable on a neighborhood of* Ω^* *for* $d_{\mathrm{W}^{1,\infty}}$*. We denote* $\ell_1 = \ell_1[J](\Omega^*)$ *and* $\ell_2 = \ell_2[J](\Omega^*)$ *(given by Theorem 2.1), and assume that* J *satisfies* $(\mathbf{C}_{\mathrm{H}^{s_2}})$ *and* $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$ *at* Ω^* *for some* $s_2 \in (0,1]$ *and* Θ *a Banach space such that* $\mathrm{C}^\infty(\mathbb{R}^d,\mathbb{R}^d) \subset \Theta \subset \mathrm{W}^{1,\infty}$*.*
  Then if Ω^* *is a critical and strictly stable shape for* J*, that is to say*

  $$\ell_1 = 0, \qquad \text{and } \ell_2 > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega,\mathbb{R}) \setminus \{0\}, \tag{1.11}$$

  then Ω^* *is an* H^{s_2}**-stable local minimum of** J **in a** Θ**-neighborhood***, that is to say there exists* $\eta > 0$ *and* $c = c(\eta) > 0$ *such that*

  $$\forall\, \Omega \text{ such that } d_\Theta(\Omega^*,\Omega) \leq \eta, \quad J(\Omega) \geq J(\Omega^*) + c\widetilde{d}_{\mathrm{H}^{s_2}}(\Omega^*,\Omega)^2 \tag{1.12}$$

  where $\widetilde{d}_{\mathrm{H}^{s_2}}$ *is defined in* (1.4)*.*

  With the help of the previous remarks, the proof of this result is rather easy, and its main interest lies in the fact that its hypotheses (more accurately the one of the constrained version) are valid in practice for many examples, as we show in Section 4, see also the end of this introduction.

- **Constraints and invariance:** as in the isoperimetric problem, whose quantitative version is recalled in (1.2), we often have to handle two difficulties: first, the functional is translation invariant, and second, there is a volume constraint in the optimization problem. Therefore one cannot expect (1.11) to be satisfied, and it should be replaced by

  $$\ell_1[J](\Omega^*) = \mu\ell_1[\mathrm{Vol}](\Omega^*), \qquad \text{and } \ell_2[J](\Omega^*) - \mu\ell_2[\mathrm{Vol}](\Omega^*) > 0 \text{ on } T(\partial\Omega^*)\setminus\{0\}$$
  $$\text{where } T(\partial\Omega^*) := \left\{\varphi \in \mathrm{H}^{s_2}(\partial\Omega^*),\ \int_{\partial\Omega^*}\varphi = 0 \text{ and } \int_{\partial\Omega^*}\varphi\overrightarrow{x} = \overrightarrow{0}\right\}. \tag{1.13}$$

Here $\mu \in \mathbb{R}$ is the Lagrange multiplier, handling the notion of criticality when there is a volume constraint, and $T(\partial\Omega^*)$ can be seen as the tangent space to the constraint ($\int_{\partial\Omega^*}\varphi$ being the first order derivative of the volume) and to the invariance ($(\int_{\partial\Omega^*}\varphi\overrightarrow{x})$ being the first order derivative of the barycenter functional). In [12, 11], especially when dealing with the two-norm discrepancy issue, the author carefully handle the volume constraint by building a path preserving the volume and being almost normal, and prove that an estimate like $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$ is valid for this more involved path. In [2], a very similar approach (adapted to $\mathrm{W}^{2,p}$-spaces) is given, and they also handle the translation-invariance (which is not there in the example of [11]) which implies a lot a technicalities.

Inspired by the strategy of [25] who deals with the volume constraint for area minimizing surfaces, we drastically simplify the presentation of [12, 11, 2] by using an exact penalization method. More precisely (see also **Theorem 3.2***), we prove that under the assumptions* $(\mathbf{C}_{\mathrm{H}^{s_2}})$ *for* $\ell_2[J](\Omega^*)$*, the constrained optimality conditions* (1.13) *implies the unconstrained conditions* (1.11) *when* J *is replaced by*

$$J_{\mu,C} = J - \mu\mathrm{Vol} + C\left(\mathrm{Vol} - V_0\right)^2 + C\left\|\mathrm{Bar} - \mathrm{Bar}(\Omega^*)\right\|^2, \tag{1.14}$$

where μ *is the Lagrange multiplier and* $C \in (0,\infty)$ *is large enough. We can therefore apply the unconstrained result for stability to* $J_{\mu,C}$*, see the proof of Theorem 3.2, and this clearly implies the constrained local minimality. It is clear, looking at the proofs of our results, that the situation we aim to describe in this paper is quite general and can be applied to many other constraints or invariance; in particular if there is a volume constraint but no translation invariance, the same strategy applies.*

Old and new applications:

In order to justify the interest of our previous general statement, we provide several examples of functionals for which Theorems 1.1 or 3.2 apply. We give here a short list of them, see Section 4 for more details.

- First, we want to show that our Theorem allows to retrieve classical statements already existing in the literature. Mainly relying on the computation of the first and second derivatives of the functionals (see Section 2), and the fact that they satisfy conditions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$, we believe that despite the degree of generality of our approach, the proofs are less technical and more straightforward than the existing literature. This includes the examples of [12, 11, 2, 7], see Section 4.1 for a more detailed description.

- Second, we notice ealier that if we were only interested in linear stability, namely that for every smooth path $t \mapsto \Omega_t$, there exists t_0 small enough and c such that $J(\Omega_t) - J(\Omega) \geq ct^2$ for every $t \in]-t_0, t_0[$, then most of the work relies only on the computation of derivatives, and proving the positivity of the second order derivative of $\frac{d^2}{dt^2}J(\Omega_t)_{|t=0}$. In other words, most of the previous difficulties we described earlier do not need to be handled. But it is in general much more satisfying (in particular in view of the second step of the global strategy described before to obtain (1.2) for weak distances) to obtain a uniformity in t_0, c as we require in (1.2). *On a couple of explicit examples, the contribution of this paper is to get uniform stability results, while only directional results have been obtained for these examples. This includes the result of [31] (see Proposition 4.1).*

- We also provide new examples, which comes with minor cost thanks to our results. One example we have in mind is the following generic example: if Ω^* is a ball of volume $V_0 \in (0,\infty)$, $P(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$ denotes the perimeter of Ω (for reasonably smooth sets), and E is the Dirichlet energy:

$$E(\Omega) = \min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u, \;\; u \in \mathrm{H}^1_0(\Omega)\right\}, \tag{1.15}$$

  then the conditions of Theorem 3.2 (version of Theorem 1.1 taking into account the translation invariance and the volume constraint) are fulfilled for the functional $J = P + \gamma E$ when $\gamma \geq \gamma_0$ and $\gamma_0 \in (-\infty, 0)$ (whose optimal value can be explicitly computed), and we can conclude from our strategy that the ball is a local minimizer (in a smooth neighborhood, say $C^{2,\alpha}$ or even $W^{2,p}$ for p large enough) of the following optimization problem

$$\min\left\{P(\Omega)+\gamma E(\Omega),\quad |\Omega|=V_0\right\}. \tag{1.16}$$

In other words, (1.2) is valid for $J=P+\gamma E$, d_1 is the H^1-distance, and $d_2=d_\Theta$ for $\Theta=W^{2,p}$ with p large enough.

For $\gamma\geq 0$ this result is not surprising, since the ball minimizes both the perimeter and the Dirichlet energy, so a stability version is a direct consequence of (1.3) *(if we replace d_1 with the Fraenkel asymmetry) or [19], but this result is new and surprising when γ is nonpositive (but small enough), even considering only $d_1=0$: indeed there is a competition between minimizing the perimeter and maximizing the Dirichlet energy. Another way to state the result is to say that*

$$\frac{P(\Omega)-P(B)}{E(\Omega)-E(B)}\geq|\gamma_0|,\qquad\forall\Omega\in\mathcal{V}(B), \tag{1.17}$$

where $\mathcal{V}(B)=\{\Omega, d_{W^{2,p}}(\Omega,B)<\eta\}$, for some $\eta>0$.

For a problem related to (1.16) when $\gamma<0$, see also [24]. It is also interesting to notice that (nonlinear)-stability (and even optimality of the ball) is no longer valid when one consider a neighborhood of Ω^* for a weak norm, for example the L^1-norm. A counterexample is given in Section 5.1. Especially it means that the second step of the strategy described before to handle non-smooth deformations, do not apply to (1.16) if $\gamma<0$, despite the fact that sets are minimizing the perimeter. This shows in which extend the two steps defers concerning their degree of generality.

In addition to this example, we obtain several new local isoperimetric inequalities, see **Proposition 4.1***, in Section 4.2.*

Going deeper in the computations, it is possible, as it is done in [31] who obtains a directional version of (1.17) when replacing E with λ_1 (the first eigenvalue of the Dirichlet-Laplacian), to compute the optimal value of the constant γ_0 such that the ball is a smooth local minimizer for (1.16) (or equivalently compute the optimal γ_0 in (1.17) when η goes to 0), see **Proposition 4.4**. As we were mentioning in the previous item, we insist on the fact that even in this particular case, we improve the result of [31], since our analysis provides a uniform neighborhood where we have an isoperimetric inequality, while this author make an asymptotic analysis on each path. However we do not compute the optimal value for this case, it is indeed more involved because of the form of the second order derivative of λ_1, we therefore only obtain an estimate and refer to [31] for the optimal value.

In Section 2, we start with a new proof of the Structure Theorem for second order shape derivatives. We also recall the classical examples of second order shape derivatives, noticing in particular in which norms they are continuous (which leads to the value of s_2 from assumption $(\mathbf{C}_{\mathrm{H}^{s_2}})$), and focus on the case of the ball for which we diagonalize the shape hessians (which leads to the classical stability properties of the ball for these functionals). Section 3 contains the main results of this work: we state the version of Theorem 1.1 adapted to the constrained/invariant case, we discuss the coercivity assumptions proving (1.6) (Lemma 3.3), and precise the known results on assumption $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$, in particular we recall and improve the existing results, and provide new ones, for example the first eigenvalue of the Dirichlet Laplacian for which it seems this condition was not proven in the existing literature. In Section 4, we explain how our general results allow to retrieve known results, and then prove some local isoperimetric inequalities, some of which are known, some of which are new, see Proposition 4.1. All these applications are simple corollaries of our main results, combined with the computations reminded in Section 2. In the last Section, we show counterexamples related to nonlinear-stability, which helps to understand the hypotheses of our results.

2 On Second order shape derivatives.

In this section, we recall classical facts on second order shape derivatives, and give a new proof of their structure. As for all the examples of this paper, J is a shape functional such that $\theta\in\Theta\mapsto J((Id+\theta)(\Omega))$

is of class C^2 in a neighborhood of 0 in Θ if Ω is smooth enough and $\Theta = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, we focus on this framework, though similar proofs can be adapted to other functional spaces. In Section 3.2 we give remarks about other functional spaces in order to handle PDE functionals.

2.1 Structure theorem

It is well-known since Hadamard's work that the shape gradient is a distribution supported on the moving boundary and acting on the normal component of the deformation field. The second order shape derivative also has a specific structure as stated by A. Novruzi and M. Pierre in [32]. We quote their result, and provide a new proof:

Theorem 2.1 (Structure Theorem of first and second shape derivatives) *Let $\Theta = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, Ω an open bounded domain of $\mathbb{R}^d$ and J a real-valued shape function defined on $\mathcal{V}(\Omega) = \{(Id+\theta)(\Omega), \|\theta\|_\Theta < 1\}$. Let us define the function $\mathcal{J}_\Omega$ on $\{\theta \in \Theta, \|\theta\|_\Theta < 1\}$ by*

$$\mathcal{J}_\Omega(\theta) = J[(Id+\theta)(\Omega)].$$

(i) *If $\mathcal{J}_\Omega$ is differentiable at 0 and Ω is C^2, then there exists a continuous linear form ℓ_1 on $\mathrm{C}^1(\partial\Omega)$ such that $\mathcal{J}'_\Omega(0)\xi = \ell_1(\xi_{|\partial\Omega} \cdot \boldsymbol{n})$ for all $\xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, where $\boldsymbol{n}$ denotes the unit exterior normal vector on $\partial\Omega$.*

(ii) *If moreover $\mathcal{J}_\Omega$ is twice differentiable at 0 and Ω is C^3, then there exists a continuous symmetric bilinear form ℓ_2 on $\mathrm{C}^1(\partial\Omega) \times \mathrm{C}^1(\partial\Omega)$ such that for all $(\xi, \zeta) \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)^2$*

$$\mathcal{J}''_\Omega(0)(\xi, \zeta) = \ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) + \ell_1(\boldsymbol{B}(\zeta_\tau, \xi_\tau) - \nabla_\tau(\zeta \cdot \boldsymbol{n}) \cdot \xi_\tau - \nabla_\tau(\xi \cdot \boldsymbol{n}) \cdot \zeta_\tau), \tag{2.1}$$

where ∇_τ is the tangential gradient, ξ_τ and ζ_τ stands for the tangential components of ξ and ζ, and $\boldsymbol{B}$ is the second fondamental form of $\partial\Omega$.

