DESCRIBING LACK OF COMPACTNESS IN SOBOLEV SPACES

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Abstract. We review the concentration-compactness method and the profile decomposition in Sobolev spaces, which allow to describe the possible lack of compactness of sequences in these spaces.

In these notes we consider a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $L^2(\mathbb{R}^d)$ and we describe its possible behavior (up to subsequences) in a rather detailed way. We typically think of a sequence u_n which converges to u weakly in $L^2(\mathbb{R}^d)$ but which does not converge for the strong topology. One says that $\boldsymbol{u} = \{u_n\}$ exhibits a *lack of compactness*. Considering instead $u_n - u$, we can think of a sequence $u_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^d)$ but such that, for instance, $\int_{\mathbb{R}^d} |u_n|^2 = \lambda > 0$ for all n.

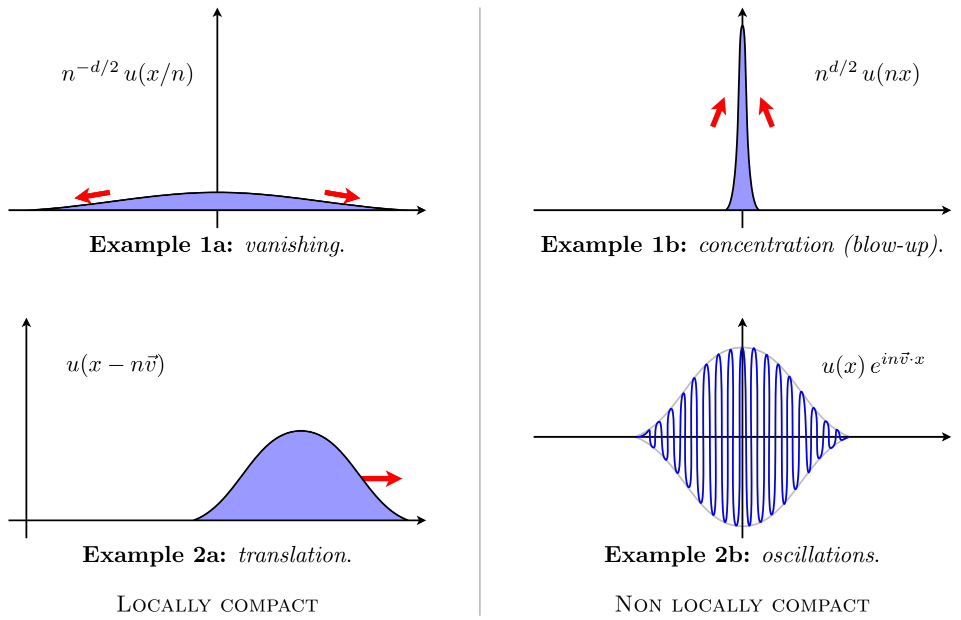

Figure 1. Four typical behaviors of a sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ with $u_n \rightharpoonup 0$ while $\int_{\mathbb{R}^d} |u_n|^2 = \lambda$ for all n. In the examples, u is a fixed (smooth) function in $L^2(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} |u|^2 = \lambda$.

In Figure 1, we display four typical examples of such a sequence $\boldsymbol{u} = \{u_n\}$ (the examples can easily be adapted to fix any $L^p(\mathbb{R}^d)$ norm). In Example 1a, the sequence $\{u_n\}$ spreads out everywhere in space but since its mass is conserved, locally the mass must go to zero. The sequence is said to "vanish". In Example 1b, the sequence concentrates at one point (to be more precise $|u_n|^2$ converges to a delta measure located at zero). For n large enough we essentially detect all the mass in

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any small neighborhood of the origin. In Example 2a, the sequence $\{u_n\}$ keeps a constant shape for all n but it runs off to infinity. If we want to see anything, we have to follow the sequence by "running after it". Finally, in Example 2b, the function oscillates so fast that (by the Riemann-Lebesgue Lemma) it converges weakly to zero in $L^2(\mathbb{R}^d)$.

Examples 1a and 1b on the one side, and 2a and 2b on the other side are in some sense dual to each other. We can go from 1a to 1b (resp. 2a to 2b) by applying a Fourier transform. For instance, a sequence which oscillates very fast has a lack of compactness due to translations in Fourier space, and conversely. Similarly, a sequence which concentrates in Fourier space vanishes in direct space and conversely.

Another important remark is that the examples are all associated to the action of a non-compact group over $L^2(\mathbb{R}^d)$. The group is $(\mathbb{R}_+, \times)$ for Examples 1a and 1b and its action consists in dilating a function in the manner $(\lambda \cdot u)(x) = \lambda^{d/2} u(\lambda x)$. The non-compactness is obtained by taking either $\lambda \to 0^+$ or $\lambda \to +\infty$. Writing $\lambda = e^t$ we can also see this as an action of the additive group $(\mathbb{R}, +)$. In Examples 2a and 2b the group is $(\mathbb{R}^d, +)$ and it acts either by translating the function or by multiplying it by a phase factor, the two actions being the same up to a Fourier transform. The non-compactness of the sequence $\{u_n\}$ then arises from the non-compactness of the group itself.

One difference between the examples from the first line and the ones from the second is the behavior with respect to L^p norms with $p \neq 2$. In the vanishing case (Example 1a) we have $\|u_n\|_{L^p(\mathbb{R}^d)} \to 0$ for all $2 < p \leq \infty$ and $\|u_n\|_{L^p(\mathbb{R}^d)} \to \infty$ for all $1 \leq p < 2$. In Example 1b the situation is exactly reversed. On the other hand, in Examples 2a and 2b the L^p norms are all conserved for all p. This says that we can expect to use other L^p norms to detect a sequence with vanishes or concentrates, but not the lack of compactness due to translations and oscillations.

There is a crucial difference between the examples of the first column and that of the second one. Examples 1a and 2a cannot happen in a bounded domain $\Omega \subset \mathbb{R}^d$, it is important that the problem is settled over the infinite space $\mathbb{R}^d$ and the non-compactness arises from the non-compactness of $\mathbb{R}^d$ itself. Examples 1b and 2b can however perfectly happen in a bounded set Ω. Examples 1a and 2a are called *locally compact*, whereas Examples 1b and 2b are not locally compact. The difference can also be detected by looking at the derivative. In the first column, the derivative of the sequence $\{u_n\}$ stays bounded uniformly in n: For instance the sequence is bounded in the Sobolev space $H^1(\mathbb{R}^d)$. In the cases of the second column, the function oscillates or blows up so fast that the derivative has to explode: We have $\|\nabla u_n\|_{L^2(\Omega)} \to \infty$, even in a (well-chosen) bounded domain Ω.

Of course, it is clear that we can combine the four examples as we want. We can add functions behaving differently or even compose the corresponding group actions. Think of a sequence u_n which concentrates about a point x_n such that $|x_n| \to \infty$.

In the beginning of the 80s, there has been a high activity in trying to describe the possible behavior of sequences which undergo a lack of compactness. The main idea was to prove that, in an appropriate sense, the four above examples are *universal.* Stating differently, a non-compact sequence should, up to a subsequence, be a (possibly infinite) sum of sequences having one or several of the above behaviors, up to a small error. Knowing this fact is very useful either in a contradiction argument (when we want to prove that the sequence has to be compact), or when the studied lack of compactness has a special physical meaning and has to be described more accurately.

The first to address these issues on a specific example were Sacks and Uhlenbeck [28] in 1981 who dealt with a concentration phenomenon for harmonic maps.

Brezis and Nirenberg [5] then faced similar difficulties for some elliptic Partial Differential Equations (PDE) with a critical Sobolev exponent in 1983. The same year, Lieb proved in [17] a useful lemma dealing with lack of compactness due to translations in the locally compact case. The most general method for dealing with locally compact problems was published by Lions [22, 23] in 1984 under the name "concentration-compactness" (it was announced before in [20]). Later in 1984-85, Struwe [30] and, independently, Brezis and Coron [3] have provided the first "bubble decompositions" (that is they have shown that the sequence under study can be written as a sum of functions which concentrate at some points in space). In 1985, Lions adapted his concentration-compactness method to the nonlocal case [24, 25] (see also [21]). These fundamental works from the 80s have generated many interesting further developments. The tools invented at that time are now of widespread use in the analysis of PDEs.

In these notes, we present a simple "Hilbert" version of this theory. First we consider a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $H^1(\mathbb{R}^d)$ and describe in a rather detailed way its behavior in $L^p(\mathbb{R}^d)$ for a subcritical $2 \le p < p^*$ (with p^* being the critical Sobolev exponent to be recalled below). More precisely, we will show that $\{u_n\}$ can be written as a sum of functions retaining their shape and escaping to infinity at different speeds (like in Example 2a), plus a remainder which can vanish in the sense of Example 1a. This is enough for many of the physical models encountered in practice. Most of the arguments of this section can be generalized to a Sobolev space $W^{k,p}(\mathbb{R}^d)$ with $k \ge 1$ and $1 < p < \infty$. It is not essential that the underlying space is a Hilbert space, but several arguments are easier in this setting. An important point is that the sequence is locally compact, by the Rellich-Kondrachov Theorem. Then, in Section 6, we quickly touch upon the critical case where both Examples 1a, 1b and 2a can happen.

Our approach in these notes is a combination of ideas of Lieb [17], of the concentration-compactness method of Lions [22, 23] and of the bubble decomposition as used for instance in [30, 10]. For a more general version of the concentration-compactness method (in $L^1(\mathbb{R}^d)$ or for measures), we refer to the original papers of Lions, or to the book of Struwe [31]. For a more abstract presentation, see the book of Tintarev and Fieseler [33]. The french speaking reader can refer to the book of Kavian [11], or to that of Cancès, Le Bris and Maday [6] for an intuitive presentation of the method.

1. Finding bubbles

We consider a sequence $\boldsymbol{u} = \{u_n\}$ bounded in $L^2(\mathbb{R}^d)$ (later our sequence will be bounded in $H^1(\mathbb{R}^d)$). Our goal is to detect pieces of mass which retain their shape for n large and, possibly, escape to infinity in the spirit of Example 2a in Figure 1. We call them 'bubbles'. For this reason, we consider all the possible weak limits, up to translations, of subsequences of $\{u_n\}$ and define the largest possible mass that these weak limits can have.

Definition 1 (Highest local mass of a sequence)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. We define the following number*

$$(1)\quad \boldsymbol{m}(\boldsymbol{u}) = \sup\left\{\int_{\mathbb{R}^d} |u|^2 \ :\ \exists \{x_k\} \subset \mathbb{R}^d,\ u_{n_k}(\cdot + x_k) \rightharpoonup u \text{ weakly in } L^2(\mathbb{R}^d)\right\}.$$

Here we are interested in understanding the lack of compactness due to space translations which corresponds to the group action of $(\mathbb{R}^d, +)$ on $L^2(\mathbb{R}^d)$. There is a similar definition of $\boldsymbol{m}$ for any other group action. The combination of translations and dilations will be considered later in Section 6.

