DESCRIBING LACK OF COMPACTNESS

MATHIEU LEWIN

We consider a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $L^2(\mathbb{R}^d)$ and we describe its possible behavior (up to subsequences) in a rather detailed way. We typically think of a sequence u_n which converges to u weakly in $L^2(\mathbb{R}^d)$ but which does not converge for the strong topology. One says that $\boldsymbol{u} = \{u_n\}$ exhibits a *lack of compactness*. Considering instead $u_n - u$, we can think of a sequence $u_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$ but such that, for instance, $\int_{\mathbb{R}^d} |u_n|^2 = \lambda > 0$ for all n.

In Figure 1, we display four typical examples of such a sequence $\{u_n\}$ (the examples can easily be adapted to fix any $L^p(\mathbb{R}^d)$ norm). In Example 1a, the sequence $\{u_n\}$ spreads out everywhere in space but since its mass is conserved, locally the mass must go to zero. The sequence is said to "vanish" (Y. Meyer likes to think of a melting snowman: If we wait sufficiently long, there is essentially no remaining water anywhere on the floor). In Example 1b, the sequence concentrates at one point (to be more precise $|u_n|^2$ converges to a delta measure located at zero). For n large enough we essentially detect all the mass in any small neighborhood of the origin. In Example 2a, the sequence $\{u_n\}$ keeps a constant shape for all n but it runs off to infinity. If we want to see anything, we have to follow the sequence by "running after it". Finally, in Example 2b, the function oscillates so fast that (by the Riemann-Lebesgue Lemma) it converges weakly to zero in $L^2(\mathbb{R}^d)$.

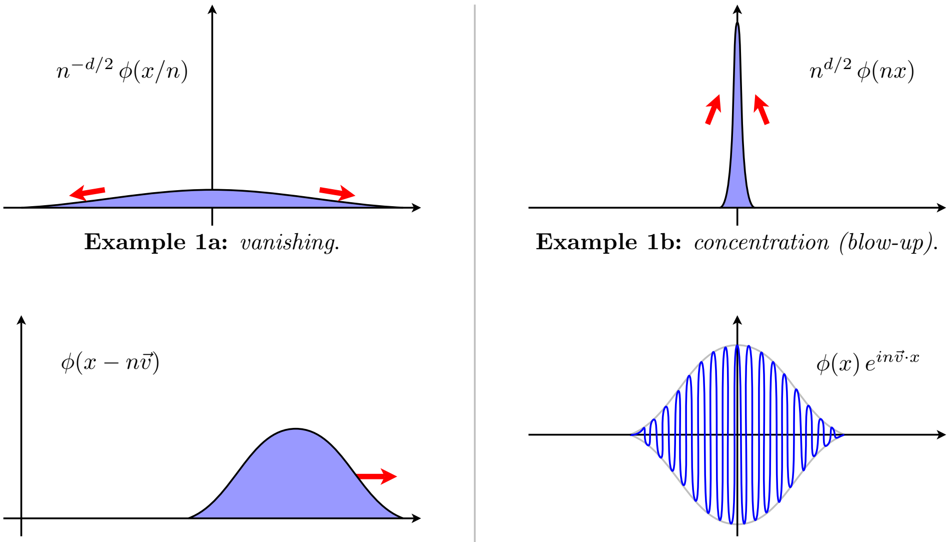
Examples 1a and 1b on the one side, and 2a and 2b on the other side are in some sense dual to each other: We can go from 1a to 1b (resp. 2a to 2b) by applying a Fourier transform. For instance, a sequence which oscillates very fast has a lack of compactness due to translations in Fourier space, and conversely. Similarly, a sequence which concentrates in Fourier space vanishes in direct space and conversely.

There is a crucial difference between the examples of the first column and that of the second one. In the first column, the derivative of the sequence $\{u_n\}$ stays bounded uniformly in n: For instance the sequence is bounded in the Sobolev space $H^1(\mathbb{R}^d)$. In the cases of the second column, the function oscillates or blows up so fast that the derivative has to explode: We have $\|\nabla u_n\|_{L^2(\Omega)} \to \infty$, even in a (well-chosen) bounded domain Ω. Examples 1a and 2a are called *locally compact*, whereas Examples 1b and 2b are not locally compact.

Of course, it is clear that we can combine the four examples before as we want by adding functions behaving differently. For instance we can have a sequence composed of several pieces receding from each other but which keep a constant shape.

In the beginning of the 80s, there has been a high activity in trying to describe the possible behavior of sequences which undergo a lack of compactness. The main idea was to prove that, in an appropriate sense, the four behaviors mentioned before are *universal*. Stating differently, a non-compact sequence should, up to a subsequence, be a (possibly infinite) sum of sequences having one of the above behaviors, up to a small error. Knowing this fact is very useful either in a contradiction argument (when we want to prove that the sequence has to be compact), or when the studied lack of compactness has a special physical meaning and has to be described more accurately.

Date: January 10, 2019. Taken from: M. Lewin *Variational Methods in Quantum Mechanics*, Unpublished Lecture Notes, University of Cergy-Pontoise (2010).



Example 1a: *vanishing.* **Example 1b:** *concentration (blow-up).*

Example 2a: *translation.* **Example 2b:** *oscillations.*

Locally compact Non locally compact

Figure 1. Four typical behaviors of a sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ with $u_n \rightharpoonup 0$ while $\int_{\mathbb{R}^d} |u_n|^2 = \lambda$ for all n. In the examples, ϕ is a fixed (smooth) function in $L^2(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} |\phi|^2 = \lambda$.

The first to tackle these issues on a specific example were Sacks and Uhlenbeck [20] in 1981 who dealt with a concentration phenomenon for harmonic maps. Brezis and Nirenberg [3] then faced similar difficulties for some elliptic Partial Differential Equations (PDE) with a critical Sobolev exponent in 1983. The same year, Lieb proved in [11] a useful lemma dealing with lack of compactness due to translations in the locally compact case. The most general method for dealing with locally compact problems was published by Lions [15, 16] in 1984 under the name "concentration-compactness" (it was announced before in [13]). Later in 1984-85, Struwe [21] and, independently, Brezis and Coron [2] have provided the first "bubble decompositions" (that is they have shown that the sequence under study can be written as a sum of functions which concentrate at some points in space). In 1985, Lions adapted his concentration-compactness method to the nonlocal case [17, 18] (see also [14]).

The tools developed in the 80s are now of widespread use in the analysis of PDEs. In this section, we present a simple theory in the locally compact case. We consider a bounded sequence $\{u_n\}$ in $H^1(\mathbb{R}^d)$ and describe in a rather detailed way its behavior in $L^p(\mathbb{R}^d)$ for a subcritical $2 \leq p < p^*$ (with p^* being the critical Sobolev exponent to be recalled below). More precisely, we will show that $\{u_n\}$ can be written as a sum of functions retaining their shape and escaping to infinity at different speeds (like in Example 2a), plus a remainder which can vanish in the sense of Example 1a. This is enough for most of the examples we encounter in the following. Most of the arguments of this section can be generalized to a Sobolev space $W^{k,p}(\mathbb{R}^d)$ with $k \geq 1$ and $1 < p < \infty$. It is not essential that the underlying space is a Hilbert space, but only that the sequence is locally compact, by the Rellich-Kondrachov Theorem.

Our approach in this section is a combination of ideas of Lieb [11] and of the concentration-compactness method of Lions [15, 16]. For a more general version of the concentration-compactness method (in $L^1(\mathbb{R}^d)$ or for measures), we refer to the original papers of Lions, or to the book of Struwe [22]. The french speaking reader can refer to the book of Kavian [7], or to that of Cancs, Le Bris and Maday [4] for an intuitive presentation of the method.

1. Finding bubbles

We consider a sequence $\boldsymbol{u} = \{u_n\}$ bounded in $L^2(\mathbb{R}^d)$ (later our sequence will be bounded in $H^1(\mathbb{R}^d)$). Our goal is to detect pieces of mass which retain their shape for n large and, possibly, escape to infinity (in the spirit of Example 2a in Figure 1, we call them 'bubbles'). For this reason, we consider all the possible weak limits, up to translations, of subsequences of $\{u_n\}$ and define the largest possible mass that these weak limits can have.

Definition 1 (Highest local mass of a sequence)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. We define the following number*

$$\boldsymbol{m}(\boldsymbol{u}) = \sup\left\{\int_{\mathbb{R}^d} |u|^2 \ : \ \exists\{x_k\} \subset \mathbb{R}^d,\ u_{n_k}(\cdot + x_k) \rightharpoonup u \text{ weakly in } L^2(\mathbb{R}^d)\right\}. \tag{1}$$

Here we are interested in understanding the lack of compactness due to space translations. Translations form a subgroup of all the isometries of $L^2(\mathbb{R}^d)$, which is isomorphic to $\mathbb{R}^d$. It is because this group is not compact itself, that it can induce a lack of compactness on functions in $L^2(\mathbb{R}^d)$. The other important subgroup is that consisting of *dilations* which are responsible for the lack of compactness in Examples 1a and 1b of Figure 1. One can consider a similar number $\boldsymbol{m}(\boldsymbol{u})$ for any group of isometries acting on $L^2(\mathbb{R}^d)$, like the group generated by translations and dilations.

