On the energy asymptotics of a Bose gas in the dilute limit.

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Mean-field and other effective models in mathematical physics

Based on works with Birger Brietzke and Søren Fournais

Outline of Talk

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The Bose gas

We consider N bosons moving in a box $\Omega = [0, L]^3$ (Dirichlet or periodic b.c.)

2-body potential: $v : \mathbb{R}^3 \to [0, \infty]$ measurable, spherically symmetric, compact support, e.g., **hard core potential:** $v(x) = \infty$ if $|x| < a$ and zero otherwise.

Hamiltonian:

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \le i < j \le N} v(x_i - x_j)$$

Hilbert Space: $\bigvee^N L^2(\Lambda)$ (the symmetric tensor product)

Thermodynamic limit of ground state energy:

$$e(\rho) = \lim_{\substack{L \to \infty \\ N/L^3 \to \rho}} e_L(N), \qquad e_L(N) = L^{-3} \inf \mathrm{Spec}(H_N)$$

The scattering length and dilute limit

The Scattering Solution for the 2-body potential: $u : \mathbb{R}^3 \to \mathbb{R}$

$$-\Delta u + \frac{1}{2} v u = 0, \qquad \lim_{x\to\infty} u(x) = 1, \qquad u = 1 - \omega$$

$$0 \leq \omega \leq 1$$

Scattering length:

$$a = \lim_{x\to\infty} |x|(1 - u(x)) = \frac{1}{8\pi} \int_{\mathbb{R}^3} vu$$

The dilute limit of the Bose gas:

$$\rho a^3 \to 0$$

The Lee-Huang-Yang formula and main results

Theorem (Fournais-Solovej 2019 LHY Order)

$$e(\rho) \geq 4\pi\rho^2 a(1 - C\sqrt{\rho a^3}),$$

$C > 0$ depends on support and scattering length of v.

Note it is enough to prove this for L^1 potentials.

Theorem (Lee-Huang-Yang (1957) Formula for dilute limit $\rho a^3 \to 0$, Fournais-Solovej 2019)

If we also have $v \in L^1(\mathbb{R}^3)$ then

$$e(\rho) \geq 4\pi\rho^2 a(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3})).$$

Upper bound needs stronger assumptions on v (Yau-Yin (2009)).

Previous results

- **Lee-Huang-Yang (1957)** derived formula by summing selected terms in pertubation series and an uncontrolled pseudo potential approximation
- **Dyson (1957)** Got leading upper bound with error $(\rho a^3)^{1/3}$. His lower bound was not correct to leading order
- **Lieb (1963)** derived the formula under assumptions on the structure of the ground state
- **Lieb-Yngvason (1998)** established the leading order $4\pi\rho^2 a$ with error bound $(\rho a^3)^{1/17}$
- **Erdős-Schlein-Yau (2008)** had the LHY order as an upper bound under additional assumptions on v
- **Yau-Yin (2009)** established the LHY formula as an upper bound under additional assumptions on v
- **Giuliani-Seiringer (2009)** derived LHY for soft potential with radius of support $R \gg \rho^{-1/3}$, i.e., requirements on potential depend on density
- **Brietzke-Solovej (2018)** derived LHY for soft potentials with $a \ll R \ll \rho^{-1/3}$
- **Boccato-Brennecke-Cenatiempo-Schlein (2018)** derived the LHY formula in the confined case with additional assumptions on v

Localization of the kinetic energy

We localize to boxes

$$\Lambda_u = \ell_0(u + [-1/2, 1/2]^3), \quad u \in \mathbb{R}^3$$

of size $\ell_0 = K^{-1}(\rho a)^{-1/2}$ with K large.
Consider the **projections**

$$P_u = |\Lambda_u|^{-1}|\mathbb{1}_{\Lambda_u}\rangle\langle\mathbb{1}_{\Lambda_u}|, \qquad Q_u = \mathbb{1}_{\Lambda_u} - P_u.$$

and the **localization function**

$$\chi_u(x) = \chi(x/\ell_0 - u), \quad \chi \in C^M(\mathbb{R}^3) \text{ support in } [-1/2, 1/2]^3.$$

$$\int \chi_u^2(x)dx = \ell_0^3, \quad \int \chi_u^2(x)du = 1.$$

We have the **kinetic energy localization** (Brietzke-Fournais-Solovej)

$$-\Delta \geq \int \left(Q_u\chi_u[-\Delta - (s\ell_0)^{-2}]_+\chi_u Q_u + b\ell_0^{-2}Q_u\right) du,$$