Remark 2. *We have $\boldsymbol{m}(\boldsymbol{u}) = \boldsymbol{m}(\boldsymbol{u}(\cdot + \boldsymbol{x}))$ for any sequence of translations $\boldsymbol{x} = \{x_n\} \subset \mathbb{R}^d$ (here $\boldsymbol{u}(\cdot + \boldsymbol{x})$ is the sequence $\{u_n(\cdot + x_n)\}$). Also, if $\boldsymbol{u}' = \{u_{n_k}\}$ is a subsequence of $\boldsymbol{u} = \{u_n\}$, then $\boldsymbol{m}(\boldsymbol{u}') \leq \boldsymbol{m}(\boldsymbol{u})$.*

Remark 3. *If $\{u_n\}$ is not only bounded in $L^2(\mathbb{R}^d)$ but also in the Sobolev space $H^1(\mathbb{R}^d)$, then we can replace the weak convergence in $L^2(\mathbb{R}^d)$ by weak convergence in $H^1(\mathbb{R}^d)$ in the definition of $\boldsymbol{m}(\boldsymbol{u})$ without changing anything.*

Exercise 4. *There does not necessarily exist a $u \in L^2(\mathbb{R}^d)$ realizing the above supremum (that is such that $u_{n_k}(\cdot + x_{n_k}) \rightharpoonup u$ for a subsequence with $\int_{\mathbb{R}^d} |u|^2 = \boldsymbol{m}(\boldsymbol{u})$). We provide a counter-example. Let $\psi_n = n^{-d/2}\sqrt{1-1/n}\,u(x/n)$ for some fixed $u \in L^2(\mathbb{R}^d)$. Define a new sequence $\{u_n\}$ as follows:*

$$\begin{array}{lll} u_1 = \psi_1, & & \\ u_2 = \psi_1, & u_3 = \psi_2, & \\ u_4 = \psi_1, & u_5 = \psi_2, & u_6 = \psi_3, \\ u_7 = \psi_1, & \dots & \end{array}$$

and so on. Verify that $\boldsymbol{m}(\boldsymbol{u}) = \int_{\mathbb{R}^d} |u|^2$ but that it is not "attained" by any weakly convergent subsequence.

Exercise 5. *Verify that $\boldsymbol{m}(\boldsymbol{u}) = 0$ if $u_n(x) = n^{-d/2}u(x/n)$ (Example 1a), $u_n(x) = n^{d/2}u(nx)$ (Example 1b), or $u_n(x) = u(x)\,e^{in\vec{v}\cdot x}$ (Example 2b). Here u is a fixed function in $L^2(\mathbb{R}^d)$.*

The purpose of $\boldsymbol{m}(\boldsymbol{u})$ is to detect the largest piece of mass in the sequence $\boldsymbol{u} = \{u_n\}$, which possibly escapes to infinity (when $|x_{n_k}| \to \infty$). If $\boldsymbol{m}(\boldsymbol{u}) > 0$, then we can find

- a subsequence $\{u_{n_k}\}$;
- a sequence of translations $\{x_k^{(1)}\} \subset \mathbb{R}^d$;
- a function $u^{(1)} \in L^2(\mathbb{R}^d)$ such that $\boldsymbol{m}(\boldsymbol{u}) \geq \int_{\mathbb{R}^d} |u^{(1)}|^2 \geq \boldsymbol{m}(\boldsymbol{u}) - \varepsilon > 0$ (in particular $u^{(1)} \neq 0$),

such that

$$u_{n_k}(\,\cdot + x_k^{(1)}) \rightharpoonup u^{(1)} \qquad \text{weakly in } L^2(\mathbb{R}^d).$$

Once we have found the first 'bubble' $u^{(1)}$, we can go on and try to find the next one by considering the sequence $r_k^{(2)} := u_{n_k} - u^{(1)}(\,\cdot - x_k^{(1)})$ and the corresponding $\boldsymbol{m}(\boldsymbol{r}^{(2)})$. Arguing by induction we can find by induction all the bubbles contained in the original sequence $\boldsymbol{u} = \{u_n\}$.

Lemma 6 (Extracting bubbles)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$). Then there exists a (possibly empty or finite) sequence of functions $\{u^{(1)}, u^{(2)}, ...\}$ in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$) such that the following holds true: For any fixed $\varepsilon > 0$, there exists*

- *an integer J,*
- *a subsequence $\{u_{n_k}\}$ of $\{u_n\}$,*
- *space translations $\{x_k^{(j)}\}_{k\geq 1} \subset \mathbb{R}^d$ for $j = 1, ..., J$ with $|x_k^{(j)} - x_k^{(j')}| \to \infty$ as $k \to \infty$, for each $j \neq j'$,*

such that we can write

$$u_{n_k} = \sum_{j=1}^{J} u^{(j)}(\,\cdot - x_k^{(j)}) + r_k^{(J+1)} \tag{2}$$

where

$$r_k^{(J+1)}(\cdot + x_k^{(j)}) \underset{k\to\infty}{\rightharpoonup} 0$$

weakly in $L^2(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$) for all $j = 1, ..., J$ and

$$\boldsymbol{m}(\boldsymbol{r}^{(J+1)}) \leq \varepsilon.$$

In particular we have

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^d}|u_{n_k}|^2-\int_{\mathbb{R}^d}|r_k^{(J+1)}|^2\right)=\sum_{j=1}^J\int_{\mathbb{R}^d}|u^{(j)}|^2. \tag{3}$$

If $\boldsymbol{u}=\{u_n\}$ *is bounded in* $H^1(\mathbb{R}^d)$, *then we also have*

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^d}|\nabla u_{n_k}|^2-\int_{\mathbb{R}^d}|\nabla r_k^{(J+1)}|^2\right)=\sum_{j=1}^J\int_{\mathbb{R}^d}|\nabla u^{(j)}|^2 \tag{4}$$

and

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^d}|u_{n_k}|^p-\int_{\mathbb{R}^d}|r_k^{(J+1)}|^p\right)=\sum_{j=1}^J\int_{\mathbb{R}^d}|u^{(j)}|^p \tag{5}$$

for every

$$2\le p\begin{cases}<\infty & \text{for } d=1,2,\\ \le\frac{2d}{d-2} & \text{for } d\ge 3.\end{cases}$$

Note that the lemma is essentially an abstract result. It can be stated similarly in any Hilbert space on which is acting a non-compact group. Only the limit (5) is specific to the space $H^1(\mathbb{R}^d)$ for $p>2$. See [33] for a more abstract presentation in the same spirit.

The functions $u^{(j)}$ are all the possible weak limits of $\boldsymbol{u}$ up to translations and extraction of subsequences. The lemma says that any bounded sequence $\boldsymbol{u}=\{u_n\}$ in $L^2(\mathbb{R}^d)$ can be written as a linear combination of these limits translated in space, up to an error term $r_k^{(J+1)}$. This error term is not necessarily small in norm, because the remainder $\boldsymbol{r}^{(J+1)}$ can exhibit all sorts of other compactness issues. But we know that $\boldsymbol{m}(\boldsymbol{r}^{(J+1)})$ is small. When $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$ then $\boldsymbol{m}(\boldsymbol{r}^{(J+1)})\le\varepsilon$ will tell us a lot since Examples 1b and 2b cannot happen in a locally compact setting. We will explain this in the next section.

Let us emphasize the fact that the bubbles $u^{(j)}$ do not depend on ε. They can be constructed once and for all for any fixed sequence $\boldsymbol{u}=\{u_n\}$. On the contrary, the number J of bubbles, the space translations $x_k^{(j)}$ and the subsequence n_k needed to achieve a given accuracy in the decomposition of u_{n_k}, all depend on ε.

Proof of Lemma 6. We start with the case where $\boldsymbol{u}=\{u_n\}$ is bounded in $L^2(\mathbb{R}^d)$. If $\boldsymbol{m}(\boldsymbol{u})=0$, there is nothing to prove and we may therefore assume that $\boldsymbol{m}(\boldsymbol{u})>0$. In this case we deduce from the definition of $\boldsymbol{m}(\boldsymbol{u})$ that there exists a subsequence n_k, a sequence of translations $\{x_k^{(1)}\}\subset\mathbb{R}^d$, and a function $u^{(1)}\in L^2(\mathbb{R}^d)$ such that

$$\frac{\boldsymbol{m}(\boldsymbol{u})}{2}\le\int_{\mathbb{R}^d}|u^{(1)}|^2\le\boldsymbol{m}(\boldsymbol{u})$$

and

$$u_{n_k}(\cdot+x_k^{(1)})\rightharpoonup u^{(1)}$$

weakly in $L^2(\mathbb{R}^d)$. Defining $r_k^{(2)}:=u_{n_k}-u^{(1)}(\cdot-x_k^{(1)})$, we obviously get that $r_k^{(2)}(\cdot+x_k^{(1)})\rightharpoonup 0$. We also have

$$\|u_{n_k}\|_{L^2}^2=\left\|u^{(1)}\right\|_{L^2}^2+\left\|r_k^{(2)}\right\|_{L^2}^2+2\Re\left\langle r_k^{(2)}(\cdot+x_k^{(1)}),u^{(1)}\right\rangle$$

where the last term tends to 0 by the weak convergence of $r_k^{(2)}$. So we conclude that

$$\lim_{k\to\infty}\left(\|u_{n_k}\|_{L^2}^2-\left\|r_k^{(2)}\right\|_{L^2}^2\right)=\left\|u^{(1)}\right\|_{L^2}^2.$$

If $\boldsymbol{m}(\boldsymbol{r}^{(2)}) = 0$, we can stop here. Otherwise we go on and extract the next bubble. We find a subsequence and space translations such that $r^{(2)}_{k_\ell}(\cdot + x^{(2)}_\ell) \rightharpoonup u^{(2)} \neq 0$, with

$$\frac{\boldsymbol{m}(\boldsymbol{r}^{(2)})}{2} \leq \int_{\mathbb{R}^d} |u^{(2)}|^2 \leq \boldsymbol{m}(\boldsymbol{r}^{(2)}).$$

We also extract a further subsequence from u_{n_k} and $x^{(1)}_k$. To simplify the exposition, we use the same notation for all these subsequences. So we can write

$$u_{n_k} = u^{(1)}(\cdot - x^{(1)}_k) + u^{(2)}(\cdot - x^{(2)}_k) + r^{(3)}_k$$

where $r^{(3)}_k(\cdot + x^{(2)}_k) \rightharpoonup 0$. If $|x^{(2)}_k - x^{(1)}_k|$ does not diverge, then we can extract another subsequence such that $x^{(1)}_k - x^{(2)}_k \to v$. We would then obtain $u_{n_k}(\cdot + x^{(1)}_k) \rightharpoonup u^{(1)} + u^{(2)}(\cdot + v)$, since $r^{(3)}_k(\cdot + x^{(1)}_k) = r^{(3)}_k(\cdot + v + x^{(2)}_k) \rightharpoonup 0$. This contradicts the fact that $u_{n_k}(\cdot + x^{(1)}_k) \rightharpoonup u^{(1)}$ by construction, unless $u^{(2)} = 0$ which is not the case for us here. So we conclude that $|x^{(2)}_k - x^{(1)}_k| \to \infty$. Now we can use that $u_{n_k}(\cdot + x^{(1)}_k) \rightharpoonup u^{(1)}$ and deduce that $r^{(3)}_k(\cdot + x^{(1)}_k) \rightharpoonup 0$, since $u^{(2)}(\cdot + x^{(1)}_k - x^{(2)}_k) \rightharpoonup 0$. Similarly as before, it can be checked that

$$\lim_{k\to\infty} \left(\|u_{n_k}\|^2_{L^2} - \left\| r^{(3)}_k \right\|^2_{L^2} \right) = \left\| u^{(1)} \right\|^2_{L^2} + \left\| u^{(2)} \right\|^2_{L^2}.$$