With respect to this work, it is important to notice that at a critical domain for J, the shape hessian is reduced to ℓ_2 and hence does not see the tangential components of the deformations fields.

Remark 2.2 The requirement that Ω is bounded is made only to simplify the presentation: the result remains valid replacing $\mathrm{C}^1(\partial\Omega)$ with $\mathrm{C}^1_c(\partial\Omega)$ and localizin the test functions.

Remark 2.3 As noticed in [26, p. 225], with this degree of generality, the regularity assumption on $\partial\Omega$ are sharp as ℓ_1, ℓ_2 are a priori defined only on $\mathrm{C}^1(\partial\Omega)$. We could wonder if ℓ_1 (for example) could be extended as a continuous linear form on $\mathrm{C}^0(\partial\Omega)$; this is not true in general if Ω is only assumed to be C^1, as the example of the perimeter shows (as it would mean that the mean curvature is a Radon measure, which is not true for a C^1 domain). However, as it is shown in [32, Remark 2.8, Corollary 2.9], if we assume that ℓ_1 can be extended as a continuous linear form on $\mathrm{C}^0(\partial\Omega)$, then the point (**ii**) is valid assuming Ω of class C^2 instead of C^3. It is easy to notice that our proof also recovers this case.

Remark 2.4 Compare to the result in [32], we restricted ourself to the space $\Theta = \mathrm{W}^{1,\infty}$ (or similarly $\mathrm{C}^{1,\infty}$, see the proof below), as all the functional of this paper are differentiable in this space. Of course, the same proof can be adapted to spaces like $\mathrm{W}^{k,\infty}$ for $k \geq 2$, which is important to handle higher order geometric or PDE functional, but we do not work with such examples in this paper.

Remark 2.5 When $\xi = \zeta$, we get

$$\mathcal{J}''_\Omega(0)(\xi, \zeta) = \ell_2(\xi \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_1(Z_\xi), \text{ where } Z_\xi = \boldsymbol{B}(\xi_\tau, \xi_\tau) - 2\nabla_\tau(\xi \cdot \boldsymbol{n}) \cdot \xi_\tau.$$

As noticed in [2, Equation 7.5], the term Z_ξ can have be written in a different way:

$$Z_\xi = (\xi \cdot \boldsymbol{n})\mathrm{div}(\xi) - \mathrm{div}_\tau\left(\xi_\tau(\xi \cdot \boldsymbol{n})\right) - H(\xi.\boldsymbol{n})^2.$$

The advantage of Z_ξ is usually that it clearly vanishes when $\xi_\tau = 0$, but this second formulation can also have advantages, especially when ξ has a vanishing divergence (as it is the case in [2]) or when there are simplifications as it is the case for the volume (see Lemma 2.7 for the first equality):

$$\mathrm{Vol}''(\Omega)\cdot(\xi,\xi) = \int_{\partial\Omega} H(\xi\cdot\boldsymbol{n})^2 + \int_{\partial\Omega} Z_\xi = \int_{\partial\Omega}(\xi\cdot\boldsymbol{n})\mathrm{div}(\xi). \tag{2.2}$$

In that case, the last formula is indeed more common.

Remark 2.6 It is sometimes considered that first and second order derivatives described in the previous theorem cannot handle the differentiation of $t \mapsto J(T_t(\Omega))$ where $T \in \mathrm{C}^2([0,\alpha[,\Theta)$ when T_t is not of the form $T_t = Id + t\xi$. This is not true, as the chain rule formula easily gives (and is allowed when we have proven the Fréchet-differentiability of the functionals, which is valid for all the functionals of this paper):

$$\frac{d^2}{dt^2}J(T_t(\Omega)) = \mathcal{J}''_\Omega(T_t - Id)\left(\frac{d}{dt}T_t, \frac{d}{dt}T_t\right) + \mathcal{J}'_\Omega(T_t - Id)\left(\frac{d^2}{dt^2}T_t\right)$$

and the structure result can then be applied. For example, if T_t is the flow of the vector field ξ as it is usually done in the speed method, we obtain:

$$\frac{d^2}{dt^2}J(T_t(\Omega))_{|t=0} = \mathcal{J}''_\Omega(0)\,(\xi,\xi) + \mathcal{J}'_\Omega(0)\,((D\xi)\cdot\xi)\,.$$

Another interesting case is that if Ω is a critical shape for J, namely $\mathcal{J}'_\Omega(0) \equiv 0$, and if $T_t = Id + t\xi + \frac{t^2}{2}\eta + o(t^2)$ where $o(t^2)$ has to be understood with the norm $\|\cdot\|_\Theta$, then we always have

$$\frac{d^2}{dt^2}J(T_t(\Omega))_{|t=0} = \ell_2(\xi\cdot\boldsymbol{n}, \xi\cdot\boldsymbol{n}).$$

This fact is often observed through computations, but it is always true, when functionals and shapes are smooth enough.

Proof. We only focus on the second order derivative, as the first order one is classical (see for example [26, 14, 29]). For a few technical reasons, we replace Θ by $\mathrm{C}^{1,\infty} := \mathrm{C}^1 \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ equipped with the same norm as $\mathrm{W}^{1,\infty}$, which is also a Banach space. This does not affect the result as we stated (2.1) for smooth vector fields. Let $\xi,\zeta \in \mathrm{C}^\infty$ compactly supported in a neighborhood of $\partial\Omega$, and denote γ,δ their respective flow, namely

$$\begin{cases} \frac{d}{dt}\gamma_t(x) &= \xi(\gamma_t(x)) \\ \gamma_0(x) &= x \end{cases} \qquad \begin{cases} \frac{d}{dt}\delta_t(x) &= \zeta(\delta_t(x)) \\ \delta_0(x) &= x \end{cases}$$

As the function $\gamma \in \Theta \mapsto \xi\circ\gamma \in \Theta$ is locally Lipschitz and C^2 (thanks to our assumptions on ξ,ζ, see ???), these ODE admits solutions defined on $(-t_0,t_0)$ and $[t\mapsto \gamma_t - Id, t\mapsto \delta_t - Id]$ are in $\mathrm{C}^2((-t_0,t_0),\Theta)$.

Let now assume that $\zeta\cdot\boldsymbol{n} = 0$. Then from classical criterion of invariance of sets with the flow, we have $\delta_t(\Omega) = \Omega$ for every t small enough, so $J(\gamma_s\circ\delta_t(\Omega)) = \mathcal{J}_\Omega(\gamma_s\circ\delta_t - Id)$ is independent of t. Differentiating successively with respect to t and s at $(0,0)$, we obtain:

$$\mathcal{J}''_\Omega(0)\cdot(\xi,\zeta) + \mathcal{J}'_\Omega(0)\cdot(D\xi\cdot\zeta) = 0, \quad \forall\xi\in\Theta, \quad \forall\zeta\in K,$$

where $K = \mathrm{Ker}(\Phi)$ and $\Phi : \xi \in \Theta \mapsto \xi_{|\partial\Omega}\cdot\boldsymbol{n}$.

We define $b : (\xi,\zeta) \in \Theta\times\Theta \mapsto \mathcal{J}''_\Omega(0)\cdot(\xi,\zeta) + \mathcal{J}'_\Omega(0)\cdot(D\xi\cdot\zeta)$ which is a bilinear functional that vanishes for $\zeta \in K$, for any fixed ξ. Therefore we can write, using quotient properties, $b(\xi,\zeta) = \widetilde{b}(\xi,\zeta_{|\partial\Omega}\cdot\boldsymbol{n})$ where $\widetilde{b} : \Theta\times\mathrm{C}^1(\partial\Omega) \to \mathbb{R}$ is continuous (a priori we only get that $\widetilde{b}$ is separately continuous but with Banach-Steinhaus Theorem, it implies continuity), as Φ induces an isomorphism between Θ/K and $\Phi(\Theta) = \mathrm{C}^1(\partial\Omega)$ equipped with the C^1 norm (using that Ω is of class C^2). Moreover by construction we have:

$$\mathcal{J}''_\Omega(0)\cdot(\xi,\zeta) + \mathcal{J}'_\Omega(0)\cdot(D\xi\cdot\zeta) = \widetilde{b}(\xi,\zeta_{|\partial\Omega}\cdot\boldsymbol{n}), \quad \forall\,\xi,\zeta\in\Theta.$$

Using the symmetry of $\mathcal{J}''_\Omega(0)$, we can write

$$\tilde{b}(\zeta, \xi_{|\Gamma} \cdot \boldsymbol{n}) - \tilde{b}(\xi, \zeta_{|\Gamma} \cdot \boldsymbol{n}) = \mathcal{J}'_\Omega(0) \cdot (D\zeta \cdot \xi - D\xi \cdot \zeta)$$

Our goal is now to apply this formula to $\zeta_{\boldsymbol{n}}$ the normal component of ζ, which needs to be extended as a vector field on $\mathbb{R}^d$. To that end, we introduce $P_{\partial\Omega}$ the projection on $\partial\Omega$, which is well-defined and C^1 in a neighborhood of $\partial\Omega$ (see [14]). Then if φ is defined on $\partial\Omega$, we set $\widetilde{\varphi}(x) = \varphi(P_{\partial\Omega}x)$ (in other words, φ is extended so that it is constant in the normal direction). This operator $\varphi \mapsto \widetilde{\varphi}$ is continuous from $\mathrm{C}^1(\partial\Omega)$ to $\mathrm{C}^{1,\infty}$. Let us define then $\zeta_{\boldsymbol{n}} := \widetilde{(\zeta \cdot \boldsymbol{n})}\boldsymbol{n}$ the extension of the normal component of ζ. Defining the bilinear form $\ell_0(\varphi_1, \varphi_2) = \tilde{b}(\widetilde{\varphi_1 \boldsymbol{n}}, \varphi_2)$, defined and continuous on $\mathrm{C}^1(\partial\Omega)^2$ (but a priori non symmetric), we obtain

$$\begin{aligned}
\mathcal{J}''_\Omega(0) \cdot (\xi, \zeta) &= \tilde{b}(\xi, \zeta \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0) \cdot (D\xi \cdot \zeta) \\
&= \tilde{b}(\zeta_{\boldsymbol{n}}, \xi \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0) \cdot (D\zeta_{\boldsymbol{n}}\, \xi - D\xi \cdot \zeta_{\boldsymbol{n}}) - \mathcal{J}'_\Omega(0) \cdot (D\xi \cdot \zeta) \\
&= \ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0) \cdot (D\zeta_{\boldsymbol{n}} \cdot \xi - D\xi \cdot \zeta_{\boldsymbol{n}} + D\xi \cdot \zeta) \\
&= \ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0) \cdot (D\zeta_{\boldsymbol{n}} \cdot \xi + D\xi \cdot \zeta_\tau)
\end{aligned}$$

where $\zeta_\tau = \zeta - \zeta_{\boldsymbol{n}}$. We now use $D\zeta_{\boldsymbol{n}} = D_\tau \zeta_{\boldsymbol{n}}$, because thanks to our choice of extension operator, $\zeta_{\boldsymbol{n}}$ is constant in the direction $\boldsymbol{n}$ (by definition, $D_\tau a = Da - (Da \cdot \boldsymbol{n})\boldsymbol{n}$), and therefore $D\zeta_{\boldsymbol{n}}\, \xi = D_\tau \zeta_{\boldsymbol{n}}\, \xi_\tau$. Moreover, $D\xi\, \zeta_\tau = D_\tau \xi\, \zeta_\tau$.

Using a symmetrization of the previous formula, we obtain

$$\begin{aligned}
\mathcal{J}''_\Omega(0) \cdot (\xi, \zeta) &= \frac{1}{2}\Big[\ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_0(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0) \cdot (D_\tau \zeta_{\boldsymbol{n}} \cdot \xi_\tau + D_\tau \xi \cdot \zeta_\tau + D_\tau \xi_{\boldsymbol{n}} \cdot \zeta_\tau + D_\tau \zeta \cdot \xi_\tau)\Big] \\
&= \ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) - \frac{1}{2}\mathcal{J}'_\Omega(0) \cdot \Big(2D_\tau \zeta \cdot \xi_\tau + 2D_\tau \xi \cdot \zeta_\tau - D_\tau \xi_\tau \cdot \zeta_\tau - D_\tau \zeta_\tau \cdot \xi_\tau\Big)
\end{aligned}$$

where we defined $\ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) = \frac{1}{2}(\ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_0(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}))$, which is a continuous bilinear form on $\mathrm{C}^1(\partial\Omega)^2$.

