Chapter 1
A polaron model with point interactions in three dimensions

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Abstract We discuss a model in which a nonrelativistic particle can absorb and emit bosonic particles on contact. The bosons themselves do not propagate, as in the related Fröhlich polaron model. We determine explicitly the domain of the Hamiltonian for finitely many bosons in terms of singular boundary conditions. The singularities occurring in this model are essentially the same as in the model with propagating bosons, and simplifications in the formulas highlight their key features.

1.1 Introduction

Polaron models, in which nonrelativistic particles that can absorb and emit bosons, are commonly used in physics. For example, they can serve as an effective model for the interactions of electrons with phonons in a solid. Since the bosons are quasiparticles representing elementary excitations out of some equilibrium, a large class of dispersion relations and interactions can be relevant, depending on the properties of the equilibrium of the material. In particular, the case in which the bosons do not propagate is of interest. The Hamiltonian for such a model, in the case of one particle in three dimensions, is formally given by

$$H = -\Delta_x + N + a(v_x) + a^*(v_x),$$

where $N = d\Gamma(1)$ is the boson-number operator, v is a distribution, x denotes the variable in the configuration space of the particle, and $v_x(y) = v(y-x)$. A proper definition of the Hamiltonian should yield and unbounded self-adjoint operator $(H, D(H))$ on the Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}^3) \otimes \Gamma_{\mathrm{sym}}(L^2(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} L^2\left(\mathbb{R}^3, L^2_{\mathrm{sym}}((\mathbb{R}^3)^n)\right),$$

and clarify its relation to the formal expression.

In the well-known Fröhlich model, the interaction is given by $\hat{v}(k) = |k|^{-1}$. In this case, the expression for H is not well defined as an operator, since $v \notin L^2(R^3)$, but it is defined as a quadratic form (see e.g. [4]). The domain of the operator associated to this quadratic form can be described using generalised boundary conditions, as shown in [6].

In this note we will discuss the construction of a self-adjoint operator H in a more singular variant of this model, where the interaction is a point interaction, that is $v = \delta$. For this model, the formal expression for H makes sense neither as an operator nor a quadratic form, and no rigorous definition of a self-adjoint Hamiltonian seems to be known to date. A similar model, in which the bosons have a nonrelativistic dispersion $\omega(k) = k^2 + 1$, was recently defined in [5] using similar boundary conditions as in [6]. Here we will use the same techniques to define the Hamiltonian for bosons with a constant dispersion. Part of the purpose of this discussion is to illustrate the singularities that arise, for example for elements of the domain of H. The constant dispersion relation of the bosons simplifies many calculations and allows for a somewhat more explicit description of the singularities as compared with [5]. However, it gives less control over the growth of certain quantities with increasing boson-number, so we will restrict ourselves to the model with at most $N_{\max}$ bosons. In fact, we will focus mainly on the case with at most $N_{\max} = 2$ bosons, which is the simplest case that displays the same structure as the full problem, and then indicate how the results are obtained for arbitrary $N_{\max} < \infty$.

As in the recent works [7, 6, 5] our approach is to define the Hamiltonian of our model using special boundary conditions, called interior-boundary conditions, that relate sectors of the Hilbert space with different numbers of bosons.

1.2 Extension of the free operator and boundary conditions

In order to gain the ability to impose boundary conditions, we consider an extension of the free kinetic energy operator $L = -\Delta_x$. The domain of this operator, as an operator on $\mathscr{H}$, is explicitly given by

$$D(L) = \bigoplus_{n=0}^{\infty} H^2(\mathbb{R}^3, L^2_{\mathrm{sym}}((\mathbb{R}^3)^n)).$$

Elements of this domain are continuous functions of x and may thus be evaluated at $x = y_n$, for appropriate n. We take L_0 to be the restriction of this operator to the kernel of the annihilation operator $a(\delta_x)$, that is, using the symmetry in the y-variables,

$$D(L_0) = \{\psi \in D(L) : \psi^{(n)}(x, y_1, \dots, y_n)|_{x=y_n} = 0\}.$$

The extension of L we are interested in is now given by the adjoint L_0^*, which will allow for boundary conditions on the sets $\{x = y_j\}$. The domain of L_0^* can be parametrised using the map

$$G_\mu \psi = -(L+\mu^2)^{-1} a^*(\delta_x)\psi.$$

Explicitly, we have

$$\begin{aligned}\left(G_\mu \psi^{(n-1)}\right)(x,Y) &= -\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(-\Delta_x+\mu^2)^{-1}\delta(y_j-x)\psi^{(n-1)}(x,\hat{Y}_j)\\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} f_\mu(x-y_j)\psi^{(n-1)}(y_j,\hat{Y}_j),\end{aligned}$$

where $f_\mu(x) = -(-\Delta_x+\mu^2)^{-1}\delta(x) = -\frac{e^{-\mu|x|}}{4\pi|x|}$, and $\hat{Y}_j \in \mathbb{R}^{3(n-1)}$ denotes the vector formed by the $y_1,\dots,y_n$ without y_j. From this explicit formula one easily deduces the important mapping properties if G.

