Nonlinear Gibbs measures as the limit of equilibrium quantum Bose gases

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Ideal Bose gas: homogeneous case

Gibbs state at a positive temperature $T > 0$

$$\Gamma_0 = \mathcal{Z}_0^{-1} e^{-\frac{\mathbb{H}_0}{T}}, \quad \mathbb{H}_0 = \int_\Omega a_x^*(-\Delta_x - \nu) a_x dx, \quad [a_x, a_y^*] = \delta(x-y)$$

In the **thermodynamic limit**

$$\Omega = L\mathbb{T}^d, \quad \nu = -\kappa L^{-2}, \quad L \to \infty$$

we approach **critical density** from below

$$\frac{\langle \mathcal{N} \rangle_{\Gamma_0}}{L^d} = \frac{1}{L^d} \sum_{k \in 2\pi\mathbb{Z}^d} \frac{1}{e^{\frac{k^2+\kappa}{L^2 T}} - 1} \quad \nearrow \quad \rho_c(T) = \begin{cases} \int_{\mathbb{R}^3} \frac{T^{\frac{d}{2}}}{e^{|2\pi k|^2}-1} dk & d = 3 \\ +\infty & d = 1, 2 \end{cases}$$

Equivalently, we can fix density and approach **critical temperature** from above

Equivalently, we can consider the **rescaled model**

$$\Omega = \mathbb{T}^d, \quad \nu = -\kappa, \quad T \to \infty$$

$$\langle \mathcal{N} \rangle_{\Gamma_0} = \sum_{k \in 2\pi\mathbb{Z}^d} \frac{1}{e^{\frac{k^2+\kappa}{T}} - 1} \sim \begin{cases} T & \text{in } d = 1 \\ T \log T & \text{in } d = 2 \\ T^{3/2} & \text{in } d = 3 \end{cases}$$

Ideal Bose gas: general case

Gibbs state at a positive temperature $T > 0$

$$\Gamma_0 = \mathcal{Z}_0^{-1} e^{-\frac{\mathbb{H}_0}{T}}, \quad \mathbb{H}_0 = \int_\Omega a_x^* h_x a_x dx, \quad h > 0 \text{ on } L^2(\Omega)$$

Theorem (Density matrices of ideal Bose gas)

If $\mathrm{Tr}(h^{-p}) < \infty$, then $\forall k \geq 1$, $\Gamma_0^{(k)}(x_1, ..., x_k; y_1, ..., y_k) = \langle a_{x_1}^* ... a_{x_k}^* a_{y_1} ... a_{y_k} \rangle_{\Gamma_0}$

$$\frac{k!}{T^k} \Gamma_0^{(k)} = \frac{k!}{T^k} \left(\frac{1}{e^{h/T} - 1} \right)^{\otimes k} \longrightarrow k!(h^{-1})^{\otimes k} = \int |u^{\otimes k}\rangle \langle u^{\otimes k}| d\mu_0(u)$$

strongly in Schatten space $\mathfrak{S}^p$, with μ_0 Gaussian measure with covariance h^{-1}

$$d\mu_0(u) = \text{“} z_0^{-1} e^{-\langle u, hu \rangle} du \text{”} := \bigotimes_{i=1}^{\infty} \left(\frac{\lambda_i}{\pi} e^{-\lambda_i |\langle u_i, u \rangle|^2} d\langle u_i, u \rangle \right), \quad h = \sum_{i=1}^{\infty} \lambda_i |u_i\rangle \langle u_i|,$$

which is supported on Sobolev-type space $H^{1-p} = D(h^{\frac{1-p}{2}})$.

Example: $h = -\Delta + |x|^s$ on $L^2(\mathbb{R}^d)$, then $\mathrm{Tr}(h^{-p}) < \infty$ with $p > \frac{d}{2} + \frac{d}{s}$.