From the structure of the first order derivative, and using the formula

$${}^t D_\tau \xi_\tau \cdot \boldsymbol{n} + {}^t D_\tau \boldsymbol{n} \cdot \xi_\tau = 0$$

(obtained by tangentially differentiating $\xi_\tau \cdot \boldsymbol{n} = 0$), we finally obtain (using the C^3 regularity of $\partial\Omega$ so that $D_\tau \boldsymbol{n}$ belongs to the space of definition of ℓ_1)

$$\begin{aligned}
\mathcal{J}''_\Omega(0).(\xi, \zeta) &= \ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) - \frac{1}{2}\ell_1\Big((2D_\tau \zeta \cdot \xi_\tau + 2D_\tau \xi \cdot \zeta_\tau) \cdot \boldsymbol{n} - \zeta_\tau \cdot ({}^t D_\tau \xi_\tau \cdot \boldsymbol{n}) - \xi_\tau \cdot ({}^t D_\tau \zeta_\tau \cdot \boldsymbol{n})\Big) \\
&= \ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) + \ell_1\Big((D_\tau \boldsymbol{n} \cdot \zeta_\tau) \cdot \xi_\tau - \nabla_\tau(\zeta \cdot \boldsymbol{n}) \cdot \xi_\tau - \nabla_\tau(\xi \cdot \boldsymbol{n}) \cdot \zeta_\tau\Big)
\end{aligned}$$

(where we used that $D_\tau \boldsymbol{n}$ is symmetric), which concludes the proof (a priori, ℓ_0 depends on the extension operator that has been chosen, but as in the final formula the extension only appears in ℓ_2, this last one do not depend, in fact, of the extension operator). □

2.2 Examples of shapes derivatives on general domains.

For a domain $\Omega \subset \mathbb{R}^d$, we consider in this section (and in the rest of the paper) its volume $|\Omega|$, its perimeter $P(\Omega)$, its Dirichlet energy $E(\Omega)$ defined as

$$E(\Omega) = -\frac{1}{2}\int_{\partial\Omega} |\nabla u_\Omega|^2,$$

where u_Ω is the solution of $-\Delta u = 1$ in $\mathrm{H}^1_0(\Omega)$ (this is equivalent to (1.15)) and λ_1 the first eigenvalue of the Dirichlet Laplace operator. The existence and computations of these shape derivatives of these functionals are well known, see for example [26, Chapter 5].

We need first to precise some geometrical definitions: the mean curvature (understood as the sum of the principal curvatures of $\partial\Omega$) is denoted by H. We recall that $\boldsymbol{B} = D_\tau \boldsymbol{n}$ is the second fundamental form of $\partial\Omega$ and that $\|B\|^2$ is the sum of the squares of the principal curvatures of $\partial\Omega$.

Lemma 2.7 (Expression of shape derivatives) *If Ω is C^2, one has, for any $\varphi \in \mathrm{C}^\infty(\partial\Omega)$,*

- $\ell_1[\mathrm{Vol}](\Omega).\varphi = \int_{\partial\Omega} \varphi, \qquad \ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} H\varphi^2.$

- $\ell_1[P](\Omega).\varphi = \int_{\partial\Omega} H\varphi, \qquad \ell_2[P](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} |\nabla_\tau \varphi|^2 + \int_{\partial\Omega} \left[H^2 - \|B\|^2\right]\varphi^2$

- $\ell_1[E](\Omega).\varphi = -\frac{1}{2}\int_{\partial\Omega} (\partial_n u)^2\varphi,$

  $\ell_2[E](\Omega).(\varphi,\varphi) = \langle \partial_n u\ \varphi, \Lambda(\partial_n u\ \varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} + \int_{\partial\Omega}\left[\partial_n u + \frac{1}{2}H(\partial_n u)^2\right]\varphi^2$

  where $\Lambda :\ \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ is the Dirichlet-to-Neumann map defined as $\Lambda(\psi) = \partial_n V(\psi)$ with $V(\psi)$ is the solution of

$$-\Delta V(\psi) = 0 \text{ in } \Omega, \quad V(\psi) = \psi \text{ on } \partial\Omega, \tag{2.3}$$

- $\ell_1[\lambda_1](\Omega).\varphi = -\int_{\partial\Omega} (\partial_n v)^2\varphi, \qquad \ell_2[\lambda_1](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} 2w(\varphi)\ \partial_n w(\varphi) + H(\partial_n v)^2\varphi^2$

  where v is the normalized eigenfunction (solution in $\mathrm{H}^1_0(\Omega)$ of $-\Delta v = \lambda_1 v$ with $\|v\|_{\mathrm{L}^2(\Omega)} = 1$ and $v > 0$ in Ω) and $w(\varphi)$ is the solution of

$$\left\{\begin{array}{rcl} -\Delta w(\varphi) &=& \lambda_1 w(\varphi) - v\int_{\partial\Omega}(\partial_n v)^2\varphi \text{ in } \Omega, \\ w(\varphi) &=& -\varphi\partial_n v \text{ on } \partial\Omega, \\ \int_\Omega v\ w(\varphi) &=& 0. \end{array}\right. \tag{2.4}$$

A fundamental fact for this work appears here in the expression of the shape hessians. Even if they are defined and derived for regular perturbations, they are naturally defined and continuous on different Sobolev spaces on $\partial\Omega$. The hessian of the perimeter is defined on $\mathrm{H}^1(\partial\Omega)$, the hessian of Dirichlet energy on $\mathrm{H}^{1/2}(\partial\Omega)$ while the hessian of the volume is defined on $\mathrm{L}^2(\partial\Omega)$ as expressed in the following continuity properties:

Lemma 2.8 (Continuity of shape Hessians) *If Ω is C^2, there is a constant $C > 0$ such that*

$$|\ell_2[P](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^1(\partial\Omega)}, \quad |\ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{L}^2(\partial\Omega)},$$
$$|\ell_2[E](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}, \quad |\ell_2[\lambda_1](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

Therefore, from this Lemma, it is natural to consider the extension of these bilinear forms to their space of continuity. Note that the C^2 assumption on Ω is sufficient here, as explained in Remark 2.3.

2.3 The case of balls

When proving the second order optimality condition, one needs to explicit the shape derivatives of these functionals on the balls B_R (of radius R). For the Dirichlet energy E, we need to remark that $u(x) = (R^2 - |x|^2)/2d$ solves $-\Delta u = 1$ in $\mathrm{H}^1_0(B_R)$ and satisfies $\partial_n u = -R/d$ on ∂B_R. For λ_1, we recall that the eigenvalue and eigenfunction are

$$\lambda_1(B_R) = \frac{j^2_{d/2-1}}{R^2} \text{ associated to } v(x) \ = \ \alpha_d\ |x|^{1-d/2}\ J_{d/2-1}\left(\frac{j_{d/2-1}}{R}\ |x|\right),$$

where the normalization constant is defined as

$$\alpha_d = \left[|\partial B_1|\int_0^R rJ^2_{d/2-1}\left(\frac{j_{d/2-1}}{R}r\right)\ dr\right]^{-1/2},$$

and where $j_{d/2-1}$ is the first zero of Bessel's function $J_{d/2-1}$. On the unit ball, the eigenfunction satisfies

$$\partial_n v = \sqrt{\frac{2}{P(B_1)}}\, j_{d/2-1} := \gamma_d, \text{ so that } \gamma_d^2 = \frac{2\lambda_1(B_1)}{P(B_1)}; \tag{2.5}$$

from [27, p. 35]. We obtain the **shape gradients**:

$$\ell_1[\mathrm{Vol}](B_R).\varphi = \int_{\partial B_R} \varphi, \qquad \ell_1[P](B_R).\varphi = \frac{d-1}{R}\int_{\partial B_R} \varphi,$$
$$\ell_1[E](B_R).\varphi = -\frac{R^2}{2d^2}\int_{\partial B_R} \varphi, \qquad \ell_1[\lambda_1](B_R).\varphi = -\gamma_d^2\int_{\partial B_R} \varphi.$$

Let us notice that these four shape gradients at balls are colinear. As a consequence, the balls are critical domains for the perimeter, the Dirichlet energy and λ_1 (or any sum of these functionals) under a volume constraint, and these formula easily provide the value of the Lagrange-multiplier.

Let us turn our attention to the **hessians**. The value of $\ell_2[\lambda_1]$ is a bit more involved, so we deal with it in the next Lemma. For the other functionals, it is known that:

$$\ell_2[\mathrm{Vol}](B_R).(\varphi,\varphi) = \frac{d-1}{R}\int_{\partial B_R} \varphi^2,$$
$$\ell_2[P](B_R).(\varphi,\varphi) = \int_{\partial B_R} |\nabla_\tau \varphi|^2 + \frac{(d-1)(d-2)}{R^2}\int_{\partial B_R} \varphi^2,$$
$$\ell_2[E](B_R).(\varphi,\varphi) = \frac{R^2}{d^2}\langle \varphi, \Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times \mathrm{H}^{-1/2}} - \frac{d+1}{2d^2}R\int_{\partial B_R} \varphi^2.$$

It is well known in the literature (see for example [11]) that on balls the quadratic forms associated to the Lagrangian are coercive on their natural spaces. Let us make this point precise by diagonalizing the Hessian. The useful tool to explicit the shape hessian under consideration is spherical harmonics defined as the restriction to the unit sphere of harmonic polynomials.

We recall here facts from [37, pages 139-141]. We let $\mathcal{H}_k$ denote the space of spherical harmonics of degree k. It is also the eigenspace of the Laplace-Beltrami operator on the unit sphere associated with the eigenvalue $-k(k+d-2)$. Let $(Y^{k,l})_{1\le l\le d_k}$ be an orthonormal basis of $\mathcal{H}_k$ with respect to the $\mathrm{L}^2(\partial B_1)$ scalar product. The $(\mathcal{H}_k)_{k\in\mathbb{N}}$ spans a vector space dense in $\mathrm{L}^2(\partial B_1)$ and the family $(Y^{k,l})_{k\in\mathbb{N},1\le l\le d_k}$ is a Hilbert basis of $\mathrm{L}^2(\partial B_1)$. Hence, any function φ in $\mathrm{L}^2(\partial B_1)$ can be decomposed as the Fourier series:

$$\varphi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}(x), \quad \text{for } |x| = 1. \tag{2.7}$$

Then, by construction, the function h defined by

$$h(x) = \sum_{k=0}^{\infty} |x|^k \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}\left(\frac{x}{|x|}\right), \quad \text{for } |x| \le 1,$$

is harmonic in B_1 and satisfies $h = \varphi$ on ∂B_1. Moreover, the sequence of coefficients $\alpha_{k,l}$ characterizes the Sobolev regularity of φ: indeed $\varphi \in \mathrm{H}^s(\partial B_1)$ if and only if the sum $\sum_k (1+k^2)^s \sum_l |\alpha_{k,l}|^2$ converges. Let us now prove the following lemma expressing the fact that the shape hessian of the volume, the perimeter, the Dirichlet energy and the first eigenvalue are diagonal on the basis of spherical harmonics.

Lemma 2.9 *Assume that φ is decomposed on the basis of spherical harmonics as in* (2.7)*, then*

$$\ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)\ \alpha_{k,l}(\varphi)^2,$$

$$\ell_2[E](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{1}{d^2}\ k\ -\frac{d+1}{2d^2}\right]\alpha_{k,l}(\varphi)^2,$$

$$\ell_2[P](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[k^2+(d-2)k+(d-1)(d-2)\right]\ \alpha_{k,l}(\varphi)^2,$$

$$\ell_2[\lambda_1](B_1)(\varphi,\varphi) = \gamma_d^2\left(3P(B_1)^2\alpha_{0,1}^2(\varphi)+\sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\left[k-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)\right).$$

where γ_d is the constant defined in (2.5)*.*

This result is very useful to look for the sign of $\ell_2[J-\mu\mathrm{Vol}](B_1)$ where J is the functional we minimize and μ is a Lagrange multiplier, see Section 4. For the computation of $\ell_2[\lambda_1](B_1)$, we use the presentation given by D. Henry in [27, p. 35], though the computation was performed by Lord Rayleigh in [35] when $d=2$ (see also [36]).

Proof. We decompose $\varphi\in \mathrm{L}^2(\partial B_1)$ on the spherical harmonics basis as

$$\varphi(x) = \sum_{k=0}^{\infty}\left(\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)Y^{k,l}(x)\right),\quad \text{for } |x|=1. \tag{2.8}$$

and let us express the various integrals arising in the shape hessian in terms of the spherical harmonics decomposition. First we check that

$$\int_{\partial B_1}\varphi^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

$$\int_{\partial B_1}|\nabla_\tau\varphi|^2 = -\int_{\partial B_1}\varphi\ \Delta_\tau\varphi = \sum_{k=0}^{\infty}k(k+d-2)\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

Then, we precise the term involving the Dirichlet-to-Neumann map that appears in the shape hessian of the Dirichlet energy. The series defining h is normally convergent inside B_1, we cannot directly differentiate with respect to r up to the boundary. Though, by Green formula, we have:

$$\begin{aligned}
\langle\varphi,\Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} &= \int_{\partial B_1}\varphi\partial_n h = \int_{B_1}|\nabla h|^2\\
&= \int_0^1\left(\int_{\partial B_r}\left((\partial_n h)^2+|\nabla_\tau h|^2\right)d\sigma\right)dr = \int_0^1\left(\int_{\partial B_r}\left((\partial_n h)^2-h\Delta_\tau h\right)d\sigma\right)dr\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\int_0^1 r^{d-1}\left[k^2r^{2(k-1)}+\frac{k(k+d-2)}{r^2}\ r^{2k}\right]dr\ \alpha_{k,l}(\varphi)^2\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{k^2}{2k+d-2}+\frac{k(k+d-2)}{2k+d-2}\right]\alpha_{k,l}(\varphi)^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}k\ \alpha_{k,l}(\varphi)^2.
\end{aligned}$$

We obtain $\ell_2[\mathrm{Vol}]$, $\ell_2[P]$ and $\ell_2[E]$ by gathering these elementary terms.