Lemma 1. *For any positive $n\in\mathbb{N}$, $\mu>0$ and $0\le s<1/2$ the operator G_μ is bounded from $\mathscr{H}^{(n-1)}$ to $H^s(\mathbb{R}^3, L^2_{\mathrm{sym}}((\mathbb{R}^3)^n))$.*

Proof. The statement follows immediately from the fact that $(-\Delta_x+\mu^2)^{-1+s}\delta \in L^2(\mathbb{R}^3)$ for $0\le s<1/2$. Note, however, that the estimate for the norm of $G_\mu : \mathscr{H}^{(n-1)} \to \mathscr{H}^{(n)}$ will grow like $\sqrt{n/(2\mu)}$.

We then have a characterisation of $D(L_0^*)$, by standard arguments from the theory of abstract boundary value problems (see e.g. [1]). We only sketch the proof since, for the construction of the operator H, we do not really need this characterisation and it is sufficient to work on a subset suitably parametrised by G_μ.

Proposition 1. *For any positive $n\in\mathbb{N}$ and $\mu>0$ we have*

$$D(L_0^*)\cap\mathscr{H}^{(n)} = (D(L)\cap\mathscr{H}^{(n)})\oplus G_\mu\mathscr{H}^{(n-1)},$$

i.e., for every $\psi\in D(L_0^*)$ there is a unique $\varphi_\mu^{(n-1)}\in\mathscr{H}^{(n-1)}$ so that $\psi^{(n)}-G_\mu\varphi_\mu^{(n-1)}$ is an element of $H^2(\mathbb{R}^3, L^2(\mathbb{R}^{3n}))$.

Proof (sketch). First observe that this characterisation is independent of μ, since $f_\mu(x)-f_\nu(x)\in H^2(\mathbb{R}^3)$.

Since L is a positive self-adjoint extension of L_0, we have

$$D(L_0^*) = D(L_0)\oplus\ker(L_0^*+i)\oplus\ker(L_0^*-i) = D(L)\oplus\ker(L_0^*+\mu^2)$$

for $\mu>0$. Now let $\varphi^{(n)}\in D(L_0)\cap\mathscr{H}^{(n)}$ and $\psi^{(n-1)}\in\mathscr{H}^{(n-1)}$, then

$$\begin{aligned}\langle G_\mu\psi^{(n-1)}, (L_0+\mu^2)\varphi^{(n)}\rangle_{\mathscr{H}^{(n)}} &= \langle\psi^{(n-1)}, G_\mu^*(L+\mu^2)\varphi^{(n)}\rangle_{\mathscr{H}^{(n)}}\\ &= -\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\langle\psi^{(n-1)}(y_j,\hat{Y}_j), \varphi^{(n)}|_{x=y_j}\rangle_{\mathscr{H}^{(n-1)}}\\ &= 0.\end{aligned}$$

Thus $G_\mu \psi^{(n-1)}$ is in the kernel of $L_0^* + \mu^2$. The point is now to show that we have equality, $\ker(L_0^* + \mu^2) = G_\mu \mathscr{H}^{(n-1)}$. To achieve this, first observe that, by the same logic as above, for any $\xi \in \ker(L_0^* + \mu^2) \cap \mathscr{H}^{(n)}$, $(L + \mu^2)\xi$ annihilates $D(L_0)$ in the pairing of $D(L^{-1}) \times D(L)$. The domain $D(L_0)$ is exactly the kernel of $a(\delta_x)$ in $D(L)$, so its annihilator is the closure of the range of $a^*(\delta_x) : \mathscr{H}^{(n-1)} \to D(L^{-1})$ (cf. [2, Chap. 2.7]), i.e. $\xi = (L^2 + \mu^2)^{-1}\eta$ for some $\eta \in \overline{\operatorname{ran} a^*(\delta_x)}$. To complete the argument, one shows that the range of $a^*(\delta_x)$ is closed in $D(L^{-1})$ by proving that $\|G_\mu \psi\| \geq C\|\psi\|$ for some $C > 0$. We will not go into this technical point here. An argument, for a slightly different situation, that can be adapted to our case is given in [3, Lem. B2].

1.2.1 Boundary values

On $D(L_0^*)$ we can define a boundary value operator that extracts the singular part of a function by

$$\left(B\psi^{(n+1)}\right)(x,Y) = -4\pi\sqrt{n+1} \lim_{y_{n+1}\to x} |x - y_{n+1}| \psi^{(n+1)}(y_{n+1}, Y, x). \tag{1.1}$$

Clearly, we have $B\psi = 0$ for $\psi \in D(L)$ and $BG\psi = \psi$. This shows that B is a well-defined operator from $D(L_0^*) \cap \mathscr{H}^{(n+1)}$ to $\mathscr{H}^{(n)}$, by the characterisation of the domain in Prop. 1.

