# Gibbs measures of nonlinear Schrödinger equations and many-body quantum mechanics

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# Classical mechanics and Gibbs measures

A Hamiltonian system consists of the following ingredients.

- Linear phase space $\Gamma \ni \phi$.
- Hamilton (or energy) function $H \in C^\infty(\Gamma)$.
- Poisson bracket $\{\cdot\,,\cdot\}$ on $C^\infty(\Gamma) \times C^\infty(\Gamma)$.

  (Properties: antisymmetric, bilinear, Leibnitz rule in both arguments, Jacobi identity.)

Classical dynamics is given by Hamiltonian flow $\phi \mapsto S^t \phi$ on $\Gamma$ defined by the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} f(S^t \phi) = \{H, f\}(S^t \phi)$$

for any $f \in C^\infty(\Gamma)$.

Standard example: classical system of $n$ degrees of freedom.

- Phase space $\Gamma = \mathbb{R}^{2n} \ni (p, q)$.
- Hamilton function $H(p, q) = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q)$.
- Poisson bracket $\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$.

Hamiltonian flow reads

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i = -\frac{\partial H}{\partial q_i} = -\partial_i V(q)\,, \qquad \frac{\mathrm{d}}{\mathrm{d}t} q_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i}\,.$$

The Gibbs measure at temperature $\beta$ is

$$\mathbb{P}(\mathrm{d}\phi) := \frac{1}{Z} \mathrm{e}^{-\beta H(\phi)}\, \mathrm{d}\phi\,, \qquad Z := \int \mathrm{e}^{-\beta H(\phi)}\, \mathrm{d}\phi\,.$$

$\mathbb{P}$ is invariant under the flow $S^t$.

# Nonlinear Schrödinger equations

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-dimensional torus.

- Phase space $\Gamma$ is some appropriate subspace of $\{\phi : \mathbb{T}^d \to \mathbb{C}\}$.
- Hamilton function

$$H(\phi) = \int \mathrm{d}x\, \bar{\phi}(x)(\kappa - \Delta)\phi(x) + \frac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, w(x-y)|\phi(x)|^2|\phi(y)|^2\,,$$

  where $\kappa > 0$.
- Poisson bracket

$$\{\phi(x), \bar{\phi}(y)\} = \mathrm{i}\delta(x-y)\,, \qquad \{\phi(x), \phi(y)\} = \{\bar{\phi}(x), \bar{\phi}(y)\} = 0\,.$$

Hamiltonian flow given by time-dependent nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \phi(x) = (\kappa - \Delta)\phi(x) + \int \mathrm{d}y\, w(x-y)|\phi(y)|^2 \phi(x)\,.$$

Time-dependent nonlinear Schrödinger equation

$$\mathrm{i}\partial_t \phi(x) = (\kappa - \Delta)\phi(x) + \int \mathrm{d}y \, w(x-y)|\phi(y)|^2 \phi(x) \,. \tag{1}$$

Gibbs measure of nonlinear Schrödinger equation is formally

$$\mathbb{P}(\mathrm{d}\phi) = \frac{1}{Z} \mathrm{e}^{-H(\phi)} \mathrm{d}\phi \,.$$

Formally, $\mathbb{P}$ is invariant under the flow generated by (1).

Rigorous results: Lebowitz–Rose–Speer, Bourgain, Bourgain–Bulut, Tzvetkov, Thomann–Tzvetkov, Nahmod–Oh–Rey-Bellet–Staffilani, Oh–Quastel, Deng–Tzvetkov–Visciglia, Cacciafesta–de Suzzoni, Genovese–Lucá–Valeri, ...

Important application: $\mathbb{P}$-almost sure well-posedness of (1) for rough initial data.

# Goal

Analyse $\mathbb{P}$ and $(S^t)$ through

- the moments of $\mathbb{P}$ (which determine $\mathbb{P}$),
- the time-dependent correlation functions

$$\int X^1(S^{t_1}\phi) \cdots X^m(S^{t_m}\phi)\, \mathrm{d}\mathbb{P}(\phi)\,,$$

for $X^i \in C^\infty(\Gamma)$ and $t_i \in \mathbb{R}$.