Let us now consider the case of the first eigenvalue. Again we decompose φ on the basis of spherical harmonics according to (2.8) and we apply [27, p. 35]: it is proven there that for a second order volume preserving path, that is $t \mapsto T_t$ such that $|T_t(\Omega)| = |\Omega| + o(t^2)$ for small t, we have

$$\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} = 2\sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\gamma_d^2\ \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)$$

where $\varphi = (\frac{d}{dt}T_t)_{|t=0}\cdot \boldsymbol{n}$ and we have used the recurrence formula for Bessel function $J_\nu'(z) = (\nu/z)J_\nu(z) - J_{\nu+1}(z)$ to adapt his expression to our notations ([1, 9.1.27, p 361]). To deduce $\ell_2[\lambda_1]$ from this computation, we introduce θ a smooth vector field which is normal on ∂B_1 and denote $\varphi = \theta\cdot\boldsymbol{n}$. We assume that $\int_{\partial B_1}\varphi = \alpha_{0,1}(\varphi) = 0$. It is then clear that there exists ξ such that $T_t := Id + t\theta + \frac{t^2}{2}\xi$ is volume preserving at the second order that is such that

$$\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi) + \ell_1[\mathrm{Vol}](B_1)(\psi) = 0.$$

Then we observe that for a smooth shape functional J and for such $t\mapsto T_t$,

$$\left(\frac{d^2}{dt^2}J(T_t(B_1))\right)_{|t=0} = \ell_2[J](B_1)(\varphi,\varphi) + \ell_1[J](B_1)(\psi),$$

where $\psi := \xi\cdot\boldsymbol{n}$, and therefore, denoting μ the Lagrange multiplier such that $\ell_1[\lambda_1 - \mu\mathrm{Vol}](B_1) = 0$, we obtain

$$\begin{aligned}\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} &= \ell_2[\lambda_1](B_1)(\varphi,\varphi) + \ell_1[\lambda_1](B_1)(\psi) = \ell_2[\lambda_1](B_1)(\varphi,\varphi) + \mu\ell_1[\mathrm{Vol}](B_1)(\psi)\\ &= \ell_2[\lambda_1](B_1)(\varphi,\varphi) - \mu\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi)\end{aligned}$$

Then, we get, as here $\mu = -\gamma_d^2$:

$$\begin{aligned}\ell_2[\lambda_1](\varphi,\varphi) &= \left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} + \mu\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi),\\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\gamma_d^2\ \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi) - \gamma_d^2\sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)a_{k,l}^2(\varphi),\\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}\gamma_d^2\ \left[k-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi).\end{aligned}$$

It remains to compute the coefficient associated to the mode $k=0$. It suffices to consider the deformations as $T_t(x) = x + t\|Y^{0,1}\|x$ mapping the ball B_1 onto the ball of radius $1 + tP(B_1)^{1/2}$. Since λ_1 is homogeneous of degree -2, we get $\lambda(t) = (1+tP(B_1)^{1/2})^{-2}\lambda_1(B_1)$ so that $\lambda''(0) = 6P(B_1)\lambda_1(B_1)$. $\square$

3 Main Theorem

In the introduction, we gave the unconstrained version of the main result of this paper. As in most applications we need to deal with a volume constraint and a translation invariance of the functional, we describe here the corresponding statement. In the rest of the section, we discuss condition $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and prove (1.6) about coercivity, then condition $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$ with old and new examples, and we finally prove Theorems 1.1 and 3.2.

We start with the suitable definitions of critical and stable domains for problems with volume constraint and translation invariant functionals:

Definition 3.1 *Let Ω^* be a shape and J a shape functional defined and twice shape differentiable at Ω^*, where Θ is the Banach space for differentiability.*

- *We say that Ω^* is a critical domain for J under volume constraint if*

$$\forall \varphi \in \mathrm{C}^\infty(\partial\Omega^*) \text{ such that } \ell_1[\mathrm{Vol}](\Omega^*).\varphi = \int_{\partial\Omega^*} \varphi = 0, \qquad \ell_1[J](\Omega^*).(\varphi) = 0. \tag{3.1}$$

  It is well-known that it is equivalent to the existence of $\mu \in \mathbb{R}$ such that $(\ell_1[J] - \mu\ell_1[\mathrm{Vol}])(\Omega^*) = 0$ on $\mathrm{C}^\infty(\partial\Omega^*)$; in that case, μ is called a Lagrange multiplier associated to J.

- *When Ω^* is a critical domain for J under volume constraint, we say that Ω^* is a stable shape for J under volume constraint and up to translations if*

$$\forall \varphi \in T(\partial\Omega) \setminus \{0\},\ (\ell_2[J] - \mu\ell_2[\mathrm{Vol}])(\Omega).(\varphi,\varphi) > 0 \tag{3.2}$$

  where

$$T(\partial\Omega) := \left\{ \varphi \in \mathrm{H}^s(\partial\Omega),\ \int_{\partial\Omega} \varphi = 0,\ \int_{\partial\Omega} \varphi \overrightarrow{x} = \overrightarrow{0} \right\}, \tag{3.3}$$

  μ is the Lagrange multiplier associated to J and $s \geq 0$ is the lowest index so that $\ell_2[J](\Omega)$ is continuous on $\mathrm{H}^s(\partial\Omega)$ (see Lemma 2.8).

Here is the main result of this paper:

Theorem 3.2 *Let Ω^* of class C^3, and J a shape functional, translation invariant and twice Fréchet differentiable on a neighborhood of Ω^* for $d_{\mathrm{W}^{1,\infty}}$. We assume:*

- **Structural hypotheses:** *there exists $s_2 \in (0,1]$ and Θ a Banach space with $\mathrm{C}^\infty(\mathbb{R}^d,\mathbb{R}^d) \subset \Theta \subset \mathrm{W}^{1,\infty}$ such that J satisfies* $(\mathbf{C}_{\mathrm{H}^{s_2}})$ *and* $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$ *at Ω^*,*
- **Necessary optimality conditions:**
  - *Ω^* is a critical shape under volume constraint for J,*
  - *Ω^* is a stable shape for J under volume constraint and up to translations:*

Then Ω^* is an H^{s_2}**-stable local minimum of J in a Θ-neighborhood under volume constraint**, *that is to say there exists $\eta > 0$ and $c = c(\eta) > 0$ such that:*

$$\forall\ \Omega \text{ such that } d_\Theta(\Omega,\Omega^*) \leq \eta \text{ and } |\Omega| = |\Omega^*|, \quad J(\Omega) \geq J(\Omega^*) + c\bar{d}_{\mathrm{H}^{s_2}}(\Omega^*,\Omega)^2.$$

where $\bar{d}_{\mathrm{H}^{s_2}}$ is defined in (1.5).

3.1 About coercivity and condition $(\mathbf{EC}_{\mathrm{H}^s})$

Usually the coercivity property for the second order derivative (or the one of the Lagrangian; we notice here that thanks to our exact penalization procedure (see (1.14) and the proof of Theorem 3.2) this section and the following can indifferently be used for both the unconstrained and the constrained result) has to be proved by hand on each specific example by studying the lower bound of the spectrum of the bilinear form ℓ_2 defined in Theorem 2.1, typically thanks to Lemma 2.9. Nevertheless, when ℓ_2 enjoys some structural property, coercivity can be more easily checked as a consequence of the following general lemma.

Lemma 3.3 *Let X be a Lipschitz manifold, $s_2 \in [0,1]$, and V a vectorial subspace of $\mathrm{H}^{s_2}(X)$, closed for the weak convergence in $\mathrm{H}^{s_2}(X)$. If ℓ, a quadratic form defined on $\mathrm{H}^{s_2}(X)$ satisfies condition* $(\mathbf{C}_{\mathrm{H}^{s_2}})$, *namely*

$(\mathbf{C}_{\mathrm{H}^{s_2}})$ *there exists $0 \leq s_1 < s_2$ and $c_1 > 0$ such that $\ell = \ell_m + \ell_r$ with*

$$\begin{cases} \ell_m \text{ is lower semi-continuous in } \mathrm{H}^{s_2}(X) \text{ and } \ell_m(\varphi,\varphi) \geq c_1|\varphi|^2_{\mathrm{H}^{s_2}},\ \forall \varphi \in \mathrm{C}^\infty(X), \\ \ell_r \text{ continuous in } \mathrm{H}^{s_1}(X). \end{cases}$$

then the following propositions are equivalent:

(i) $\ell(\varphi, \varphi) > 0$ *for any* $\varphi \in V \setminus \{0\}$.

(ii) $\exists \lambda > 0, \quad \ell(\varphi, \varphi) \geq \lambda \|\varphi\|^2_{\mathrm{H}^{s_1}}$ *for any* $\varphi \in V$.

(iii) $\exists \lambda > 0, \quad \ell(\varphi, \varphi) \geq \lambda \|\varphi\|^2_{\mathrm{H}^{s_2}}$ *for any* $\varphi \in V$.

Remark 3.4 In practice, we apply this lemma to the spaces $\mathrm{H}^s(\partial\Omega)$ where $\partial\Omega$ is smooth enough, and V is either $\mathrm{H}^{s_2}(\partial\Omega)$ or $T(\partial\Omega)$ defined in (3.3).

Proof. Since the implications $(iii) \implies (ii)$ and $(ii) \implies (ii)$ are trivial, it suffices to prove $(i) \implies (iii)$. To that end, let $(\varphi_k)_k$ a minimizing sequence for the problem

$$\inf \left\{ \ell(\varphi, \varphi), \ \varphi \in V, \|\varphi\|_{\mathrm{H}^{s_2}} = 1 \right\}.$$

Up to a subsequence, φ_k weakly converges in $\mathrm{H}^{s_2}(X)$ to some $\varphi_\infty \in V$. By the compactness of the embedding of $\mathrm{H}^{s_2}(X)$ into $\mathrm{H}^{s_1}(X)$, $\varphi_k \to \varphi_\infty$ in $\mathrm{H}^{s_1}(X)$ so that $\ell_r(\varphi_k, \varphi_k) \to \ell_r(\varphi_\infty, \varphi_\infty)$. We distinguish two cases: if $\varphi_\infty \neq 0$, $\liminf_k \ell_m(\varphi_k, \varphi_k) \geq \ell_m(\varphi_\infty, \varphi_\infty)$ by the lower semi continuity of ℓ_m, so that $\liminf_k \ell(\varphi_k, \varphi_k) \geq \ell(\varphi_\infty, \varphi_\infty) > 0$ by assumption **(i)**. If $\varphi_\infty = 0$, then as the norm $\|\cdot\|_{\mathrm{H}^{s_2}}$ is equivalent to the norm $\|\cdot\|_{\mathrm{H}^{s_1}} + |\cdot|_{\mathrm{H}^{s_2}}$, we know that $|\varphi_k|_{\mathrm{H}^{s_2}}$ is bounded from below by a positive constant, and using $(\mathbf{C}_{\mathrm{H}^{s_2}})$, $\liminf_k \ell(\varphi_k, \varphi_k) = \liminf_k \ell_m(\varphi_k, \varphi_k) \geq c_1 \liminf_k |\varphi_k|^2_{\mathrm{H}^{s_2}} > 0$. $\square$

Remark 3.5 The equivalence between coercivity in L^2 and H^1 was already known in the context of stable minimal surface it appears in the work [25] of Grosse-Brauckmann. Also in [2], the previous lemma is proven in the particular case of the functional they study.

Remark 3.6 When, one applies this lemma to a shape hessian, assumption (i) is not natural. Indeed, shape derivatives are defined for regular perturbations that are dense subsets of $\mathrm{H}^s(\partial\Omega)$ and one could expect: $\ell(\varphi, \varphi) > 0$ for $\varphi \in V \setminus \{0\}$ smooth enough. But, in that case, our proof is not valid since φ_∞ may not be smooth and therefore not admissible to test the positivity property. Therefore, the bilinear form ℓ has to be extended by continuity to the whole $\mathrm{H}^s(\partial\Omega)$ (see assumption (1.11) in Theorem 1.1 and (3.2) for Theorem 3.2). Notice that this extension is for free once the expression of the shape derivative has been computed as illustrated by Lemma 2.7.

