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RESEARCH ARTICLE

Wassim Daher · V. Filipe Martins-da-Rocha
Yiannis Vailakis

Asset market equilibrium with short-selling and differential information

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Abstract We introduce differential information in the asset market model studied by Cheng J Math Econ 20(1):137–152,1991, Dana and Le Van J Math Econ 25(3):263–280,1996 and Le Van and Truong Xuan J Math Econ 36(3): 241–254, 2001. We prove an equilibrium existence result assuming that the economy's information structure satisfies the conditional independence property. If private information is not publicly verifiable, agents have incentives to misreport their types and therefore contracts may not be executed in the second period. We also show that under the conditional independence property equilibrium contracts are always executable.

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W. Daher (✉) · Y. Vailakis
CES, Université Paris-1, 106-112 Boulevard de l'Hôpital,
75647 Paris Cedex 13, France
E-mail: daher@univ-paris1.fr
E-mail: vailakis@univ-paris1.fr
Tel.: +33-1-44078275
Fax: +33-1-44078301

V. F. Martins-da-Rocha
Ceremade, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny,
75775 Paris Cedex 16, France
E-mail: martins@ceremade.dauphine.fr
Tel.: +33-1-44054683
Fax: +33-1-44054599

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1 Introduction

It is well known that in asset market models equilibrium prices may fail to exist, since unbounded and mutually compatible arbitrage opportunities can arise. The first equilibrium existence result was proven by Hart (1974). In a specific context of an economy with finitely many securities, Hart has exhibited a list of conditions sufficient to limit the arbitrage opportunities present in the economy. Much later, Hammond (1983), Page (1987), Werner (1987), Chichilnisky (1995) and Page et al. (2000) reconsidered the problem by providing variations of Hart's list of conditions. In all cases, the role played by those no-arbitrage conditions was to bound the economy endogenously. For a review and comparison of the different concepts of no-arbitrage conditions, refer to Dana et al. (1999) and Allouch et al. (2002).

Developments in continuous trading models and portfolio analysis (Black and Scholes 1973; Kreps 1981; Duffie and Huang 1985) have simultaneously motivated the study of existence problem in infinite dimensional economies with unbounded sets (due to the possibility of short-selling). In the infinite dimensional case, the notions of no-arbitrage are not sufficient to ensure the existence of equilibrium prices. An alternative assumption in this context that ensures existence is to assume that the individually rational utility set is compact (in the finite dimensional case, this assumption is equivalent to no-arbitrage conditions, see Dana et al. 1999). This is the case in Dana et al. (1997) and Dana and Le Van (2000). The problem then turns out to find conditions under which the compactness of the utility set holds. When markets for contingent claims are complete, agents' consumption sets coincide with the space $L^p(\Omega, \mathscr{F}, \mathbb{P})$, and preferences take the usual von Neuman–Morgenstern form, Cheng (1991) and Dana and Le Van (1996) have proved (using two different approaches) that this assumption is satisfied. As Le Van and Truong Xuan (2001) have shown, the result is still valid in economies with separable but not necessarily von Neuman–Morgenstern utility functions.

Economies with uncertainty are encompassed by the general equilibrium model of Arrow-Debreu as long as there is a complete market of elementary securities or contingent claims. The formal extension of the Arrow-Debreu model to include uncertainty has been criticized for requiring a large number of necessary markets. Introducing differences among the informational structures of several agents is a way to diminish the force of the aforemention criticism, since it has the effect of drastically reducing the number of required contracts present in the economy.

The first study that introduced differential information in a general equilibrium setup was Radner (1968). Radner studies an environment where the structure of information is fixed in advance and all contracts are negotiated at the beginning of the history of the economy. For such an economy Radner defined a notion of equilibrium (Walrasian expectations equilibrium), an analogous concept to the Walrasian equilibrium in Arrow-Debreu model with symmetric information. As in (Debreu, 1959, Chapter 7), agents arrange contracts that may be contingent on the realized state of nature at the second period, but after the realization of the state, they do not necessarily know which state of nature has actually occurred. Therefore, they are restricted to sign contracts that are compatible with their private information. It is important to notice that in sharp contrast with the rational expectations equilibrium model (see Radner 1972), markets are open only at the first period and prices are observed and not expected.[1] In our case, prices do not reveal any private information ex ante. They rather reflect agents' informational asymmetries as they have been obtained by maximizing utility taking into account the private information of each agent.

Recently, there has been a resurgent interest in models with differential information (see Einy et al. 2001; Herves-Beloso et al. 2005a,b). At issue are questions concerning the existence and characterization of Walrasian expectations equilibria in terms of the private core introduced by Yannelis (1991).

The paper follows this growing literature. Its aim is to explore the Walrasian expectations equilibrium concept in infinite dimensional economies with a complete asset market structure and possible short-selling. We introduce differential information in the asset market model studied by Cheng (1991), Dana and Le Van (1996) and Le Van and Truong Xuan (2001). We prove an equilibrium existence result assuming that the economy's information structure satisfies the conditional independence property, i.e. individuals' information are assumed to be independent conditionally to the common information.

Our existence proof follows in two steps. In the first step, we show that the individually rational utility set associated with our differential information economy coincides with the individually rational utility set of a symmetric information economy. Using this fact and the results established in Le Van and Truong Xuan (2001), we subsequently show that the individually rational utility set of our differential economy is compact. In that respect, we show that the compactness result established for symmetric information economies (Cheng 1991; Dana and Le Van 1996; Le Van and Truong Xuan 2001) is still valid in a differential information setting.

It is known, due to Allouch and Florenzano (2004), that the compactness of the individually rational utility set implies the existence of an Edgeworth equilibrium. The second step then amounts to show that there exist prices supporting Edgeworth allocations as competitive equilibria. In that respect, we provide a characterization of Walrasian expectations equilibrium by means of cooperative solutions.