We can apply the previous argument *ad infinitum*, constructing the sequences $u^{(j)}$ and $r^{(j)}$ in the same fashion, except when we reach a remainder $r^{(J+1)}_k$ which is such that $\boldsymbol{m}(\boldsymbol{r}^{(J+1)}) = 0$. Our construction satisfies

$$\lim_{k\to\infty} \left(\|u_{n_k}\|^2_{L^2} - \left\| r^{(J+1)}_k \right\|^2_{L^2} \right) = \sum_{j=1}^{J} \left\| u^{(j)} \right\|^2_{L^2}$$

for all J, and in particular

$$\sum_j \left\| u^{(j)} \right\|^2_{L^2} \leq \limsup_{n\to\infty} \|u_n\|^2_{L^2} \leq C.$$

Therefore we see that $\left\| u^{(j)} \right\|_{L^2} \to 0$ as $j \to \infty$. Recalling that

$$\boldsymbol{m}(\boldsymbol{r}^{(j)}) \leq 2 \int_{\mathbb{R}^d} |u^{(j)}|^2$$

by construction, we deduce that $\boldsymbol{m}(\boldsymbol{r}^{(j)}) \to 0$ as well. Hence, for J large enough, $\boldsymbol{m}(\boldsymbol{r}^{(J)})$ must be smaller than any ε fixed in advance, and this concludes the proof of Theorem 6 in the L^2 case.

When the sequence is bounded in $H^1(\mathbb{R}^d)$, any weak limit in $L^2(\mathbb{R}^d)$ is automatically also a weak limit in $H^1(\mathbb{R}^d)$, hence (3) and (4) follow easily. For (5), however, we have to use the strong local compactness (Rellich-Kondrachov theorem). That is, when $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$, we can find a subsequence such that $u_n \to u$ strongly locally in L^2 and almost everywhere. Then (5) follows from the *missing term in Fatou's lemma*, also called the *Brezis-Lieb lemma* [4, 18], which precisely states that

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^d} |u_n(x)|^p\,dx - \int_{\mathbb{R}^d} |u(x) - u_n(x)|^p\,dx \right) = \int_{\mathbb{R}^d} |u(x)|^p\,dx \tag{6}$$

when $u_n(x) \to u(x)$ almost everywhere. □

Exercise 7. *Write the full details of the proof in the $H^1(\mathbb{R}^d)$ case.*

Remark 8. *The proof of* (3) *relies on the Hilbert space structure of $L^2(\mathbb{R}^d)$ and the theory is much less nice in $L^p(\mathbb{R}^d)$ with $1 \le p < \infty$ and $p \neq 2$. For instance, in this case it is* ***not true in general*** *that $u_n \rightharpoonup u$ implies* (6)*. From the Brezis-Lieb lemma the limit* (6) *follows from almost everywhere convergence, but the latter cannot be obtained from weak convergence in $L^p(\mathbb{R}^d)$ without more information on the sequence. In Sobolev spaces, almost everywhere convergence is obtained in our proof by using strong local compactness. For a theory in $L^p(\mathbb{R}^d)$ with $p \neq 2$, the best is to use concentration functions as in* [22]*, see Section 3 below.*

Remark 9. *As is obvious from the above proof of Lemma 6, we can also choose the sequence $\{u^{(j)}\}$ to have that*

$$\max_{j\geq 0} \int_{\mathbb{R}^d} |u^{(j)}|^2 \geq \boldsymbol{m}(\boldsymbol{u}) - \eta \tag{7}$$

for any $\eta > 0$. It suffices to choose $u^{(1)}$ satisfying this property. Note that (7) *is a maximum, since $\int_{\mathbb{R}^d} |u^{(j)}|^2 \to 0$ as $j \to \infty$, by construction.*

As we have mentioned in Exercise 4, $\boldsymbol{m}(\boldsymbol{u})$ is not necessarily 'attained'. We now prove that it is always attained for a subsequence.

Corollary 10 ($m(\{u_n\})$ is attained for a subsequence)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$ and $\eta > 0$. There exists a subsequence $\boldsymbol{u'} = \{u_{n_k}\}$, $\{x_k\} \subset \mathbb{R}^d$, $u \in L^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |u|^2 = \boldsymbol{m}(\boldsymbol{u'}) \geq \boldsymbol{m}(\boldsymbol{u}) - \eta$ and $u_{n_k}(\,\cdot + x_k) \rightharpoonup u$.*

Proof. We apply Lemma 6, choosing specifically the sequence $\{u^{(j)}\}$ such as to have

$$\max_{j\geq 0} \int_{\mathbb{R}^d} |u^{(j)}|^2 \geq \boldsymbol{m}(\boldsymbol{u}) - \eta$$

by Remark 9. Then we choose J large enough such that

$$\boldsymbol{m}\big(\boldsymbol{r}^{(J+1)}\big) < \max\Big(\int_{\mathbb{R}^d} |u^{(j)}|^2,\ j = 1, ..., J\Big).$$

It is then clear that $\boldsymbol{m}(\boldsymbol{u}) = \max\big(\int_{\mathbb{R}^d} |u^{(j)}|^2,\ j = 1, ..., J\big)$. □

2. Vanishing

In the previous section we have defined the highest mass that weak limits can have up to space translations. We have shown in Lemma 6 that any bounded sequence in $L^2(\mathbb{R}^d)$ can be written as a linear combination of these bumps escaping to infinity, plus a remainder $r_k^{(J+1)}$ which is such that $\boldsymbol{m}(\boldsymbol{r}^{(J+1)})$ is small. If we continue this construction infinitely many times we will have found all the bubbles and essentially arrive at a sequence $\boldsymbol{r}^{(\infty)}$ such that $\boldsymbol{m}(\boldsymbol{r}^{(\infty)}) = 0$. What does this information tell us? Under some additional assumption on the derivatives, we will show that the remainder can only vanish, in a proper sense. We use here the Sobolev space $H^1(\mathbb{R}^d)$ and other L^p norms, as we have explained after the examples from Figure 1.

Lemma 11 (A subcritical estimate involving $\boldsymbol{m}(\boldsymbol{u})$)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. There exists a universal constant $C = C(d)$ such that the following holds:*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \leq C\, \boldsymbol{m}(\boldsymbol{u})^{\frac{2}{d}}\ \limsup_{n\to\infty} \|u_n\|^2_{H^1(\mathbb{R}^d)}\,. \tag{8}$$

The previous result gives an estimate on the size of $\|u_n\|_{L^{2+4/d}(\mathbb{R}^d)}$ in terms of $\boldsymbol{m}(\boldsymbol{u})$. Note that the exponent $2 + 4/d$ is always smaller than the critical Sobolev exponent p^*, which equals $2 + 4/(d-2)$ in dimensions $d \geq 3$ and $+\infty$ in dimensions $d = 1, 2$. The proof relies on the Sobolev inequality. Lemma 11 is due to Lions [23, Lemma I.1].

Proof of Lemma 11. We only write the proof for $d \geq 3$ and let the (very similar) cases $d = 1, 2$ as an exercise. We consider a tiling of the whole space $\mathbb{R}^d$ by means of cubes, say $\mathbb{R}^d = \cup_{z\in\mathbb{Z}^d} C_z$ with $C_z = \prod_{j=1}^d [z_j, z_j + 1)$. We then calculate by Hölder's inequality

$$\int_{\mathbb{R}^d} |u_n|^q = \sum_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^q \leq \sum_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)}^{\theta q} \|u_n\|_{L^{p^*}(C_z)}^{(1-\theta)q}, \tag{9}$$

where $1/q = \theta/2 + (1-\theta)/2^*$. We choose q in such a way that $(1-\theta)q = 2$. A simple calculation shows that $q = 2 + 4/d$ and $\theta q = 4/d$. Note that this choice satisfies $2 < q < 2^* = 2 + 4/(d-2)$ (recall we are in the case $d \geq 3$). By the Sobolev embedding in the cube C_z, there exists a constant C such that

$$\|u_n\|_{L^{2^*}(C_z)}^2 \leq C \left(\|u_n\|_{L^2(C_z)}^2 + \|\nabla u_n\|_{L^2(C_z)}^2 \right).$$

Note that the constant C only depends on the volume of the cube, hence it is independent of $z \in \mathbb{Z}^d$. Inserting in (9), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} |u_n|^q &\leq C \left(\sup_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)} \right)^{\frac{4}{d}} \sum_{z\in\mathbb{Z}^d} \left(\int_{C_z} |u_n|^2 + \int_{C_z} |\nabla u_n|^2 \right) \\ &= C \left(\sup_{z\in\mathbb{Z}^d} \|u_n\|_{L^2(C_z)} \right)^{\frac{4}{d}} \|u_n\|_{H^1(\mathbb{R}^d)}^2 .\end{aligned}$$

Passing to the limit $n \to \infty$, we deduce that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{2+\frac{4}{d}} \leq C \left(\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 \right)^{\frac{2}{d}} \limsup_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^d)}^2 . \tag{10}$$

We now claim that

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 \leq \boldsymbol{m}(\boldsymbol{u}), \tag{11}$$

which will end the proof of Lemma 11. Indeed, consider a sequence $\boldsymbol{z} = \{z_n\} \subset \mathbb{R}^d$ such that

$$\lim_{n\to\infty} \int_{C_{z_n}} |u_n|^2 = \limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2.$$

The sequence $u_n(\cdot + z_n)$ is bounded in $H^1(\mathbb{R}^d)$. Up to extraction of a subsequence, we have $u_{n_k}(\cdot + z_{n_k}) \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and, by the Rellich-Kondrachov compactness theorem, strongly in $L^2(C_0)$. We deduce that

$$\lim_{n\to\infty} \int_{C_{z_n}} |u_n|^2 = \lim_{n_k\to\infty} \int_{C_0} |u_{n_k}(\cdot + z_{n_k})|^2 = \int_{C_0} |u|^2 \leq \int_{\mathbb{R}^d} |u|^2 \leq \boldsymbol{m}(\boldsymbol{u}).$$

This concludes the proof of Lemma 11. □

We now use the previous result to characterize when $\boldsymbol{m}(\boldsymbol{u}) = 0$ holds, which is equivalent to saying that $u_n(\cdot + x_n) \rightharpoonup 0$ for all $\{x_n\} \subset \mathbb{R}^d$. Intuitively this corresponds to vanishing as displayed in Example 1a of Figure 1 since u_n must then converge to zero strongly in all $L^p(\mathbb{R}^d)$, for all the sub-critical exponents $2 < p < 2^*$.