We conclude this section noticing that the shape hessians of the model functionals from Section 2 satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$:

- The perimeter satisfies $(\mathbf{C}_{\mathrm{H}^1})$ with

$$\ell_m[P](\varphi, \varphi) = \int_{\partial\Omega} |\nabla_\tau \varphi|^2 \quad \text{and} \quad \ell_r[P](\varphi, \varphi) = \int_{\partial\Omega} \left[H^2 - \|B\|^2\right] \varphi^2 \qquad \text{(here we can choose } s_1 = 0\text{)}.$$

- The Dirichlet energy and λ_1 satisfy $(\mathbf{C}_{\mathrm{H}^{1/2}})$ (again $s_1 = 0$):

$$\ell_m[E](\varphi, \varphi) = \langle \partial_n u \varphi, \Lambda(\partial_n u \varphi) \rangle_{\mathrm{H}^{1/2} \times \mathrm{H}^{-1/2}} \quad \text{and} \quad \ell_r[E](\varphi, \varphi) = \int_{\partial\Omega} \left[\partial_n u + \frac{1}{2} H (\partial_n u)^2\right] \varphi^2,$$

$$\ell_m[\lambda_1](\Omega).(\varphi, \varphi) = \int_{\partial\Omega} 2 w(\varphi) \ \partial_n w(\varphi) \quad \text{and} \quad \ell_r[\lambda_1](\varphi, \varphi) = \int_{\partial\Omega} H (\partial_n v)^2 \varphi^2.$$

See Section 5.2 for an example where $(\mathbf{C}_{\mathrm{H}^s})$ is not satisfied.

3.2 About Condition $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$

In this section, we show that our main examples satisfy condition $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$ where s is given in Section 3.1, and Θ is hoped to be as small as possible.

Given Θ a Banach space of vector fields, we fix Ω^* a C^2 open set (the regularity insures the uniqueness of the projection on $\partial\Omega^*$ in its neighborhood). We recall that for $\Omega = (Id+\theta)(\Omega^*)$ with $\|\theta\|_\Theta$ small enough, we can write

$$\partial\Omega = \{x + h(x)\boldsymbol{n}(x), x \in \partial\Omega^*\}$$

for some function $h : \partial\Omega^* \to \mathbb{R}$, and we then focus on $(\Omega_t)_{t\in[0,1]}$ the path of open sets so that

$$\partial\Omega_t = \{x + th(x)\boldsymbol{n}(x), x \in \partial\Omega^*\}.$$

It can be useful to find a path of vector field that can describe this path of shapes. One convenient way to do this is to extend h and $\boldsymbol{n}$ in a neighborhood of $\partial\Omega^*$ so that it is constant in the direction given by $\boldsymbol{n}$ (then we should be careful not to mix $\boldsymbol{n}$ with $\boldsymbol{n}_t$ which will denote the normal vector to $\partial\Omega_t$):

$$h(x) = h(\pi_{\partial\Omega^*}(x)) \qquad \text{and} \qquad \boldsymbol{n}(x) = \boldsymbol{n}(\pi_{\partial\Omega^*}(x)),$$

where $\pi_{\partial\Omega^*}$ is the projection on $\partial\Omega^*$, well-defined in a neighborhood of $\partial\Omega^*$. Then we define $\xi_h = h(x)\boldsymbol{n}(x)$ in this neighborhood, and extend it smoothly to $\mathbb{R}^d$, so that $\xi_h \in \Theta$ (assuming Ω^* smooth enough). Therefore, denoting $T_h = Id + \xi_h$, we have $\Omega = T_h(\Omega^*)$, and $j(t) = \mathcal{J}_{\Omega^*}(t\xi_h) = J(\Omega_t)$ for any $t \in [0,1]$ (where J is the shape functional under study). Therefore $j''(t) = \mathcal{J}''_{\Omega^*}(t\xi_h)\cdot(\xi_h,\xi_h) = J''(\Omega_t)\cdot(\xi_h,\xi_h)$.

3.2.1 Geometric quantities

• The volume:

Proposition 3.7 *If Ω is C^2, then* Vol *satisfies* $(\mathbf{EC}_{\mathrm{L}^2,\mathrm{W}^{1,\infty}})$ *condition.*

Before proving this result, we give a geometric Lemma, inspired by the results in [11] in the context where $\Theta = \mathrm{C}^{2,\alpha}$. We recall that $J(h) := \det DT_h \|({}^t DT_h^{-1})\boldsymbol{n}\|$ is the surface jacobian, appearing when changing variables between $\partial\Omega_h$ and $\partial\Omega^*$. In this section, the notation $\widehat{w}_h$ stands for $w_h \circ T_h$ where w_h is defined on Ω_h or $\partial\Omega_h$.

Lemma 3.8 *We have the following Taylor expensions, where $\mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}})$ is a domination uniform in x,*

- $J(h) = 1 + \ell_1^J(h) + \frac{1}{2}\ell_2^J(h,h) + \mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}})(|h|^2 + |\nabla h|^2)$,
- $\widetilde{\boldsymbol{n}}_h = \boldsymbol{n} + \ell_1^{\boldsymbol{n}}(h) + \frac{1}{2}\ell_2^{\boldsymbol{n}}(h,h) + \mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}})(|h|^2 + |\nabla h|^2)$.

where $(\ell_1^J, \ell_1^{\boldsymbol{n}})$, $(\ell_2^J, \ell_2^{\boldsymbol{n}})$ are respectively linear and quadratic form, acting on $(h, \nabla h)$.

Proof of Lemma 3.8: The first part follows simply from the fact that $A \in M_d(\mathbb{R}) \mapsto \det(A)\left\|({}^t A^{-1})\boldsymbol{n}\right\|$ is smooth in a neighborhood of Id, and the fact that $D\xi_h = h(D\boldsymbol{n}) + \nabla h \otimes \boldsymbol{n}$.
For the second part, we use a level-set parametrization: there exists ϕ of class C^2 such that $\Omega^* = \{\phi < 0\}$ and $\nabla\phi$ do not vanishes, and then $\Omega = \{\phi \circ T_h^{-1} < 0\}$. Therefore

$$\widetilde{\boldsymbol{n}}_h - \boldsymbol{n} = \frac{\nabla(\phi\circ T_h^{-1})}{|\nabla(\phi\circ T_h^{-1})|}\circ T_h - \frac{\nabla\phi}{|\nabla\phi|} = \frac{{}^t DT_h^{-1}.\nabla\phi}{|{}^t DT_h^{-1}.\nabla\phi|} - \frac{\nabla\phi}{|\nabla\phi|},$$

and we conclude using the smoothness of $A \mapsto {}^t A^{-1}$ and $w \in \mathbb{R}^d \mapsto \frac{w}{|w|}$ in the neighborhood of Id and $\nabla\phi$ respectively. $\square$

Proof of Proposition 3.7: We denote $H = \mathrm{div}(\boldsymbol{n})$ the mean curvature of $\partial\Omega^*$ (extended to $\mathbb{R}^d$ and constant in the normal direction of $\partial\Omega^*$). We use (2.2), and the fact that $\mathrm{div}(\xi) = \nabla h \cdot \boldsymbol{n} + h\mathrm{div}(\boldsymbol{n}) = h\mathrm{div}(\boldsymbol{n})$ as h is constant in the direction of $\boldsymbol{n}$. Therefore if $v(t) = \mathrm{Vol}(\Omega_t)$, we have

$$v''(t) = \int_{\partial\Omega_t} \xi \cdot \boldsymbol{n}_t \mathrm{div}(\xi) = \int_{\partial\Omega_t} \mathrm{div}(\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{n}_t)h^2 = \int_{\partial\Omega^*} H(\boldsymbol{n}\cdot\widehat{\boldsymbol{n}}_t)h^2 J(t).$$

With Lemma **??**, we easily obtain

$$|v''(t) - v''(0)| \leq C\|T_t - I\|_{\mathrm{W}^{1,\infty}}\|h\|^2_{\mathrm{L}^2} = Ct\|\xi\|_{\mathrm{W}^{1,\infty}}\|h\|^2_{\mathrm{L}^2}.$$

□

Remark 3.9 In the spirit of [34, Lemma 4.1], we could try a direct proof using the divergence formula: we indeed obtain

$$|\Omega| = \frac{1}{d}\int_{\partial\Omega} x\cdot\boldsymbol{n}_h = \frac{1}{d}\int_{\partial\Omega^*}(x+\xi_h(x))\cdot\widehat{\boldsymbol{n}}_h J(h) = \int_{\partial\Omega^*}(x+h\boldsymbol{n}(x)).(\boldsymbol{n}+\ell_1(h)+\frac{1}{2}\ell_2(h,h)+\mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}})(|h|^2+|\nabla h|^2))$$

but this only leads to the fact that the volume satisfy $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$.

- **The perimeter:**

Proposition 3.10 *If Ω is C^2, then P satisfies $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ condition.*

Proof. We follow exactly the proof suggested in Remark 3.9:

$$P(\Omega_h) = \int_{\partial\Omega_h} 1 = \int_{\partial\Omega} J(h) = P(\Omega) + \ell_1[P](\Omega)(h) + \frac{1}{2}\ell_2[P](\Omega)(h,h) + \mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}})\|h\|^2_{\mathrm{H}^1(\mathbb{S}^{d-1})},$$

where we used Lemma 3.8. This result is actually a strong version of $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$ condition: indeed, denoting $p(t) = P(\Omega_t)$, we apply the previous formula to th, and differentiate in t to get:

$$p''(t) = p''(0) + t\mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}})\|h\|^2_{\mathrm{H}^1(\mathbb{S}^{d-1})}$$

and the property follows. □

Remark 3.11 It is interesting to compare the two strategies used for the volume and for the perimeter: indeed, for the volume we prefered to use the structure theorem, which lead to the best estimate (see Remark 3.9), while a similar strategy for the perimeter, as it is done in [11] or in [2, Proof of Theorem 3.9] (but for a different path of shapes) lead to weaker results (in the sense that the space Θ is smaller, namely $\mathrm{C}^{2,\alpha}$ and $\mathrm{W}^{2,p}$ respectively). Therefore, we prefered here to directly prove a Taylor expension in h for the functional P, with a refined control of the remainder term, that is an expansion of the form

$$J(\Omega_h) = J(\Omega^*) + \ell_1[J](\Omega^*)(h) + \frac{1}{2}\ell_2[J](\Omega^*)(h,h) + \omega(\|h\|_\Theta)\|h\|^2_{\mathrm{H}^s(\partial\Omega^*)}. \tag{3.4}$$

This condition could be considered as another version of condition $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$. As we will see in the proof of Theorem 1.1, our initial condition indeed implies (3.4), while it is easy to see, as it is done in the previous proof when $\omega(x) = x$, that under very weak assumption on ω (including for example $\omega(x) = x^\alpha, \alpha > 0$), (3.4) implies $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$.

3.2.2 PDE energy

For PDE energies, a condition of the type $(\mathbf{EC}_{\mathrm{H}^s,\Theta})$ was studied first in [12] where it is proven that in dimension two the Dirichlet energy satisfy $(\mathbf{EC}_{\mathrm{H}^{1/2},\mathrm{C}^{2,\alpha}})$ (for a volume preserving path instead of a normal path), then a similar result is proven for general PDE functionals in any dimension in [11], either for the path (1.10) or a volume preserving path. More recently in [2], it was proven that the functional described in (4.1) involving the sum of the perimeter and a PDE functional (of a different kind than in [11]) satisfy $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{2,p}})$ for p large enough, also for a volume preserving path. Thanks to our method to handle the volume constraint (see Section 3.3), we only need to deal with the path (1.10), which is easier than considering volume preserving pathes.

In this section, we chose to focus on E and λ_1 (the former is yet not handled in the literature), and we will improve the space Θ compare to [11] (the smaller the space Θ is, the better the result is). For the functional in [2] we refer to Section 4.1. Note that condition $(\mathbf{EC}_{\mathrm{H}^{1/2},\mathrm{C}^{2,\alpha}})$ is also established for the drag in a Stokes flow in [8].

In other words, the aim of this section is to prove:

Proposition 3.12 *Let Ω^* be a bounded domain. If Ω^* is $\mathrm{C}^?$, then E and λ_1 satisfies $(\mathbf{EC}_{\mathrm{H}^{1/2},\mathrm{W}^{2,p}})$ for $p > d+2$.*

We treat the case of E (see [11] and also [7, Appendix]) where it is proven that E satisfy $(\mathbf{EC}_{\mathrm{H}^{1/2},\mathrm{C}^{2,\alpha}})$ with an emphasis on how to treat $\mathrm{W}^{2,p}$ deformations. We sketch the proof except when the treatment differs from the known case. We recall the expression of the second derivative along the path that we split for convienience into : $E''(t) = \mathcal{T}_1(t) + \mathcal{T}_2(t)$ with

$$\begin{aligned}
\mathcal{T}_1(t) &= \int_{\partial\Omega_t} \partial_{\boldsymbol{n}(t)} u'(t) \partial_{\boldsymbol{n}(t)} u(t) \xi \cdot \boldsymbol{n}(t), \\
\mathcal{T}_2(t) &= \int_{\partial\Omega_t} \nabla|\nabla u|^2 \cdot \xi(\xi \cdot \boldsymbol{n}(t)) + H(t)(\xi \cdot \boldsymbol{n}(t))^2 (\partial_{\boldsymbol{n}(t)} u(t))^2
\end{aligned}$$

The idea is to estimate separately the variations of each terms. In order to replace $\mathrm{C}^{2,\alpha}$ with $\mathrm{W}^{2,p}$, two steps have to be adapted. First, one needs to estimate $\|\widehat{u}_t - u\|_{\mathrm{W}^{2,p}}$ using the Sobolev theory of elliptic PDE. We provide here a new proof based on the method used to prove existence of shape derivation. Second, one then takes advantage of the fact that we seek for an estimate in $\mathrm{H}^{1/2}$ of the normal component of the deformation, so any term naturally leading to the L^2-norm can be improved.