Remark 2. *It is clear that $\boldsymbol{m}(\boldsymbol{u}) = \boldsymbol{m}(\boldsymbol{u}(\cdot + \boldsymbol{x}))$ for any sequence of translations $\boldsymbol{x} = \{x_n\} \subset \mathbb{R}^d$ (here $\boldsymbol{u}(\cdot + \boldsymbol{x})$ is the sequence $\{u_n(\cdot + x_n)\}$). Also, if $\boldsymbol{u}' = \{u_{n_k}\}$ is a subsequence of $\boldsymbol{u} = \{u_n\}$, then $\boldsymbol{m}(\boldsymbol{u}') \leq \boldsymbol{m}(\boldsymbol{u})$.* $\diamondsuit$

Remark 3. *If $\{u_n\}$ is not only bounded in $L^2(\mathbb{R}^d)$ but also in the Sobolev space $H^1(\mathbb{R}^d)$, then we can replace the weak convergence in $L^2(\mathbb{R}^d)$ by weak convergence in $H^1(\mathbb{R}^d)$ in the definition of $\boldsymbol{m}(\boldsymbol{u})$ without changing anything.*

Exercise 4. *There does not necessarily exist a $u \in L^2(\mathbb{R}^d)$ realizing the above supremum (that is such that $u_{n_k}(\cdot + x_{n_k}) \rightharpoonup u$ for a subsequence with $\int_{\mathbb{R}^d} |u|^2 = \boldsymbol{m}(\boldsymbol{u})$). We provide a counter-example. Let $\psi_n = n^{-d/2}\sqrt{1-1/n}\, u(x/n)$ for some fixed $u \in L^2(\mathbb{R}^d)$. Define a new sequence $\{u_n\}$ as follows:*

$$\begin{array}{lll} u_1 = \psi_1, & & \\ u_2 = \psi_1, & u_3 = \psi_2, & \\ u_4 = \psi_1, & u_5 = \psi_2, & u_6 = \psi_3, \\ u_7 = \psi_1, & \dots & \end{array}$$

and so on. Verify that $\boldsymbol{m}(\boldsymbol{u}) = \int_{\mathbb{R}^d} |u|^2$ but that it is not "attained" by any weakly convergent subsequence.

Exercise 5. *Verify that $\boldsymbol{m}(\boldsymbol{u}) = 0$ if $u_n(x) = n^{-d/2}u(x/n)$ (Example 1a), $u_n(x) = n^{d/2}u(nx)$ (Example 1b), or $u_n(x) = u(x)\, e^{in\vec{v}\cdot x}$ (Example 2b). Here u is a fixed function in $L^2(\mathbb{R}^d)$.*

The purpose of $\boldsymbol{m}(\boldsymbol{u})$ is to detect the largest piece of mass in the sequence $\boldsymbol{u} = \{u_n\}$, which possibly escapes to infinity (when $|x_{n_k}| \to \infty$). If $\boldsymbol{m}(\boldsymbol{u}) > 0$, then we can find

- a subsequence $\{u_{n_k}\}$;
- a sequence of translations $\{x_k^{(1)}\} \subset \mathbb{R}^d$;
- a function $u^{(1)} \in L^2(\mathbb{R}^d)$ such that $\boldsymbol{m}(\boldsymbol{u}) \geq \int_{\mathbb{R}^d} |u^{(1)}|^2 \geq \boldsymbol{m}(\boldsymbol{u}) - \varepsilon > 0$ (in particular $u^{(1)} \neq 0$),

such that

$$u_{n_k}(\,\cdot+x_k^{(1)})\rightharpoonup u^{(1)}\qquad\text{weakly in }L^2(\mathbb{R}^d).$$

Once we have found the first 'bubble' $u^{(1)}$, we can go on and try to find the next one by considering the sequence $r_k^{(2)}:=u_{n_k}-u^{(1)}(\,\cdot-x_k^{(1)})$ and the corresponding $\boldsymbol{m}(\boldsymbol{r}^{(2)})$. Arguing by induction we can find by induction all the bubbles contained in the original sequence u_n.

Lemma 6 (Extracting bubbles)**.** *Let $\boldsymbol{u}=\{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$). Then there exists a (possibly empty or finite) sequence of functions $\{u^{(1)},u^{(2)},...\}$ in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$) such that the following holds true: For any fixed $\varepsilon>0$, there exists*

- *an integer J,*
- *a subsequence $\{u_{n_k}\}$ of $\{u_n\}$,*
- *space translations $\{x_k^{(j)}\}_{k\geq 1}\subset\mathbb{R}^d$ for $j=1,...,J$ with $|x_k^{(j)}-x_k^{(j')}|\to\infty$ as $k\to\infty$, for each $j\neq j'$,*

such that we can write

$$u_{n_k}=\sum_{j=1}^{J}u^{(j)}(\,\cdot-x_k^{(j)})+r_k^{(J+1)}\tag{2}$$

where

$$r_k^{(J+1)}(\cdot+x_k^{(j)})\underset{k\to\infty}{\rightharpoonup}0$$

weakly in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$) for all $j=1,...,J$ and

$$\boldsymbol{m}(\boldsymbol{r}^{(J+1)})\leq\varepsilon.$$

In particular we have

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^d}|u_{n_k}|^2-\int_{\mathbb{R}^d}|r_k^{(J+1)}|^2\right)=\sum_{j=1}^{J}\int_{\mathbb{R}^d}|u^{(j)}|^2.\tag{3}$$

The functions $u^{(j)}$ are all the possible weak limits of $\boldsymbol{u}$ up to translations and extraction of subsequences. The theorem says that any bounded sequence $\boldsymbol{u}$ in $L^2(\mathbb{R}^d)$ can be written as a linear combination of these limits translated in space, up to an error term $r_k^{(J+1)}$. This error term is not necessarily small in norm, but it is such that $\boldsymbol{m}(\boldsymbol{r}^{(J+1)})$ is small. If $\{u_n\}$ is only bounded in $L^2(\mathbb{R}^d)$ then lots of things can happen to $r_k^{(J+1)}$, think of Examples 1a, 1b and 2b of Figure 1. When $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$ then $\boldsymbol{m}(\boldsymbol{r}^{(J+1)})\leq\varepsilon$ will tell us a lot since Examples 1b and 2b cannot happen in a locally compact setting. We will explain this in the next section.

Let us emphasize the fact that the bubbles $u^{(j)}$ do not depend on ε. They can be constructed once and for all for any fixed sequence $\boldsymbol{u}=\{u_n\}$. On the contrary, the number J of bubbles, the space translations $x_k^{(j)}$ and the subsequence n_k needed to achieve a given accuracy in the decomposition of u_{n_k}, all depend on ε.

Proof of Lemma 6. We only write the proof when $\{u_n\}$ is bounded in $L^2(\mathbb{R}^d)$. When it is bounded in $H^1(\mathbb{R}^d)$, any weak limit in $L^2(\mathbb{R}^d)$ is automatically also a weak limit in $H^1(\mathbb{R}^d)$.

If $\boldsymbol{m}(\boldsymbol{u})=0$, there is nothing to prove and we may therefore assume that $\boldsymbol{m}(\boldsymbol{u})>0$. In this case we deduce from the definition of $\boldsymbol{m}(\boldsymbol{u})$ that there exists a subsequence n_k, a sequence of translations $\{x_k^{(1)}\}\subset\mathbb{R}^d$, and a function $u^{(1)}\in L^2(\mathbb{R}^d)$ such that

$$\frac{\boldsymbol{m}(\boldsymbol{u})}{2}\leq\int_{\mathbb{R}^d}|u^{(1)}|^2\leq\boldsymbol{m}(\boldsymbol{u})$$

and

$$u_{n_k}(\cdot + x_k^{(1)}) \rightharpoonup u^{(1)}$$

weakly in $L^2(\mathbb{R}^d)$. Defining $r_k^{(2)} := u_{n_k} - u^{(1)}(\cdot - x_k^{(1)})$, we obviously get that $r_k^{(2)}(\cdot + x_k^{(1)}) \rightharpoonup 0$. We also have

$$\|u_{n_k}\|_{L^2}^2 = \left\|u^{(1)}\right\|_{L^2}^2 + \left\|r_k^{(2)}\right\|_{L^2}^2 + 2\Re\left\langle r_k^{(2)}(\cdot + x_k^{(1)}), u^{(1)}\right\rangle$$

where the last term tends to 0 by the weak convergence of $r_k^{(2)}$. So we conclude that

$$\lim_{k\to\infty}\left(\|u_{n_k}\|_{L^2}^2 - \left\|r_k^{(2)}\right\|_{L^2}^2\right) = \left\|u^{(1)}\right\|_{L^2}^2.$$