In addition to the equilibrium existence result, we deal with another issue: the execution of contracts. If private information is not publicly verifiable, agents have incentives to misreport their types and therefore contracts may not be executed in the second period. We show that conditional independence is crucial to address this problem. Indeed, it ensures that equilibrium allocations are individually incentive compatible. This in turn implies that, at equilibrium, contracts are always executed.

The structure of the paper is as follows. Section 2 presents the theoretical framework and outlines the basic model. In Section 3, we prove the compactness of the individually rational utility set. Section 4 deals with the decentralization of Edgeworth allocations. Section 5 is devoted to economies with finitely many states of nature. The reason for paying special attention to the finite case stems

[1] We believe that the term Walrasian expectations equilibrium may be misleading since, contrary to the rational expectations equilibrium concept (Radner 1972), agents do not form expectations about prices.

from the fact that the conditional independence property is not needed anymore to prove existence of equilibrium prices. In this case, under our assumptions, it can be shown directly that a no-arbitrage price always exists. Existence of a supporting price system then follows from standard arguments in the literature (see Allouch et al. (2002) and the references therein). However, it should be stressed that dropping out conditional independence leaves open the problem of enforceability of contracts. In that respect, we believe that conditional independence is a relevant restriction even in the finite dimensional case.

2 The model

We consider a pure exchange economy with a finite set I of agents. The economy extends over two periods with uncertainty on the realized state of nature in the second one.

2.1 Uncertainty and information structure

Each agent $i \in I$ has an individual perception of the uncertainty at the second period, represented by a probability space $(T^i, \mathscr{T}^i, \mathbb{P}^i)$. Each element t^i in T^i may be interpreted as an individual type or signal, observed at the second period. We do not assume that all type profiles in $\prod_{i \in I} T^i$ are possible at the second period. This allows to capture aspects of uncertainty which may be commonly known. The set of possible types is captured by a couple $\{(\Omega, \mathscr{F}, \mathbb{P}), (\mathrm{pr}^i, i \in I)\}$ where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space representing the underlying uncertainty, related to individual perception through the measurable mappings $\mathrm{pr}^i : (\Omega, \mathscr{F}) \longrightarrow (T^i, \mathscr{T}^i)$, satisfying

$$\forall i \in I, \quad \forall A \in \mathscr{T}^i, \quad \mathbb{P}^i(A) = \mathbb{P}\left[(\mathrm{pr}^i)^{-1}(A)\right].$$

A profile of types $(t^i, i \in I) \in \prod_{i \in I} T^i$ is said to be possible if there exists $\omega \in \Omega$ such that $t^i = \mathrm{pr}^i(\omega)$ for each $i \in I$. It should be clear that this framework captures the one in which agent i's private information is formulated by a sub σ-algebra of $\mathscr{F}$. Indeed, let us denote by $\mathscr{F}^i$ the sub σ-algebra of $\mathscr{F}$ generated by pr^i. For every $x \in L^1(T^i, \mathscr{T}^i, \mathbb{P}^i)$, denote by $\Delta^i(x)$ the mapping in $L^1(\Omega, \mathscr{F}^i, \mathbb{P})$ defined by $\Delta^i(x) = x \circ \mathrm{pr}^i$. The mapping Δ^i is a bijection from $L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ to $L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ satisfying

$$\forall z \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i), \quad \mathbb{E}^{\mathbb{P}^i}[z] = \mathbb{E}^{\mathbb{P}}[\Delta^i(z)].^2$$

By abusing notations, we assimilate a function $x \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ with $\Delta^i(x)$, writing indifferently $x(t^i)$ or $x(\omega)$ with $\omega \in (\mathrm{pr}^i)^{-1}(t^i)$.

Remark 1 In what follows, we provide a general framework where the underlying uncertainty is derived from the individual perception of uncertainty. Consider probability spaces $(S, \mathscr{S}, \mathbb{Q})$ and $(N^i, \mathscr{N}^i, \eta^i)$ for each $i \in I$. The probability

[2] If (A, Σ, μ) is a probability space and $z \in L^1(A, \Sigma, \mu)$ then $\mathbb{E}^{\mu}[z]$ denotes the expectation of z under μ.

space $(S, \mathscr{S}, \mathbb{Q})$ represents uncertainty that is commonly observed, while each N^i represents agent i's private information. We do not specify if agent i knows the private information N^k of another agent $k \neq i$. Agent i's perception of uncertainty is defined by

$$T^i = S \times N^i, \quad \mathscr{T}^i = \mathscr{S} \otimes \mathscr{N}^i, \quad \text{and} \quad \mathbb{P}^i = \mathbb{Q} \otimes \eta^i.$$

The underlying uncertainty $(\Omega, \mathscr{F}, \mathbb{P})$ is then represented by

$$\Omega = S \times \prod_{i \in I} N^i, \quad \mathscr{F} = \mathscr{S} \otimes \otimes_{i \in I} \mathscr{N}^i, \quad \mathbb{P} = \mathbb{Q} \otimes \otimes_{i \in I} \eta^i,$$

and $\mathrm{pr}^i(s, (n^k, k \in I)) = (s, n^i)$. Observe that if agent i is not aware of the private information $\prod_{k \neq i} N^k$ of other agents, then he is not aware of Ω.

Remark 2 In what follows, we provide a general framework where the individual perception of uncertainty is derived from the underlying uncertainty. Consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ representing states of nature that are commonly known to be possible at the second period. Each agent i has an incomplete information about the true state of nature, in the sense that he will only observe the output of a random variable $\tau^i : \Omega \to \mathbb{R}$. Denote by T^i the range $\tau^i(\Omega)$ of τ^i. Then the individual perception of uncertainty is represented by $(T^i, \mathscr{B}, \mathbb{P}^i)$ where $\mathscr{B}$ is the Borelian σ-algebra and $\mathbb{P}^i$ is defined by

$$\forall A \in \mathscr{B}, \quad \mathbb{P}^i(A) = \mathbb{P}\{\tau^i \in A\}.$$

Note that here, the mappings τ^i and pr^i coincide.