Derivation as a (high-temperature) limit of a microscopic $n$-body quantum theory of bosons.

# Rigorous construction of Gibbs measure

Spectral decomposition

$$\kappa - \Delta = \sum_{k\in\mathbb{N}} \lambda_k u_k u_k^* \,, \qquad \lambda_k > 0\,, \qquad \|u_k\|_{L^2} = 1\,.$$

Let $\omega = (\omega_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ be i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$ random variables with joint law $\mu_0$.

Define the Gaussian free field

$$\phi^\omega \equiv \phi := \sum_{k\in\mathbb{N}} \frac{\omega_k}{\sqrt{\lambda_k}} u_k \,.$$

The sum converges in $\|\phi\|_{\mathcal{H}^s} := \|(\kappa-\Delta)^{s/2}\phi\|_{L^2}$ in the sense of $L^p(\mu_0)$ for all $p \in (1,\infty)$, provided that

$$\sum_{k\in\mathbb{N}} \lambda_k^{s-1} < \infty\,.$$

For example,

$$\mathbb{E}^{\mu_0}\|\phi\|_{\mathcal{H}^s}^2 = \sum_{k\in\mathbb{N}} \mathbb{E}^{\mu_0}|\omega_k|^2 \frac{\lambda_k^s}{\lambda_k} = \sum_{k\in\mathbb{N}} \lambda_k^{s-1}\,.$$

Under $\mu_0$, $\phi = \sum_{k \in \mathbb{N}} \frac{\omega_k}{\sqrt{\lambda_k}} u_k$ is a Gaussian free field with covariance $(\kappa - \Delta)^{-1}$:

$$\mathbb{E}^{\mu_0} \langle f \, , \phi \rangle \langle \phi \, , g \rangle = \langle f \, , (\kappa - \Delta)^{-1} g \rangle \, .$$

We find that

$$\mu_0[\phi \in \mathcal{H}^0] = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } d > 1 \, . \end{cases}$$

Define the measure

$$\mu(\mathrm{d}\omega) := \frac{1}{Z} \mathrm{e}^{-W(\phi^\omega)} \mu_0(\mathrm{d}\omega) \, , \qquad W(\phi) = \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, w(x-y) |\phi(x)|^2 |\phi(y)|^2 \, .$$

$\mu$ is well-defined for instance if

$$d = 1 \, , \qquad w \in L^\infty \, , \qquad w \text{ positive definite} \, ,$$

since then $0 \leqslant W(\phi) < \infty$ $\mu_0$-a.s.

(Then $\mathbb{P}$ is defined as $\phi_* \mu$.)

# Quantum many-body theory

Define the one-particle space $\mathfrak{H} := L^2(\mathbb{T}^d;\mathbb{C})$ and the $n$-particle space

$$\mathfrak{H}^{(n)} := \mathfrak{H}^{\otimes_{\mathsf{sym}} n} = L^2_{\mathsf{sym}}\big((\mathbb{T}^d)^n\big)\,.$$

Hamilton operator

$$H^{(n)} := H_0^{(n)} + \lambda \sum_{1 \leqslant i < j \leqslant n} w(x_i - x_j)\,, \qquad H_0^{(n)} := \sum_{i=1}^{n} (\kappa - \Delta_{x_i})$$

Canonical thermal state at temperature $\tau > 0$ is $P_\tau^{(n)} := \mathrm{e}^{-H^{(n)}/\tau}$.

Expectation of an observable $A \in \mathfrak{B}(\mathfrak{H}^{(n)})$ is

$$\rho_\tau^{(n)}(A) := \frac{\mathrm{Tr}(A P_\tau^{(n)})}{\mathrm{Tr}(P_\tau^{(n)})}\,.$$

What happens as $n \to \infty$?

In order to obtain a nontrivial limit, we set $\lambda = 1/n$.