If $\boldsymbol{m}(\boldsymbol{r}^{(2)}) = 0$, we can stop here. Otherwise we go on and extract the next bubble. We find a subsequence and space translations such that $r_{k_\ell}^{(2)}(\cdot + x_\ell^{(2)}) \rightharpoonup u^{(2)} \neq 0$, with

$$\frac{\boldsymbol{m}(\boldsymbol{r}^{(2)})}{2} \leq \int_{\mathbb{R}^d} |u^{(2)}|^2 \leq \boldsymbol{m}(\boldsymbol{r}^{(2)}).$$

We also extract a further subsequence from u_{n_k} and $x_k^{(1)}$. To simplify the exposition, we use the same notation for all these subsequences. So we can write

$$u_{n_k} = u^{(1)}(\,\cdot - x_k^{(1)}) + u^{(2)}(\,\cdot - x_k^{(2)}) + r_k^{(3)}$$

where $r_k^{(3)}(\cdot + x_k^{(2)}) \rightharpoonup 0$. If $|x_k^{(2)} - x_k^{(1)}|$ does not diverge, then we can extract another subsequence such that $x_k^{(1)} - x_k^{(2)} \to v$. We would then obtain $u_{n_k}(\cdot + x_k^{(1)}) \rightharpoonup u^{(1)} + u^{(2)}(\cdot + v)$, since $r_k^{(3)}(\cdot + x_k^{(1)}) = r_k^{(3)}(\cdot + v + x_k^{(2)}) \rightharpoonup 0$. This contradicts the fact that $u_{n_k}(\cdot + x_k^{(1)}) \rightharpoonup u^{(1)}$ by construction, unless $u^{(2)} = 0$ which is not the case for us here. So we conclude that $|x_k^{(2)} - x_k^{(1)}| \to \infty$. Now we can use that $u_{n_k}(\cdot + x_k^{(1)}) \rightharpoonup u^{(1)}$ and deduce that $r_k^{(3)}(\cdot + x_k^{(1)}) \rightharpoonup 0$, since $u^{(2)}(\cdot + x_k^{(1)} - x_k^{(2)}) \rightharpoonup 0$. Similarly as before, it can be checked that

$$\lim_{k\to\infty}\left(\|u_{n_k}\|_{L^2}^2 - \left\|r_k^{(3)}\right\|_{L^2}^2\right) = \left\|u^{(1)}\right\|_{L^2}^2 + \left\|u^{(2)}\right\|_{L^2}^2.$$

We can apply the previous argument *ad infinitum*, constructing the sequences $u^{(j)}$ and $r^{(j)}$ in the same fashion, except when we reach a remainder $r_k^{(J+1)}$ which is such that $\boldsymbol{m}(\boldsymbol{r}^{(J+1)}) = 0$. Our construction satisfies

$$\lim_{k\to\infty}\left(\|u_{n_k}\|_{L^2}^2 - \left\|r_k^{(J+1)}\right\|_{L^2}^2\right) = \sum_{j=1}^{J}\left\|u^{(j)}\right\|_{L^2}^2$$

for all J, and in particular

$$\sum_j \left\|u^{(j)}\right\|_{L^2}^2 \leq \limsup_{n\to\infty}\|u_n\|_{L^2}^2 \leq C.$$

Therefore we see that $\left\|u^{(j)}\right\|_{L^2} \to 0$ as $j \to \infty$. Recalling that

$$\boldsymbol{m}(\boldsymbol{r}^{(j)}) \leq 2\int_{\mathbb{R}^d} |u^{(j)}|^2$$

by construction, we deduce that $\boldsymbol{m}(\boldsymbol{r}^{(j)}) \to 0$ as well. Hence, for J large enough, $\boldsymbol{m}(\boldsymbol{r}^{(J)})$ must be smaller than any ε fixed in advance, and this concludes the proof of Theorem 6. $\square$

Remark 7. *As is obvious from the above proof of Lemma 6, we can also choose the sequence $\{u^{(j)}\}$ to have that*

$$\max_{j\geq 0}\int_{\mathbb{R}^d}|u^{(j)}|^2\geq \boldsymbol{m}(\boldsymbol{u})-\eta \tag{4}$$

for any $\eta>0$. It suffices to choose $u^{(1)}$ satisfying this property. Note that (4) is a maximum, since $\int_{\mathbb{R}^d}|u^{(j)}|^2\to 0$ as $j\to\infty$, by construction. $\Diamond$

As we have mentioned in Exercise 4, $\boldsymbol{m}(\boldsymbol{u})$ is not necessarily 'attained'. We now prove that it is always attained for a subsequence.

Corollary 8 ($m(\{u_n\})$ is attained for a subsequence)**.** *Let $\boldsymbol{u}=\{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ and $\eta>0$. There exists a subsequence $\boldsymbol{u}'=\{u_{n_k}\}$, $\{x_k\}\subset\mathbb{R}^d$, $u\in L^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|u|^2=\boldsymbol{m}(\boldsymbol{u}')\geq \boldsymbol{m}(\boldsymbol{u})-\eta$ and $u_{n_k}(\cdot - x_k)\rightharpoonup u$.*

Proof. We apply Lemma 6, choosing specifically the sequence $\{u^{(j)}\}$ such as to have

$$\max_{j\geq 0}\int_{\mathbb{R}^d}|u^{(j)}|^2\geq \boldsymbol{m}(\boldsymbol{u})-\eta$$

by Remark 7. Then we choose J large enough such that

$$\boldsymbol{m}\big(\boldsymbol{r}^{(J+1)}\big)<\max\Big(\int_{\mathbb{R}^d}|u^{(j)}|^2,\ j=1,...,J\Big).$$

It is then clear that $\boldsymbol{m}(\boldsymbol{u})=\max\big(\int_{\mathbb{R}^d}|u^{(j)}|^2,\ j=1,...,J\big)$. $\square$

2. Vanishing

In the previous section we have defined the highest mass that weak limits can have up to space translations. We have shown in Lemma 6 that any bounded sequence in $L^2(\mathbb{R}^d)$ can be written as a linear combination of these bumps escaping to infinity, plus a remainder $r_k^{(J+1)}$ which is such that $\boldsymbol{m}(\boldsymbol{r}^{(J+1)}$ is small. Our purpose here is to better understand what this means for a sequence which is bounded in the Sobolev space $H^1(\mathbb{R}^d)$. In this case, the following essential lemma allows to relate $\boldsymbol{m}(\boldsymbol{u})$ to subcritical L^p norms:

Lemma 9 (A subcritical estimate involving $\boldsymbol{m}(\boldsymbol{u})$)**.** *Let $\boldsymbol{u}=\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. There exists a universal constant $C=C(d)$ such that the following holds:*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^d}|u_n|^{2+\frac{4}{d}}\leq C\,\boldsymbol{m}(\boldsymbol{u})^{\frac{2}{d}}\,\limsup_{n\to\infty}\|u_n\|^2_{H^1(\mathbb{R}^d)}\,. \tag{5}$$

The previous result gives an estimate on the size of $\|u_n\|_{L^{2+4/d}(\mathbb{R}^d)}$ in terms of $\boldsymbol{m}(\boldsymbol{u})$. Note that the exponent $2+4/d$ is always smaller than the critical Sobolev exponent p^*, which equals $2+4/(d-2)$ in dimensions $d\geq 3$ and $+\infty$ in dimensions $d=1,2$. The proof of course relies on the Sobolev inequality which we discuss later in Section 5. Lemma 9 is essentially due to Lions (see the proof of Lemma I.1 in [16]).

Proof of Lemma 9. We only write the proof for $d\geq 3$ and let the (very similar) cases $d=1,2$ as an exercise. We consider a tiling of the whole space $\mathbb{R}^d$ by means of cubes, say $\mathbb{R}^d=\cup_{z\in\mathbb{Z}^d}C_z$ with $C_z=\prod_{j=1}^d[z_j,z_j+1)$. We then calculate by Hlder's inequality

$$\int_{\mathbb{R}^d}|u_n|^q=\sum_{z\in\mathbb{Z}^d}\int_{C_z}|u_n|^q\leq\sum_{z\in\mathbb{Z}^d}\|u_n\|^{\theta q}_{L^2(C_z)}\,\|u_n\|^{(1-\theta)q}_{L^{p^*}(C_z)}\,, \tag{6}$$

where $1/q=\theta/2+(1-\theta)/p^*$. We choose q in such a way that $(1-\theta)q=2$. A simple calculation shows that $q=2+4/d$. Note that this choice satisfies $2<q<$

$p^* = 2 + 4/(d-2)$ (recall we are in the case $d \geq 3$). By the Sobolev embedding in the cube C_z, there exists a constant C such that

$$\|u_n\|^2_{L^{p^*}(C_z)} \leq C \left(\|u_n\|^2_{L^2(C_z)} + \|\nabla u_n\|^2_{L^2(C_z)} \right).$$