2.2 Contracts and incentive compatibility

In the first period, agents sign contracts contingent to their perception of uncertainty. In the second period, individual uncertainty is resolved and contracts are executed. Since agents trade with an anonymous market, they only sign contracts that are contingent to events they can observe at the second period. To be specific, for given $p \in [1, +\infty]$, a contract for agent i is represented by a function in $L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ or $L^p(\Omega, \mathscr{F}^i, \mathbb{P})$. This implies that short-selling is allowed in the sense that agents can sell an arbitrary large number of contracts.

Each agent $i \in I$ is endowed with a type dependent initial endowment represented by a contingent claim $e^i \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ and a type dependent utility function $U^i : T^i \times \mathbb{R} \to \mathbb{R}$. Alternatively, we may consider e^i to be a function in $L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ and U^i to be a function from $\Omega \times \mathbb{R} \to \mathbb{R}$ such that for each $z \in \mathbb{R}$, the function $\omega \mapsto U^i(\omega, z)$ is $\mathscr{F}^i$-measurable.

Following our equilibrium concept defined below, each agent i signs a contract $x^i \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ such that the allocation $(x^i, i \in I)$ is exact feasible, i.e.

$$\sum_{i \in I} x^i(\omega) = \sum_{i \in I} e^i(\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$[3]

[3] Formally, we should write $\sum_{i \in I} x^i(\mathrm{pr}^i(\omega)) = \sum_{i \in I} e^i(\mathrm{pr}^i(\omega))$.

If private information does not become publicly verifiable, then there is an issue concerning the execution of contracts. In order to clarify this issue, we need to introduce the σ-algebra $\mathscr{F}^c$ defined by

$$\mathscr{F}^c = \bigcap_{i \in I} \mathscr{F}^i$$

which represents the common knowledge information to all agents. An event A belongs to $\mathscr{F}^c$ if and only if it belongs to each $\mathscr{F}^i$, i.e. the σ-algebra $\mathscr{F}^c$ is the collection of all events that are observable by all agents.

2.2.1 Finitely many types

We first consider the particular case where each agent has finitely many types, i.e. the set T^i is finite for each $i \in I$ and $\mathscr{T}^i$ is the collection of all subsets of T^i. The induced sub σ-algebra $\mathscr{F}^i$ is defined by the partition $\{(\mathrm{pr}^i)^{-1}(t^i) : t^i \in T^i\}$. The atom $(\mathrm{pr}^i)^{-1}(t^i)$ is denoted by $\Omega^i(t^i)$ and $\Omega^c(t^i)$ is defined as the unique atom in $\mathscr{F}^c$ containing $\Omega^i(t^i)$.

Assume that the realized state of nature at the second period is $\sigma \in \Omega$. Each agent i has an incomplete information in the sense that he only observes his type $t^i = \mathrm{pr}^i(\sigma)$. Individual information may not be publicly verifiable. Therefore, an agent $i \in I$ may have an incentive to misreport his type by announcing a type ζ^i instead of t^i if the following condition is satisfied

$$U^i(t^i, e^i(t^i) + x^i(\zeta^i) - e^i(\zeta^i)) > U^i(t^i, x^i(t^i)).$$

Since we allow for aspects of uncertainty to be known to all agents, agent i can only misreport if t^i and ζ^i are compatible with respect to the common knowledge information, i.e.

$$\Omega^c(t^i) = \Omega^c(\zeta^i).$$

Assume that each agent $i \in I$ reports type ζ^i. If there exists $\omega \in \Omega$ (ω may be different from the realized state of nature σ) such that $\mathrm{pr}^i(\omega) = \zeta^i$ for each $i \in I$, then the profile of types $(\zeta^i, i \in I)$ is such that contracts can be executed, i.e. each agent i will receive $x^i(\zeta^i)$ and markets clear. Indeed,

$$\sum_{i \in I} x^i(\zeta^i) = \sum_{i \in I} x^i(\omega) = \sum_{i \in I} e^i(\omega) = \sum_{i \in I} e^i(\zeta^i).$$

However, if there does not exist $\omega \in \Omega$ such that $\mathrm{pr}^i(\omega) = \zeta^i$ for each $i \in I$, then contracts may not be executable.

A sufficient condition to ensure that an allocation $(x^i, i \in I)$ is executable, is that each contract x^i is incentive compatible as defined below.

Definition 1 *Given an agent $i \in I$, a contract $x^i \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ is said to be (individually) incentive compatible if the following is not true: there exist t^i and ζ^i in T^i with $\mathbb{P}^i\{t^i\} > 0$, $\mathbb{P}^i\{\zeta^i\} > 0$ such that*

(a) agent i has an incentive to misreport t^i by announcing ζ^i, i.e.

$$U^i(t^i, e^i(t^i) + x^i(\zeta^i) - e^i(\zeta^i)) > U^i(t^i, x^i(t^i)) \tag{1}$$

(b) the misreporting is compatible with the common knowledge information, i.e.

$$\Omega^c(t^i) = \Omega^c(\zeta^i). \tag{2}$$

Remark 3 Alternative concepts of incentive compatibility have been proposed in the literature (see for instance Koutsougeras and Yannelis 1993; Krasa and Yannelis 1994; Hahn and Yannelis 1997).