**Theorem** [Lewin-Nam-Serfaty-Solovej, 2012; Lewin-Nam-Rougerie, 2013]. For $\lambda = 1/n$ and $\tau$ fixed, the state $\rho_\tau^{(n)}(\cdot)$ converges to the atomic measure $\delta_\Phi$ in the sense of $p$-particle correlation functions (see later), where $\Phi$ is the minimizer of the energy function $H$.

Complete Bose-Einstein condensation for fixed $\tau$.

In order to obtain the Gibbs measure $\mu$, we need to let

- $\tau$ grow with $n$ (high-temperature limit),
- $n$ fluctuate. ($n/\tau$ will correspond to $\|\phi\|_2^2$.)

# High-temperature limit and Fock space

Define the Fock space $\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathfrak{H}^{(n)}$ and the grand canonical thermal state

$$P_\tau := \bigoplus_{n \in \mathbb{N}} P_\tau^{(n)} = \mathrm{e}^{-H_\tau}\,, \qquad H_\tau := \frac{1}{\tau} \bigoplus_{n \in \mathbb{N}} H^{(n)}\,.$$

Rescaled particle number operator $\mathcal{N}_\tau := \frac{1}{\tau} \bigoplus_{n \in \mathbb{N}} nI$. Expectation of an observable $A \in \mathfrak{B}(\mathcal{F})$ is

$$\rho_\tau(A) := \frac{\mathrm{Tr}(AP_\tau)}{\mathrm{Tr}(P_\tau)}\,.$$

Explicit computation for $d = 1$ and $\lambda = 0$:

$$\lim_{\tau \to \infty} \rho_\tau(\mathcal{N}_\tau^k) = \mathbb{E}^\mu \|\phi\|_{L^2}^{2k}\,, \qquad k = 1, 2, \dots\,.$$

Number of particles is of order $\tau$. Thus, set $\lambda := \tau^{-1}$ to obtain nontrivial interacting limit.

# Second quantization

Let $b, b^*$ be the bosonic annihilation and creation operators on $\mathcal{F}$ and set $\phi := \tau^{-1/2} b$. Hence,

$$[\phi_\tau(x), \phi_\tau^*(y)] = \frac{1}{\tau}\delta(x-y)\,, \qquad [\phi_\tau(x), \phi_\tau(y)] = [\phi_\tau^*(x), \phi_\tau^*(y)] = 0\,.$$

Thus, we can write $H_\tau = H_{\tau,0} + W$, where

$$H_{\tau,0} = \int \mathrm{d}x\, \phi_\tau^*(x)(\kappa - \Delta)\phi_\tau(x)\,,$$
$$W_\tau = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\phi_\tau^*(x)\phi_\tau(x)\,w(x-y)\,\phi_\tau^*(y)\phi_\tau(y)\,,$$

as well as $P_\tau = \mathrm{e}^{-H_\tau}$, $\rho_\tau(A) = \frac{\mathrm{Tr}(AP_\tau)}{\mathrm{Tr}(P_\tau)}$.

# High-temperature limit for $d = 1$

Define the $p$-particle reduced density matrix

$$\gamma_{\tau,p}(x_1, \dots, x_p; y_1, \dots, y_p) \;:=\; \rho_\tau\big(\phi_\tau^*(y_1) \cdots \phi_\tau^*(y_p)\phi_\tau(x_1) \cdots \phi_\tau(x_p)\big)\,.$$

Analogously, we define the classical $p$-particle correlation function

$$\gamma_p(x_1, \dots, x_p; y_1, \dots, y_p) \;:=\; \mathbb{E}^\mu\big(\bar{\phi}(y_1) \cdots \bar{\phi}(y_p)\phi(x_1) \cdots \phi(x_p)\big)\,.$$

The family $(\gamma_p)_{p \in \mathbb{N}}$ completely determines all moments of the field $\phi$.