Lemma 12 (Vanishing)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. The following assertions are equivalent:*

(i) $\boldsymbol{m}(\boldsymbol{u}) = 0$;

(ii) for all $R > 0$, we have $\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(x,R)} |u_n|^2 = 0$;

(iii) $u_n \to 0$ strongly in $L^p(\mathbb{R}^d)$ for all $2 < p < 2^*$, where $2^* = 2d/(d-2)$ if $d \geq 3$, $2^* = \infty$ if $d = 1, 2$.

The interpretation of (ii) is that there is essentially no mass remaining for n large enough in any ball of fixed radius R, independently of the location of the center of the ball.

Proof of Lemma 12. We start by proving that $(i) \Rightarrow (ii)$. Indeed, from the estimate (11) of the proof of Lemma 11, we deduce that

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{Z}^d} \int_{C_z} |u_n|^2 = 0$$

where $C_z = \prod_{j=1}^d [z_j, z_j + 1)$ are cubes covering the whole space. Since a ball of radius R can be covered by a finite union of such cubes, the result follows.

To show that $(iii) \Rightarrow (i)$, we consider $\{x_{n_k}\} \subset \mathbb{R}^d$ and u such that $u_{n_k}(\cdot + x_{n_k}) \rightharpoonup u$. Since

$$\|u_{n_k}(\cdot + x_{n_k})\|_{L^p(\mathbb{R}^d)} = \|u_{n_k}\|_{L^p(\mathbb{R}^d)} \to 0$$

when $2 < p < 2^*$, we have $u_{n_k}(\cdot + x_{n_k}) \to 0$ strongly in $L^p(\mathbb{R}^d)$, hence (by uniqueness of the weak limit) $u = 0$. Therefore $\boldsymbol{m}(\boldsymbol{u}) = 0$.

Finally, the proof that $(ii) \Rightarrow (iii)$ is a consequence of (10). We have $\|u_n\|_{L^{2+4/d}(\mathbb{R}^d)} \to 0$ and the rest follows by interpolation, using that $\{u_n\}$ is bounded in $L^p(\mathbb{R}^d)$ for $2 \le p \le 2^*$ by the Sobolev embedding. □

Exercise 13. *Write the proof for $d = 1, 2$. Generalize also Lemmas 11 and 12 to the case of a bounded sequence $\{u_n\}$ in $W^{k,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and k a positive integer, with an appropriate definition of $\boldsymbol{m}(\boldsymbol{u})$.*

3. Isolating bubbles in space

In applications it is sometimes useful to "isolate" the bubbles, which means that we write u_{n_k} as a sum of functions $u_k^{(j)}$ of compact supports, each of them converging strongly to the weak limit $u^{(j)}$ of Lemma 6, and where the distance between the supports diverges. This step is not always necessary in practice in $L^2(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$ but it is sometimes convenient. In addition, this is the preferred technique to be used in other spaces such as $L^p(\mathbb{R}^d)$ or for non-negative measures. For instance when all the functions u_n are non-negative, it might be convenient to ensure as well that the remainder $r_k^{(J+1)}$ is non-negative (in the proof of Lemma 6 the remainder $r_k^{(J+1)}$ is defined as a difference). Using localization functions is more in the spirit of Lions' concentration-compactness principle and this is an occasion for us to learn how to use the concentration functions of Levy.

The localization procedure can be done for any chosen J in the profile decomposition (2) of Lemma 6. In order to illustrate the main idea, we start with the case $J = 1$, that is when we have one sequence u_n which converges weakly to some $u^{(1)} := u$. We will come back to the general case later in Corollary 20.

Lemma 14 (Extracting the locally convergent part)**.** *Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and let $0 \le R_k \le R_k'$ such that $R_k \to \infty$. Then there exists a subsequence $\{u_{n_k}\}$ such that*

$$\int_{|x|\le R_k} |u_{n_k}(x)|^2\,dx \to \int_{\mathbb{R}^d} |u(x)|^2\,dx \tag{12}$$

and

$$\int_{R_k\le|x|\le R_k'} \left(|u_{n_k}(x)|^2 + |\nabla u_{n_k}(x)|^2\right) dx \to 0 \tag{13}$$

as $k \to \infty$. In particular, we have that $u_{n_k}\mathbb{1}_{B(0,R_k)} \to u$ strongly in $L^p(\mathbb{R}^d)$ for all $2 \le p < 2^*$ (where 2^* is like in Lemma 12).

The interpretation of Lemma 14 is that we can isolate by means of a growing ball $B(0,R_k)$ the part which converges strongly to u in sub-critical spaces $L^p(\mathbb{R}^d)$ with $2\leq p<2^*$. In the annulus $\{R_k\leq |x|\leq R'_k\}$, we have essentially nothing (by the convergence to zero in $H^1(\mathbb{R}^d)$). In practice, we choose R'_k such that $R'_k-R_k\to\infty$. Since the original sequence $\{u_{n_k}\}$ has been decomposed into two parts (the convergent part in the ball $B(0,R_k)$ and the rest in the domain $\mathbb{R}^d\setminus B(0,R'_k)$), Lions used the word *dichotomy* [22, 23] to describe this procedure.

The intuition behind the decomposition of Lemma 14 is that we have to choose R_k and R'_k growing slowly enough that we essentially only retain the mass of the weak limit u and not more. However, our statement is for convenience written the other way: We can choose R_k and R'_k essentially as we want, but then the sequence n_k has to go to infinity so fast that u_{n_k} has almost already converged to its local limit u in the ball $B(0,R'_k)$.

Proof of Lemma 14. We follow Lions [19, 22, 23] and introduce the so-called *Levy concentration functions* [16]

$$Q_n(R):=\int_{B(0,R)}|u_n|^2,\qquad\text{and}\qquad K_n(R):=\int_{B(0,R)}|\nabla u_n|^2.$$

Note that Q_n and K_n are continuous non-decreasing functions on $[0,\infty)$, such that

$$\forall n\geq 1,\quad \forall R>0,\qquad Q_n(R)+K_n(R)\leq\int_{\mathbb{R}^d}|u_n|^2+|\nabla u_n|^2\leq C$$

since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$. By the Rellich-Kondrachov Theorem, we have

$$Q_n(R)=\int_{B(0,R)}|u_n|^2\to\int_{B(0,R)}|u|^2:=Q(R)$$

for all $R\geq 0$. We now recall a very useful result dealing with sequences of monotone functions.

Lemma 15 (Sequences of monotone functions)**.** *Let I be a (possibly unbounded) interval of $\mathbb{R}$, and $\{f_n\}$ be a sequence of non-increasing non-negative functions on I. We assume that there exists a fixed function g, locally bounded on I, such that $0\leq f_n\leq g$ on I. Then there exists a subsequence $\{f_{n_k}\}$ and a non-increasing function f such that $f_{n_k}(x)\to f(x)$ for all $x\in I$.*

In the applications, g is often a constant on the whole interval I. By Lemma 15, up to extraction of a subsequence (for the sake of clarity we do not change notation), we may assume that $K_n(R)\to K(R)$ for all $R\geq 0$, and for some non-decreasing function K. We denote $\ell:=\lim_{R\to\infty}K(R)$.

It can be shown that $\ell\geq\int_{\mathbb{R}^d}|\nabla u|^2$ but there is not always equality, except if we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^d)$ for some $s>1$.

Consider now the given sequences R_k and R'_k which diverge to ∞. We have $Q_n(R_k)\to Q(R_k)$ and $Q_n(R'_k)\to Q(R'_k)$ as $n\to\infty$, and a similar property for K_n. Extracting a subsequence, we may assume for instance that

$$|Q_{n_k}(R_k)-Q(R_k)|+|Q_{n_k}(R'_k)-Q(R'_k)|+|K_{n_k}(R_k)-K(R_k)|+|K_{n_k}(R'_k)-K(R'_k)|\leq\frac{1}{k}.$$

We then deduce that

$$\left|\int_{B(0,R_k)}|u_{n_k}|^2-\int_{\mathbb{R}^d}|u|^2\right|=|Q_{n_k}(R_k)-Q(\infty)|\leq\frac{1}{k}+\int_{|x|\geq R_k}|u|^2\underset{k\to\infty}{\longrightarrow}0$$

and that

$$\int_{R_k\leq|x|\leq R'_k}|u_{n_k}|^2=Q_{n_k}(R'_k)-Q_{n_k}(R_k)\leq\frac{1}{k}+Q(R'_k)-Q(R_k)\underset{k\to\infty}{\longrightarrow}0,$$

$$\int_{R_k \le |x| \le R'_k} |\nabla u_{n_k}|^2 = K_{n_k}(R'_k) - K_{n_k}(R_k) \le \frac{1}{k} + K(R'_k) - K(R_k) \underset{k\to\infty}{\longrightarrow} 0,$$

where we have used that $K(R'_k) - K(R_k) \to \ell - \ell = 0$ when $k \to \infty$.

Finally, we have $\mathbb{1}_{B(0,R_k)} u_{n_k} \rightharpoonup u$ weakly in $L^2(\mathbb{R}^d)$. Since the norm also converges, we get that the convergence must be strong. By the Sobolev embeddings, $\{u_n\}$ is bounded in $L^p(\mathbb{R}^d)$ for all $2 \le p < 2^*$, hence so is $\mathbb{1}_{B(0,R_k)} u_{n_k}$. By interpolation, $\mathbb{1}_{B(0,R_k)} u_{n_k}$ converges towards u strongly in $L^p(\mathbb{R}^d)$ for all $2 \le p < 2^*$. This concludes the proof of Lemma 14. □

Exercise 16. *Prove that for any bounded sequence $\boldsymbol{u} = \{u_n\}$ in $H^1(\mathbb{R}^d)$,*

$$\boldsymbol{m}(\boldsymbol{u}) = \lim_{R\to\infty} \limsup_{n\to\infty} \sup_{x \in \mathbb{R}^d} \int_{B(x,R)} |u_n|^2. \tag{14}$$

Exercise 17. *Generalize Lemma 14 to the case of a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $W^{k,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and k a positive integer.*

In the previous lemma, we have isolated in the ball $B(0, R_k)$ the part of the sequence u_{n_k} which converges to u strongly, by retaining only the appropriate mass. Unfortunately, the function $u_{n_k} \mathbb{1}_{B(0,R_k)}$ is not in $H^1(\mathbb{R}^d)$ and it is in practice convenient to replace the characteristic function $\mathbb{1}_{B(0,R_k)}$ by a smooth cut-off. This is the purpose of the next corollary.