Estimate of $\mathcal{T}_1(t) - \mathcal{T}_1(0)$. This part is unaffected by the passage to the $\mathrm{W}^{2,p}$. Only first order derivatives are involved and the L^∞ bounds on the first order derivatives are deduced by the Sobolev injections as soon as $p > d$.

• **First modified step: Estimate of $\|\widehat{u}_t - u\|_{\mathrm{W}^{2,q}}$ by the $\mathrm{W}^{2,p}$ norm of the deformations.** Such an estimate is a direct consequence of the regularity of the map $\theta \mapsto \widehat{u}_\theta$ from X to Y in well chosen spaces X and Y. The analyticity of the map from $X = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ into $Y = \mathrm{H}^1_0(\Omega)$ is proved in [26, Proof of Theorem 5.3.2]. The same conclusion is reached for the map from $X = \mathcal{C}^{2,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ into $Y = \mathcal{C}^{2,\alpha}(\Omega)$ in [27, Example 3.1, p. 28]. As a clear consequence of the $\mathcal{C}^1$ regularity, there exist two constants $\eta > 0$ and C such that

$$\|\widehat{v}_\theta - v_0\|_Y \leq C\|\theta\|_X, \tag{3.5}$$

when $\|\theta - Id\|_Y \leq \eta$. Both proofs apply the Implicit Function Theorem to $\mathcal{F}: X \times Y \to Z$ defined by

$$\mathcal{F}(\theta, v) = -\mathrm{div} A(\theta)\nabla v - J(\theta), \tag{3.6}$$

where $Z = \mathrm{H}^{-1}(\Omega)$ for $Y = \mathrm{H}^1_0(\Omega)$ and $J(\theta) = \det(Id + D\theta)$ and $A(\theta) = J(\theta)(Id + D\theta)^{-1}(Id + {}^tD\theta)^{-1}$. We extend the previous results to the case $X = \mathrm{W}^{2,p}(\mathbb{R}^d, \mathbb{R}^d)$ for p large.

Lemma 3.13 *Let Ω be a bounded $\mathcal{C}^2$ domain in $\mathbb{R}^d$. For $q > d$ and $p > max(q, d+2)$, the map $\theta \mapsto (v_\theta, \lambda_\theta)$ from $\mathrm{W}^{2,p}(\mathbb{R}^d, \mathbb{R}^d)$ with values in $\mathrm{W}^{2,q}(\Omega) \times \mathbb{R}$ is analytic around 0.*

Proof. By Neumann series expansion and Hölder inequality, the map J is analytic around 0 from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ into $\mathrm{W}^{1,p/d}(\mathbb{R}^d,\mathbb{R}^d)$. In a similar manner, the map A is analytic from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ into $\mathrm{W}^{1,p/(d+2)}(\mathbb{R}^d,\mathbb{R}^{d\times d})$. As a consequence, by Sobolev's injection, the map $\mathcal{F}$ is analytic around (Id, v_0) from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)\times \mathrm{W}^{2,q}(\Omega)$ into $\mathrm{L}^q(\Omega)$. Ones checks that $\mathcal{F}(0,v_0)=(0,0)$ and that the differential

$$\partial_v\mathcal{F}(0,v_0).[u] = -\Delta u$$

is an isomorphism from $\mathrm{W}^{2,q}(\Omega)$ into $\mathrm{L}^q(\Omega)\times\mathbb{R}$ and the conclusion follows from the Implicit Function Theorem. $\square$

• **Second modified step: Estimate of $\mathcal{T}_2(t)-\mathcal{T}_2(0)$.** We first transport the integral $\mathcal{T}_2(t)$ on the fixed boundary and rewrite it as

$$\mathcal{T}_2(t) = \int_{\partial\Omega^*}\sigma(t)\varphi^2,$$

where we have set

$$\sigma(t) = \Big[M(t)\nabla|M(t)\nabla\tilde{u}(t)|^2\cdot\boldsymbol{n} \;+\; \tilde{H}(t)\left(\boldsymbol{n}\cdot M(t)\tilde{\boldsymbol{n}}(t)\right)\right)\left(M(t)\nabla\tilde{u}(t)\cdot M(t)\tilde{\boldsymbol{n}}(t)\right)^2\Big]\left(\boldsymbol{n}\cdot M(t)\tilde{\boldsymbol{n}}(t)\right)J_\tau(t)$$

From the previous geometric estimates and the estimates on the states, there is a constant C such that

$$\|\sigma(t)-\sigma(0)\|_{\mathrm{L}^p}\leq C\|T_t - I\|_{\mathrm{W}^{2,p}}.$$

Notice that the control holds only in L^p and not in L^∞ as in [11] and also [7, Appendix]. Hence, we do not obtain a control with the L^2 norm of φ. By Hölder inequality, it comes $|\mathcal{T}_2(t)-\mathcal{T}_2(0)|\leq\|\sigma(t)-\sigma(0)\|_{\mathrm{L}^p}\|\varphi\|^2_{\mathrm{L}^{\tilde{p}}}$ for any $\tilde{p}\geq 2p/(p-1)$. Since $\|\varphi\|_{\mathrm{L}^{\tilde{p}}}\geq C\|\varphi\|_{\mathrm{H}^{1/2}}$ when $\tilde{p}<2d/(d-1)$ by Sobolev's injection, such a $\tilde{p}$ can be chosen provided that for $p>d$. Then, it holds

$$|\mathcal{T}_2(t)-\mathcal{T}_2(0)|\leq\|\sigma(t)-\sigma(0)\|_{\mathrm{L}^p}\|\varphi\|^2_{\mathrm{H}^{1/2}}\leq C\|T_t - Id\|^2_{\mathrm{W}^{2,p}}\|\varphi\|^2_{\mathrm{H}^{1/2}}.$$

3.3 Proof of Theorem 1.1

We are now in position to prove Theorem 1.1 corresponding to the unconstrained case. Let Ω^* be a domain satisfying the assumption of Theorem 1.1. Let $\eta>0$ and let Ω be such that $d_\Theta(\Omega,\Omega^*)<\eta$. Then, there exists h such that the boundary of Ω is the set $\{x+h(x)\boldsymbol{n}(x),\ x\in\partial\Omega^*\}$. Consider the path $(\Omega_t)_{t\in[0,1]}$ defined in (1.10), j the restriction of J to the path Ω_t. We write Taylor formula along this path:

$$J(\Omega)-J(\Omega^*) = \int_0^1 j''(t)(1-t)dt = \frac{1}{2}j''(0)+\int_0^1[j''(t)-j''(0)](1-t)dt \geq \frac{1}{2}j''(0)-\int_0^1|j''(t)-j''(0)|dt.$$

From $(\mathbf{C}_{\mathrm{H}^{s_2}})$, we can apply Lemma 3.3 and there is a constant $\lambda>0$ such that

$$\ell_2[J](\Omega^*).(h,h)\geq\lambda\|h\|^2_{\mathrm{H}^{s_2}}.$$

Applying the $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$ assumption, we obtain that for η small enough,

$$\left|j''(t)-j''(0)\right|\leq\frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}},\ \ \forall t\in[0,1],\ \text{and therefore } J(\Omega)-J(\Omega^*)\geq\frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}.$$

$\square$

3.4 Proof of Theorem 3.2

We denote μ the Lagrange multiplier associated to J. Therefore we consider $J_\mu = J - \mu\mathrm{Vol}$ and Ω^* satisfies $J'_\mu(\Omega^*) = 0$.

Step 1: Stability under volume and barycenter constraint: The bilinear form ℓ_2 associated to the second order derivative of the Lagragian J_μ is $\ell_2[J] - \mu\ell_2[\mathrm{Vol}]$. Under the structural hypotheses on $\ell_2[J](\Omega^*) = \ell_m + \ell_r$ and the fact that $\ell_2[\mathrm{Vol}](\Omega^*)$ is continuous in the L^2-norm, we can applied Lemma 3.3 to $\ell_2[J_\mu](\Omega^*)$, so there are constants c_1, c_2, c_3 and $c_4 > 0$ such that

$$\forall\varphi \in \mathrm{H}^{s_2}(\partial\Omega^*),\quad |\ell_m(\varphi,\varphi)| \geq c_1|\varphi|^2_{\mathrm{H}^{s_1}} \quad |\ell_r(\varphi,\varphi)| \leq c_2\|\varphi\|^2_{\mathrm{H}^{s_1}}, \quad |\ell_2[\mathrm{Vol}](\Omega^*).(\varphi,\varphi)| \leq c_3\|\varphi\|^2_{\mathrm{L}^2}, \tag{3.7}$$

$$\forall\varphi \in T(\partial\Omega^*),\quad \ell_2[J - \mu\mathrm{Vol}](\Omega^*).(\varphi,\varphi) \geq c_4\|\varphi\|^2_{\mathrm{H}^{s_2}}. \tag{3.8}$$

Step 2: Stability without constraint: In order to deal with the volume constraint and the invariance with respect to translations, we use an idea of [40, 25] by considering

$$J_{\mu,C} = J - \mu\mathrm{Vol} + C\left(\mathrm{Vol} - V_0\right)^2 + C\left\|\mathrm{Bar} - \mathrm{Bar}(\Omega^*)\right\|^2,$$

where $\mathrm{Bar}(\Omega) := \int_\Omega x$ and $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^d$. The shape Ω^* still satisfies $J'_{\mu,C}(\Omega^*) = 0$. We claim that Ω^* is astable shape for $J_{\mu,C}$ on the entire space $\mathrm{H}^{s_2}(\partial\Omega^*)$ when C is big enough, that is to say for all φ in $\mathrm{H}^{s_2} \setminus \{0\}$,

$$\ell_2[J_{\mu,C}](\Omega^*).(\varphi,\varphi) > 0. \tag{3.9}$$

Indeed, if it was not the case, we would have the existence of $\varphi_n \in \mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\}$ such that

$$\ell_2[J_{\mu,n}](\Omega^*).(\varphi_n,\varphi_n) \leq 0. \tag{3.10}$$

Using (3.7), this leads to

$$c_1|\varphi_n|^2_{\mathrm{H}^{s_2}} - c_2\|\varphi_n\|^2_{\mathrm{H}^{s_1}} - |\mu|c_3\|\varphi_n\|^2_{\mathrm{L}^2} + 2n\left(\int_{\partial\Omega^*}\varphi_n\right)^2 + 2n\left\|\int_{\partial\Omega^*}\varphi_n x\right\|^2 \leq 0. \tag{3.11}$$

Assuming by homogeneity that $\|\varphi_n\|_{\mathrm{H}^{s_1}} = 1$ for every n, (3.11) implies that $(\varphi_n)_n$ is bounded in H^{s_2} and using the compactness of $\mathrm{H}^{s_2}(\partial\Omega^*)$ in $\mathrm{H}^{s_1}(\partial\Omega^*)$, we have, up to a subsequence, that φ_n converges to φ, weakly in H^{s_2} and strongly in H^{s_1} and L^2. Therefore, (3.11) implies first that $2n[\mathrm{Vol}'(\varphi_n)^2 + \mathrm{Bar}'(\varphi_n)^2]$ is bounded, then that $\varphi \in T(\partial\Omega^*)$, that is to say

$$\int_{\partial\Omega^*}\varphi = 0 \text{ and } \int_{\partial\Omega^*}\varphi x = 0,$$

and then the semi-lower continuity assumption in $(\mathbf{C}_{\mathrm{H}^{s_2}})$ implies

$$\ell_2[J_\mu](\Omega^*).(\varphi,\varphi) \leq 0, \quad \text{with } \|\varphi\|_{\mathrm{H}^{s_1}} = 1$$

which contradicts (3.8), since $\varphi \neq 0$.

Step 3: Stability: It is now easy to see that $J_{\mu,C}$ satisfies both $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$ at Ω^*, and for C large enough, we have (3.9), so applying Theorem 1.1, there exists $\lambda > 0$ and $\eta > 0$ such that for every Ω with $d_\Theta(\Omega,\Omega^*) < \eta$,

$$J_{\mu,C}(\Omega) - J_{\mu,C}(\Omega^*) \geq \lambda\|h\|^2_{\mathrm{H}^{s_2}},$$

Writing this inequality in particular for shapes Ω of volume V_0 and having the same barycenter as Ω^*,

$$J(\Omega) - J(\Omega^*) \geq \frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}.$$

We conclude using the invariance of J with translations. ☐

4 Applications

4.1 Retrieving some examples from the literature

Isoperimetric inequalities

According to the previous sections, the perimeter satisfy conditions $(\mathbf{C}_{\mathrm{H}^1})$ and $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ at any smooth enough set, and in particular for the ball. Moreover, as shows Section 2.3, we have

$$\ell_1[P](B_1) = (d-1)\ell_1[\mathrm{Vol}](B_1), \qquad \text{and} \qquad \ell_2[P-(d-1)\mathrm{Vol}](B_1)(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(k-1)(k+d-1)\ \alpha_{k,l}(\varphi)^2.$$

Moreover, $\varphi \in T(\partial B_1)$ if and only if $\alpha_{0,1}(\varphi) = \alpha_{1,i}(\varphi) = 0$ for $i \in \{1,\dots,d\}$. Therefore B_1 is a critical and stable shape for P under volume constraint, and up to translations: Theorem 3.2 applies, and we retrieve Fuglede's result from [19].