Remark 4 To clarify condition (2), return to the framework studied in Remark 1. In that case, given an agent $i \in I$, a contract $x^i \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ is incentive compatible if and only if there do not exist $s \in S$, n^i and μ^i in N^i with $\eta^i(n^i) > 0$, $\eta^i(\mu^i) > 0$, $\mathbb{Q}(s) > 0$ such that

$$U^i((s, n^i), e^i(s, n^i) + x^i(s, \mu^i) - e^i(s, \mu^i)) > U^i((s, n^i), x^i(s, n^i)). \tag{3}$$

Observe that in this framework, if an allocation $(x^i, i \in I)$ is exact feasible then for each $i \in I$, there exists a function $f^i \in L^p(S, \mathscr{S}, \mathbb{Q})$ such the net trade function $z^i = x^i - e^i$ satisfies $z^i(s, n^i) = f^i(s)$ for $\mathbb{Q} \otimes \eta^i$-a.e. $(s, n^i) \in S \times N^i$. It then follows that each contract x^i is incentive compatible. Indeed, if it is not the case, there exist $s \in S$, n^i and μ^i in N^i with $\eta^i(n^i) > 0$, $\eta^i(\mu^i) > 0$, $\mathbb{Q}(s) > 0$ such that

$$e^i(s, n^i) + x^i(s, \mu^i) - e^i(s, \mu^i) > x^i(s, n^i).$$

A contradiction to the fact that $z^i(s, n^i) = f^i(s) = z^i(s, \mu^i)$.

2.2.2 The general case

We do not anymore assume that each agent has finitely many types. In order to extend the concept of incentive compatibility to the general case, we first introduce some notations. For each $i \in I$, we denote by $\mathscr{T}^i_c$ the σ-algebra of events in $\mathscr{T}^i$ that are common knowledge to all agents, defined by

$$\mathscr{T}^i_c = \left\{ H \in \mathscr{T}^i \ : \ (\mathrm{pr}^i)^{-1}(H) \in \mathscr{F}^c \right\}.$$

Observe that if x^i is a function in $L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ then

$$x^i \in L^p(T^i, \mathscr{T}^i_c, \mathbb{P}^i) \Longleftrightarrow x^i \circ \mathrm{pr}^i \in L^p(\Omega, \mathscr{F}^c, \mathbb{P}).$$

We may assume that each $i \in I$ only knows his individual perception of uncertainty $(T^i, \mathscr{T}^i, \mathbb{P}^i)$ and what is commonly observed $\mathscr{T}^i_c$. In that respect, we do not restrict the model to the case where each agent $i \in I$ knows the underlying uncertainty $(\Omega, \mathscr{F}, \mathbb{P})$.

We propose now a generalization of Definition 1 to the general case.

Definition 2 *Given an agent $i \in I$, a contract $x^i \in L^p(T^i, \mathscr{T}^i, \mathbb{P}^i)$ is said to be (individually) incentive compatible if the following is not true: there exist $E^i \in \mathscr{T}^i$ and $G^i \in \mathscr{T}^i$ with $\mathbb{P}^i(E^i) > 0$, $\mathbb{P}^i(G^i) > 0$ such that*

(a) *for every* $(t^i, \zeta^i) \in E^i \times G^i$, *agent* i *has an incentive to misreport* t^i *by announcing* ζ^i, *i.e.*

$$U^i(t^i, e^i(t^i) + x^i(\zeta^i) - e^i(\zeta^i)) > U^i(t^i, x^i(t^i)) \tag{4}$$

(b) *the misreporting is compatible with the common knowledge information, i.e.*

$$\frac{1}{\mathbb{P}^i(E^i)}\mathbb{E}^{\mathbb{P}^i}\left[\mathbf{1}_{E^i}|\mathscr{T}_c^i\right] = \frac{1}{\mathbb{P}^i(G^i)}\mathbb{E}^{\mathbb{P}^i}\left[\mathbf{1}_{G^i}|\mathscr{T}_c^i\right]. \tag{5}$$

Remark 5 Observe that when agents have finitely many types, then Definition 2 reduces to Definition 1. Indeed, if T^i is finite and $\mathscr{T}^i$ is the σ-algebra of all subsets of T^i, then for every $t^i \in T^i$ with $\mathbb{P}^i(\{t^i\}) > 0$, one has

$$\mathbb{E}^{\mathbb{P}^i}\left[\mathbf{1}_{\{t^i\}}|\mathscr{T}_c^i\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\Omega^i(t^i)}|\mathscr{F}^c\right] = \frac{\mathbb{P}(\Omega^i(t^i))}{\mathbb{P}(\Omega^c(t^i))}\mathbf{1}_{\Omega^c(t^i)}.$$

2.2.3 Conditional independence: a sufficient condition

Example in Remark 4 highlights that some kind of restriction in the information structure is sufficient to ensure that an exact feasible allocation is incentive compatible.

Definition 3 *The information structure* $\mathscr{F}^\bullet = (\mathscr{F}^i, i \in I)$ *of an economy* $\mathscr{E} = (U^i, \mathscr{F}^i, e^i, i \in I)$ *is said to be conditionally independent (given the common information), if for every* $i \neq j$ *in* I, *the* σ*-algebra* $\mathscr{F}^i$ *and* $\mathscr{F}^j$ *are independent given* $\mathscr{F}^c$.[4]

Example 2.1 The information structure in Remark 1 is conditionally independent.

Remark 6 The information structure is conditionally independent if and only if for every pair $i \neq j$ in I we have

$$\forall x \in L^1(\Omega, \mathscr{F}^i, \mathbb{P}), \quad \mathbb{E}[x|\mathscr{F}^j] = \mathbb{E}[x|\mathscr{F}^c].$$

Conditional independence also appears in McLean and Postlewaite (2002) and Sun and Yannelis (2006).

The main result of this section is the following proposition.

Proposition 1 *If the information structure is conditionally independent then every exact feasible allocation* $(x^i, i \in I)$ *with* $x^i \in L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ *is incentive compatible.*

[4] That is, for every pair of events $A^i \in \mathscr{F}^i$ and $A^j \in \mathscr{F}^j$, we have that $\mathbb{P}(A^i \cap A^j|\mathscr{F}^c) = \mathbb{P}(A^i|\mathscr{F}^c)\mathbb{P}(A^j|\mathscr{F}^c)$ almost surely.