Corollary 18 (Splitting of a weakly convergent sequence in $H^1(\mathbb{R}^d)$)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and let $0 \le R_k \le R'_k$ such that $R_k \to \infty$. Then we have for a subsequence $\{u_{n_k}\}$*

$$\lim_{k\to\infty} \left\| u_{n_k} - u_k^{(1)} - \psi_k^{(2)} \right\|_{H^1(\mathbb{R}^d)} = 0$$

where $\boldsymbol{u}^{(1)} = \{u_k^{(1)}\}$ and $\boldsymbol{\psi}^{(2)} = \{\psi_k^{(2)}\}$ are sequences in $H^1(\mathbb{R}^d)$ such that

- *$u_k^{(1)} \to u$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^p(\mathbb{R}^d)$ for all $2 \le p < 2^*$;*
- *$\mathrm{supp}(u_k^{(1)}) \subset B(0, R_k)$ and $\mathrm{supp}(\psi_k^{(2)}) \subset \mathbb{R}^d \setminus B(0, R'_k)$;*
- *$\boldsymbol{m}(\boldsymbol{r}^{(2)}) \le \boldsymbol{m}(\{u_{n_k}\}) \le \boldsymbol{m}(\boldsymbol{u})$.*

Proof. We apply Lemma 14 with $R_k/2$ and $4R'_k$. We obtain a subsequence $\{u_{n_k}\}$ such that

$$\int_{|x| \le R_k/2} |u_{n_k}|^2 \to \int_{\mathbb{R}^d} |u|^2, \qquad \int_{R_k/2 \le |x| \le 4R'_k} |u_{n_k}|^2 + |\nabla u_{n_k}|^2 \to 0. \tag{15}$$

Let $\chi : \mathbb{R}^+ \to [0,1]$ be a smooth function such that $0 \le \chi' \le 2$, $\chi_{|[0,1]} \equiv 1$ and $\chi_{|[2,\infty)} \equiv 0$. We denote $\chi_k(x) := \chi(2|x|/R_k)$ and $\zeta_k(x) = 1 - \chi(|x|/R'_k)$ and introduce $u_k^{(1)} := \chi_k u_{n_k}$ and $\psi_k^{(2)} := \zeta_k u_{n_k}$. By (15), we clearly have $u_{n_k} - u_k^{(1)} - \psi_k^{(2)} \to 0$ in $H^1(\mathbb{R}^d)$ since this function has its support in the annulus $\{R_k/2 \le |x| \le 2R'_k\}$. Furthermore

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} |u_k^{(1)}|^2 = \lim_{k\to\infty} \int_{|x| \le R_k/2} |u_k^{(1)}|^2 = \int_{\mathbb{R}^d} |u|^2,$$

hence $u_k^{(1)} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^p(\mathbb{R}^d)$ for all $2 \le p < 2^*$. Note that by construction $\mathbb{1}_{B(0,4R'_k)} \psi_k^{(2)} \to 0$ strongly in $L^2(\mathbb{R}^d)$.

It remains to show that $\boldsymbol{m}(\boldsymbol{\psi}^{(2)}) \le \boldsymbol{m}(\{u_{n_k}\})$. If $\boldsymbol{m}(\boldsymbol{\psi}^{(2)}) = 0$ there is nothing to prove. Assume that $\psi_{k_j}^{(2)}(\cdot - x_j) \rightharpoonup \psi$ for some subsequence and some $\psi \ne 0$. If for a subsequence (denoted the same for simplicity), we have $|x_j| \le 3R'_{k_j}$, then $B(x_j, R'_{k_j}) \subset B(0, 4R'_{k_j})$, hence $\psi_{k_j}^{(2)}(\cdot - x_j) \mathbb{1}_{B(0,R'_{k_j})} \rightharpoonup 0 = \psi$, a contradiction.

Hence it must hold $|x_j| \geq 3R'_{k_j}$ for all j sufficiently large. Note then that for j large enough, $\zeta_{k_j} \equiv 1$ on the ball $B(x_j, R'_{k_j})$, hence

$$\psi^{(2)}_{k_j}(\,\cdot - x_j)\mathbb{1}_{B(0,R'_{k_j})} = u_{n_{k_j}}(\,\cdot - x_j)\mathbb{1}_{B(0,R'_{k_j})} \rightharpoonup \psi,$$

thus $u_{n_{k_j}}(\,\cdot - x_j) \rightharpoonup \psi$ weakly in $H^1(\mathbb{R}^d)$. This proves that $\int_{\mathbb{R}^d} |\psi|^2 \leq \boldsymbol{m}(\{u_{n_k}\})$ and the proof is complete. $\square$

Remark 19. *Here $\psi^{(2)}_k = \zeta_k u_{n_k}$ is different from $r^{(2)}_k$ used in Lemma 6, although they behave essentially the same in the limit $k \to \infty$. The fact that $\psi^{(2)}_k$ is proportional to u_{n_k} is sometimes useful in practical applications. For instance when $u_{n_k} \geq 0$ then $\psi^{(2)}_k$ is non-negative as well.*

We now consider a sequence $\boldsymbol{u} = \{u_n\}$ and its weak limits $u^{(j)}$ up to translations, obtained from Lemma 6. We want to localize all the bubbles in disjoint balls receeding from each other, in the same spirit as in the previous result. The following result is proved in a very similar fashion as Lemma 14 and Corollary 18, using two concentration functions per bubble.

Theorem 20 (Splitting in arbitrarily many localized bubbles)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ and $\{u^{(j)}\} \subset H^1(\mathbb{R}^d)$ be the sequence given by Lemma 6. For any $\varepsilon > 0$ and any fixed sequence $0 \leq R_k \to \infty$, there exist*

- *$J \geq 0$,*
- *a subsequence $\{u_{n_k}\}$,*
- *sequences of functions $\boldsymbol{u}^{(1)} = \{u^{(1)}_k\}, ..., \boldsymbol{u}^{(J)} = \{u^{(J)}_k\}, \boldsymbol{\psi}^{(J+1)} = \{\psi^{(J+1)}_k\}$ in $H^1(\mathbb{R}^d)$,*
- *space translations $\boldsymbol{x}^{(1)} = \{x^{(1)}_k\}, ..., \boldsymbol{x}^{(J)} = \{x^{(J)}_k\}$ in $\mathbb{R}^d$,*

such that

$$\lim_{k\to\infty} \left\| u_{n_k} - \sum_{j=1}^{J} u^{(j)}_k(\,\cdot - x^{(j)}_k) - \psi^{(J+1)}_k \right\|_{H^1(\mathbb{R}^d)} = 0 \tag{16}$$

where

- *$u^{(j)}_k \to u^{(j)} \neq 0$ weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^p(\mathbb{R}^d)$ for all $2 \leq p < 2^*$;*
- *$\mathrm{supp}(u^{(j)}_k) \subset B(0, R_k)$ for all $j = 1, ..., J$ and all k;*
- *$\mathrm{supp}(\psi^{(J+1)}_k) \subset \mathbb{R}^d \setminus \cup_{j=1}^J B(x^j_k, 2R_k)$ for all k;*
- *$|x^{(i)}_k - x^{(j)}_k| \geq 5R_k$ for all $i \neq j$ and all k;*
- *$\boldsymbol{m}(\boldsymbol{\psi}^{(J+1)}) \leq \varepsilon$.*

We emphasize again that the error term $\psi^{(J+1)}_k$ is small in the sense that $\boldsymbol{m}(\boldsymbol{\psi}^{(J+1)})$ is small, that is $\{\psi^{(J+1)}_k\}$ does not contain a local mass larger than ε. In general the mass $\int_{\mathbb{R}^d} |\psi^{(J+1)}_k|^2$ of $\{\psi^{(J+1)}_k\}$ is not necessarily small since the sequence can still undergo vanishing. However, by Lemma 11, the subcritical norms of $\psi^{(J+1)}_k$ are all small.

Results taking the same form as Corollary 20 are ubiquitous in the literature and they are often called *profile* or *bubble decompositions* of the sequence $\boldsymbol{u} = \{u_n\}$, see for instance [30, 3, 26, 10].

Exercise 21. *Write the proof of Corollary 20, using $2J$ concentration functions (two per bubble), that is for the functions $|u_{n_k}(\,\cdot + x^{(j)}_k)|^2$ and $|\nabla u_{n_k}(\,\cdot + x^{(j)}_k)|^2$ for $j = 1, ..., J$.*

4. The concentration-compactness principle

In this section, we explain how to use the tools of Sections 1–3 in practice, following Lions [22, 23, 26]. The "concentration-compactness principle" is a general method to prove the compactness of some specific sequences $\boldsymbol{u} = \{u_n\}$ (possibly up to translations and/or dilations), which are usually minimizing sequences for some variational problem, Palais-Smale sequences for some nonlinear equations, or even time-dependent solutions to some dispersive nonlinear PDEs [2, 12]. Concentration-compactness is a general strategy which has to be developed in each practical situation, not an abstract theorem that can be applied as a black box. The main idea is to prove the compactness of the sequence $\boldsymbol{u} = \{u_n\}$ by showing that it must stay "concentrated", meaning that it stays in one piece and does not split into two or more non-trivial bubbles, nor vanish. The way to prove this fact is by studying what would happen in the case of non-compactness, usually by showing that the corresponding energy would become the sum of the energies of the bubbles and that it would then be too high.

To describe the strategy, we assume that we are given an energy functional $\mathcal{E}$ defined on $H^1(\mathbb{R}^d)$, which is continuous, bounded from below and coercive on

$$\mathcal{S}_{\leq}(\lambda) := \left\{ u \in H^1(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} |u|^2 \leq \lambda \right\}. \tag{17}$$

for all $\lambda \geq 0$. The corresponding minimization principle reads

$$I(\lambda) := \inf_{u \in \mathcal{S}(\lambda)} \mathcal{E}(u). \tag{18}$$

where, this time,

$$\mathcal{S}(\lambda) := \left\{ u \in H^1(\mathbb{R}^3) \ : \ \int_{\mathbb{R}^3} |u|^2 = \lambda \right\}. \tag{19}$$

The constraint on the L^2 norm is only for illustration, other norms can be fixed. By coercivity, all the minimizing sequences $\boldsymbol{u} = \{u_n\}$ are bounded in $H^1(\mathbb{R}^d)$. The goal is to prove the existence of a minimizer for (18) and to give a criterion for the precompactness of all the minimizing sequences. As we have explained, a given minimizing sequence can undergo lack of compactness: It can vanish in the sense that $\boldsymbol{m}(\boldsymbol{u}) = 0$, or it can split into several pieces as in Corollary 20. The main idea is to describe in a rather detailed way the behavior of minimizing sequences in the case of lack of compactness and to find what would be the total energy of the system when this happens. If we can show that the energy is above $I(\lambda)$, we arrive at a contradiction, hence the minimizing sequence must be precompact.