The other relevant boundary value in our problem is the annihilation operator, i.e. the evaluation at $x = y_{n+1}$. Since functions in the range of G_μ diverge at $x = y_{n+1}$, this is naturally defined on $D(L)$, but not $D(L_0^*)$. We thus need to find an appropriate extension A of this operator to $D(L_0^*)$. As in [7, 5] this extension to singular functions is obtained by considering the expansion near the set $\{x = y_{n+1}\}$,

$$f_\mu(x - y_{n+1}) = -\frac{1}{4\pi|x - y_{n+1}|} + \frac{\mu}{4\pi} + \mathscr{O}(|x - y_{n+1}|),$$

and taking only the value of the constant part. That is, the extended annihilation operator A acts on $G_\mu \psi^{(n-1)}$ as

$$AG_\mu \psi^{(n)}(x,Y) = \frac{\mu}{4\pi}\psi^{(n)}(x,Y) + \sum_{j=1}^{n} f_\mu(x - y_j)\psi^{(n)}(y_j, \hat{Y}_j, x). \tag{1.2}$$

The action of A is equivalently given by the formula

$$\left(A\psi^{(n+1)}\right)(x,Y) = \sqrt{n+1} \lim_{r\to 0} \left(\psi^{(n+1)}(x + r, Y, x) + \frac{1}{4\pi r}(B\psi^{(n+1)})(x,Y) \right). \tag{1.3}$$

One easily sees that this formula yields the action of $a(\delta_x)$ for $\psi \in D(L)$ and (1.2) on the range of G_μ. The extension A of $a(\delta_x)$ given by (1.3) is clearly local, in the sense that the value of $\left(A\psi^{(n+1)}\right)(x,Y)$ depends only on the values of $\psi^{(n+1)}$ in a small neighbourhood of the point $(x,Y,x) \in \mathbb{R}^{3(n+2)}$, on the "boundary" $\{x = y_{n+1}\}$.

1.2.2 The Hamiltonian for $N_{\max} = 1$

We will now give a short exposition of the construction of the operator H_1 with $N_{\max} = 1$. This is considerably easier than the other cases, and very similar models were already discussed by Lévy-Leblond [8] and Thomas [9].

Let H_1 be the operator given by

$$(H_1\psi)^{(n)} = \begin{cases} 0 & n > 1 \\ L_0^*\psi^{(1)} & n = 1 \\ -\Delta\psi^{(0)} + A\psi^{(1)} & n = 0 \end{cases}$$

with the domain

$$D(H_1) = \{\psi \in \mathscr{H} : \psi^{(0)} \in H^2(\mathbb{R}^3), \psi^{(1)} \in D(L_0^*), B\psi^{(1)} = \psi^{(0)}\}.$$

Note that the boundary condition $B\psi^{(1)} = \psi^{(0)}$ can equivalently be written as $\psi^{(1)} - G_\mu\psi^{(0)} \in D(L)$, by Prop. 1.

Proposition 2. *The operator $(H_1, D(H_1))$ is self-adjoint and non-negative.*

Proof. Let $G_\mu^{(0)} : \mathscr{H} \to \mathscr{H}$ be the operator given by $G_\mu^{(0)}\psi^{(0)} = G_\mu\psi^{(0)}$, $G_\mu^{(0)}\psi^{(n)} = 0$, $n > 0$. Since $G_\mu^{(0)}\psi^{(0)}$ is in the kernel of $L_0^* + \mu^2$, we have

$$L_0^*\psi^{(1)} = (L+\mu^2)(\psi^{(1)} - G_\mu^{(0)}\psi^{(0)}) - \mu^2\psi^{(1)}. \tag{1.4}$$

Using that $(G_\mu^{(0)})^*(L+\mu^2) = -a(\delta_x) = -A$ on $D(L)$, we also obtain

$$-(G_\mu^{(0)})^*(L+\mu^2)(\psi^{(1)} - G_\mu^{(0)}\psi^{(0)}) = A(\psi^{(1)} - G_\mu^{(0)}\psi^{(0)}) = A\psi^{(1)} - \frac{\mu}{4\pi}\psi^{(0)}.$$

Inserting these identities for $L_0^*\psi^{(1)}$ and $A\psi^{(1)}$ in the definition of H_1, we obtain the formula

$$H_1\psi = (1 - G_\mu^{(0)})^*(L+\mu^2)(1 - G_\mu^{(0)}) - \mu^2\psi + \frac{\mu}{4\pi}\psi^{(0)}.$$

Because $(G_\mu^{(0)})^2 = 0$, the operator $1 - G_\mu^{(0)}$ is invertible (with inverse $1 + G_\mu^{(0)}$). Consequently, $(1 - G_\mu^{(0)})^*(L+\mu^2)(1 - G_\mu^{(0)})$, and thus also H_1, are self-adjoint. It is clearly bounded from below by $-\mu^2$, for arbitrary $\mu > 0$, so it is non-negative.