Proof Let $(x^i, i \in I)$ be an allocation with $x^i \in L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ such that

$$\sum_{i \in I} x^i(\omega) = \sum_{i \in I} e^i(\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

We claim that under conditional independence of the information structure, for each $i \in I$, the net trade $z^i := x^i - e^i$ is $\mathscr{F}^c$-measurable.[5] Indeed, for each $i \in I$, one has $z^i = -\sum_{j \neq i} z^j$ and in particular

$$z^i = \mathbb{E}[z^i | \mathscr{F}^i] = -\sum_{j \neq i} \mathbb{E}[z^j | \mathscr{F}^i] = -\sum_{j \neq i} \mathbb{E}[z^j | \mathscr{F}^c].$$

Fix $i \in I$ and assume that the contract x^i is not incentive compatible. Then there exist $E^i \in \mathscr{T}^i$ and $G^i \in \mathscr{T}^i$ with $\mathbb{P}^i(E^i) > 0$, $\mathbb{P}^i(G^i) > 0$ and satisfying conditions (4) and (5). Then

$$\mathbb{P}^i(G^i)\mathbb{E}^{\mathbb{P}^i}[z^i \mathbf{1}_{E^i}] - \mathbb{P}^i(E^i)\mathbb{E}^{\mathbb{P}^i}[z^i \mathbf{1}_{G^i}] = \int_{E^i \times G^i} [z^i(t) - z^i(\zeta)] \mathbb{P}^i \otimes \mathbb{P}^i(\mathrm{d}t, \mathrm{d}\zeta) < 0.$$

Since z^i is $\mathscr{F}^c$-measurable, it follows from (5) that

$$\begin{aligned}
\mathbb{P}^i(G^i)\mathbb{E}^{\mathbb{P}^i}\left[z^i \mathbf{1}_{E^i}\right] &= \mathbb{P}^i(G^i)\mathbb{E}^{\mathbb{P}^i}\left[z^i \mathbb{E}^{\mathbb{P}^i}\left[\mathbf{1}_{E^i} | \mathscr{T}_c^i\right]\right] \\
&= \mathbb{P}^i(G^i)\mathbb{E}^{\mathbb{P}^i}\left[z^i \frac{\mathbb{P}^i(E^i)}{\mathbb{P}^i(G^i)} \mathbb{E}^{\mathbb{P}^i}\left[\mathbf{1}_{G^i} | \mathscr{T}_c^i\right]\right] \\
&= \mathbb{P}^i(E^i)\mathbb{E}^{\mathbb{P}^i}\left[z^i \mathbf{1}_{G^i}\right]
\end{aligned}$$

which yields a contradiction.

2.3 Equilibrium concept

To relate our equilibrium concept to the traditional one proposed in the literature with symmetric information (see Cheng (1991)), we choose to reprensent agent i's individual uncertainty by $(\Omega, \mathscr{F}^i, \mathbb{P})$.

The preference relation of agent i is represented by the utility function u^i from $L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ to $\mathbb{R}$ defined by

$$\forall x \in L^p(\Omega, \mathscr{F}^i, \mathbb{P}), \quad u^i(x) = \int_\Omega U^i(\omega, x(\omega)) \mathbb{P}(\mathrm{d}\omega).$$

If $x \in L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ is a contingent claim then $P^i(x)$ is the set of strictly preferred contingent claims for agent i, i.e.

$$P^i(x) = \{y \in L^p(\Omega, \mathscr{F}^i, \mathbb{P}) : u^i(y) > u^i(x)\}.$$

[5] When there are only two agents, the conditional independence of the information structure is no more needed.

An economy $\mathscr{E}$ is defined by a family $(U^i, \mathscr{F}^i, e^i, i \in I)$. A vector $x^\bullet = (x^i, i \in I)$ with $x^i \in L^p(\Omega, \mathscr{F}^i, \mathbb{P})$ is called an *allocation*. The space $\mathscr{A}$ of *individually rational attainable allocation* is defined by

$$\mathscr{A} = \left\{ x^\bullet = (x^i, i \in I) \in \prod_{i \in I} L^p(\Omega, \mathscr{F}^i, \mathbb{P}) : \sum_{i \in I} x^i = e \quad \text{and} \quad u^i(x^i) \geqslant u^i(e^i) \right\}$$

where $e = \sum_{i \in I} e^i$ is the aggregate initial endowment. If $\psi \in L^q(\Omega, \mathscr{F}, \mathbb{P})$[6] then we can define the value functional $\langle \psi, . \rangle$ on $L^p(\Omega, \mathscr{F}, \mathbb{P})$ by

$$\forall x \in L^p(\Omega, \mathscr{F}, \mathbb{P}), \quad \langle \psi, x \rangle = \int_\Omega \psi(\omega) x(\omega) \mathbb{P}(\mathrm{d}\omega).$$

The vector ψ is called a price (system) and $\langle \psi, x \rangle$ represents the value (in the sense of Debreu 1959) at the first period of the contingent claim x under the price ψ. We also denote the value $\langle \psi, x \rangle$ by $\mathbb{E}[\psi x]$.

Definition 4 *A competitive equilibrium is a pair $(x^\bullet, \psi)$ where $x^\bullet = (x^i, i \in I)$ is an attainable allocation and ψ is a price such that each agent maximizes his utility in his budget set, i.e.*

$$\forall i \in I, \quad x^i \in \operatorname{argmax} \left\{ u^i(y) : \; y \in L^p(\Omega, \mathscr{F}^i, \mathbb{P}) \; \text{ and } \; \langle \psi, y \rangle \leqslant \langle \psi, e^i \rangle \right\}.$$

An allocation $x^\bullet$ is said to be a competitive allocation if there exists a price ψ such that $(x^\bullet, \psi)$ is a competitive equilibrium.

To prove the existence of a competitive equilibrium, we first show that, under suitable assumptions, the *individually rational utility set* (as defined below) is compact.

Definition 5 *We denote by $\mathscr{U}$ the individually rational utility set defined by*

$$\mathscr{U} = \left\{ v^\bullet = (v^i, i \in I) \in \mathbb{R}^I : \; \exists x^\bullet \in \mathscr{A}, \quad u^i(e^i) \leqslant v^i \leqslant u^i(x^i) \right\}.$$

For any $1 \leqslant p \leqslant +\infty$ and for any sub σ-algebra $\mathscr{G} \subset \mathscr{F}$, the space $L^p(\Omega, \mathscr{G}, \mathbb{P})$ is denoted by $L^p(\mathscr{G})$.