To deal with vanishing, we introduce a functional $\mathcal{E}^{\rm van}$ which is the original energy $\mathcal{E}$ to which all the subcritical terms which go to zero when $u_n \to 0$ in $L^p(\mathbb{R}^d)$ for $2 < p < 2^*$ (by Lemma 12) are removed. In the applications, $\mathcal{E}^{\rm van}$ usually only contains the gradient terms. One then defines the corresponding minimization principle $I^{\rm van}(\lambda) := \inf_{u \in \mathcal{S}(\lambda)} \mathcal{E}^{\rm van}(u)$. Another equivalent way to define $I^{\rm van}(\lambda)$ is as follows:

$$I^{\rm van}(\lambda) := \inf_{\substack{\boldsymbol{u} = \{u_n\} \subset \mathcal{S}(\lambda) \\ \boldsymbol{m}(\boldsymbol{u}) = 0}} \liminf_{n \to \infty} \mathcal{E}(u_n). \tag{20}$$

To deal with the phenomenon of splitting ("dichotomy"), we introduce an energy $\mathcal{E}^\infty$ which is the original energy $\mathcal{E}$ to which we remove all the compact terms that converge to zero when $u_n \rightharpoonup 0$ (but without assuming *a priori* that $u_n \to 0$ strongly in some $L^p(\mathbb{R}^d)$ space), thinking of $u_n = u(\cdot - n\vec{v})$. In practical examples, this usually means removing the potential energy terms when they involve an external potential V tending to zero at infinity, and to only keep the terms which are translation-invariant. The corresponding minimization principle reads $I^\infty(\lambda) := \inf_{u\in\mathcal{S}(\lambda)} \mathcal{E}^\infty(u)$. Another way to define $I^\infty(\lambda)$ is:

$$I^\infty(\lambda) := \inf_{\substack{\{u_n\}\subset\mathcal{S}(\lambda)\\ u_n\rightharpoonup 0}} \liminf_{n\to\infty} \mathcal{E}(u_n). \tag{21}$$

Note that $I^\infty(\lambda) \le I^{\rm van}(\lambda)$. By taking appropriate test functions, it can usually be proven that

$$\forall 0\le\lambda'\le\lambda, \qquad I(\lambda)\le I(\lambda-\lambda')+I^{\infty/{\rm van}}(\lambda').$$

If $I(0) = I^{\rm van}(0) = I^\infty(0) = 0$ (which we always assume), we deduce that $I(\lambda) \le I^{\infty/{\rm van}}(\lambda)$.

For models which are translation-invariant, we always get $\mathcal{E}^\infty = \mathcal{E}$. In this case we can only hope to prove the precompactness of minimizing sequences *up to translations* (that is, we want to show that there exists $\{x_k\}\subset\mathbb{R}^d$ and a subsequence such that $u_{n_k}(\,\cdot - x_k)$ is compact). We start by explaining the general strategy when $\mathcal{E}$ is translation-invariant.

The first step is to show that *vanishing does not occur*, by proving that

$$\boxed{I(\lambda) = I^\infty(\lambda) < I^{\rm van}(\lambda) \qquad \text{for all } \lambda>0.} \tag{22}$$

This is our first energetic inequality to be proven, usually by means of a well-chosen test function. This implies that $\boldsymbol{m}(\boldsymbol{u}) > 0$ for every minimizing sequence $\boldsymbol{u} = \{u_n\}$ of $I(\lambda)$. Then, by definition of $\boldsymbol{m}(\boldsymbol{u})$, there exists a subsequence (denoted the same for clarity) and translations $\boldsymbol{x}^{(1)} = \{x_n^{(1)}\}\subset\mathbb{R}^d$ such that $u_n(\,\cdot - x_n^{(1)}) \rightharpoonup u^{(1)} \ne 0$. Using Lemma 6 and its proof, we can write $u_n(\,\cdot - x_n^{(1)}) = u^{(1)} + r_n^{(2)}$. One then shows that

$$\mathcal{E}(u_n) = \mathcal{E}(u_n(\,\cdot - x_n^{(1)})) = \mathcal{E}(u^{(1)}) + \mathcal{E}(r_n^{(2)}) + {\rm o}(1)$$

using the splitting properties in (3), (4), (5) or similar results for other kinds of terms in the energy. Let us denote

$$\lambda^{(1)} := \int_{\mathbb{R}^d} |u^{(1)}|^2 > 0$$

and remark that

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}|r_n^{(2)}|^2 = \lambda-\lambda^{(1)}$$

by (3). Using

$$\mathcal{E}(r_n^{(2)}) \ge I\left(\int_{\mathbb{R}^d}|r_n^{(2)}|^2\right)$$

and passing to the limit (we need here the continuity of $\lambda\mapsto I(\lambda)$), we obtain

$$I(\lambda) \ge \mathcal{E}(u^{(1)}) + I(\lambda-\lambda^{(1)}) \ge I(\lambda^{(1)}) + I(\lambda-\lambda^{(1)}). \tag{23}$$

Since the converse inequality is always true, there must be equality:

$$I(\lambda) = \mathcal{E}(u^{(1)}) + I(\lambda-\lambda^{(1)}) = I(\lambda^{(1)}) + I(\lambda-\lambda^{(1)}).$$

In particular, $u^{(1)}$ is a minimizer of the problem corresponding to its own mass $I(\lambda^{(1)})$ and $\lim_{n\to\infty}\mathcal{E}(r_n^{(2)}) = I(\lambda-\lambda^{(1)})$, that is $\boldsymbol{r}^{(2)} = \{r_n^{(2)}\}$ is a minimizing sequence for $I(\lambda-\lambda^{(1)})$.

If we know already that

$$\boxed{I(\lambda) < I(\lambda') + I(\lambda-\lambda') \qquad \text{for all } 0<\lambda'<\lambda,} \tag{24}$$

then we conclude (since $\lambda^{(1)} > 0$) that $\lambda = \lambda^{(1)}$. Therefore $\{u_n\}$ is compact in $L^2(\mathbb{R}^d)$ and $u^{(1)}$ is the sought-after minimizer. The compactness in $H^1(\mathbb{R}^d)$ usually follows from the convergence $\mathcal{E}(u_n)\to\mathcal{E}(u^{(1)})$.

The "binding inequality" (24) is our second inequality to be proven in order to get compactness up to translations. In practice (24) may be hard to prove without more information on $I(\lambda')$ and $I(\lambda-\lambda')$. One possible strategy is to go on with our extraction of bubbles and apply the whole argument to $\boldsymbol{r}^{(2)}=\{r_n^{(2)}\}$, which is a minimizing sequence for $I(\lambda-\lambda^{(1)})$, assuming of course that $\lambda^{(1)}<\lambda$. The sequence $\boldsymbol{r}^{(2)}$ cannot vanish by (22), hence there is a $u^{(2)}\neq 0$ such that, up to a subsequence, $r_n^{(2)}(\,\cdot-x_n^{(2)})\rightharpoonup u^{(2)}\neq 0$. As before $u^{(2)}$ must be a minimizer of the problem corresponding to its own mass, $\lambda^{(2)}=\int_{\mathbb{R}^d}|u^{(2)}|^2>0$ and we can write $r_n^{(2)}=u^{(2)}+r_n^{(3)}$ where $\{r_n^{(3)}\}$ is a minimizing sequence for $I(\lambda-\lambda^{(1)}-\lambda^{(2)})$. We obtain

$$I(\lambda)=I(\lambda^{(1)})+I(\lambda^{(2)})+I(\lambda-\lambda^{(1)}-\lambda^{(2)})$$

where $I(\lambda^{(1)})$ and $I(\lambda^{(2)})$ admit $u^{(1)}$ and $u^{(2)}$ as minimizers. In principle one could go on and extract as many bubbles as possible (or even apply Lemma 6 directly). It is however usually enough to stop as soon as there is a contradiction, which is often the case when there are two bubbles. The argument is as follows. We have

$$I(\lambda)\leq I(\lambda^{(1)}+\lambda^{(2)})+I(\lambda-\lambda^{(1)}-\lambda^{(2)})\leq I(\lambda^{(1)})+I(\lambda^{(2)})+I(\lambda-\lambda^{(1)}-\lambda^{(2)})$$

and since the term on the left equals the one on the right, there must be equality everywhere. In particular

$$I(\lambda^{(1)}+\lambda^{(2)})=I(\lambda^{(1)})+I(\lambda^{(2)}).$$

In order to get a contradiction we have to prove the strict inequality. We have gained here that both $I(\lambda^{(1)})$ and $I(\lambda^{(2)})$ are known to have minimizers. Therefore we see that it suffices to prove (24) for any $\lambda>0$ **under the additional assumption that both $I(\lambda')$ and $I(\lambda-\lambda')$ admit minimizers**. In practice this last step requires to place the two minimizers far away at a distance R and to evaluate rather precisely the energy expansion as $R\to\infty$ in order to show that the two bubbles attract each other.

The previous method was devoted to the case of a translation-invariant system, $\mathcal{E}=\mathcal{E}^\infty$. When $\mathcal{E}\neq\mathcal{E}^\infty$ (for instance when there is an external potential V), there is an additional step at the beginning of the method. One start by showing that

$$\boxed{I(\lambda)<I^\infty(\lambda)\qquad\text{for all }\lambda>0}$$

which implies that a given minimizing sequence $\boldsymbol{u}=\{u_n\}$ cannot have a vanishing weak limit up to subsequences: $u_n\rightharpoonup u^{(1)}\neq 0$. Here there is no space translation at the first step. We then go on as before and write $u_n=u^{(1)}+r_n^{(2)}$ and show that the energy splits as

$$\mathcal{E}(u_n)=\mathcal{E}(u^{(1)})+\mathcal{E}^\infty(r_n^{(2)})+\mathrm{o}(1).$$

Note that $r_n^{(2)}\rightharpoonup 0$ hence the local terms disappear and we get $\mathcal{E}^\infty$. Arguing as before we find $I(\lambda)=I(\lambda^{(1)})+I^\infty(\lambda-\lambda^{(1)})$. The rest of the proof is similar to what we have said before and the binding inequality that must be proven now reads:

$$\boxed{I(\lambda^{(1)}+\lambda^{(2)})<I(\lambda^{(1)})+I^\infty(\lambda^{(2)})}$$

where $I(\lambda^{(1)})$ and $I^\infty(\lambda^{(2)})$ can both be assumed to have minimizers.

For many concrete examples of the strategy described in this section, see [22, 23].