Note that the constant C only depends on the volume of the cube, hence it is independent of $z \in \mathbb{Z}^d$. Inserting in (6), we obtain

$$\int_{\mathbb{R}^d} |u_n|^q \leq C \left(\sup_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)} \right)^{\theta q} \sum_{z\in\mathbb{Z}^d} \left(\int_{C_z} |u_n|^2 + \int_{C_z} |\nabla u_n|^2 \right)$$
$$= C \left(\sup_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)} \right)^{\theta q} \|u_n\|^2_{H^1(\mathbb{R}^d)}.$$

Passing to the limit $n \to \infty$, we deduce that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \leq C \left(\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 \right)^{\frac{2}{d}} \limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}. \tag{7}$$

We now claim that

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 \leq \boldsymbol{m}(\boldsymbol{u}), \tag{8}$$

which will end the proof of Lemma 9. Indeed, consider a sequence $\boldsymbol{z} = \{z_n\} \subset \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \int_{C_{z_n}} |u_n|^2 = \limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2.$$

The sequence $u_n(\cdot + z_n)$ is bounded in $H^1(\mathbb{R}^d)$. Up to extraction of a subsequence, we have $u_{n_k}(\cdot + z_{n_k}) \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and, by the Rellich-Kondrachov compactness Theorem, strongly in $L^2(C_0)$. We deduce that

$$\lim_{n\to\infty} \int_{C_{z_n}} |u_n|^2 = \lim_{n_k\to\infty} \int_{C_0} |u_{n_k}(\cdot + z_{n_k})|^2 = \int_{C_0} |u|^2 \leq \int_{\mathbb{R}^d} |u|^2 \leq \boldsymbol{m}(\boldsymbol{u}).$$

This concludes the proof of Lemma 9. □

We now use the previous result to characterize when $\boldsymbol{m}(\boldsymbol{u}) = 0$ holds, which is equivalent to saying that $u_n(\cdot + x_n) \rightharpoonup 0$ for all $\{x_n\} \subset \mathbb{R}^d$. Intuitively this corresponds to vanishing as displayed in Example 1a of Figure 1 since u_n must then converge to zero strongly in all $L^p(\mathbb{R}^d)$, for all sub-critical exponents $2 < p < p^*$.

Lemma 10 (Vanishing)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. The following assertions are equivalent:*

(i) $\boldsymbol{m}(\boldsymbol{u}) = 0$*;*

(ii) for all $R > 0$, we have $\displaystyle\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(x,R)} |u_n|^2 = 0$*;*

(iii) $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for all $2 < p < p^*$, where $p^* = 2d/(d-2)$ if $d \geq 3$, $p^* = \infty$ if $d = 1, 2$.

The interpretation of (ii) is that there is essentially no mass remaining for n large enough in any ball of fixed radius R, independently of the location of the center of the ball.

Proof of Lemma 10. We start by proving that $(i) \Rightarrow (ii)$. Indeed, from the estimate (8) of the proof of Lemma 9, we deduce that

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 = 0$$

where $C_z = \prod_{j=1}^d [z_j, z_j+1)$ are cubes covering the whole space. Since a ball of radius R can be covered by a finite union of such cubes, the result follows.

To show that $(iii) \Rightarrow (i)$, we consider $\{x_{n_k}\} \subset \mathbb{R}^d$ and u such that $u_{n_k}(\cdot + x_{n_k}) \rightharpoonup u$. Since

$$\|u_{n_k}(\cdot + x_{n_k})\|_{L^p(\mathbb{R}^d)} = \|u_{n_k}\|_{L^p(\mathbb{R}^d)} \to 0$$

when $2 < p < p^*$, we have $u_{n_k}(\cdot + x_{n_k}) \to 0$ strongly in $L^p(\mathbb{R}^d)$, hence (by uniqueness of the weak limit) $u = 0$. Therefore $\boldsymbol{m}(\boldsymbol{u}) = 0$.

Finally, the proof that $(ii) \Rightarrow (iii)$ is a consequence of (7). We have $\|u_n\|_{L^{2+4/d}(\mathbb{R}^d)} \to 0$ and the rest follows by interpolation, using that $\{u_n\}$ is bounded in $L^p(\mathbb{R}^d)$ for $2 \le p \le p^*$ by the Sobolev embedding. $\square$

Exercise 11. *Write the proof for $d = 1, 2$. Generalize also Lemmas 9 and 10 to the case of a bounded sequence $\{u_n\}$ in $W^{k,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and k a positive integer, with an appropriate definition of $\boldsymbol{m}(\boldsymbol{u})$.*

3. Localizing bubbles

In applications it is often useful to "localize" the bubbles, which means that we write u_{n_k} as a sum of functions $u_k^{(j)}$ of compact supports, each of them converging strongly to the weak limit $u^{(j)}$ of Lemma 6, and where the distance between the supports diverges. This step is not always necessary in practice but it is sometimes convenient. It is also more in the spirit of Lions' concentration-compactness principle and this is an occasion for us to learn how to use the concentration functions of Levy.

The localization procedure can be done for any chosen J in the bubble decomposition (2) of Lemma 6. In order to illustrate the main idea, we start with the case $J = 1$, that is when we have one sequence u_n which converges weakly to some $u^{(1)} := u$. We will come back to the general case later in Corollary 17.

Lemma 12 (Extracting the locally convergent part)**.** *Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and let $0 \le R_k \le R_k'$ such that $R_k \to \infty$. Then there exists a subsequence $\{u_{n_k}\}$ such that*

$$\int_{|x| \le R_k} |u_{n_k}(x)|^2 \, dx \to \int_{\mathbb{R}^d} |u(x)|^2 \, dx \tag{9}$$

and

$$\int_{R_k \le |x| \le R_k'} \left(|u_{n_k}(x)|^2 + |\nabla u_{n_k}(x)|^2 \right) dx \to 0 \tag{10}$$

as $k \to \infty$. In particular, we have that $u_{n_k} \mathbb{1}_{B(0,R_k)} \to u$ strongly in $L^p(\mathbb{R}^d)$ for all $2 \le p < p^*$ (where p^* is like in Lemma 10).

The interpretation of Lemma 12 is that we can isolate by means of a growing ball $B(0, R_k)$ the part which converges strongly to u in sub-critical spaces $L^p(\mathbb{R}^d)$ with $2 \le p < p^*$. In the annulus $\{R_k \le |x| \le R_k'\}$, we have essentially nothing (by the convergence to zero in $H^1(\mathbb{R}^d)$). In practice, we choose R_k' such that $R_k' - R_k \to \infty$. Since the original sequence $\{u_{n_k}\}$ has been decomposed into two parts (the convergent part in the ball $B(0, R_k)$ and the rest in the domain $\mathbb{R}^d \setminus B(0, R_k')$), Lions used the word *dichotomy* [15, 16] to describe this procedure.

The intuition behind the decomposition of Lemma 12 is that we have to choose R_k and R_k' growing slowly enough that we essentially only retain the mass of the weak limit u and not more. However, our statement is for convenience written the other way: We can choose R_k and R_k' essentially as we want, but then the sequence n_k has to go to infinity so fast that u_{n_k} has almost already converged to its local limit u in the ball $B(0, R_k')$.

Proof of Lemma 12. We follow Lions [12, 15, 16] and introduce the so-called *Levy concentration functions* [10]

$$Q_n(R) := \int_{B(0,R)} |u_n|^2, \quad \text{and} \quad K_n(R) := \int_{B(0,R)} |\nabla u_n|^2.$$

Note that Q_n and K_n are continuous non-decreasing functions on $[0, \infty)$, such that

$$\forall n \geq 1, \quad \forall R > 0, \qquad Q_n(R) + K_n(R) \leq \int_{\mathbb{R}^d} |u_n|^2 + |\nabla u_n|^2 \leq C$$

since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$. By the Rellich-Kondrachov Theorem, we have

$$Q_n(R) = \int_{B(0,R)} |u_n|^2 \to \int_{B(0,R)} |u|^2 := Q(R)$$

for all $R \geq 0$. We now recall a very useful result dealing with sequences of monotone functions.

Lemma 13 (Sequences of monotone functions)**.** *Let I be a (possibly unbounded) interval of $\mathbb{R}$, and $\{f_n\}$ be a sequence of non-increasing non-negative functions on I. We assume that there exists a fixed function g, locally bounded on I, such that $0 \leq f_n \leq g$ on I. Then there exists a subsequence $\{f_{n_k}\}$ and a non-increasing function f such that $f_{n_k}(x) \to f(x)$ for all $x \in I$.*

In the applications, g is often a constant on the whole interval I. By Lemma 13, up to extraction of a subsequence (for the sake of clarity we do not change notation), we may assume that $K_n(R) \to K(R)$ for all $R \geq 0$, and for some non-decreasing function K. We denote $\ell := \lim_{R\to\infty} K(R)$.

It can be shown that $\ell \geq \int_{\mathbb{R}^d} |\nabla u|^2$ but there is not always equality, except if we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^d)$ for some $s > 1$.