- $d = 1, s > 2$: $\mathrm{Tr}(h^{-1}) < \infty$, μ_0 supported on L^2
- $d \geq 2$: μ_0 always supported on **negative** Sobolev spaces
- $d \leq 3$ needed for $\mathrm{Tr}(h^{-2}) < \infty$

Interacting Bose gas

Gibbs state at a positive temperature $T > 0$ with pair interaction $\lambda w(x-y)$

$$\Gamma_\lambda = \mathcal{Z}_\lambda^{-1} e^{-\frac{\mathbb{H}_\lambda}{T}}, \quad \mathbb{H}_\lambda = \int_\Omega a_x^* h_x a_x dx + \frac{\lambda}{2} \iint_{\Omega^2} a_x^* a_y^* w(x-y) a_x a_y dxdy$$

Heuristically, **mean-field limit** $\lambda \sim T^{-1}$ leads to **semiclassical approximation**

$$\frac{\mathbb{H}_\lambda}{T} = \int_\Omega b_x^* h_x b_x dx + \frac{1}{2} \iint_{\Omega^2} b_x^* b_y^* w(x-y) b_x b_y dxdy, \quad b_x = \frac{a_x}{\sqrt{T}}$$

$$[b_x, b_y^*] \approx 0 \quad \rightsquigarrow \quad \mathcal{E}_{\rm NLS}(u) = \langle u, hu \rangle + \frac{1}{2} \iint_\Omega |u(x)|^2 w(x-y) |u(y)|^2 dxdy$$

Conjecture (Density matrices of interacting Bose gas)

If $\mathrm{Tr}(h^{-p}) < \infty$ and $\lambda \sim T^{-1} \to 0$, then for all $k \geq 1$

$$\frac{k!}{T^k} \Gamma_\lambda^{(k)} \quad \longrightarrow \quad \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)$$

in Schatten space $\mathfrak{S}^p$, with the **nonlinear Gibbs measure**

$$d\mu(u) = \text{``}z^{-1} e^{-\mathcal{E}_{\rm NLS}(u)} du\text{''}$$

1D case

Let $d = 1$ and

$$h = -\Delta + V(x), \quad V(x) \geq |x|^{2+} \text{ as } |x| \to \infty$$

Then $\mathrm{Tr}(h^{-1}) < \infty$ and the interacting Gibbs measure is well defined on L^2

$$d\mu(u) = \text{``}z^{-1} e^{-\mathcal{E}_{\mathrm{NLS}}(u)} du\text{''} := z_r^{-1} e^{-\mathcal{D}(u)} d\mu_0(u)$$

where

$$\mathcal{D}(u) = \frac{1}{2} \iint |u(x)|^2 w(x-y) |u(y)|^2 dx dy$$

Theorem (Trace class case [Lewin-N-Rougerie '15])

When $\mathrm{Tr}(h^{-1}) < \infty$, $0 \leq w \in a\delta_0 + L^\infty$, and $\lambda = T^{-1} \to 0$,

$$\mathrm{Tr} \left| \frac{k!}{T^k} \Gamma_\lambda^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u) \right| = 0, \quad \forall k \geq 1.$$

Remark: if $V(x) \geq |x|$, then $\mathrm{Tr}(h^{-2}) < \infty$ and μ is supported on $L^4(\mathbb{R})$
$\rightsquigarrow$ Hilbert-Schmidt convergence holds

2D and 3D cases: renormalization

Classical theory: Gaussian measure μ_0 is supported on negative Sobolev spaces $\rightsquigarrow$ the interacting measure μ is defined via a **Wick renormalization**

$$d\mu(u) = z_r^{-1} e^{-\mathcal{D}(u)} d\mu_0(u)$$

with

$$\mathcal{D}(u) = \frac{1}{2} \iint \Big(|u(x)|^2 - \langle |u(x)|^2 \rangle_{\mu_0}\Big) w(x-y) \Big(|u(y)|^2 - \langle |u(y)|^2 \rangle_{\mu_0}\Big) dxdy$$

This is well-defined if $\mathrm{Tr}(h^{-2}) < \infty$ and $0 \leq \widehat{w} \in L^1$