Recently in [34], different improved versions (with a better distance than the Fraenkel asymmetry for d_1 in (1.2)) of the quantitative isoperimetric inequality has been achieved for the anisotropic perimeter

$$P_f(\Omega) = \int_{\partial\Omega} f(\boldsymbol{n}_{\partial\Omega})$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a convex positively 1-homogeneous function, whose minimizer under volume constraint is an homothetic version of the Wulff shape $K = \{f_* < 1\}$ where f_* is the gauge function of f. In particular in [34, Theorem 1.3 and Section 4] focused on the case where K is assumed to be C^2 and uniformly convex, a strategy based on the second variation is used: in particular a Fuglede type result is obtained in [34, Proposition 1.9], and this requires [34, Lemma 4.1], which in our framework asserts that P_f satisfies conditions $(\mathbf{C}_{\mathrm{H}^1})$ and $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$. It is interesting to notice though that in order to show that K is strictly stable in the sense that $\ell_2[P_f - \mu\mathrm{Vol}](K) > 0$ on $T(\partial K)\setminus\{0\}$, the author needs to use the quantitative Wulff isoperimetric inequality from [17] (and obtained with optimal transport method); they deduce then, in the spirit of Lemma 3.3 that $\ell_2[P_f - \mu\mathrm{Vol}](K)$ is coercive for the H^1-norm to complete the proof of the Fuglede-type result. They apply then the regularization procedure mentioned in the introduction. Therefore, up to our knowledge, there is no proof "from scratch" of the quantitative anisotropic isoperimetric inequality using a result similar to Theorem 3.2.

The Ohta-Kawasaki model

In the paper [2], both steps of the strategy described in the introduction are achieved in order to deal with the following functional, formulated in the periodic sense, and which includes a non-local term:

$$J(\Omega) = P_{\mathbb{T}^N}(\Omega) + \gamma G(\Omega) \quad \text{where } G(\Omega) = \int_{\mathbb{T}^N} |\nabla w_\Omega|^2 dx \text{ and } w_\Omega \text{ solves } \left\{\begin{array}{rcll} -\Delta w_\Omega &=& \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c} - m & \text{in } \mathbb{T}^N \\ \displaystyle\int_{\mathbb{T}^N} w\,dx &=& 0 & \end{array}\right. \tag{4.1}$$

where $\mathbb{T}^N$ is the N-dimensional flat torus of unit volume, and $m = |\Omega| - |\Omega^c| \in (-1,1)$ is fixed. Again, there is an invariance with translation and a volume constraint.

In order to handle the first step of the strategy, the authors in [2] prove a stability result for the $\mathrm{W}^{2,p}$-topology, for p large enough. Namely, if Ω^* is a critical domain for J that is stable under volume constraint and up to translations, (see (3.1)) then there exists $\eta > 0$, $c > 0$ such that

$$\forall\Omega \text{ such that } d_{W^{2,p}}(\Omega^*,\Omega) < \eta, \quad J(\Omega) \geq J(\Omega^*) + c\bar{d}_{\mathrm{H}^{s_2}}(\Omega^*,\Omega). \tag{4.2}$$

The strategy is very similar to [11], but in the framework of $\mathrm{W}^{2,p}$-spaces rather than $\mathrm{C}^{2,\alpha}$ spaces. Note that the difference in the choice of spaces ($\mathrm{W}^{2,p}$ instead of $\mathrm{C}^{2,\alpha}$) is not just a detail as it is relevant for the second step of the strategy when proving stability in an L^1-neighborhood as it is done in [2, Section 4]: their regularization procedure needs to allow discontinuity of the mean curvature, see equation (4.9) in the proof of [2, Theorem 4.3]. This difficulty is handled in another way for the Faber-Krahn inequality, see below.

From the computations of [9], we obtain

$$\ell_1[G](\Omega)(\varphi) = 4\int_{\partial\Omega} w\varphi,$$

$$\ell_2[G](\Omega)(\varphi,\varphi) = 8\int_{\mathbb{T}^N} |\nabla z_\varphi|^2 dx + 4\int_{\partial\Omega}(\partial_{\boldsymbol{n}} w + H)\varphi^2, \text{ where } -\Delta z_\varphi = \varphi\mathcal{H}^{N-1}\lfloor\partial\Omega$$

therefore G satisfies $(\mathbf{C}_{\mathrm{H}^{1/2}})$ and J satisfies $(\mathbf{C}_{\mathrm{H}^1})$, the dominant term being contained in the perimeter term. As we have seen that the perimeter satisfies $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ condition, it just remains to handle functional G, which is proven to satisfy $(\mathbf{EC}_{\mathrm{H}^1,\mathrm{W}^{2,p}})$ for $p > d$. Therefore Theorem 3.2 applies, and we retrieve [2, Theorem 3.9], as $\bar{d}_{\mathrm{H}^1}(\Omega^*, \Omega)$ easily dominates the Fraenkel asymmetry.

The Faber-Krahn inequality

In [7] (see also [21]) a quantitative version of the Faber-Krahn inequality is achieved, using again the two steps mentioned in the introduction: in order to achieve the first step, they use the Kohler-Jobin inequality ([28]), which implies that the Faber-Krahn deficit is controlled by the deficit of the Dirichlet energy E. However, as we show here, it is possible to achieve this step without this “trick”. Indeed, we have seen that λ_1 satisfies $(\mathbf{EC}_{\mathrm{H}^{1/2}})$ and $(\mathbf{EC}_{\mathrm{H}^{1/2},\mathrm{W}^{2,p}})$ for $p > d$, and for any $\varphi \in C^\infty(\partial B_1)$ such that $\int_{\partial B_1}\varphi = 0$, we have

$$\ell_1[\lambda_1](B_1) = -\gamma_d^2\ell_1[\mathrm{Vol}](B_1), \qquad \text{and} \qquad \ell_2[\lambda_1 + \gamma_d^2\mathrm{Vol}](B_1)(\varphi,\varphi) = 2\gamma_d^2\sum_{k=0}^{\infty}\sum_{l=1}^{d_k} Q_k\ \alpha_{k,l}(\varphi)^2.$$

where (using [1, 9.1.27, p 361])

$$Q_k = j_{d/2-1}\frac{J'_{k+d/2-1}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} + \frac{d}{2} = k + d - 1 - j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} = j_{d/2-1}\frac{J_{k+d/2-2}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} - k + 1.$$

With the last formula, we easily notice that $Q_1 = 0$. The sign of Q_k can be obtained using the argument of [33, 6.5 page 133] (done when $d = 2$, but as noticed in [27], valid for any d): indeed, their computations imply

$$j_{d/2-1}\frac{J'_{k+d/2-1}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} \geq k - d/2 - 1, \forall n \in \mathbb{N}^*,$$

which leads to

$$\forall k \geq 2, Q_k \geq k - 1.$$

Therefore Theorem 3.2 applies, and we retrieve a Faber-Krahn quantitative inequality for the $\bar{d}_{\mathrm{H}^{1/2}}$ distance in a $\mathrm{W}^{2,p}$ neighborhood of the ball.

4.2 Examples with competition

Combining the general Theorem 3.2 to the computations of shape derivatives from Section 2.1, we easily obtain the following:

Proposition 4.1 *Let $V_0 \in (0,\infty)$, and B a ball of volume V_0, and $\Theta = \mathrm{W}^{2,p}$ for p large enough. Then there exists $\gamma_0 \in (0,\infty)$ such that for every $\gamma \in [-\gamma_0,\infty)$, there exists $\eta = \eta(\gamma) > 0$ and $c = c(\gamma) > 0$ such that for every $\Omega \in \mathcal{V}_\eta := \{\Omega,\ d_\Theta(\Omega, B) < \eta, \mathrm{Vol}(\Omega) = \mathrm{Vol}(B)$ and $\mathrm{Bar}(\Omega) = \mathrm{Bar}(B)\}$,*

$$(P + \gamma E)(\Omega)\ \geq\ (P + \gamma E)(B) + c\|h\|^2_{\mathrm{H}^1}, \quad (P + \gamma\lambda_1)(\Omega)\ \geq\ (P + \gamma\lambda_1)(B) + c\|h\|^2_{\mathrm{H}^1}$$

$$(E + \gamma\lambda_1)(\Omega)\ \geq\ (E + \gamma\lambda_1)(B) + c\|h\|^2_{\mathrm{H}^{1/2}}, \quad (\lambda_1 + \gamma E)(\Omega)\ \geq\ (\lambda_1 + \gamma E)(B) + c\|h\|^2_{\mathrm{H}^{1/2}},$$

where $h = h_{B,\Omega}$ is such that $\partial\Omega = \{x + h(x)\boldsymbol{n}(x), x \in \partial B\}$.

Proof of Proposition 4.1: It suffices to prove that Theorem 3.2 can be applied to $\Omega^* = B$ and

$$J \in \{P + \gamma E, P + \gamma\lambda_1, E + \gamma\lambda_1, \lambda_1 + \gamma E)\}.$$

It is explained in Sections 3.1 and 3.2 that (P, E, λ_1) satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{EC}_{\mathrm{H}^{s_2},\Theta})$ for suitable values of s_2, and with Lemmata 2.9 and 2.8 we easily check that the ball is a critical stable domain for J under volume constraint and up to translations, either if $\gamma \geq 0$ or if $\gamma < 0$ is small enough. $\square$

Corollary 4.2 *With the same notations as in Proposition 4.1, we have, with $\eta_0 = \eta(\gamma_0)$:*

$$\forall \Omega \in \mathcal{V}_{\eta_0}, \quad \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \geq \gamma_0, \qquad \frac{P(\Omega) - P(B)}{\lambda_1(\Omega) - \lambda_1(B)} \geq \gamma_0$$

$$\gamma_0 \leq \frac{\lambda_1(\Omega) - \lambda_1(B)}{E(\Omega) - E(B)} \leq \gamma_0^{-1}.$$

Remark 4.3 In [31], the second inequality in Corollary 4.2 is also investigated, and the author computes the optimal value γ_0 when the size of the neighborhood $\mathcal{V}_{\eta_0}$ goes to 0, which we use in the proof of Proposition 4.4. We also refer to [33] for some result of this kind.
To the contrary to the last two-sided inequality, it is not possible to bound the first two ratio from above. Indeed, for every $\gamma \in (0, \infty)$, there exists $\Omega_\gamma = (Id + \theta_\gamma)(B)$ of class C^∞ such that

$$|\Omega_\gamma| = |B|, \ \ \|\theta_\gamma\|_\Theta \leq \gamma^{-1} \text{ and } \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} > \gamma.$$

This is due to the fact that the functionals P and (E, λ_1) satisfy conditions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ for different values of s_2.

Explicit constants

We want to go further and compute explicit numbers γ such that the inequalities of Proposition 4.1 holds. To simplify the expressions, we restrict ourselves to the case of the unit ball. In the first two cases, we explicit the optimal constant, see Remark 4.5.

Proposition 4.4 *Using notations of Proposition 4.1 and γ_d defined in* (2.5)*,*

(i) *if $\gamma > -(d+1)d^2$, then B_1 is a local strict minimizer of $P + \gamma E$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (P + \gamma E)(\Omega) \ \geq \ (P + \gamma E)(B).$$

Moreover, when $\gamma = -(d+1)d^2$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -(d+1)d^2$, the ball is a saddle shape for $P + \gamma E$.

(ii) *if $\gamma > -\dfrac{d(d+1)}{\gamma_d^2(d + j_{d/2-1}^2)}$, then B_1 is a local strict minimizer of $P + \gamma\lambda_1$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (P + \gamma\lambda_1)(\Omega) \ \geq \ (P + \gamma\lambda_1)(B);$$

Moreover, when $\gamma = -\dfrac{d(d+1)}{\gamma_d^2(d + j_{d/2-1}^2)}$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -\dfrac{d(d+1)}{\gamma_d^2(d + j_{d/2-1}^2)}$, the ball is a saddle shape for $P + \gamma\lambda_1$.

(iii) *if $\gamma > -\dfrac{1}{d^2(d+1)\gamma_d^2}$, then B_1 is a local strict minimizer of $E + \gamma\lambda_1$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (E + \gamma\lambda_1)(\Omega) \ \geq \ (E + \gamma\lambda_1)(B);$$

(iv) *if $\gamma > -\gamma_d^2 d^2$, then B_1 is a local strict minimizer of $\lambda_1 + \gamma E$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (\lambda_1 + \gamma E)(\Omega) \;\geq\; (\lambda_1 + \gamma E)(B).$$

Note that the additional term $\|h\|^2_{\mathrm{H}^{s_2}}$ can be added in the former inequalities with $s_2 = 1$ for the cases (i)-(ii) and with $s_2 = 1/2$ for the cases (iii)-(iv).