Consider now the given sequences R_k and R'_k which diverge to ∞. We have $Q_n(R_k) \to Q(R_k)$ and $Q_n(R'_k) \to Q(R'_k)$ as $n \to \infty$, and a similar property for K_n. Extracting a subsequence, we may assume for instance that

$$|Q_{n_k}(R_k) - Q(R_k)| + |Q_{n_k}(R'_k) - Q(R'_k)| + |K_{n_k}(R_k) - K(R_k)| + |K_{n_k}(R'_k) - K(R'_k)| \leq \frac{1}{k}.$$

We then deduce that

$$\left| \int_{B(0,R_k)} |u_{n_k}|^2 - \int_{\mathbb{R}^d} |u|^2 \right| = |Q_{n_k}(R_k) - Q(\infty)| \leq \frac{1}{k} + \int_{|x|\geq R_k} |u|^2 \underset{k\to\infty}{\longrightarrow} 0$$

and that

$$\int_{R_k \leq |x| \leq R'_k} |u_{n_k}|^2 = Q_{n_k}(R'_k) - Q_{n_k}(R_k) \leq \frac{1}{k} + Q(R'_k) - Q(R_k) \underset{k\to\infty}{\longrightarrow} 0,$$

$$\int_{R_k \leq |x| \leq R'_k} |\nabla u_{n_k}|^2 = K_{n_k}(R'_k) - K_{n_k}(R_k) \leq \frac{1}{k} + K(R'_k) - K(R_k) \underset{k\to\infty}{\longrightarrow} 0,$$

where we have used that $K(R'_k) - K(R_k) \to \ell - \ell = 0$ when $k \to \infty$.

Finally, it is clear that $\mathbb{1}_{B(0,R_k)} u_{n_k} \rightharpoonup u$ weakly in $L^2(\mathbb{R}^d)$. Since the norm also converges, we get that the convergence must be strong. By the Sobolev embeddings, $\{u_n\}$ is bounded in $L^p(\mathbb{R}^d)$ for all $2 \leq p < p^*$, hence so is $\mathbb{1}_{B(0,R_k)} u_{n_k}$. By interpolation, $\mathbb{1}_{B(0,R_k)} u_{n_k}$ converges towards u strongly in $L^p(\mathbb{R}^d)$ for all $2 \leq p < p^*$. This ends the proof of Lemma 12. □

Exercise 14. *Prove that for any bounded sequence $\boldsymbol{u} = \{u_n\}$ in $H^1(\mathbb{R}^d)$,*

$$\boldsymbol{m}(\boldsymbol{u}) = \lim_{R\to\infty} \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(x,R)} |u_n|^2. \tag{11}$$

Exercise 15. *Generalize Lemma 12 to the case of a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $W^{k,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and k a positive integer.*

In the previous lemma, we have isolated in the ball $B(0, R_k)$ the part of the sequence u_{n_k} which converges to u strongly, by retaining only the adequate mass. Unfortunately, the function $u_{n_k}\mathbb{1}_{B(0,R_k)}$ is not in $H^1(\mathbb{R}^d)$ and it is in practice convenient to replace the characteristic function $\mathbb{1}_{B(0,R_k)}$ by a smooth cut-off. This is the purpose of the next Corollary.

Corollary 16 (Splitting of a weakly convergent sequence in $H^1(\mathbb{R}^d)$)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and let $0 \leq R_k \leq R'_k$ such that $R_k \to \infty$. Then we have for a subsequence $\{u_{n_k}\}$*

$$\lim_{k\to\infty} \left\| u_{n_k} - u_k^{(1)} - \psi_k^{(2)} \right\|_{H^1(\mathbb{R}^d)} = 0$$

where $\boldsymbol{u}^{(1)} = \{u_k^{(1)}\}$ and $\boldsymbol{\psi}^{(2)} = \{\psi_k^{(2)}\}$ are sequences in $H^1(\mathbb{R}^d)$ such that

- *$u_k^{(1)} \to u$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^p(\mathbb{R}^d)$ for all $2 \leq p < p^*$;*
- *$\operatorname{supp}\big(u_k^{(1)}\big) \subset B(0, R_k)$ and $\operatorname{supp}\big(\psi_k^{(2)}\big) \subset \mathbb{R}^d \setminus B(0, R'_k)$;*
- *$\boldsymbol{m}\big(\boldsymbol{r}^{(2)}\big) \leq \boldsymbol{m}\big(\{u_{n_k}\}\big) \leq \boldsymbol{m}(\boldsymbol{u})$.*

Proof. We apply Lemma 12 with $R_k/2$ and $4R'_k$. We obtain a subsequence $\{u_{n_k}\}$ such that

$$\int_{|x|\leq R_k/2} |u_{n_k}|^2 \to \int_{\mathbb{R}^d} |u|^2, \qquad \int_{R_k/2 \leq |x| \leq 4R'_k} |u_{n_k}|^2 + |\nabla u_{n_k}|^2 \to 0. \tag{12}$$

Let $\chi : \mathbb{R}^+ \to [0, 1]$ be a smooth function such that $0 \leq \chi' \leq 2$, $\chi_{|[0,1]} \equiv 1$ and $\chi_{|[2,\infty)} \equiv 0$. We denote $\chi_k(x) := \chi(2|x|/R_k)$ and $\zeta_k(x) = 1 - \chi(|x|/R'_k)$ and introduce $u_k^{(1)} := \chi_k u_{n_k}$ and $\psi_k^{(2)} := \zeta_k u_{n_k}$. By (12), we clearly have $u_{n_k} - u_k^{(1)} - \psi_k^{(2)} \to 0$ in $H^1(\mathbb{R}^d)$ since this function has its support in the annulus $\{R_k/2 \leq |x| \leq 2R'_k\}$. Furthermore

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |u_k^{(1)}|^2 = \lim_{k\to\infty} \int_{|x|\leq R_k/2} |u_k^{(1)}|^2 = \int_{\mathbb{R}^d} |u|^2,$$

hence $u_k^{(1)} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^p(\mathbb{R}^d)$ for all $2 \leq p < p^*$. Note that by construction $\mathbb{1}_{B(0,4R'_k)}\psi_k^{(2)} \to 0$ strongly in $L^2(\mathbb{R}^d)$.

It remains to show that $\boldsymbol{m}\big(\boldsymbol{\psi}^{(2)}\big) \leq \boldsymbol{m}\big(\{u_{n_k}\}\big)$. If $\boldsymbol{m}\big(\boldsymbol{\psi}^{(2)}\big) = 0$ there is nothing to prove. Assume that $\psi_{k_j}^{(2)}(\,\cdot - x_j) \rightharpoonup \psi$ for some subsequence and some $\psi \neq 0$. If for a subsequence (denoted the same for simplicity), we have $|x_j| \leq 3R'_{k_j}$, then $B(x_j, R'_{k_j}) \subset B(0, 4R'_{k_j})$, hence $\psi_{k_j}^{(2)}(\,\cdot - x_j)\mathbb{1}_{B(0,R'_{k_j})} \rightharpoonup 0 = \psi$, a contradiction. Hence it must hold $|x_j| \geq 3R'_{k_j}$ for all j sufficiently large. Note then that for j large enough, $\zeta_{k_j} \equiv 1$ on the ball $B(x_j, R'_{k_j})$, hence

$$\psi_{k_j}^{(2)}(\,\cdot - x_j)\mathbb{1}_{B(0,R'_{k_j})} = u_{n_{k_j}}(\,\cdot - x_j)\mathbb{1}_{B(0,R'_{k_j})} \rightharpoonup \psi,$$

thus $u_{n_{k_j}}(\,\cdot - x_j) \rightharpoonup \psi$ weakly in $H^1(\mathbb{R}^d)$. This proves that $\int_{\mathbb{R}^d} |\psi|^2 \leq \boldsymbol{m}\big(\{u_{n_k}\}\big)$ and the proof is complete. □

We now consider a sequence $\boldsymbol{u} = \{u_n\}$ and its weak limits $u^{(j)}$ up to translations, obtained from Lemma 6. We want to localize all the bubbles in disjoint balls receeding from each other, in the same spirit as in the previous result. The following result is proved in a very similar fashion as Lemma 12 and Corollary 16, using two concentration functions per bubble.