Quantum theory: Renormalized interaction

$$\mathbb{W}^{\mathrm{ren}} = \frac{1}{2} \iint \Big(a_x^* a_x - \langle a_x^* a_x \rangle_{\Gamma_0}\Big) w(x-y) \Big(a_y^* a_y - \langle a_y^* a_y \rangle_{\Gamma_0}\Big) dxdy$$

$$= \frac{1}{2} \iint a_x^* a_x w(x-y) a_y^* a_y \, dxdy - \int (w * \langle a_x^* a_x \rangle_{\Gamma_0}) a_x^* a_x \, dx + E_0$$

with

$$\langle a_x^* a_x \rangle_{\Gamma_0} = \Gamma_0^{(1)}(x;x) = \left[\frac{1}{e^{h/T} - 1}\right](x;x)$$

Homogeneous Bose gas

Let $h = -\Delta + \kappa$ on $L^2(\mathbb{T}^d)$ with $\kappa > 0$ fixed. Then

$$\Gamma_0^{(1)}(x;x) = \left[\frac{1}{e^{h/T}-1}\right](x;x) = \sum_{k \in 2\pi\mathbb{Z}^d} \frac{1}{e^{\frac{|k|^2+\kappa}{T}}-1} =: N_0(T)$$

$\rightsquigarrow$ Renormalization simply amounts to **shifting the chemical potential**

Theorem (Lewin-N-Rougerie '19)

Let $d \leq 3$ and

$$h = -\Delta + \kappa \quad \text{on } L^2(\mathbb{T}^d), \quad 0 \leq \widehat{w}(k)(1+|k|^2) \in \ell^1(2\pi\mathbb{Z}^d).$$

Consider the Gibbs state $\Gamma_\lambda = \mathcal{Z}_\lambda^{-1} e^{-\frac{\mathbb{H}_\lambda}{T}}$ where

$$\mathbb{H}_\lambda = \int_{\mathbb{T}^d} a_x^*(h_x - \nu) a_x \, dx + \frac{\lambda}{2} \iint_{\mathbb{T}^d \times \mathbb{T}^d} a_x^* a_y^* w(x-y) a_x a_y \, dx dy, \quad \nu = \lambda \widehat{w}(0) N_0(T).$$

When $\lambda = T^{-1} \to 0$, we have

$$\operatorname{Tr}\left|\frac{k!}{T^k}\Gamma_\lambda^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)\right|^2 = 0, \quad \forall k \geq 1.$$

Inhomogeneous Bose gas

In general when $h = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$, the free density is **not** a constant

$$\rho_V(x) = \Gamma_0^{(1)}(x;x) = \left[\frac{1}{e^{h/T}-1}\right](x;x)$$

$\rightsquigarrow$ Renormalization amounts to **adjusting the external potential**

Theorem (Fröhlich-Knowles-Schlein-Sohinger '17)

Assume $\mathrm{Tr}(h^{-2}) < \infty$. Take chemical potential $\nu = \lambda \hat{w}(0)\rho_\kappa - \kappa$, $\kappa > 0$ fixed.

- The **counter term problem** $V - \nu = V_T - \lambda w * \rho_{V_T}$ has a unique solution V_T, and $V_T \to V_\infty$ when $T = \lambda^{-1} \to \infty$
- For any fixed $\varepsilon \in (0,1)$ the density matrices of the quantum state

$$Z_\varepsilon^{-1} e^{-\varepsilon H_0/T} e^{-(H_\lambda - 2\varepsilon H_0)/T} e^{-\varepsilon H_0/T}$$

  converge to the Gibbs measure μ_∞ associated with $h_\infty = -\Delta + V_\infty$ and w

Theorem (Lewin-N-Rougerie '19)

If $\mathrm{Tr}(h^{-5/3}) < \infty$, then the convergence to μ_∞ holds with the Gibbs state ($\varepsilon = 0$).