1.2.3 The Hamiltonians for $N_{\max} > 1$

Having understood the Hamiltonian for $N_{\max} = 1$, there is an obvious generalisation for $N_{\max} > 1$. However, this turns out not to give a well-defined operator, due to additional singularities on the sets where the positions of more than one boson and the particle coincide. To see this, let us consider the case $N_{\max} = 2$. The obvious guess would be to set $H\psi^{(n)} = L_0^*\psi^{(n)} + A\psi^{(n+1)}$ for $n = 0,1$ and $H\psi^{(2)} = L_0^*\psi^{(2)}$, as above, on the domain where $B\psi^{(n+1)} = \psi^{(n)}$ for $n = 0,1$. If $\psi^{(0)} \neq 0$, this boundary condition implies that $\psi^{(1)}$ has the singularity $1/|x-y|$. More precisely, it can be written as

$$\psi^{(1)} = \phi^{(1)} + G_\mu\psi^{(0)} = \phi^{(1)}(x,y) + f_\mu(x-y)\psi^{(0)}(y)$$

with a regular function $\phi^{(1)} \in H^2(\mathbb{R}^3, L^2(\mathbb{R}^3))$. The same condition also relates the functions $\psi^{(2)}$ and $\psi^{(1)}$, and we have that

$$\begin{aligned}\psi^{(2)} &= \phi^{(2)} + G_\mu\psi^{(1)} \\ &= \phi^{(2)} + G_\mu\phi^{(1)} + G_\mu^2\psi^{(0)}\end{aligned}$$

with $\phi^{(2)} \in H^2(\mathbb{R}^3, L^2(\mathbb{R}^6))$. Now, spelling out the last term,

$$\begin{aligned}&\left(G_\mu^2\psi^{(0)}\right)(x,y_1,y_2) \\ &= \frac{1}{\sqrt{2}}\Big(f_\mu(x-y_2)f_\mu(y_1-y_2)\psi^{(0)}(y_1) + f_\mu(x-y_1)f_\mu(y_1-y_2)\psi^{(0)}(y_2)\Big),\end{aligned}$$

we see that $\psi^{(2)}$ has a singularity like $\frac{1}{|x-y_1||x-y_2|}$ near the set $\{x = y_1 = y_2\}$ whenever $\psi^{(0)} \neq 0$. This singularity is square-integrable, of course, so it does not pose a problem for the satisfiability of he boundary conditions. However, if we apply the extended annihilation operator A using (1.2) to such a term, we obtain

$$\begin{aligned}AG_\mu^2\psi^{(0)}(x,y) &= \frac{\mu}{4\pi}G_\mu\psi^{(0)}(x,y) + f_\mu(x-y)G_\mu\psi^{(0)}(y,x) \\ &= \frac{\mu}{4\pi}f_\mu(x-y)\psi^{(0)}(y) + f_\mu^2(x-y)\psi^{(0)}(x).\end{aligned}$$

The last term here has the singularity $1/|x-y|^2$ which is not square-integrable. Hence, our tentative operator does not map its domain into the Hilbert space $\mathscr{H}$, and our guess cannot be correct. Note that this ansatz does not even define a quadratic form, since f_μ^3 is not integrable.

Another way to look at this is that the operator A, together with the boundary condition, introduces (up to a permutation of arguments) an interaction $f_\mu(x-y)$ between the bosons and the particle. Such a potential does not map the extended domain $D(L_0^*)$ to $\mathscr{H}$, so $L_0^* + A$ is not an operator from $D(L_0^*)$ to $\mathscr{H}$, either. However, multiplication by $f_\mu(x-y)$ does map the free domain $D(L)$ to $\mathscr{H}$ and is infinitesimally bounded by L, so we will be able to address the problem by grouping these interactions together with the free operator in a certain way.

Let $T_\mu = AG_\mu$ be the operator given by (1.2). It is infinitesimally L-bounded, so the operator

$$K_\mu := L + T_\mu$$

is self-adjoint on $D(L)$, and for, μ sufficiently large (depending on n), K_μ is invertible on $\mathscr{H}^{(n)}$.

The construction now proceeds in a similar way to the case $N_{\max} = 1$, but replacing the free operator L by K_μ. We first restrict this operator to the functions in its domain for which $\psi^{(n)}(x,Y)|_{x=y_j} = 0$ for any $j = 1,\dots,n$, defining

$$(K_\mu)_0 := K_\mu|_{D(L_0)}.$$

We then pass to the extension $(K_\mu)_0^*$, and parametrise elements of its domain using the following modification of G_μ for $n < N_{\max}$

$$\widetilde{G}_\mu \psi^{(n)} = -(K_\mu + \mu^2)^{-1} a^*(\delta_x)\psi^{(n)},$$

where μ is chosen large enough for $K_\mu + \mu^2$ to be invertible on $\mathscr{H}^{(n+1)}$ for all $n < N_{\max}$.