Definition 6 *A function $U : \Omega \times \mathbb{R} \to \mathbb{R}$ is said*

(i) *integrable if for every $t \in \mathbb{R}$, the function $\omega \mapsto U(\omega, t)$ is integrable,*
(ii) *p-integrable bounded if there exists $\alpha \in L^1_+(\mathscr{F})$ and $\beta > 0$ such that*

$$|U(\omega, t)| \leqslant \alpha(\omega) + \beta |t|^p$$

for every $(\omega, t) \in \Omega \times \mathbb{R}$.

[6] The real number $q \in [1, +\infty]$ is the conjugate of p defined by $1/p + 1/q = 1$ with the convention that $1/+\infty = 0$ and $1/0 = +\infty$.

Remark 7 Obviously, every p-integrable bounded function is integrable. Following Krasnoselskii (1964), if $p \in [1, +\infty)$ and $(\Omega, \mathscr{F}, \mathbb{P})$ is atomless then a function U is p-integrable bounded, if and only if, for every $x \in L^p(\mathscr{F})$, the function $\omega \mapsto U(\omega, x(\omega))$ is integrable. If $p = \infty$ then a utility function is integrable if and only if $\omega \mapsto U(\omega, x(\omega))$ is integrable for every $x \in L^p(\mathscr{F})$.

Remark 8 Observe that if $\mathscr{E} = (U^i, \mathscr{F}^i, e^i, i \in I)$ is an economy with an atomless measure space $(\Omega, \mathscr{F}, \mathbb{P})$ then a necessary condition for the separable utility function u^i to be well-defined is that for each $i \in I$, the function U^i is p-integrable bounded.

Definition 7 *An economy $\mathscr{E} = (U^i, \mathscr{F}^i, e^i, i \in I)$ is said standard if, for every $i \in I$, the following properties are satisfied:*

(S.1) *the function $t \mapsto U^i(\omega, t)$ is differentiable, concave and strictly increasing for $\mathbb{P}$-a.e. $\omega \in \Omega$;*
(S.2) *the function U^i is p-integrable bounded;*
(S.3) *the function $t \mapsto U^i_\star(\omega, t)$ is continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$, where*

$$U^i_\star(\omega, t) = \frac{\partial U^i}{\partial t}(\omega, t);$$

(S.4) *there exist two functions α^i and β^i from $\mathbb{R}$ to $\mathbb{R}$ such that for every $t \in \mathbb{R}$,*

$$\alpha^i(t) < t < \beta^i(t),$$

$$\operatorname*{ess\,sup}_{\omega \in \Omega} U^i_\star(\omega, \beta^i(t)) < \operatorname*{ess\,inf}_{\omega \in \Omega} U^i_\star(\omega, t)$$

and

$$\operatorname*{ess\,sup}_{\omega \in \Omega} U^i_\star(\omega, t) < \operatorname*{ess\,inf}_{\omega \in \Omega} U^i_\star(\omega, \alpha^i(t)).$$

Remark 9 If the information structure is symmetric, then an economy satisfies Assumptions S.1–S.4 if and only if it satisfies assumptions H.0–H.4 in Le Van and Truong Xuan (2001). In particular, all economies considered in Cheng (1991), Dana and Le Van (1996) and Le Van and Truong Xuan (2001), satisfy Assumptions S.1–S.4.

As in Le Van and Truong Xuan (2001, Section 2.1), we provide hereafter examples of utility functions satisfying Assumptions S.1–S.4.

Example 2.2 If for every $i \in I$, there exists $F^i : \mathbb{R} \to \mathbb{R}$ concave, continuously differentiable and strictly increasing and such that the function $U^i(\omega, t) = F^i(t)$ is p-integrable bounded,[7] then the economy is standard.

[7] Such a utility function has the well known von Neumann–Morgenstern form.

Example 2.3 If for every $i \in I$, there exists $F^i : \mathbb{R} \to \mathbb{R}$ concave, continuously differentiable, strictly increasing satisfying

$$\lim_{t \to -\infty} F_\star(t) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} F_\star(t) = 0;$$

and $h^i \in L^\infty_{++}(\mathscr{F}^i)$ such that the function $U^i(\omega, t) = F^i(t)h^i(\omega)$ is p-integrable bounded, then the economy is standard. In this case, agents' preference relations are represented by von Neumann–Morgenstern utility functions with heterogenous expectations. Indeed, if we denote by $\mathbb{P}^i$ the probability measure on $(\Omega, \mathscr{F})$ defined by $\mathrm{d}\mathbb{P}^i = h^i \mathrm{d}\mathbb{P}$, then the preference relation defined by $u^i : x \mapsto \mathbb{E}[U^i(x)]$ coincide with the von Neumann–Morgenstern preference relation defined by the felicity (or index) function F^i and the private belief $\mathbb{P}^i$, i.e.

$$\forall x \in L^p(\mathscr{F}^i), \quad u^i(x) = \mathbb{E}^{\mathbb{P}^i}[F^i(x)] = \int_\Omega F^i(x^i(\omega))\mathbb{P}^i(\mathrm{d}\omega).$$

We do not assume that aggregate initial endowment is measurable with respect to the common information $\mathscr{F}^c$. As the following two remarks show, in our framework such an assumption makes the existence issue a trivial one.