**Theorem** [Lewin-Nam-Rougerie, 2015]. For $d = 1$ and $w$ positive definite, for any $p \in \mathbb{N}$ we have $\gamma_{\tau,p} \to \gamma_p$ in trace class as $\tau \to \infty$.

# Time-dependent correlations

Introduce Hamiltonian time evolution:

(Cl) for a random variable $X \equiv X(\phi)$ set $\Psi^t X(\phi) := X(S^t \phi)$;

(Qu) for an operator $\mathbf{X}$ on $\mathcal{F}$ set $\Psi^t_\tau \mathbf{X} := \mathrm{e}^{\mathrm{i}t\tau H_\tau} \mathbf{X} \, \mathrm{e}^{-\mathrm{i}t\tau H_\tau}$.

For a $p$-particle operator $\xi$ on $\mathfrak{H}^{(p)}$ introduce the observables

$$\text{(Cl)} \quad \Theta(\xi) := \int \mathrm{d}x_1 \cdots \mathrm{d}x_p \, \mathrm{d}y_1 \cdots \mathrm{d}y_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \times \bar{\phi}(x_1) \cdots \bar{\phi}(x_p) \phi(y_1) \cdots \phi(y_p) \, ;$$

$$\text{(Qu)} \quad \Theta_\tau(\xi) := \int \mathrm{d}x_1 \cdots \mathrm{d}x_p \, \mathrm{d}y_1 \cdots \mathrm{d}y_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \times \phi^*_\tau(x_1) \cdots \phi^*_\tau(x_p) \phi_\tau(y_1) \cdots \phi_\tau(y_p) \, .$$

**Theorem** [Fröhlich-K-Schlein-Sohinger, 2017]. Let $d = 1$ and $w \in L^\infty$ be pointwise positive. Given $m \in \mathbb{N}$, $p_1, \dots, p_m \in \mathbb{N}$, $\xi^1 \in \mathcal{L}(\mathfrak{H}^{(p_1)}), \dots, \xi^m \in \mathcal{L}(\mathfrak{H}^{(p_m)})$ and $t_1, \dots, t_m \in \mathbb{R}$, we have

$$\lim_{\tau \to \infty} \rho_\tau\big(\Psi_\tau^{t_1}\,\Theta_\tau(\xi^1) \cdots \Psi_\tau^{t_m}\,\Theta_\tau(\xi^m)\big) \;=\; \mathbb{E}^\mu\big(\Psi^{t_1}\,\Theta(\xi^1) \cdots \Psi^{t_m}\,\Theta(\xi^m)\big)\,.$$

Remarks:

- Also works on $\mathbb{R}$ instead of $\mathbb{T}$, with sufficiently confining potential $v$ in free Hamiltonian $\kappa - \Delta + v(x)$.
- We also prove that there exists a null sequence $\varepsilon = \varepsilon_\tau$ such that, with a quantum two-body potential $\frac{1}{\varepsilon} w\big(\frac{x}{\varepsilon}\big)$ the limit is that of the cubic NLS with local nonlinearity, $w = \delta$.

# Higher dimensions

If $d > 1$ then $\phi$ has $\mu_0$-a.s. negative regularity, $\phi \notin L^2$, since $\sum_{k \in \mathbb{N}} \lambda_k^{-1} = \infty$.

Consequences:

- $W(\phi) = \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, w(x-y) |\phi(x)|^2 |\phi(y)|^2$ ill-defined even for $w \in L^\infty$.
- $p$-particle correlation functions $\gamma_p$ are not in trace class, since

$$\mathrm{Tr}(\gamma_1) = \mathbb{E}^\mu \|\phi\|_{L^2}^2 = \infty \,.$$

- On the quantum side, rescaled number of particles $\mathcal{N}_\tau$ is no longer bounded. Explicit computation for noninteracting case $w = 0$:

$$\rho_\tau(\mathcal{N}_\tau) = \sum_{k \in \mathbb{N}} \frac{1}{\tau} \frac{1}{\mathrm{e}^{\lambda_k / \tau} - 1} \to \infty$$

  as $\tau \to \infty$. Quantum model has intrinsic cutoff at energies $\lambda_k \approx \tau$.