Potential energy localization

Introduce the **localized pontetial**

$$w_u(x,y) = \chi_u(x) W(x-y) \chi_u(y), \qquad W(x) = \frac{v(x)}{\chi * \chi(x/\ell_0)}$$

Then

$$\sum_{i<j} v(x_i - x_j) = \int \sum_{i<j} w_u(x_i, x_j) du$$

In the localization we will use a **grand canonical formalism** and introduce a **chemical potential** term

$$\begin{aligned} -8\pi a \rho N &= -N\rho \int v(1-\omega) \\ &= -\int \left[\sum_{i=1}^{N} \rho \int w_u(x_i, y)(1-\omega(x_i - y)) dy \right] du. \end{aligned}$$

Localized Hamiltonian

The **localized Hamiltonians** are translates of

$$H_0 = \sum_i (\tau_i - \mu_i) + \sum_{i<j} w_{ij}$$

where, with $P_{u=0} = P$, $Q_{u=0} = Q$,

$$\tau = b\ell_0^{-2} Q + Q\chi_0[-\Delta - (s\ell_0)^{-2}]_+ \chi_0 Q$$

$$\mu = \mu(x) = \rho \int w(x,y)(1 - \omega(x-y))dy$$

and $w = w_{u=0}(x,y)$. The first term in τ is a **Neumann gap**. We have to show that

$$\ell_0^{-3} H_0 \geq -4\pi\rho^2 a - C\rho^2 a\sqrt{\rho a^3}.$$

The potential decomposition

A key idea is to decompose the potential

$$\begin{aligned} w_{ij} &= (P_i+Q_i)(P_j+Q_j)w_{ij}(P_i+Q_i)(P_j+Q_j) \\ &= (Q_iQ_j+(P_iP_j+Q_iP_j+P_iQ_j)\omega)w_{ij}(Q_iQ_j+\omega(P_iP_j+\cdots)) \\ &\quad +\mathcal{Q}_3^{\rm ren}+\mathcal{Q}_2^{\rm ren}+\mathcal{Q}_1^{\rm ren}+\mathcal{Q}_0^{\rm ren} \end{aligned}$$

where $\mathcal{Q}_q^{\rm ren}$ denotes terms with $q=0,\ldots,3$ number of Q's.

- Notice the first term is positive.
- For a $O(\rho^2 a\sqrt{\rho a^3})$ estimate the $\mathcal{Q}_3^{\rm ren}$ may be controlled by this positive term and a CS-inequality.
- The remaining $1Q$ terms can also be controlled by a CS inequality leading to the final estimate with only no-Q and $2Q$ terms.

The main estimate

With $n_0 = \sum_i P_i$, $n_+ = \sum_i Q_i$ being the number of **condensate** and **excited** particles:

$$-\sum_i \mu_i + \sum_{i<j} w_{ij} \geq \widetilde{\mathcal{Q}}_0^{\rm ren} + \widetilde{\mathcal{Q}}_2^{\rm ren} - Ca(\rho + n_0 \ell_0^{-3}) n_+$$

The error term can be absorbed in the Neumann gap and

$$\widetilde{\mathcal{Q}}_0^{\rm ren} = \frac{n_0(n_0-1)}{2\ell_0^6} \int v(1-\omega^2)(x)\,dx - 8\pi a \left(\rho \frac{n_0}{\ell_0^3} + \frac{1}{4}\left(\rho - \frac{n_0+1}{\ell_0^3}\right)^2 \right)$$

The term $\widetilde{\mathcal{Q}}_2^{\rm ren}$ has 2 Q's and with the kinetic energy will be treated by a **Bogolubov diagonalization**. This replaces $1-\omega^2$ above to $1-\omega$ and gives errors of order $\rho^2 a \sqrt{\rho a^3}$ and hence exactly what we want.