Theorem 17 (Splitting in arbitrarily many localized bubbles)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ and $\{u^{(j)}\} \subset H^1(\mathbb{R}^d)$ be the sequence given by Lemma 6. For any $\varepsilon > 0$ and any fixed sequence $0 \leq R_k \to \infty$, there exist*

- $J \geq 0$,
- *a subsequence* $\{u_{n_k}\}$,
- *sequences of functions* $\boldsymbol{u}^{(1)} = \{u_k^{(1)}\}, ..., \boldsymbol{u}^{(J)} = \{u_k^{(J)}\}, \boldsymbol{\psi}^{(J+1)} = \{\psi_k^{(J+1)}\}$ *in* $H^1(\mathbb{R}^d)$,
- *space translations* $\boldsymbol{x}^{(1)} = \{x_k^{(1)}\}, ..., \boldsymbol{x}^{(J)} = \{x_k^{(J)}\}$ *in* $\mathbb{R}^d$,

such that

$$\lim_{k\to\infty} \left\| u_{n_k} - \sum_{j=1}^{J} u_k^{(j)}(\,\cdot - x_k^{(j)}) - \psi_k^{(J+1)} \right\|_{H^1(\mathbb{R}^d)} = 0 \tag{13}$$

where

- $u_k^{(j)} \to u^{(j)} \neq 0$ *weakly in* $H^1(\mathbb{R}^d)$ *and strongly in* $L^p(\mathbb{R}^d)$ *for all* $2 \leq p < p^*$;
- $\operatorname{supp}(u_k^{(j)}) \subset B(0, R_k)$ *for all* $j = 1, ..., J$ *and all* k;
- $\operatorname{supp}(\psi_k^{(J+1)}) \subset \mathbb{R}^d \setminus \cup_{j=1}^J B(x_k^j, 2R_k)$ *for all* k;
- $|x_k^{(i)} - x_k^{(j)}| \geq 5R_k$ *for all* $i \neq j$ *and all* k;
- $\boldsymbol{m}\big(\boldsymbol{\psi}^{(J+1)}\big) \leq \varepsilon$.

We emphasize again that the error term $\psi_k^{(J+1)}$ is small in the sense that $\boldsymbol{m}\big(\boldsymbol{\psi}^{(J+1)}\big)$ is small, that is $\{\psi_k^{(J+1)}\}$ does not contain a local mass larger than ε. In general the mass $\int_{\mathbb{R}^d} |\psi_k^{(J+1)}|^2$ of $\{\psi_k^{(J+1)}\}$ is not necessarily small since the sequence can still undergo vanishing. However, by Lemma 9, the subcritical norms of $\psi_k^{(J+1)}$ are all small.

Results taking the same form as Corollary 17 are ubiquitous in the literature and they are often called “bubble decompositions” of the sequence $\boldsymbol{u} = \{u_n\}$, see, e.g., [21, 2, 19, 6].

Exercise 18. *Write the proof of Corollary 17, using $2J$ concentration functions (two per bubble), that is for the functions $|u_{n_k}(\,\cdot + x_k^{(j)})|^2$ and $|\nabla u_{n_k}(\,\cdot + x_k^{(j)})|^2$ for $j = 1, ..., J$.*

4. A general method

In this section, we vaguely explain how to use the tools of Sections 1–3 in practice, following Lions [15, 16, 19].

We assume that we are given an energy functional $\mathcal{E}$ defined on $H^1(\mathbb{R}^d)$, which is continuous, bounded from below and coercive on

$$\mathcal{S}_{\leq}(\lambda) := \left\{ u \in H^1(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} |u|^2 \leq \lambda \right\}. \tag{14}$$

for all $\lambda \geq 0$. The corresponding minimization principle reads

$$I(\lambda) := \inf_{u \in \mathcal{S}(\lambda)} \mathcal{E}(u). \tag{15}$$

where, this time,

$$\mathcal{S}(\lambda) := \left\{ u \in H^1(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} |u|^2 = \lambda \right\}. \tag{16}$$

By coercivity, all the minimizing sequences $\boldsymbol{u} = \{u_n\}$ are bounded in $H^1(\mathbb{R}^d)$. The goal is to prove the existence of a minimizer for (15) and to give a criterion for the precompactness of all the minimizing sequences. As we have explained, a given minimizing sequence can undergo lack of compactness: It can vanish in the sense that $\boldsymbol{m}(\boldsymbol{u}) = 0$, or it can split into several pieces as in Corollary 17. The main idea is to describe in a rather detailed way the behavior of minimizing sequences in the case of lack of compactness and to find what would be the total energy of the system when this happens. If we can show that the energy is too high (above $I(\lambda)$), we arrive at a contradiction, hence the minimizing sequence must be precompact.

To deal with vanishing, we introduce a functional $\mathcal{E}^{\rm van}$ which is the original energy $\mathcal{E}$ to which all the subcritical terms which go to zero when $u_n \to 0$ in $L^p(\mathbb{R}^d)$ for $2 < p < p^*$ (by Lemma 10) are removed. In the applications, $\mathcal{E}^{\rm van}$ usually only contains the gradient terms. One then defines the corresponding minimization principle $I^{\rm van}(\lambda) := \inf_{u\in\mathcal{S}(\lambda)} \mathcal{E}^{\rm van}(u)$. Another equivalent way to define $I^{\rm van}(\lambda)$ is as follows:

$$I^{\rm van}(\lambda) := \inf_{\substack{\boldsymbol{u}=\{u_n\}\subset\mathcal{S}(\lambda)\\ \boldsymbol{m}(\boldsymbol{u})=0}} \liminf_{n\to\infty} \mathcal{E}(u_n). \tag{17}$$

To deal with the phenomenon of splitting, we introduce an energy $\mathcal{E}^\infty$ which is the original energy $\mathcal{E}$ to which we remove all the compact terms that converge to zero when $u_n \rightharpoonup 0$ (but without assuming *a priori* that $u_n \to 0$ strongly in some $L^p(\mathbb{R}^d)$ space), thinking of $u_n = u(\,\cdot - n\vec{v})$. In our examples, this usually means to remove the external potential term by letting $V = 0$ and to only keep the terms which are invariant by translation. The corresponding minimization principle reads $I^\infty(\lambda) := \inf_{u\in\mathcal{S}(\lambda)} \mathcal{E}^\infty(u)$. Another way to define $I^\infty(\lambda)$ is:

$$I^{\infty}(\lambda) := \inf_{\substack{\{u_n\}\subset\mathcal{S}(\lambda)\\ u_n\rightharpoonup 0}} \liminf_{n\to\infty} \mathcal{E}(u_n). \tag{18}$$

By taking appropriate test functions, it can usually be proven that

$$\forall 0\leq\lambda'\leq\lambda, \qquad I(\lambda)\leq I(\lambda-\lambda')+I^{\infty/\rm van}(\lambda').$$

If $I(0) = I^{\rm van}(0) = I^\infty(0) = 0$ (which we always assume), we deduce that $I(\lambda) \leq I^{\infty/\rm van}(\lambda)$.

For models which are invariant by translations, we always get $\mathcal{E}^\infty = \mathcal{E}$. In this case we can only hope to prove the precompactness of minimizing sequences *up to translations* (that is there exists $\{x_n\} \subset \mathbb{R}^d$ such that $u_{n_k}(\,\cdot - x_k)$ is compact). We start by explaining the general strategy when $\mathcal{E}$ is invariant by translations.

The first step is to show that *vanishing does not occur*, by proving that $I(\lambda) = I^\infty(\lambda) < I^{\rm van}(\lambda)$ for all $\lambda > 0$. This can usually be done by means of a well-chosen test function.

Once we know this, it must hold $\boldsymbol{m}(\boldsymbol{u}) > 0$ for all minimizing sequences $\boldsymbol{u}$ of $I(\lambda)$. Then, by definition of $\boldsymbol{m}(\boldsymbol{u})$, there exists a subsequence (denoted the same for clarity) and translations $\boldsymbol{x}^{(1)} = \{x_n^{(1)}\} \subset \mathbb{R}^d$ such that $u_n(\,\cdot - x_n^{(1)}) \rightharpoonup u^{(1)} \neq 0$. Using Lemma 12, we can write $u_n(\,\cdot - x_n^{(1)}) = u_n^{(1)} + \psi_n^{(2)} + \varepsilon_n^{(2)}$ where $\varepsilon_n^{(2)} \to 0$ strongly in $H^1(\mathbb{R}^d)$. One then shows that

$$\mathcal{E}(u_n) = \mathcal{E}(u_n(\,\cdot - x_n^{(1)})) = \mathcal{E}(u_n^{(1)}) + \mathcal{E}(\psi_n^{(2)}) + \mathrm{o}(1).$$

Using that $u_n^{(1)} \to u^{(1)}$ strongly in $L^p(\mathbb{R}^d)$ for $2 \leq p < p^*$, we try to show that $\mathcal{E}$ is weakly lower semicontinuous along this sequence (it is not needed that $\mathcal{E}$ is wlsc, but only that all the terms which are not wlsc are continuous in $L^p(\mathbb{R}^d)$ for $2 \leq p < p^*$). This means we have

$$\liminf_{n\to\infty} \mathcal{E}(u_n^{(1)}) \geq \mathcal{E}(u^{(1)}) \geq I(\lambda^{(1)})$$

where $\lambda^{(1)} := \int_{\mathbb{R}^d} |u^{(1)}|^2$. For $\{\psi_n^{(2)}\}$, we only write that

$$\mathcal{E}(\psi_n^{(2)}) \geq I\left(\int_{\mathbb{R}^d} |\psi_n^{(2)}|^2\right) \to I\left(\lambda - \lambda^{(1)}\right),$$

using that $u_n^{(1)} \to u^{(1)}$ in $L^2(\mathbb{R}^d)$. Passing to the limit, we find that $I(\lambda) \geq I(\lambda^{(1)}) + I(\lambda - \lambda^{(1)})$. However, since the converse inequality is always true, there must be equality. One finally deduces that $\mathcal{E}(u^{(1)}) = I(\lambda^{(1)})$ (hence $u^{(1)}$ is a minimizer of the problem corresponding to its own mass $I(\lambda^{(1)})$) and that $\lim_{n\to\infty} \mathcal{E}(\psi_n^{(2)}) = I(\lambda - \lambda^{(1)})$, that is $\{\psi_n^{(2)}\}$ is a minimizing sequence for $I(\lambda - \lambda^{(1)})$.