Since K is a perturbation of L we can relate $\widetilde{G}_\mu$ to G_μ, and many important properties carry over to the modified operator. By the resolvent formula we have

$$\begin{aligned}\widetilde{G}_\mu \psi^{(n)} &= -(K_\mu + \mu^2)^{-1} a^*(\delta_x)\psi^{(n)}\\ &= G_\mu \psi^{(n)} - (-\Delta_x + \mu^2)^{-1} T_\mu \widetilde{G}_\mu \psi^{(n)}.\end{aligned} \tag{1.5}$$

From this and Lem. 1 we immediately obtain the boundedness of $\widetilde{G}_\mu$ for sufficiently large μ.

Lemma 2. *For any positive $n \in \mathbb{N}$, $\mu > 0$ large enough and $0 \le s < 1/2$ the operator $\widetilde{G}_\mu$ is bounded from $\mathscr{H}^{(n-1)}$ to $H^s(\mathbb{R}^3, L^2_{\mathrm{sym}}((\mathbb{R}^3)^n))$.*

We also have a characterisation of $D((K_\mu)_0^*)$, which follows from the same arguments as for L_0^* and the equivalence of norms on $D(K) = D(L)$ and their duals.

Proposition 3. *For any positive $n \in \mathbb{N}$ and $\mu > 0$ large enough we have*

$$D((K_\mu)_0^*) \cap \mathscr{H}^{(n)} = (D(L) \cap \mathscr{H}^{(n)}) \oplus \widetilde{G}_\mu \mathscr{H}^{(n-1)},$$

i.e., for every $\psi \in D((K_\mu)_0^*)$ there is a unique $\varphi_\mu^{(n-1)} \in \mathscr{H}^{(n-1)}$ so that $\psi^{(n)} - \widetilde{G}_\mu \varphi_\mu^{(n-1)}$ is an element of $H^2(\mathbb{R}^3, L^2(\mathbb{R}^{3n}))$.

However, functions in the range of $\widetilde{G}_\mu$ have different singularities than those in the range of G_μ, and the domains of L_0^* and K_0^* are different. We will now analyse these singularities in more detail and define the analogues of the boundary value operators A and B on the range of $\widetilde{G}_\mu$.

The operator $(-\Delta_x + \mu^2)^{-1} T_\mu$ in (1.5) is regularising, so the principal singularity is still given by the first term $f_\mu(x - y_{n+1})\psi^{n)}(y)$. In fact, one easily sees that it maps $\mathscr{H}^{(n)}$ into $H^1(\mathbb{R}^3, L^2(\mathbb{R}^{3(n+1)}))$ and that consequently $B(-\Delta_x+\mu^2)^{-1}T_\mu$ is well defined and equal to zero. Hence the boundary operator B, given by the expression (1.1) is defined on the range of $\widetilde{G}_\mu$ and $B\widetilde{G}_\mu\psi^{(n)} = \psi^{(n)}$, as for G_μ. The same is not true, however, for the extension of the annihilation operator A. The second term in $\widetilde{G}_\mu\psi$, Eq. (1.5), is not sufficiently regular to be evaluated at $x = y_{n+1}$, where it has a logarithmic singularity, as we will now see. The analogue of A is then defined by neglecting the divergent terms in the expansion of $\widetilde{G}_\mu\psi$ near $x = y$, as was done for A. More precisely, in the case $n = 0$, the singular behaviour of $\widetilde{G}_\mu\psi^{(0)}(x,y)$ at $|x-y| = r = 0$ is given by

$$\begin{aligned}\widetilde{G}_\mu\psi^{(0)}(x,y) &= G_\mu\psi^{(0)}(x,y) - (-\Delta_x+\mu^2)^{-1}\left(\frac{\mu}{4\pi} + f_\mu(x-y)\right)\widetilde{G}_\mu\psi^{(0)}(x,y)\\ &= g(r)\psi^{(0)} + \frac{\mu}{4\pi}\psi + S\psi^{(0)} + o(1),\end{aligned}$$

where

$$g(r) = -\frac{1}{4\pi r} + \frac{\log r}{16\pi^2},$$

the constant $\frac{\mu}{4\pi}$ corresponds of course to T_μ on $\mathscr{H}^{(0)}$ and S is a bounded operator from $H^\varepsilon(\mathbb{R}^3)$ to $\mathscr{H}^{(0)}$ for any $\varepsilon > 0$. The logarithmic divergence in $g(r)$ originates from the term

$$(-\Delta_x+\mu^2)^{-1} f_\mu(x-y)^2\psi^{(0)}(x)$$

that appears in $\widetilde{G}_\mu\psi^{(0)}$ after using the resolvent formula as above. We will give a proof of this asymptotic behaviour later, but we may already observe, by scaling, that this function should behave like a homogeneous function of degree zero for small values of $|x-y|$.