Heuristics:

Singularity of classical field $\quad \Longleftrightarrow \quad$ Rapid growth of number of particles .

# Renormalization

Renormalize interaction $W$ by Wick ordering. Formally, take

$$W(\phi) = \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, w(x-y)(|\phi(x)|^2 - \infty)(|\phi(y)|^2 - \infty) \, .$$

Rigorously, introduce truncated field and density

$$\phi_{[K]} := \sum_{k=0}^{K} \frac{\omega_k}{\sqrt{\lambda_k}} u_k \, , \qquad \varrho_{[K]}(x) := \mathbb{E}^{\mu_0} |\phi_{[K]}(x)|^2 \, .$$

Then

$$W_{[K]} := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, w(x-y) \big(|\phi_{[K]}(x)|^2 - \varrho_{[K]}(x)\big) \big(|\phi_{[K]}(x)|^2 - \varrho_{[K]}(x)\big)$$

has a limit in $\bigcap_{p<\infty} L^p(\mu_0)$ as $K \to \infty$, denoted by $W$.

Use this $W$ in definition of $\mu$.

Similarly, we need to renormalize the quantum interaction. The quantum Gibbs state is defined by the renormalized many-body Hamiltonian $H_\tau = H_{\tau,0} + W_\tau$, where

$$W_\tau := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, \big(\phi_\tau^*(x)\phi_\tau(x) - \varrho_\tau(x)\big) \, w(x-y) \, \big(\phi_\tau^*(y)\phi_\tau(y) - \varrho_\tau(y)\big) \,,$$

where the quantum density at $x$ $\varrho_\tau(x)$ is defined as

$$\varrho_\tau(x) \; := \; \rho_{\tau,0}\big(\phi_\tau^*(x)\phi_\tau(x)\big) \,.$$

# Convergence of moments for $d = 2, 3$

For technical reasons, instead of $P_\tau = \mathrm{e}^{-H_{\tau,0} - W_\tau}$, we consider a family of modified thermal quantum states

$$P_\tau^\eta \;:=\; \mathrm{e}^{-\eta H_{\tau,0}} \mathrm{e}^{-(1-2\eta) H_{\tau,0} - W_\tau} \mathrm{e}^{-\eta H_{\tau,0}}\,, \qquad \eta \in [0,1)\,.$$

**Theorem** [Fröhlich-K-Schlein-Sohinger, 2016]. Let $d = 2, 3$, $w \in L^\infty$ positive definite, $\eta > 0$, and $p \in \mathbb{N}$. Then $\gamma_{\tau,p}^\eta \to \gamma_p$ in Hilbert-Schmidt as $\tau \to \infty$.

Recent developments:

- [Sohinger, 2019] optimal integrability conditions [Bourgain, 1997] on $w$: $w \in L^1$ $(d = 1)$, $w \in L^{1+}$ $(d = 2)$, $w \in L^{3+}$ $(d = 3)$.
- [Lewin-Nam-Rougerie, 2018] $\eta = 0$ for smooth $w$ and $d = 2$.

# Counterterm problem

Also works on $\mathbb{R}^d$ with sufficiently confining potential $V$. Relation between original and renormalized problems is nontrivial. True Hamiltonian

$$\widetilde{H}_\tau := \int \mathrm{d}x\,\mathrm{d}y\,\phi_\tau^*(x)\big(\nu - \Delta + V\big)(x;y)\phi_\tau(y) + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\phi_\tau^*(x)\phi_\tau^*(y)\,w(x-y)\,\phi_\tau(x)\phi_\tau(y)$$

compared with renormalized Hamiltonian (from above),

$$H_\tau = \int \mathrm{d}x\,\phi_\tau^*(x)(\kappa - \Delta + v_\tau)\phi_\tau(x) + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\big(\phi_\tau^*(x)\phi_\tau(x) - \varrho_\tau(x)\big)\,w(x-y)\,\big(\phi_\tau^*(y)\phi_\tau(y) - \varrho_\tau(y)\big)$$

Essentially, $\widetilde{H}_\tau$ and $H_\tau$ are related by a shift in a diverging chemical potential, provided that one chooses the bare one-body potential $v_\tau$ appropriately (depending on $\tau$).