The next step consists in applying the whole argument to the new sequence $\boldsymbol{\psi}^{(2)} = \{\psi_n^{(2)}\}$. Since it is a minimizing sequence for $I(\lambda - \lambda^{(1)})$, it cannot vanish, hence there is a $u^{(2)}$ such that, up to a subsequence, $\psi_n^{(2)}(\,\cdot - x_n^{(2)}) \rightharpoonup u^{(2)} \neq 0$. As before $u^{(2)}$ must be a minimizer of the problem corresponding to its own mass, $\lambda^{(2)} = \int_{\mathbb{R}^d} |u^{(2)}|^2$ and we can write $\psi_n^{(2)} = u_n^{(2)} + \psi_n^{(3)} + \varepsilon_n^{(3)}$ where $u_n^{(2)} \to u^{(2)}$ in appropriate spaces and $\{\psi_n^{(3)}\}$ is a minimizing sequence for $I(\lambda - \lambda^{(1)} - \lambda^{(2)})$. If $\lambda^{(2)} = \lambda - \lambda^{(1)}$, then $\{\psi_n^{(2)}\}$ is compact.

One could go on and in principle get infinitely many pieces of mass receding from each other (or even apply Lemma 17 directly). It is however enough to stop as soon as there is a contradiction, which is often the case when there are two pieces of mass. The argument is as follows: We have

$$I(\lambda) \leq I(\lambda^{(1)} + \lambda^{(2)}) + I(\lambda - \lambda^{(1)} - \lambda^{(2)}) \leq I(\lambda^{(1)}) + I(\lambda^{(2)}) + I(\lambda - \lambda^{(1)} - \lambda^{(2)})$$

and since $I(\lambda) = I(\lambda^{(1)}) + I(\lambda^{(2)}) + I(\lambda - \lambda^{(1)} - \lambda^{(2)})$, there must be equality everywhere. In particular

$$I(\lambda^{(1)} + \lambda^{(2)}) = I(\lambda^{(1)}) + I(\lambda^{(2)}).$$

The last step of the proof consists then in using the two minimizers $u^{(1)}$ and $u^{(2)}$ of, respectively, $I(\lambda^{(1)})$ and $I(\lambda^{(2)})$, to build a convenient trial state showing the so-called *binding inequality*

$$I(\lambda^{(1)} + \lambda^{(2)}) < I(\lambda^{(1)}) + I(\lambda^{(2)}),$$

which is a contradiction. In practice this last step often uses the fact that charged Coulomb systems attract at large distances.

The previous method was devoted to the case of a translation-invariant system, $\mathcal{E} = \mathcal{E}^\infty$. When $\mathcal{E} \neq \mathcal{E}^\infty$ (for instance when there is a potential V), there is an additional step at the beginning of the method. One proves that $I(\lambda) < I^\infty(\lambda)$ (by constructing a convenient trial state), which implies that a given minimizing sequence $\boldsymbol{u} = \{u_n\}$ cannot have a vanishing weak limit: $u_n \rightharpoonup u^{(1)} \neq 0$. One then applies Lemma 12 to write $u_n = u_n^{(1)} + \psi_n^{(2)} + \varepsilon_n^{(2)}$ and to show that the energy splits in two pieces:

$$\mathcal{E}(u_n) = \mathcal{E}(u_n^{(1)}) + \mathcal{E}^\infty(\psi_n^{(2)}) + \mathrm{o}(1).$$

Note that $\psi_n^{(2)} \rightharpoonup 0$ hence the local terms disappear and we get $\mathcal{E}^\infty$. Arguing as before we find $I(\lambda) = I(\lambda^{(1)}) + I^\infty(\lambda - \lambda^{(1)})$. The rest of the proof is similar to what we have said before and the binding inequality that must be proven now reads:

$$I(\lambda^{(1)} + \lambda^{(2)}) < I(\lambda^{(1)}) + I^\infty(\lambda^{(2)}).$$

5. The Sobolev inequality and the critical case

Up to now we have been working in $H^1(\mathbb{R}^d)$. The above results easily generalize to $H^s(\mathbb{R}^d)$, at the expense of using localization methods for fractional Laplacians as described for instance in [8, App. A] and [9, App. B]. We discuss now the critical case where the mass $\int_{\mathbb{R}^d} |u|^2$ is not finite anymore, which prevents from using the subcritical exponent $2 + 4/d$ as we did in Lemma 9. In other words, we provide a version of Lemma 9 in the critical case. But before we start with a simple proof of the Sobolev inequality, following [5], which will be useful for our argument.

5.1. **The Sobolev inequality.**

Theorem 19 (Sobolev inequality). *For every $s > 2d$ in dimension $d \geq 1$, there exists a constant $C_{d,s}$ such that*

$$\|u\|^2_{L^{2d/(d-2s)}(\mathbb{R}^d)} \leq C_{d,s} \int_{\mathbb{R}^d} |k|^{2s} |\widehat{u}(k)|^2 \, dk, \tag{19}$$

for every $u \in \mathscr{S}'(\mathbb{R}^d)$ such that $|\{|u| \geq \lambda\}| < \infty$ for all $\lambda > 0$.

Proof. For shortness we denote

$$K := \left(\int_{\mathbb{R}^d} |k|^{2s} |\widehat{u}(k)|^2 \, dk \right)^{1/2} := \|u\|_{\dot{H}^s(\mathbb{R}^d)} .$$

We start by writing

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2s}} \, dx &= \frac{d-2s}{d+2s} \int_{\mathbb{R}^d} \int_0^\infty \lambda^{\frac{2d}{d-2s}-1} \mathbb{1}(|u(x)| \geq \lambda) \, d\lambda \, dx \\ &= \frac{d-2s}{d+2s} \int_0^\infty \lambda^{\frac{2d}{d-2s}-1} |\{|u(x)| \geq \lambda\}| \, d\lambda. \end{aligned} \tag{20}$$

Now we will give an estimate on $|\{|u(x)| \geq \lambda\}|$ for any fixed λ. We write $u = v + w$ where $\widehat{v}(k) = \widehat{u}(k)\chi(|k|/a)$, with a a parameter depending on λ to be determined later and $0 \leq \chi \leq 1$ a localizing function satisfying $\chi_{|[0,1]} \equiv 1$ and $\chi_{|[2,\infty)} \equiv 0$.

We use that

$$|\{|u(x)| \geq \lambda\}| \leq |\{|v(x)| \geq \lambda/2\}| + |\{|w(x)| \geq \lambda/2\}|$$

and choose a to ensure $\|v\|_{L^\infty} \leq \lambda/2$ which will give $|\{|v(x)| \geq \lambda/2\}| = 0$. Indeed, we have

$$\begin{aligned} \|v\|_{L^\infty} &\leq (2\pi)^{-d} \int_{|k| \leq a} |\widehat{u}(k)| \, dk \\ &\leq (2\pi)^{-d} \left(\int_{|k| \leq a} \frac{dk}{|k|^{2s}} \right)^{1/2} \left(\int_{|k| \leq a} |k|^{2s} |\widehat{u}(k)|^2 dk \right)^{1/2} \\ &\leq C \, a^{\frac{d-2s}{2}} \, \|u\|_{\dot{H}^s(\mathbb{R}^d)} = C \, K \, a^{\frac{d-2s}{2}}, \end{aligned} \tag{21}$$

which suggests to take $a^{\frac{d-2s}{2}} = \lambda/(2CK) \iff a = C'(\lambda/K)^{\frac{2}{d-2s}}$. For the term involving w, we write

$$|\{|w(x)| \geq \lambda/2\}| \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^d} |w|^2 \leq \frac{1}{\lambda^2} \int_{|k| \geq a} |\widehat{u}|^2.$$

Coming back to (20), this gives

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2s}} \, dx &\leq C \int_0^\infty \lambda^{\frac{2d}{d-2s}-1} \frac{1}{\lambda^2} \int_{|k| \geq a} |\widehat{u}|^2 \, d\lambda \\ &\leq C \int_0^\infty \lambda^{\frac{6s-d}{d-2s}} \int_{|k| \geq C'(\lambda/K)^{\frac{2}{d-2s}}} |\widehat{u}|^2 \, dk \, d\lambda \\ &= C K^{\frac{4s}{d-2s}} \int |k|^{2s} |\widehat{u}|^2 \, dk = C K^{\frac{2d}{d-2s}}. \end{aligned}$$

□

5.2. **Existence of minimizers: lack of compactness in $\dot H^s(\mathbb{R}^d)$.** In order to adapt the arguments used before in the subcritical case, we now have to deal with the lack of compactness due to dilations, in addition to translations. This leads us to introduce the following concept. For a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $L^{p^*}(\mathbb{R}^d)$ we introduce the highest (critical) mass that weak limits can have, up to translations, dilations, and extraction of a subsequence:

$$\boldsymbol{m}(\boldsymbol{u}) = \left\{ \int_{\mathbb{R}^d} |u|^{p^*} \ :\ \alpha_k^{-d/p^*} u_{n_k}\left(\frac{\cdot - x_k}{\alpha_k}\right) \rightharpoonup u \right\}. \tag{22}$$

The following is the corresponding adaptation of Lemma 9 to the critical case.