5. Application: existence of Gagliardo-Nirenberg optimizers

In this section we explain how to use the previous strategy in order to show the existence of optimizers for the Gagliardo-Nirenberg inequality.[1]. We recall that we

[1] The original proof of existence is due to Weinstein [34] who used that one can restrict the problem to radial functions by rearrangement inequalities [18] and then a compactness lemma of Strauss [29] for radial functions in $H^1(\mathbb{R}^d)$. Here we give a direct argument inspired of [22, 23]

have

$$\|u\|_{L^p(\mathbb{R}^d)}^2 \leq C_{d,p} \left(\int_{\mathbb{R}^d} |u(x)|^2\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx\right)^{1-\theta} \tag{25}$$

where

$$2 \leq p < 2^* = \begin{cases} \infty & \text{for } d = 1, 2, \\ \frac{2d}{d-2} & \text{for } d \geq 3, \end{cases} \qquad \theta = \frac{2d-(d-2)p}{2p}.$$

In our convention $C_{d,p}$ is the *best constant* in (25), which means that

$$C_{d,p} = \sup_{u \in H^1(\mathbb{R}^d)} \frac{\|u\|_{L^p(\mathbb{R}^d)}^2}{\|u\|_{L^2(\mathbb{R}^d)}^{2\theta}\,\|\nabla u\|_{L^2(\mathbb{R}^d)}^{2(1-\theta)}} \tag{26}$$

The norms are all raised to the power two to simplify some later expressions. For $p = 2$ we simply have $\theta = 1$ and $C_{d,p} = 1$ and the inequality is an equality for every function. Our goal in this section is to investigate the existence and uniqueness of optimizers for $2 < p < 2^*$. The result stays valid for $p = 2^*$ in dimensions $d \geq 3$ but the existence in this critical case requires more advanced arguments discussed in the next section.

Theorem 22 (Optimizers for the Gagliardo-Nirenberg inequality)**.** *Let $2 < p < 2^*$ in dimension $d \geq 1$. Then the supremum in (26) is attained. The optimizers are unique up to translations, dilations and multiplication by a constant. That is, they all take the form*

$$x \mapsto aQ(bx - z) \tag{27}$$

for some $a, b \neq 0$ and $z \in \mathbb{R}^d$, where Q is the unique *positive radial solution to the nonlinear Schrödinger equation*

$$-\Delta Q - Q^{p-1} + Q = 0 \tag{28}$$

in $L^2(\mathbb{R}^d)$.

Here we only discuss the existence part of the theorem and give some short comments and references about the rest of the statement at the end of the section. It is possible to study directly the optimization problem (26) in the form of a quotient (see for instance [34, 9]) but we prefer to use extensive quantities, that is, to only involve integrals raised to the power one. Those behave better when a function splits into two bubbles. We introduce the following auxiliary minimization problem:

$$\boxed{I(\lambda) := \inf_{\substack{u \in H^1(\mathbb{R}^d) \\ \int_{\mathbb{R}^d} |u(x)|^p\,dx = \lambda}} \left\{\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx + \int_{\mathbb{R}^d} |u(x)|^2\,dx\right\}.} \tag{29}$$

The parameter λ is redundant and its role will appear more clearly later. Writing $u = \lambda^{1/p} v$ we immediately conclude that

$$\boxed{I(\lambda) = \lambda^{\frac{2}{p}} I(1).} \tag{30}$$

In addition $I(\lambda)$ has minimizers if and only $I(1)$ has some and they are related by the previous simple multiplication.

The following result says that the minimization problem $I(\lambda)$ in (29) is just the original Gagliardo-Nirenberg optimization problem (26), written in a different manner.

Lemma 23. *We have*

$$I(1) = \frac{1}{\theta^\theta (1-\theta)^{1-\theta} C_{d,p}}. \tag{31}$$

where $C_{d,p}$ is the Gagliardo-Nirenberg constant in (26)*. In addition, minimizers for $I(1)$ coincide exactly with the optimizers for* (26)*, appropriately rescaled.*

Proof. From Hölder's inequality for numbers we have

$$a+b \geq \frac{a^\theta b^{1-\theta}}{\theta^\theta(1-\theta)^{1-\theta}}$$

and therefore we obtain

$$I(1) \geq \frac{\|u\|_{L^2(\mathbb{R}^d)}^{2\theta}\|\nabla u\|_{L^2(\mathbb{R}^d)}^{2(1-\theta)}}{\theta^\theta(1-\theta)^{1-\theta}} \geq \frac{1}{\theta^\theta(1-\theta)^{1-\theta}C_{d,p}}. \tag{32}$$

This proves in particular that $I(1) > 0$. Let us fix any $u \in H^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|u(x)|^p\,dx = 1$ and define $u_\alpha(x) = \alpha^{d/p}u(\alpha x)$ which is still normalized in $L^p(\mathbb{R}^d)$. By definition of $I(1)$, we obtain

$$I(1) \leq \alpha^{2\theta}\int_{\mathbb{R}^d}|\nabla u(x)|^2\,dx + \alpha^{-2(1-\theta)}\int_{\mathbb{R}^d}|u(x)|^2\,dx.$$

If we minimize over α, we find

$$I(1) \leq \frac{\|u\|_{L^2(\mathbb{R}^d)}^{2\theta}\,\|\nabla u\|_{L^2(\mathbb{R}^d)}^{2(1-\theta)}}{\theta^\theta(1-\theta)^{1-\theta}},$$

with the optimal choice

$$\alpha^2 = \frac{1-\theta}{\theta}\frac{\|u\|_{L^2(\mathbb{R}^d)}^2}{\|\nabla u\|_{L^2(\mathbb{R}^d)}^2}.$$

Minimizing over u, this proves that

$$I(1) \leq \frac{1}{\theta^\theta(1-\theta)^{1-\theta}C_{d,p}}.$$

Recalling (32), we obtain the stated equality (31). Our proof also shows how to relate the optimizers of the two problems via the above rescaling. $\square$

We now discuss the existence of minimizers for $I(\lambda)$, using the techniques developed in the previous sections.

Lemma 24 (Existence)**.** *The minimization problem $I(\lambda)$ admits minimizers for all $\lambda > 0$. All the minimizing sequences are precompact in $H^1(\mathbb{R}^d)$, up to translations.*

Proof. It suffices to consider $I(1)$ by (31). Consider a minimizing sequences $\boldsymbol{u} = \{u_n\}$ for $I(1)$. Then $\|u_n\|_{H^1(\mathbb{R}^d)}^2 \to I(1)$ and therefore u_n is bounded in $H^1(\mathbb{R}^d)$. Define $\boldsymbol{m}(\boldsymbol{u})$ as in the previous sections. By definition of $I(1)$ we have $\|u_n\|_{L^p(\mathbb{R}^d)} = 1$ which can therefore not tend to zero. We conclude from Lemma 12 that $\boldsymbol{m}(\boldsymbol{u}) > 0$, that is, the sequence cannot vanish. Therefore there exists a subsequence n_k and space translations $x_k \in \mathbb{R}^d$ such that $u_{n_k}(\cdot + x_k) \rightharpoonup u \neq 0$ weakly in $H^1(\mathbb{R}^d)$. Since our problem is translation-invariant, we can assume without loss of generality that $x_k \equiv 0$. For simplicity of notation we also assume that $u_n \rightharpoonup u$ weakly, that is, we replace the original sequence by the subsequence. From the local strong compactness we may as well assume that $u_n(x) \to u(x)$ almost everywhere and strongly in $L^2_{\rm loc}(\mathbb{R}^d)$. From (5) in Lemma 6 we have

$$1 = \int_{\mathbb{R}^d}|u_n(x)|^p\,dx = \underbrace{\int_{\mathbb{R}^d}|u(x)|^p\,dx}_{:=\lambda} + \int_{\mathbb{R}^d}|u_n(x)-u(x)|^p\,dx + o(1) \tag{33}$$

and therefore, using (3) and (4),

$$\begin{aligned}\|u_n\|^2_{H^1(\mathbb{R}^d)} &= \|u\|^2_{H^1(\mathbb{R}^d)} + \|u_n - u\|^2_{H^1(\mathbb{R}^d)} + o(1)\\ &\geq \|u\|^2_{H^1(\mathbb{R}^d)} + I\left(\int_{\mathbb{R}^d} |u_n - u|^p\right) + o(1)\\ &\geq \|u\|^2_{H^1(\mathbb{R}^d)} + I(1-\lambda) + o(1).\end{aligned}$$

In the third line we have used that $\int_{\mathbb{R}^d} |u_n - u|^p \to 1-\lambda$ and the continuity of I which follows from the explicit formula (30). Passing to the limit, we obtain

$$I(1) \geq \|u\|^2_{H^1(\mathbb{R}^d)} + I(1-\lambda) \geq I(\lambda) + I(1-\lambda) = I(1)\left(\lambda^{\frac{2}{p}} + (1-\lambda)^{\frac{2}{p}}\right). \tag{34}$$

The right side is always greater than $I(1) > 0$ for $\lambda \in (0,1)$, which cannot be. On the other hand we already know that $\lambda > 0$ since $u \neq 0$. We conclude that $\lambda = 1$ and the series of inequalities in (34) then imply that $I(1) = \|u\|^2_{H^1(\mathbb{R}^d)}$, that is, u is an optimizer. Also, we have $\|u_n\|^2_{H^1(\mathbb{R}^d)} \to \|u\|^2_{H^1(\mathbb{R}^d)}$, hence the convergence is strong in $H^1(\mathbb{R}^d)$. □

Once we know that $I(1)$ has a minimizer u, usual perturbations arguments show that satisfies the nonlinear equation

$$-\Delta u + u = I(1)u^{p-1}. \tag{35}$$

Here $I(1)$ plays a role of a Lagrange multiplier due to the constraint $\int_{\mathbb{R}^d} |u|^p = 1$. In addition, we remark that $|u|$ is a minimizer if u is one, hence we may always assume that $u \geq 0$. Letting $u = \alpha Q$ with $\alpha = I(1)^{1/(2-p)}$ we see that Q solves (28). The uniqueness (up to translations) of non-negative solutions to the equation (28) is proved in [8, 13, 27, 32, 9].

The results from all the previous sections easily generalize to $H^s(\mathbb{R}^d)$, at the expense of using localization methods for fractional Laplacians as described for instance in [14, App. A] and [15, App. B].

Exercise 25. *Let u be an optimizer for $I(1)$ and define $u(t) := (u+t\chi)\|u+t\chi\|^{-1}_{L^p(\mathbb{R}^d)}$ with $\chi \in H^1(\mathbb{R}^d)$. Expand the $H^1(\mathbb{R}^d)$ norm of $u(t)$ to first order in t and deduce that u must solve (35) in $H^{-1}(\mathbb{R}^d)$.*

Exercise 26 (The NLS functional)**.** *Let us consider the minimization problem*

$$J(\lambda) := \inf_{\substack{u\in H^1(\mathbb{R}^d)\\ \int_{\mathbb{R}^d}|u|^2=\lambda}} \left\{\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx - \int_{\mathbb{R}^d} |u(x)|^p\,dx\right\}.$$

Show that $J(\lambda) > -\infty$ for $2 \leq p < 2+4/d$ but that $J(\lambda) = -\infty$ for $2+4/d < p < 2^*$. In the first case, compute $J(\lambda)$ in terms of the Gagliardo-Nirenberg constant $C_{d,p}$. Prove finally that $J(\lambda)$ admits a minimizer and that all minimizing sequences are precompact, by following a similar strategy as for $I(\lambda)$.