Proof strategy

Variational approach:

$$\Gamma_\lambda \text{ minimizes} \qquad -\log\frac{\mathcal{Z}_\lambda}{\mathcal{Z}_0} = \inf_{\Gamma\geq 0, \operatorname{Tr}\Gamma=1}\Big[\underbrace{\mathcal{H}(\Gamma,\Gamma_0)}_{\operatorname{Tr}(\Gamma(\log\Gamma-\log\Gamma_0))\geq 0} + \frac{\lambda}{T}\operatorname{Tr}(\mathbb{W}\Gamma)\Big]$$

$$\mu \text{ minimizes} \qquad -\log z_r = \inf_{\mu' \text{ prob. measure}}\Big[\underbrace{\mathcal{H}_{\rm cl}(\mu',\mu_0)}_{\int \frac{d\mu'}{d\mu_0}\log\frac{d\mu'}{d\mu_0}d\mu_0\geq 0} + \int \mathcal{D}(u)d\mu'(u)\Big]$$

Quantum de Finetti theorem

$$\frac{k!}{T^k}\Gamma_\lambda^{(k)} \rightharpoonup \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu'(u), \quad \forall k\geq 1$$

Berezin-Lieb type inequality

$$\liminf \mathcal{H}(\Gamma_\lambda,\Gamma_0) \geq \mathcal{H}_{\rm cl}(\mu',\mu_0)$$

When $d=1$, $w\geq 0$ and Fatou's lemma imply

$$T^{-2}\operatorname{Tr}(\mathbb{W}\Gamma_\lambda) = T^{-2}\operatorname{Tr}(w\Gamma_\lambda^{(2)}) \geq \int \mathcal{D}(u)d\mu'(u) + o(1)_{T\to\infty},$$

leading to

$$-\log\frac{\mathcal{Z}_\lambda}{\mathcal{Z}_0} \to -\log z_r, \quad \mu' = \mu$$

Proof strategy

When $d = 2, 3$: Renormalized interaction has no sign $\rightsquigarrow$ Fatou's does not apply!

Quantitative method: Finite dimensional reduction needs **variance estimate**

$$\langle \mathbb{A}^2 \rangle_\lambda \longrightarrow 0, \quad \mathbb{A} = \frac{\mathrm{d}\Gamma(\mathbb{1}_{h>\Lambda}) - \langle \mathrm{d}\Gamma(\mathbb{1}_{h>\Lambda})\rangle_0}{T}, \quad 1 \ll \Lambda \ll T$$

i.e. particles at high momenta don't feel interaction. It requires a new method to control particle correlations.

- One-body estimate $\langle \mathbb{A} \rangle_\lambda \to 0$: Hellmann-Feynman + new entropy inequality
- Two-body estimate can be related to a one-body problem by **linear response**

$$\partial_{\varepsilon=0}\langle \mathbb{A} \rangle_{\lambda,\varepsilon} = \int_0^1 \mathrm{Tr}(\Gamma_\lambda^s \mathbb{A} \Gamma_\lambda^{1-s} \mathbb{A}) ds - \langle \mathbb{A} \rangle_\lambda^2, \qquad \Gamma_{\lambda,\varepsilon} = \mathcal{Z}_{\lambda,\varepsilon}^{-1} e^{-\frac{\mathbb{H}_\lambda}{T} + \varepsilon \mathbb{A}}$$

  General estimate

$$0 \geq \mathrm{Tr}(\Gamma_\lambda^s \mathbb{A} \Gamma_\lambda^{1-s} \mathbb{A}) - \langle \mathbb{A}^2 \rangle_\lambda \geq -\frac{1}{4} \Big\langle [[\mathbb{H}_\lambda, \mathbb{A}], \mathbb{A}] \Big\rangle_\lambda \to 0$$

- The linear response $\varepsilon \mapsto \partial_\varepsilon \langle \mathbb{A} \rangle_{\lambda,\varepsilon}$ is **mostly convex**

$$\partial_\varepsilon^3 \langle \mathbb{A} \rangle_{\lambda,\varepsilon} \simeq \langle \mathbb{A}^4 \rangle_{\lambda,\varepsilon} - 3 \langle \mathbb{A}^2 \rangle_{\lambda,\varepsilon}^2 \geq -2 \langle \mathbb{A}^2 \rangle_{\lambda,\varepsilon}^2$$

  Elementary fact: if $f''' \geq 0$, then $f'(0) \leq \frac{f(\varepsilon) - f(-\varepsilon)}{2\varepsilon}$.