Lemma 20 (A critical estimate involving $\boldsymbol{m}(\boldsymbol{u})$)**.** *Let $d > 2s > 0$. We have for every bounded sequence $\boldsymbol{u} = \{u_n\} \subset \dot H^s(\mathbb{R}^d)$*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{\frac{2d}{d-2s}} \le C\, \boldsymbol{m}(\boldsymbol{u})^{\frac{4s}{d-2s}} \limsup_{n\to\infty} \|u_n\|^2_{\dot H^s(\mathbb{R}^d)}\,. \tag{23}$$

Proof. We follow the argument used in the proof of Theorem 19 and only give a different estimate of $\|v_n\|_{L^\infty}$ with $\widehat{v_n} = \widehat{u_n}\,\chi(\cdot/a)$, in terms of $\boldsymbol{m}(\boldsymbol{u})$ instead of K. Indeed, we have

$$\begin{aligned}
\|v_n\|_{L^\infty} &= (2\pi)^{-d/2} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} a^d \check\chi(ay) u_n(x-y)\, dy \right| \\
&= (2\pi)^{-d/2} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \check\chi(y) u_n\left(\frac{x-y}{a}\right) dy \right| \\
&\le (2\pi)^{-d/2} a^{\frac{d-2s}{2}} \sup_{a>0} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \check\chi(y) \frac{1}{a^{d/p^*}} u_n\left(\frac{x-y}{a}\right) dy \right|.
\end{aligned}$$

For every n, let a_n and x_n realizing the above supremum. Since the function

$$\frac{1}{a_n^{d/p^*}} u_n\left(\frac{x_n - y}{a_n}\right)$$

is bounded in $L^{p^*}(\mathbb{R}^d)$, we may extract a subsequence and assume that

$$\frac{1}{a_n^{d/p^*}} u_n\left(\frac{x_n - \cdot}{a_n}\right) \rightharpoonup u.$$

By definition of $\boldsymbol{m}(u)$ we have $\|u\|_{L^{p^*}(\mathbb{R}^d)} \le \boldsymbol{m}(\boldsymbol{u})^{1/p^*}$. (Note that $u_n(-x)$ has the same $\boldsymbol{m}$ as $u_n(x)$.) On the other hand,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \check\chi(y) \frac{1}{a_n^{\frac{d-2s}{2}}} u_n\left(\frac{x_n - y}{a_n}\right) dy = \int_{\mathbb{R}^d} \check\chi u$$

and therefore

$$\limsup_{n\to\infty} \|v_n\|_{L^\infty} \le \|\check\chi\|_{L^{\frac{p^*}{p^*-1}}}\, a^{\frac{d-2s}{2}}\, \boldsymbol{m}(\boldsymbol{u})^{1/p^*}.$$

The argument is then the same as before. □

Corollary 21. *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $\dot H^s(\mathbb{R}^d)$, with $d > 2s$. Then $\boldsymbol{m}(\boldsymbol{u}) = 0$ if and only if $u_n \to 0$ strongly in $L^{p^*}(\mathbb{R}^d)$.*

We can now deduce the existence of an optimizer for the Sobolev inequality (19).

Corollary 22 (Existence of a minimizer for the Sobolev inequality)**.** *The best constant $C_{d,s}$ in* (19) *is attained.*

Proof. A minimizing sequence for Sobolev

$$S(\lambda) := \inf_{\int_{\mathbb{R}^d} |u|^{p^*} = \lambda} \|u\|^2_{\dot H^s(\mathbb{R}^d)} = \lambda^{\frac{2}{p^*}} S(1)$$

must satisfy $\boldsymbol{m}(\boldsymbol{u}) > 0$. Hence we can find a translation and a dilation such that, after extracting a subsequence,

$$\frac{1}{a_n^{d/p^*}} u_n \left(\frac{\cdot - x_n}{a_n} \right) \rightharpoonup u \neq 0.$$

Writing

$$u_n = a_n^{d/p^*} u \left(a_n((\cdot - x_n)\right) + r_n$$

we have

$$\int_{\mathbb{R}^d} |u_n|^{p^*} = \underbrace{\int_{\mathbb{R}^d} |u|^{p^*}}_{:=\lambda > 0} + \int_{\mathbb{R}^d} |r_n|^{p^*} + o(1)$$

(by strong local convergence in $\dot{H}^s(\mathbb{R}^d)$), and

$$\|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)} = \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|r_n\|^2_{\dot{H}^s(\mathbb{R}^d)} + o(1)$$

(since $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert). Therefore,

$$\begin{aligned} S(1) = \lim_{n\to\infty} \|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)} &\geq \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} + S(1-\lambda) \\ &\geq \left(\lambda^{2/p^*} + (1-\lambda)^{2/p^*} \right) S(1) \end{aligned}$$

which proves that $\lambda = 1$ since $2/p^* < 1$, and hence that u is an optimizer. ☐

Arguments of this type have been used for instance in [1]. It is possible to extend several of the results of the previous sections to the critical case (for instance the bubble decomposition), with of course the addition of the possible lack of compactness due to dilations.

References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, vol. 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg, 2011.

[2] H. Brezis and J.-M. Coron, *Convergence of solutions of H-systems or how to blow bubbles*, Arch. Rational Mech. Anal., 89 (1985), pp. 21–56.

[3] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36 (1983), pp. 437–477.

[4] É. Cancès, C. Le Bris, and Y. Maday, *Méthodes mathématiques en chimie quantique. Une introduction*, vol. 53 of Collection Mathématiques et Applications, Springer, 2006.

[5] J.-Y. Chemin and C.-J. Xu, *Inclusions de Sobolev en calcul de Weyl-Hörmander et champs de vecteurs sous-elliptiques*, Ann. Sci. École Norm. Sup. (4), 30 (1997), pp. 719–751.

[6] P. Gérard, *Description du défaut de compacité de l'injection de Sobolev*, ESAIM Control Optim. Calc. Var., 3 (1998), pp. 213–233 (electronic).

[7] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, vol. 13 of Mathématiques & Applications [Mathematics & Applications], Springer-Verlag, Paris, 1993.

[8] E. Lenzmann and M. Lewin, *Minimizers for the Hartree-Fock-Bogoliubov theory of neutron stars and white dwarfs*, Duke Math. J., 152 (2010), pp. 257–315.

[9] ———, *On singularity formation for the L^2-critical Boson star equation*, Nonlinearity, 24 (2011), p. 3515.

[10] P. Levy, *Théorie de l'Addition des Variables Aléatoires*, Monographies des Probabilités, Gauthier-Villars, Paris, 2nd ed., 1954.

[11] E. H. Lieb, *On the lowest eigenvalue of the Laplacian for the intersection of two domains*, Invent. Math., 74 (1983), pp. 441–448.

[12] P.-L. Lions, *Principe de concentration-compacité en calcul des variations*, C. R. Acad. Sci. Paris Sér. I Math., 294 (1982), pp. 261–264.

[13] ———, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal., 49 (1982), pp. 315–334.

[14] ———, *Applications de la méthode de concentration-compacité à l'existence de fonctions extrémales*, C. R. Acad. Sci. Paris Sér. I Math., 296 (1983), pp. 645–648.

[15] ———, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 109–149.

[16] ———, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 223–283.
[17] ———, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana, 1 (1985), pp. 145–201.
[18] ———, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana, 1 (1985), pp. 45–121.
[19] ———, *Solutions of Hartree-Fock equations for Coulomb systems*, Commun. Math. Phys., 109 (1987), pp. 33–97.
[20] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2), 113 (1981), pp. 1–24.
[21] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., 187 (1984), pp. 511–517.
[22] ———, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, New York, 4 ed., 2008.

CNRS & Laboratoire de Mathmatiques (UMR 8088), Universit de Cergy-Pontoise, F-95000 Cergy-Pontoise, France.
E-mail address: Mathieu.Lewin@math.cnrs.fr