We now define the extension $\widetilde{A}$ of $a(\delta_x)$ to $D((K_\mu)_0^*)$ in an analogous way to A, Eq. (1.3),

$$\left(\widetilde{A}\psi^{(n+1)}\right)(x,Y) = \sqrt{n+1}\lim_{r\to 0}\left(\psi^{(n+1)}(x+r,Y,x) - g(r)(B\psi^{(n+1)})(x,Y)\right). \tag{1.6}$$

As with A, this is a local boundary value operator that restricts to $a(\delta_x)$ on $D(L)$. Its important properties as an operator on the range of G_μ are as follows.

Proposition 4. *The expression* (1.6) *defines a map from the range of* $\widetilde{G}_\mu$ *to* $H^{-1}(\mathbb{R}^3, L^2(\mathbb{R}^{3n}))$ *and* $\widetilde{A}\widetilde{G}_\mu - T_\mu =: S_\mu$ *defines a symmetric operator on the domain* $D(S) = H^\varepsilon(\mathbb{R}^3, L^2_{\mathrm{sym}}((\mathbb{R}^3)^n))$ *for any* $\varepsilon > 0$.

We will postpone the proof of this proposition, and first explain how this allows us to define the operator $H_{N_{\max}}$ and prove its self-adjointness. We set

$$(H_{N_{\max}}\psi)^{(n)} = \begin{cases} 0 & n > N_{\max} \\ L_0^*\psi^{(n)} & n = N_{\max} \\ (K_\mu)_0^*\psi^{(n)} + A\psi^{(n+1)} - T_\mu\psi^{(n)} & n = N_{\max}-1 \\ (K_\mu)_0^*\psi^{(n)} + \widetilde{A}\psi^{(n+1)} - T_\mu\psi^{(n)} & 0 < n < N_{\max}-1 \\ L\psi^{(0)} + \widetilde{A}\psi^{(1)} & n = 0. \end{cases}$$

on the domain

$$D(H_{N_{\max}}) = \left\{ \psi \in \mathscr{H} \,\middle|\, \begin{array}{ll} \psi^{(n)} \in D(L_0^*) \text{ and } B\psi^{(n)} = \psi^{(n-1)} & \text{for } n = N_{\max} \\ \psi^{(n)} \in D((K_\mu)_0^*) \text{ and } B\psi^{(n)} = \psi^{(n-1)} & \text{for } 0 < n < N_{\max} \\ \psi^{(0)} \in D(L) & \end{array} \right\}.$$

The condition $B\psi^{(n)} = \psi^{(n-1)}$ on $D((K_\mu)_0^*)$ is of course equivalent to $\psi^{(n)} - \widetilde{G}_\mu\psi^{(n-1)} \in D(L)$, as in the case $N_{\max} = 1$. In this definition μ must be taken large enough for Lem. 2 and Prop. 3 to hold but is otherwise arbitrary. On easily checks that $H_{N_{\max}}$ and its domain are independent of μ.

Theorem 1. *For any positive $N_{\max}$, the operator $H_{\max}$ is self-adjoint and bounded from below.*

Proof. To keep the notation simple, we will focus on the case $N_{\max} = 2$. The general case is a straightforward generalisation.

The operator for $N_{\max} = 2$ reads

$$(H_2\psi)^{(n)} = \begin{cases} 0 & n > 2 \\ L_0^*\psi^{(n)} & n = 2 \\ (K_\mu)_0^*\psi^{(n)} + A\psi^{(n+1)} - T_\mu\psi^{(n)} & n = 1 \\ L\psi^{(0)} + \widetilde{A}\psi^{(1)} & n = 0, \end{cases}$$

and the conditions on $D(H_2)$ are that $\psi^{(2)} - G_\mu\psi^{(1)}$, $\psi^{(1)} - \widetilde{G}_\mu\psi^{(0)}$ and $\psi^{(0)}$ be in $D(L) = D(K)$. The range of $\widetilde{G}_\mu$ is contained in the kernel of $(K_\mu)_0^* + \mu^2$ on the sector with than $n = 1$ bosons, and in the kernel of $L_0^* + \mu$ for $n = 2$ bosons. We thus have for $\psi \in D(H_2)$

$$(K_\mu)_0^*\psi^{(1)} = (L + T_\mu + \mu^2)(\psi^{(1)} - \widetilde{G}_\mu\psi^{(0)}) - \mu^2\psi^{(0)},$$

and the analogue with only L, as in Eq. (1.4), on the sector with $n = 2$ bosons. We also have

$$\begin{aligned} \widetilde{A}\psi^{(1)} &= \widetilde{A}(\psi^{(1)} - \widetilde{G}_\mu\psi^{(0)}) + \widetilde{A}\widetilde{G}_\mu\psi^{(0)} = a(\delta_x)(\psi^{(1)} - \widetilde{G}_\mu\psi^{(0)}) + (T_\mu + S_\mu)\psi^{(0)} \\ A\psi^{(2)} &= a(\delta_x)(\psi^{(2)} - G_\mu\psi^{(1)}) + T_\mu\psi^{(1)} \end{aligned}$$