Remark 10 Consider a standard economy $\mathscr{E} = (U^i, \mathscr{F}^i, e^i, i \in I)$ such that every initial endowment e^i is measurable with respect to the common information, i.e. e^i belongs to $L^p(\mathscr{F}^c)$ and such that each utility function u^i is von Neumann–Morgenstern. In this case the existence of a competitive equilibrium follows in a straightforward way. Indeed, let $\mathscr{E}^c$ denote the economy $(U^i, \mathscr{F}^c, e^i, i \in I)$. This economy is standard and symmetric. Following Cheng (1991) we can show that there exists an equilibrium $(x^\bullet, \psi)$ with

$$\psi \in L^q_+(\mathscr{F}^c) \quad \text{and} \quad x^i \in L^p(\mathscr{F}^c), \quad \forall i \in I.$$

We claim that $(x^\bullet, \psi)$ is a competitive equilibrium for the original economy $\mathscr{E}$. For this, we only need to prove that x^i is optimal in the budget set of the economy $\mathscr{E}$. Assume by way of contradiction that there exists $y^i \in L^p(\mathscr{F}^i)$ such that

$$\mathbb{E}[\psi y^i] \leqslant \mathbb{E}[\psi e^i] \quad \text{and} \quad u^i(y^i) > u^i(x^i).$$

Since ψ and e^i are measurable with respect to $\mathscr{F}^c$, the consumption plan $\mathbb{E}[y^i|\mathscr{F}^c]$ satisfies the budget constraints associated with the symmetric economy $\mathscr{E}^c$. But from Jensen's inequality, we also have that

$$u^i(\mathbb{E}[y^i|\mathscr{F}^c]) \geqslant u^i(y^i) > u^i(x^i)$$

which yields a contradiction to the optimality of x^i in the symmetric economy.

Remark 11 Now consider a standard economy $\mathscr{E} = (U^i, \mathscr{F}^i, e^i, i \in I)$ such that the aggregate endowment $e = \sum_{i \in I} e^i$ is measurable with respect to the common information, i.e. e belongs to $L^p(\mathscr{F}^c)$, each utility function u^i is von Neumann–Morgenstern and the information structure is conditionally independent. In this case, the existence of a competitive equilibrium follows also in a straightforward way. Indeed, the $\mathscr{F}^c$-measurability of e together with conditional independence imply that individual endowments e^i are also $\mathscr{F}^c$-measurable.[8] Existence follows from the previous remark.

3 Compactness of the individually rational utility set

Theorem 3.1 stated below, proves the compactness of the individually rational utility set associated with our differential information economy. In that respect it generalizes the results established for symmetric information economies found in Cheng (1991), Dana and Le Van (1996) and Le Van and Truong Xuan (2001).

Theorem 1 *For every standard economy with a conditionally independent information structure, the individually rational utility set is compact.*

Proof We let θ^c be the linear mapping from $L^p(\mathscr{F})$ to $L^p(\mathscr{F}^c)$ defined by

$$\forall z \in L^p(\mathscr{F}), \quad \theta^c(z) = \mathbb{E}[z|\mathscr{F}^c]$$

and we let ξ^i be the linear mapping from $L^p(\mathscr{F})$ to $L^p(\mathscr{F}^i)$ defined by

$$\forall z \in L^p(\mathscr{F}), \quad \xi^i(z) = \mathbb{E}[z - \theta^c(z)|\mathscr{F}^i].$$

Observe that

$$\forall i \in I, \ \theta^c \circ \xi^i = 0 \quad \text{and} \quad \forall z^i \in L^p(\mathscr{F}^i), \ \xi^i(z^i) = z^i - \theta^c(z^i).$$

Since the information structure is conditionally independent, we have

$$\forall i \neq j \in I, \ \forall z^j \in L^p(\mathscr{F}^j), \quad \xi^i(z^j) = \mathbb{E}[z^j|\mathscr{F}^i] - \mathbb{E}[z^j|\mathscr{F}^c] = 0. \quad (6)$$

We let $\mathscr{Z}$ be the set of allocations defined by

$$\mathscr{Z} = \left\{ z^\bullet \in \prod_{i\in I} L^p(\mathscr{F}^c) \ : \ \sum_{i\in I} z^i = \theta^c(e) \right\},$$

where $e = \sum_{i\in I} e^i$ is the aggregate initial endowment. Observe that from (6) we have that $\xi^i(e) = \xi^i(e^i)$.

Claim 1 An allocation $x^\bullet$ is attainable (feasible) if and only if there exists $z^\bullet \in \mathscr{Z}$ such that

$$\forall i \in I, \quad x^i = z^i + \xi^i(e).$$

[8] We have $e = \sum_{i\in I} e^i$. Fix $j \in I$, taking conditional expectations with respect to F^j, we get

$$e = \mathbb{E}[e|\mathscr{F}^j] = e^j + \sum_{i\neq j} \mathbb{E}[e^i|\mathscr{F}^j] = e^j + \sum_{i\neq j} \mathbb{E}[e^i|\mathscr{F}^c].$$

Proof If $x^\bullet$ be an attainable allocation, then

$$\forall i \in I, \quad \xi^i(e) = \sum_{j \in I} \xi^i(x^j) = \xi^i(x^i) = x^i - \theta^c(x^i).$$

If for every $i \in I$, we let $z^i = \theta^c(x^i)$ then $z^\bullet$ belongs to $\mathscr{Z}$ and satisfies $x^i = z^i + \xi^i(e)$. Reciprocally, if $x^\bullet$ is an allocation such that $x^i = z^i + \xi^i(e)$ for some $z^\bullet \in \mathscr{Z}$, then

$$\sum_{i \in I} x^i = \theta^c(e) + \sum_{i \in I} \xi^i(e) = \theta^c(e) + \sum_{i \in I} \left\{ e^i - \theta^c(e^i) \right\} = e$$

which implies that $x^\bullet$ is attainable. □

Now we consider the function $V^i : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\forall (\omega, t) \in \Omega \times \mathbb{R}, \quad V^i(\omega, t) = \mathbb{E}[U^i(\xi^i(e) + t) | \mathscr{F}^c](\omega).^9$$