Remark 4.5 In the cases (iii) and (iv), the constants we compute are not optimal, in particular we do not claim the ball is a saddle point once we go beyond the computed value. Though it is possible to compute the optimal value, one just need to compute explicitly the value of $\sup_{k\geq 2}\tau_k'$ and $\sup_{k\geq 2}\tau_k''$ (see the notations in the proof below) as it is done in the cases (i) and (ii). As it is seen in the second case handled by Nitsch in [31], this computation can be rather technical.

Proof of Proposition 4.4:

Proof of (i): We first compute the Lagrange multiplier $\mu(t)$ associated to the volume constraint at B_1: it is defined as $\ell_1[P + tE) + \mu(t)\mathrm{Vol}] = 0$ that is from the expression of the shape gradients of Vol, P and E:

$$\mu(t) = \frac{1}{2d^2}\, t \; - (d-1).$$

Let us now turn our attention to hessian of the function $P + tE + \mu\mathrm{Vol}$ on the balls B_1. As a consequence of Lemma 2.9, the shape hessian of the lagrangian $P + tE + \lambda(t)\mathrm{Vol}$ at balls is

$$\ell_2[P + tE + \mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty} c_k \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = k^2 + \left[(d-2) + \frac{1}{d^2}\,t\right]\; k - \left[(d-1) + \frac{1}{d^2}\,t\right] = (k-1)\left[k + (d-1) + \frac{1}{d^2}\,t\right].$$

Therefore, the hessian of the Lagrangian $\ell_2[P + tE + \mu(t)\mathrm{Vol}](B_1)$ is coercive in $\mathrm{H}^1(\partial B_1)$ when t solves the inequalities

$$k + (d-1) + \frac{1}{d^2}\,t > 0$$

for all $k \geq 2$. Of course, it suffices to solves that inequality in the special case $k = 2$ that provides $t > -(d+1)d^2$.

Proof of (ii): Notice that the case $t \geq 0$ is well known so we consider the case where $t < 0$. We compute the Lagrange multiplier $\mu(t)$ associated to the volume at B_1 defined by $\ell_1[P + t\lambda_1) + \mu(t)\mathrm{Vol}] = 0$ that is from the expression of the shape gradient of the volume, the perimeter and λ_1:

$$\mu(t) = \gamma_d^2\; t \; - (d-1).$$

Let us now turn our attention to the hessian of the Lagragian $P + tE + \mu(t)\mathrm{Vol}$ on the balls B_1:

$$\ell_2[P + t\lambda_1 + \mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty} c_k(t) \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = k^2 + (d - 2 + t\gamma_d^2)k - (d-1) + t\gamma_d^2\left[d - 1 - j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right].$$

We introduce the sequences $a_k = J_{k-1+d/2}(j_{d/2-1})$ and $b_k = a_{k+1}/a_k$ so that:

$$c_k(t) = k^2 + (d-2)k - (d-1) + t\gamma_d^2 \left[k + d - 1 - j_{d/2-1}b_k\right].$$

One should have $c_1(t) = 0$ for any t, as known for the invariance by translations of all the involved functions, we can attest this once we describe how one can compute the numbers b_k, see below. For a given integer $k \geq 2$, $c_k(t) > 0$ holds when $t > \tau_k$ defined as

$$\tau_k = -\frac{(k-1)(k+d-1)}{\gamma_d^2(k+d-1-j_{d/2-1}b_k)}.$$

In order to obtain to find the optimal value of t so that these inequalities are satisfied for every $k \geq 2$, we need to compute the supremum of $\{\tau_k, k \geq 2\}$. It is proven by Nitsch in [31, p. 332, proof of Lemma 2.3] that for all $k \geq 2, \tau_k \leq \tau_2$. We describe here how one can obtain a more explicit version of τ_2: from the recurrence formula for Bessel function ([1, 9.1.27, p 361])

$$(2\nu/z)J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$$

applied to $\nu = k - 1 + d/2$ and $z = j_{d/2-1}$, the sequences a_k and b_k satisfy the recurrence property

$$a_{k+1} = \frac{2(k-1)+d}{j_{d/2-1}} a_k - a_{k-1} \text{ and } b_{k+1} = \frac{2(k-1)+d}{j_{d/2-1}} - \frac{1}{b_k}$$

with the initial terms $a_0 = 0$ and $a_1 = J_{d/2}(j_{d/2-1})$ so that $b_1 = a_2/a_1 = d/j_{d/2-1}$. Therefore, we have:

$$b_2 = \frac{d}{j_{d/2-1}} - \frac{j_{d/2-1}}{d} = \frac{d^2 - j_{d/2-1}^2}{dj_{d/2-1}}$$

and as a consequence, we obtain that

$$\tau_2 = -\frac{d(d+1)}{\gamma_d^2(d+j_{d/2-1}^2)}.$$

Proof of (iii): The Lagrange multiplier is $\mu(t) = (1/d^2) + t\gamma_d^2$. The Hessian of the Lagrangian is

$$\ell_2[E + t\lambda_1 + \mu(t)\mathrm{Vol}](B_1).(\varphi, \varphi) = \sum_{k=0}^{\infty} c_k(t) \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = \left(\frac{1}{d^2} + t\gamma_d^2\right) k - \frac{1}{d^2} + t\gamma_d^2 \left[d - 1 - j_{d/2-1}b_k\right].$$

Again $c_1(t) = 0$ and $c_k(t) > 0$ if and only if

$$t > \tau_k' = -\frac{k-1}{d^2\gamma_d^2(k+d-1-j_{d/2-1}b_k)}.$$

Using that $b_1 \geq b_k > 0$, we obtain

$$\tau_k' < -\frac{1}{d^2\gamma_d^2} \frac{k-1}{k+d-1} = -\frac{1}{d^2\gamma_d^2}\left(1 - \frac{d}{k+d-1}\right) \leq -\frac{1}{d^2(d+1)\gamma_d^2}.$$

Therefore, if $t > -\dfrac{1}{d^2(d+1)\gamma_d^2}$ then for any $k \geq 2$, $t > \tau_k'$, which leads to the result.

Proof of (iv): The Lagrange multiplier is $\mu(t) = (t/d^2) + \gamma_d^2$. The Hessian of the Lagrangian is

$$\ell_2[\lambda_1 + tE + \mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty} c_k(t) \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = \left(\frac{t}{d^2} + \gamma_d^2\right) k - \frac{t}{d^2} + \gamma_d^2 \left[d - 1 - j_{d/2-1} b_k\right].$$

We check $c_1(t) = 0$ and $c_k(t) > 0$ if and only if

$$t > \tau_k'' = -\gamma_d^2 d^2 \left(1 + \frac{d - j_{d/2-1} b_k}{k-1}\right).$$

Using that $b_1 \geq b_k > 0$, we obtain

$$\tau_k \leq -\gamma_d^2 d^2,$$

and therefore, if $t > -\gamma_d^2 d^2$ then for any $k \geq 2$, $t > \tau_k''$, which leads to the result. □

5 Remarks on non-stability results

5.1 Counterexample for non smooth perturbations

Let us consider P the perimeter and E the Dirichlet energy with second right hand side 1 (defined in (1.15)), and $\Omega^* = B$ a ball of volume V_0. We have seen in Proposition 4.1 that there is a real number $\gamma_0 \in (0,\infty)$ such that for every $\gamma \in (-\gamma_0, \infty)$, B is a stable local minimum for $P + \gamma E$.

For $\gamma \geq 0$ this is not very surprising: since the ball minimizes E among sets of given volume, it is enough to prove that the ball is a stable minimizer for the perimeter, which goes back to Fuglede [19]. Moreover, it has been proven that B is an L^1-stable minimizer of the perimeter in a L^1-neighborhood of the ball, that is to say there exists $\eta > 0$ such that

$$\forall\, \Omega \text{ such that } |\Omega \Delta B| \leq \eta, |\Omega| = |B|, \quad P(\Omega) - P(B) \geq c|\Omega \Delta B|^2 \tag{5.1}$$

where we assume the barycenter of Ω to be the same as the one of B (actually this is no longer local, this inequality can be stated for every set Ω of finite perimeter, see [20]). Therefore a similar inequality is valid for $P + \gamma E$ if $\gamma \geq 0$.

However, *for $\gamma < 0$, the fact that the ball is a local minimizer is no longer trivial*, there is a competition between the minimization of the perimeter and maximization the Dirichlet energy. Though if the coefficient in E is small enough, our result state that B is still a local minimizer in a Θ-neighborhood. Nevertheless, in that case B is no longer a local minimizer in a L^1-neighborhood. In other words, for every $\gamma < 0$ and any $\varepsilon > 0$ one can find Ω_ε such that

$$d_{\mathrm{L}^1}(\Omega_\varepsilon, B) < \varepsilon, \quad |\Omega_\varepsilon| = |B|, \quad \text{and} \quad (P + \gamma E)(\Omega_\varepsilon) < (P + \gamma E)(B).$$

To prove this, we use the idea of topological derivative, it is well known that if one consider a small hole of size ε in the interior of a fixed shape. The energy will change at order ε^{d-2} if $d \geq 3$ and $1/\log(\varepsilon)$ if $d = 2$, which is strictly bigger than the change of perimeter which is of order ε^{d-1}, and therefore will strictly decrease the energy $P + \gamma E$ when $\gamma < 0$.

We compute here explicitly these estimates when the hole is at the center of the ball: let us consider a fixed ball $B_1 = B(0,1)$ of radius 1 (to simplify the computations) and define $\Omega_\varepsilon = B_1 \setminus B(0,\varepsilon)$ an annulus. Using that $\Delta u = \partial_{rr} u + \frac{d-1}{r} \partial_r u$ when u is radial, the state function is:

$$u_{\Omega_\varepsilon}(r) = \frac{(\varepsilon^{d-2} - \varepsilon^d) r^{2-d} + \varepsilon^d - 1}{2d(\varepsilon^{d-2} - 1)} - \frac{r^2}{2d}, \quad \text{if } d \geq 3$$

$$u_{\Omega_\varepsilon}(r) = \frac{1-\varepsilon^2}{-4\log(\varepsilon)}\log(r) + \frac{1-r^2}{4}, \quad \text{if } d=2$$

and therefore

$$\begin{aligned}
\text{if } d \geq 3, \quad E(\Omega_\varepsilon) &= -\frac{1}{2}\int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{d(1-\varepsilon^2)^2\varepsilon^{d-2} - 2(1-\varepsilon^d)^2}{8d^2(1-\varepsilon^{d-2})} + \frac{1-\varepsilon^{d+2}}{4d(d+2)}\right] P(B_1) \\
&= \left[-\frac{1}{2d^2(d+2)} + \frac{d-2}{8d^2}\varepsilon^{d-2} + o(\varepsilon^{d-2})\right] P(B_1),
\end{aligned}$$

$$\begin{aligned}
\text{if } d = 2, \quad E(\Omega_\varepsilon) &= -\frac{1}{2}\int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{(1-\varepsilon^2)}{-8\log(\varepsilon)}(1-\varepsilon^2(1-2\log(\varepsilon))) - \frac{1}{16}(1-\varepsilon^2+\frac{\varepsilon^4}{2})\right] P(B_{1B_1}) \\
&= \left[-\frac{1}{16} - \frac{1}{8\log(\varepsilon)} + o\left(\frac{1}{\log(\varepsilon)}\right)\right] P(B_1).
\end{aligned}$$

We now define $\widetilde{\Omega_\varepsilon} = \mu_\varepsilon\Omega_\varepsilon$ where $\mu_\varepsilon = (1-\varepsilon^d)^{-1/d}$ so that

$$|\widetilde{\Omega_\varepsilon}| = |B_1|, \quad P(\widetilde{\Omega_\varepsilon}) - P(B_1) = \left[\mu_\varepsilon^{d-1}(1+\varepsilon^{d-1}) - 1\right] P(B_1) \sim_{\varepsilon\to 0} \varepsilon^{d-1} P(B_1)$$

$$E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon\to 0} \frac{(d-2)P(B_1)}{8d^2}\varepsilon^{d-2} > 0, \quad \text{if } d\geq 3, \quad E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon\to 0} \frac{P(B_1)}{-8\log(\varepsilon)} > 0, \quad \text{if } d=2$$

so that in both cases, for any nonpositive γ, $(P+\gamma E)(\Omega_\varepsilon) - (P+\gamma E)(B_1) < 0$ for small ε.

5.2 Instability when no coercivity

The geometric inverse problems of shape reconstruction from boundary measurements leading to a overdetermination in the boundary conditions are well known to be unstable. Hence, the theory presented in this work should not apply. Indeed, shape functionals used for domain reconstruction from boundary measurements are such that (i) holds while (ii) and (iii) are false (see [3], [4]). The general situation in the general class of such inverse problems is then: for a reconstruction function J (for example the least square fitting to data), the Riesz operator corresponding to the shape Hessian $\ell_2[J]$ at a critical domain is compact. This means, roughly speaking, that, in a neighborhood of the critical domain (i.e. for t small), J behaves as its second order approximation and one cannot expect an estimate of the kind $J(\Omega_t) - J(\Omega_0) \geq ct^2$ with a constant c uniform in the deformation direction. This explains also why regularization is required in the numerical treatment of this type of problem.

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