More precisely, for any constant $\bar{\varrho}_\tau \in \mathbb{R}$ we have

$$\widetilde{H}_\tau \;=\; H_\tau + \left[\bar{\varrho}_\tau \hat{w}(0) - \frac{1}{2\tau} w(0) + \nu - \kappa\right] \mathcal{N}_\tau - \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, \varrho_\tau(x) w(x-y) \varrho_\tau(y)\,,$$

provided that $v_\tau$ solves the counterterm problem

$$v_\tau = V + w * (\varrho_\tau^{v_\tau} - \bar{\varrho}_\tau)\,. \tag{2}$$

For $\varrho_\tau^{v_\tau} - \bar{\varrho}_\tau$ to remain bounded, we need $\lim_{\tau\to\infty} \bar{\varrho}_\tau = \infty$, and hence (for bracket to vanish) $\lim_{\tau\to\infty} \nu = -\infty$. (Compensates large repulsive interaction energy.)

The counterterm problem (2) is solved in [FKSS, 2016], where we also show that the solution $v_\tau$ converges (in a suitable space) to some $v =$ the correct renormalized external potential.

# Morsels of proof

Basic approach: perturbative expansion of partition functions $\mathbb{E}^{\mu_0} \mathrm{e}^{-zW}$ and $\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0} - zW_\tau})$ in powers of $z$. Well-defined for $\mathrm{Re}\, z \geqslant 0$ but ill-defined for $\mathrm{Re}\, z < 0$: zero radius of convergence around $z = 0$.

Toy problem:

$$A(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-x^2/2}\, \mathrm{e}^{-zx^4}\,;$$

analytic for $\mathrm{Re}\, z > 0$ but zero radius of convergence, with Taylor coefficient $a_m = A^{(m)}(0)/m! \sim m!$.

However, Taylor series $\sum_{m \geqslant 0} a_m z^m$ has Borel transform $B(z) := \sum_{m \geqslant 0} \frac{a_m}{m!} z^m$ with positive radius of convergence. Formally, we can recover $A$ from

$$A(z) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} B(tz)\,.$$

Works provided we can prove good enough bounds on Taylor coefficients and remainder term of $A$ (Sokal, 1980).

Main work: control of the coefficients and remainder of quantum many-body problem. Starting point for algebra is Wick's theorem for the free states.

For time-dependent problem, we perform an expansion of $\Psi_\tau^t \Theta_\tau(\xi) = \mathrm{e}^{\mathrm{i}t\tau H_\tau} \Theta_\tau(\xi)\, \mathrm{e}^{-\mathrm{i}t\tau H_\tau}$ in powers of the interaction potential $w$.

The expansion is controlled graphically, tree graphs sum up precisely to the quantization of $\Psi^t \Theta(\xi)$.

Problem: expansion is only convergent on sector of $\mathcal{F}$ where $\mathcal{N}_\tau$ is bounded.

Introduce cutoff in rescaled number of particles $\mathcal{N}_\tau$. Need to show that for $f \in C_c^\infty(\mathbb{R})$ we have

$$\lim_{\tau \to \infty} \rho_\tau\big(\Theta_\tau(\xi) f(\mathcal{N}_\tau)\big) \;=\; \mathbb{E}^\mu\big(\Theta(\xi) f(\mathcal{N})\big) \qquad (3)$$

Problem: cutoff breaks Gaussianity, and Wick's theorem does not apply to (3).

Idea: using complex analysis, it suffices to analyse $\rho_\tau\big(\Theta_\tau(\xi)\mathrm{e}^{-\mathcal{N}_\tau}\big)$ for fixed $\nu > 0$.