6. The Sobolev inequality and its optimizers

The arguments used to prove the existence of optimizers for the Gagliardo-Nirenberg inequality do not apply to the critical case $p = 2^* = 2d/(d-2)$ in dimensions $d \geq 3$. The Sobolev inequality has an additional scaling invariance due to dilations and, since it does not involve the L^2 norm at all, it is not possible to work in $H^1(\mathbb{R}^d)$. Our goal in this section is to provide a version of Lemma 11 in the critical case and this will force us to take dilations into account, in addition to translations. But before we start with a simple proof of the Sobolev inequality taken from [7], which will guide us in our reasoning.

6.1. **The Sobolev inequality.**

Theorem 27 (Sobolev inequality)**.** *For every $s > 2d$ in dimension $d \geq 1$, there exists a constant $S_{d,s} < \infty$ such that*

$$\|u\|^2_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \leq S_{d,s} \int_{\mathbb{R}^d} |k|^{2s} |\widehat{u}(k)|^2 \, dk, \tag{36}$$

for every $u \in \mathscr{S}'(\mathbb{R}^d)$ such that $|\{|u| \geq \lambda\}| < \infty$ for all $\lambda > 0$.

Later we use the convention that $S_{d,s}$ is the best constant in this inequality. For $s = 1$ we have $S_{d,s} = C_{d,2^*}$, the constant introduced in (26).

Proof. For shortness we denote

$$K := \left(\int_{\mathbb{R}^d} |k|^{2s} |\widehat{u}(k)|^2 \, dk \right)^{1/2} := \|u\|_{\dot{H}^s(\mathbb{R}^d)} .$$

We start by writing

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2s}} \, dx &= \frac{d-2s}{d+2s} \int_{\mathbb{R}^d} \int_0^\infty \lambda^{\frac{2d}{d-2s}-1} \mathbb{1}(|u(x)| \geq \lambda) \, d\lambda \, dx \\ &= \frac{d-2s}{d+2s} \int_0^\infty \lambda^{\frac{2d}{d-2s}-1} |\{|u(x)| \geq \lambda\}| \, d\lambda. \end{aligned} \tag{37}$$

Now we will give an estimate on $|\{|u(x)| \geq \lambda\}|$ for any fixed λ. We write $u = v + w$ where $\widehat{v}(k) = \widehat{u}(k)\chi(k/a)$, with a a parameter depending on λ to be determined later and $0 \leq \chi \leq 1$ a radial localizing function satisfying $\chi_{|B_1} \equiv 1$ and $\chi_{|\mathbb{R}^d \setminus B_2} \equiv 0$.

We use that

$$|\{|u(x)| \geq \lambda\}| \leq |\{|v(x)| \geq \lambda/2\}| + |\{|w(x)| \geq \lambda/2\}|$$

and choose a to ensure $\|v\|_{L^\infty} \leq \lambda/2$ which will give $|\{|v(x)| \geq \lambda/2\}| = 0$. Indeed, we have

$$\begin{aligned} \|v\|_{L^\infty} &\leq (2\pi)^{-d} \int_{|k| \leq a} |\widehat{u}(k)| \, dk \\ &\leq (2\pi)^{-d} \left(\int_{|k| \leq a} \frac{dk}{|k|^{2s}} \right)^{1/2} \left(\int_{|k| \leq a} |k|^{2s} |\widehat{u}(k)|^2 dk \right)^{1/2} \\ &\leq C \, a^{\frac{d-2s}{2}} \, \|u\|_{\dot{H}^s(\mathbb{R}^d)} = C \, K \, a^{\frac{d-2s}{2}}, \end{aligned} \tag{38}$$

which suggests to take $a^{\frac{d-2s}{2}} = \lambda/(2CK) \iff a = C'(\lambda/K)^{\frac{2}{d-2s}}$. For the term involving w, we write

$$|\{|w(x)| \geq \lambda/2\}| \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^d} |w|^2 \leq \frac{1}{\lambda^2} \int_{|k| \geq a} |\widehat{u}|^2.$$

Coming back to (37), this gives

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2s}} \, dx &\leq C \int_0^\infty \lambda^{\frac{2d}{d-2s}-1} \frac{1}{\lambda^2} \int_{|k| \geq a} |\widehat{u}|^2 \, d\lambda \\ &\leq C \int_0^\infty \lambda^{\frac{6s-d}{d-2s}} \int_{|k| \geq C'(\lambda/K)^{\frac{2}{d-2s}}} |\widehat{u}|^2 \, dk \, d\lambda \\ &= CK^{\frac{4s}{d-2s}} \int |k|^{2s} |\widehat{u}|^2 \, dk = CK^{\frac{2d}{d-2s}}. \end{aligned}$$

$\square$

6.2. **Lack of compactness in $\dot{H}^s(\mathbb{R}^d)$.** In order to adapt the arguments used before in the subcritical case, we now have to deal with the lack of compactness due to dilations, in addition to translations. This leads us to introducing the following concept. For a bounded sequence $\boldsymbol{u} = \{u_n\}$ in $L^{p^*}(\mathbb{R}^d)$ with $p^* = \frac{2d}{d-2s}$, we introduce the highest (critical) mass that weak limits can have, up to translations, dilations, and extraction of a subsequence:

$$\boldsymbol{m}'(\boldsymbol{u}) = \left\{ \int_{\mathbb{R}^d} |u|^{p^*} \ :\ \alpha_k^{-d/p^*} u_{n_k}\left(\frac{\cdot + x_k}{\alpha_k}\right) \rightharpoonup u \right\}. \tag{39}$$

The following is the corresponding adaptation of Lemma 11 to the critical case.

Lemma 28 (A critical estimate involving $\boldsymbol{m}'(\boldsymbol{u})$)**.** *Let $d > 2s > 0$. We have for every bounded sequence $\boldsymbol{u} = \{u_n\} \subset \dot{H}^s(\mathbb{R}^d)$*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^d} |u_n|^{\frac{2d}{d-2s}} \leq C\, \boldsymbol{m}'(\boldsymbol{u})^{\frac{4s}{d-2s}} \limsup_{n\to\infty} \|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)}. \tag{40}$$

Proof. We follow the argument used in the proof of Theorem 27 and only give a different estimate of $\|v_n\|_{L^\infty}$ with $\widehat{v_n} = \widehat{u_n}\,\chi(\cdot/a)$, in terms of $\boldsymbol{m}'(\boldsymbol{u})$ instead of K. Indeed, we have

$$\begin{aligned}
\|v_n\|_{L^\infty} &= (2\pi)^{-d/2} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} a^d \check{\chi}(ay) u_n(x-y)\, dy \right| \\
&= (2\pi)^{-d/2} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \check{\chi}(y) u_n\left(\frac{x-y}{a}\right) dy \right| \\
&\leq (2\pi)^{-d/2} a^{\frac{d-2s}{2}} \sup_{a>0} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \check{\chi}(y) \frac{1}{a^{d/p^*}} u_n\left(\frac{y+x}{a}\right) dy \right|.
\end{aligned}$$

In the last line we used that $\check{\chi}$ is radial. For every n, let a_n and x_n realizing the above supremum. Since the function

$$y \mapsto \frac{1}{a_n^{d/p^*}} u_n\left(\frac{y+x_n}{a_n}\right)$$

is bounded in $L^{p^*}(\mathbb{R}^d)$, we may extract a subsequence and assume that

$$\frac{1}{a_n^{d/p^*}} u_n\left(\frac{\cdot + x_n}{a_n}\right) \rightharpoonup u.$$

By definition of $\boldsymbol{m}'(u)$ we have $\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \boldsymbol{m}'(\boldsymbol{u})^{1/p^*}$. On the other hand,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \check{\chi}(y) \frac{1}{a_n^{\frac{d-2s}{2}}} u_n\left(\frac{y+x_n}{a_n}\right) dy = \int_{\mathbb{R}^d} \check{\chi} u$$

and therefore

$$\limsup_{n\to\infty} \|v_n\|_{L^\infty} \leq \|\check{\chi}\|_{L^{\frac{p^*}{p^*-1}}} a^{\frac{d-2s}{2}} \boldsymbol{m}'(\boldsymbol{u})^{1/p^*}.$$

The argument is then the same as before. □

The following is the equivalent of Lemma 12 in the sub-critical case.

Corollary 29 (Vanishing in the critical case)**.** *Let $\boldsymbol{u} = \{u_n\}$ be a bounded sequence in $\dot{H}^s(\mathbb{R}^d)$, with $d > 2s$. Then $\boldsymbol{m}'(\boldsymbol{u}) = 0$ if and only if $u_n \to 0$ strongly in $L^{p^*}(\mathbb{R}^d)$.*

A similar result has appeared first in [10].

6.3. **Existence of optimizers for the Sobolev inequality.** Using the above result, we can deduce the existence of an optimizer for the Sobolev inequality (36) in a few lines, using the exact same argument as in Section 5 in the case of sub-critical p.

Corollary 30 (Existence of a minimizer for the Sobolev inequality)**.** *The best constant $S_{d,s}$ in* (36) *is attained and all the minimizing sequences are precompact up to translations and dilations.*

Proof. We introduce the minimization problem similar to the sub-critical case (29):

$$I(\lambda) := \inf_{\int_{\mathbb{R}^d} |u|^{p^*}=\lambda} \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{\frac{2}{p^*}} I(1).$$

A minimizing sequence $\boldsymbol{u} = \{u_n\}$ must satisfy $\boldsymbol{m'}(\boldsymbol{u}) > 0$ by Lemma 28. Hence we can find a translation and a dilation such that, after extracting a subsequence,

$$\frac{1}{a_n^{d/p^*}} u_n \left(\frac{\cdot + x_n}{a_n}\right) \rightharpoonup u \neq 0.$$

Writing

$$u_n = a_n^{d/p^*} u\left(a_n(\cdot + x_n)\right) + r_n$$

we have

$$\int_{\mathbb{R}^d} |u_n|^{p^*} = \underbrace{\int_{\mathbb{R}^d} |u|^{p^*}}_{:=\lambda>0} + \int_{\mathbb{R}^d} |r_n|^{p^*} + o(1).$$

This is because a sequence bounded in $\dot{H}^s(\mathbb{R}^d)$ is also bounded in $H^s_{\rm loc}(\mathbb{R}^d)$ hence we may assume, after extracting a subsequence, that it converges almost everywhere, so as to apply the Brezis-Lieb convergence (6). We also have

$$\|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)} = \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|r_n\|^2_{\dot{H}^s(\mathbb{R}^d)} + o(1)$$

since $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space. Therefore,

$$\begin{aligned} I(1) = \lim_{n\to\infty} \|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)} &\geq \|u\|^2_{\dot{H}^s(\mathbb{R}^d)} + I(1-\lambda) \\ &\geq \left(\lambda^{\frac{2}{p^*}} + (1-\lambda)^{\frac{2}{p^*}}\right) I(1) \end{aligned}$$

which proves again that $\lambda = 1$ since $2/p^* < 1$. Hence u is an optimizer and $u_n \to u$ strongly in $\dot{H}^s(\mathbb{R}^d)$. □

Arguments of this type have been used for instance in [1]. It is possible to extend several of the results of the previous sections to the critical case (for instance the profile decomposition, due to Gérard [10]), with of course the addition of the possible lack of compactness due to dilations.

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