When we instert these identities into the definition of H_2, the terms $\pm T_\mu \psi^{(1)}$ will cancel each other. We thus have

$$H_2 = (1-\widetilde{G}_\mu^{(1)})^*(L+T_\mu^{(1)}+\mu^2)(1-\widetilde{G}_\mu^{(1)})+S_\mu^{(0)}-\mu^2 \tag{1.7}$$

on the sectors with at most $N_{\max}=2$ bosons. Here, $\widetilde{G}_\mu^{(1)}$ is given by

$$\widetilde{G}_\mu^{(1)}\psi^{(n)} = \begin{cases} G_\mu \psi^{(1)} & n=1 \\ \widetilde{G}_\mu \psi^{(0)} & n=0 \\ 0 & \text{otherwise,} \end{cases}$$

the operator $T_\mu^{(1)}$ acts as T_μ on the sectors with at most one boson, $S_\mu^{(0)}$ acts as S_μ on $\mathscr{H}^{(0)}$, and zero on all other sectors.

The operator $\widetilde{G}_\mu^{(1)}$ is nilpotent and thus $1-\widetilde{G}_\mu^{(1)}$ is invertible. Consequently, the first term in Eq. (1.7) is a self-adjoint operator and bounded from below. The operator $S^{(0)}$ is relatively bounded w.r.t. this operator by Prop. 4. In fact, by Eq. (1.11) below, $S^{(0)}$ is essentially a Fourier multiplier of logarithmic growth. Since it acts non-trivially only on $\mathscr{H}^{(0)}$, we have $S_\mu^{(0)} = (1-\widetilde{G}_\mu^{(1)})^* S_\mu^{(0)}(1-\widetilde{G}_\mu^{(1)})$ and the relative bound is obvious. This completes the proof for $N_{\max}=2$.

For $N_{\max}>2$ one needs to use in the final step the fact that, by Prop. 4 and Lem. 1, $S_\mu \widetilde{G}_\mu$ is a bounded operator.

We finally come to the proof of Prop. 4.

Proof (of Prop. 4). From Eq. (1.5) we can immediately conclude that the sum of the first term in (1.5) and the first term in $g(x-y_{n+1})$ has a limit, and this acts in the same way as $AG_\mu = T_\mu$, mapping $\mathscr{H}^{(n)}$ to $H^{-1}(\mathbb{R}^3, L^2(\mathbb{R}^{3(n-1)}))$.

We then need to analyse the second term in (1.5), i.e. the negative of

$$(-\Delta_x+\mu^2)^{-1} T_\mu \widetilde{G}_\mu \psi^{(0)}. \tag{1.8}$$

We give the details of this analysis only in the case $n=0$ and comment on the adjustments for the general case in the end. The analysis for $n=0$ is sufficient for the construction of the model with $N_{\max}=2$.

By the regularising properties of $(-\Delta_x+\mu^2)^{-1}$, the difference of (1.8) with

$$(-\Delta_x+\mu^2)^{-1} f_\mu(x-y) G_\mu \psi^{(0)}(y,x) = (-\Delta_x+\mu^2)^{-1} f_\mu^2(x-y)\psi^{(0)}(x) \tag{1.9}$$

is an element of $H^2(\mathbb{R}^3, L^2(\mathbb{R}^3))$, and can thus be evaluated at $x=y$, yielding a bounded operator on $\mathscr{H}^{(0)}$ that will be absorbed into S_μ. We now calculate the asymptotics of (1.9) near $x=y$ using the Fourier representation. One can explicitly calculate the Fourier transform of f_μ^2,

$$\widehat{f_\mu^2}(q) = \frac{1}{4\pi(2\pi)^{3/2}} \frac{1}{|q|} \arctan(|q|/2\mu),$$

and thus the Fourier transform of (1.9) is

$$\frac{1}{4\pi(2\pi)^{3/2}}\frac{1}{(p^2+\mu)|q|}\arctan(|q|/2\mu)\,\hat{\psi}^{(0)}(p+q). \tag{1.10}$$

Taking the inverse transform then leads to

$$\begin{aligned}(1.9) &= \frac{1}{4\pi(2\pi)^{9/2}}\int dpdq\frac{e^{ipx+iqy}}{(p^2+\mu^2)|q|}\arctan(|q|/2\mu)\,\hat{\psi}^{(0)}(p+q)\\ &= \frac{1}{4\pi(2\pi)^{9/2}}\int dkdq\frac{e^{ikx+iq(y-x)}}{((q-k)^2+\mu^2)|q|}\arctan(|q|/2\mu)\,\hat{\psi}^{(0)}(k).\end{aligned}$$

Setting $x-y=0$ we would (formally) have the action of a Fourier multiplier on $\psi^{(0)}$, but this makes no sense since the q-integral does not converge. We must thus show that the difference of this expression and the logarithmic term in $g(x-y)$ has a limit, and this will act as an appropriate Fourier multiplier on $\psi^{(0)}$. As the difficulty stems from the insufficient decay of the integrand at $q\to\infty$, we may replace $\arctan(|q|/2\mu)$ by its limit $\pi/2$. We also replace $\big((q-k)^2+\mu^2\big)^{-1}$ by $\big(q^2+k^2+\mu^2\big)^{-1}$. The errors arising from both replacements give absolutely convergent integrals that evaluate to bounded Fourier multipliers. It then remains to determine the singular behaviour of

$$\frac{1}{8(2\pi)^3}\int dq\frac{e^{iq(x-y)}}{(q^2+k^2+\mu^2)|q|} = \frac{1}{16\pi^2}\int_0^\infty dt\frac{\sin(tr)}{r}\frac{1}{t^2+k^2+\mu^2},$$

as $r\to 0$. Changing variables to $s=tr$, one reduces this to the calculation of

$$\int_0^\infty ds\frac{\sin(s)}{s^2+r^2(k^2+\mu^2)} = -\log r - \tfrac{1}{2}\log(k^2+\mu^2)+\mathscr{O}(1),$$

where the remainder is convergent as $r\to 0$ and uniformly bounded in k. Consequently,

$$\begin{aligned}S_{\mathrm{sing}}(k) :=& \lim_{|x-y|\to 0}\left(-\frac{1}{4\pi(2\pi)^3}\int dq\frac{e^{iq(x-y)}}{(q^2+k^2+\mu^2)|q|}-\frac{\log|x-y|}{16\pi^2}\right)\\ =&\frac{\log(k^2+\mu^2)}{32\pi^2}+\mathscr{O}(1)\end{aligned} \tag{1.11}$$

defines a Fourier multiplier that gives rise to a symmetric operator on $D(S)$, as claimed. This completes the argument for $n=0$.

For $n\geq 1$, T_μ contains additional terms that, in the analogue of (1.9) lead to terms of the form (with $Y\in\mathbb{R}^{3(n+1)}$)

$$\begin{aligned}&(-\Delta_x+\mu^2)^{-1}f_\mu(x-y_j)f_\mu(y_j-y_i)\psi^{(n)}(y_i,\hat{Y}_{i,j},x),\\ &\qquad(-\Delta_x+\mu^2)^{-1}f_\mu^2(x-y_i)\psi^{(n)}(x,\hat{Y}_i),\end{aligned} \tag{1.12}$$

with $j \leq n+1$, $i \leq n$ and $j \neq i$. These terms are less singular than (1.9) at $x = y_{n+1}$, since at least some of the singularities in f_μ concern different directions. In fact, they can be evaluated at $x = y_{n+1}$ and this evaluation defines a bounded operator on $\mathscr{H}^{(n)}$. We will give a short proof of this statement for the term (1.12), the argument for the other one is very similar. A detailed exposition of similar arguments may be found in [5, App. A]. Using the Fourier transform (1.10) the evaluation of (1.12) at $x = y_{n+1}$ has the Fourier representation (now with $Q \in \mathbb{R}^{3n}$)

$$\frac{1}{2(2\pi)^4} \int dk \frac{\arctan(|q_i|/2\mu)}{((p-k)^2+\mu^2)|q_i|} \hat{\psi}^{(n)}(p-k+q_i, \hat{Q}_i, k).$$

This integral operator can be bounded by an argument similar to the well-known Schur test. We have for $\varphi \in \mathscr{H}^{(n)}$

$$\begin{aligned} &\int dQ dp dk \left| \overline{\hat{\varphi}(p,Q)} \frac{\arctan(|q_i|/2\mu)}{((p-k)^2+\mu^2)|q_i|} \hat{\psi}^{(n)}(p-k+q_i, \hat{Q}_i, k) \right| \qquad (1.13) \\ &\leq \frac{\pi}{4} \int d\hat{Q}_i dp dk d\ell \bigg(\frac{|\hat{\varphi}(p, \hat{Q}_i, \ell - p)|^2}{((p-k)^2+\mu^2)} \frac{|\ell - p|^{s-1}}{|\ell - k|^s} \\ &\qquad\qquad \frac{|\hat{\psi}(\ell - k, \hat{Q}_i, k)|^2}{((p-k)^2+\mu^2)} \frac{|\ell - k|^s}{|\ell - p|^{s+1}} \bigg). \end{aligned}$$

Now for $1 < s < 2$,

$$\begin{aligned} \int \frac{dk}{((p-k)^2+\mu^2)|\ell - k|^s} &\leq C|\ell - p|^{1-s} \\ \int \frac{dp}{((p-k)^2+\mu^2)|\ell - p|^{s+1}} &\leq C|\ell - k|^{-s}, \end{aligned}$$

for some constant C. This implies that (1.13) is bounded by $C(\|\varphi\|^2_{\mathscr{H}^{(n)}} + \|\psi\|^2_{\mathscr{H}^{(n)}})$, which proves boundedness of the corresponding operator by standard arguments.

The symmetry of S_μ is easy to check from the Fourier representation by changing variables and summing over the indices i, j.

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