By the Disintegration Theorem we have

$$\begin{aligned}\forall z \in L^p(\mathscr{F}^c), \quad \mathbb{E}[U^i(\xi^i(e) + z)] &= \int_\Omega U^i(\omega, \xi^i(e)(\omega) + z(\omega))\mathbb{P}(\mathrm{d}\omega) \\ &= \int_\Omega V^i(\omega, z(\omega))\mathbb{P}(\mathrm{d}\omega) \\ &= \mathbb{E}[V^i(z)]. \end{aligned} \tag{7}$$

Combining Claim 1 and equation (7), the individually rational utility set $\mathscr{U}$ coincide with the set $\mathscr{V}$ defined by

$$\mathscr{V} = \left\{ w^\bullet \in \mathbb{R}^I : \exists z^\bullet \in \mathscr{Z}, \quad v^i(\theta^c(e^i)) \leqslant w^i \leqslant v^i(z^i) \right\}$$

where v^i is the utility function defined on $L^p(\mathscr{F}^c)$ by $v^i(z) = \mathbb{E}[V^i(z)]$.

Observe that the set $\mathscr{V}$ corresponds to the individually rational utility set of a symmetric information economy $\mathscr{E}^\star$ defined by

$$\mathscr{E}^\star := (V^i, \mathscr{F}^c, \theta^c(e^i), i \in I).$$

In addition, as the following result shows, this symmetric economy is standard.

Claim 2 For every $i \in I$, the utility function V^i satisfies Assumptions S.1–S.4.

[9] In the sense that if $t \in \mathbb{R}$ then we let $f_t^i(\omega) = U^i(\omega, \xi^i(e)(\omega) + t)$. The real number $V^i(\omega, t)$ is then defined by $V^i(\omega, t) = \mathbb{E}[f_t^i | \mathscr{F}^c](\omega)$.

Proof Fix $i \in I$ and denote by ζ^i the function $\xi^i(e)$. Since the function U^i is p-integrable bounded, it follows that for each $t \in \mathbb{R}$, the function $\omega \mapsto U^i(\omega, \zeta^i(\omega)+t)$ is integrable and the function $V^i(\omega, t)$ is well-defined. It is straightforward to check that the function $t \mapsto V^i(\omega, t)$ is concave and strictly increasing for $\mathbb{P}$-a.e. ω. We propose now to prove that $t \mapsto V^i(\omega, t)$ is differentiable for $\mathbb{P}$-a.e. ω. Let (t_n) be a sequence converging to $t \in \mathbb{R}$ with $t_n \neq t$ and let

$$\forall \omega \in \Omega, \quad f_n(\omega) := \frac{U^i(\omega, \zeta^i(\omega) + t_n) - U^i(\omega, \zeta^i(\omega) + t)}{t_n - t}.$$

The sequence (f_n) converges almost surely to the function f defined by $f(\omega) = U^i_\star(\omega, \zeta^i(\omega) + t)$. Moreover, since the function $t \mapsto U^i(\omega, t)$ is concave, differentiable and strictly increasing, for every n large enough[10] we have

$$|f_n(\omega)| \leqslant U^i_\star(\omega, \zeta^i(\omega) + t - 1), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Observe that since the function $t \mapsto U^i(\omega, t)$ is concave, differentiable and strictly increasing, we have for $\mathbb{P}$-a.e. ω,

$$0 < U^i_\star(\omega, \zeta^i(\omega) + t - 1) \leqslant U^i(\omega, \zeta^i(\omega) + t - 1) - U^i(\omega, \zeta^i(\omega) + t - 2),$$

but since the function U^i is p-integrable bounded, we get

$$0 < \mathbb{E}[U^i_\star(\zeta^i + t - 1)] \leqslant |\mathbb{E}[U^i(\zeta^i + t - 1)]| + |\mathbb{E}[U^i(\zeta^i + t - 2)]| < +\infty.$$

Therefore the sequence (f_n) is integrably bounded. Applying the Lebesgue Dominated Convergence Theorem for conditional expectations, we get that the function $t \mapsto V^i(\omega, t)$ is differentiable and for every $t \in \mathbb{R}$,

$$V^i_\star(\omega, t) := \frac{\partial V^i}{\partial t}(\omega, t) = \mathbb{E}[U^i_\star(\xi^i(e) + t)|\mathscr{F}^c](\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

We have thus proved that the function V^i satisfies Assumption S.1. Assumption S.2 follows from the p-integrability of U^i and the Disintegration Theorem. We propose now to prove that the function $t \mapsto V^i_\star(\omega, t)$ is continuous for $\mathbb{P}$-a.e. ω. Let (t_n) be a sequence converging to $t \in \mathbb{R}$ and let g_n be the function defined by $g_n(\omega) := U^i_\star(\omega, \zeta^i(\omega) + t_n)$. The function (g_n) converges almost surely to the function g defined by $g(\omega) := U^i_\star(\omega, \zeta^i(\omega) + t)$. Moreover, for n large enough we have

$$0 < g_n(\omega) \leqslant U^i_\star(\omega, \zeta^i(\omega) + t - 1), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Applying the Lebesgue Dominated Convergence Theorem for conditional expectations we get that the function $t \mapsto V^i_\star(\omega, t)$ is continuous for $\mathbb{P}$-a.e. ω.

We propose now to prove that

$$\forall t \in \mathbb{R}, \quad \operatorname*{ess\,sup}_{\omega \in \Omega} V^i_\star(\omega, \beta^i(t)) < \operatorname*{ess\,inf}_{\omega \in \Omega} V^i_\star(\omega, t).$$

[10] More precisely, for every n such that $|t_n - t| \leqslant 1$.

Fix $t \in \mathbb{R}$. Since the function U^i satisfies Assumption S.4, we trivially have that

$$U^i_\star(\omega, \beta^i(t)) \leqslant \operatorname*{ess\,sup}_{\omega\in\Omega} U^i_\star(\omega, \beta^i(t)), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Therefore for $\mathbb{P}$-a.e. ω,

$$V^i_\star(\omega, \beta^i(t)) = \mathbb{E}[U^i_\star(\xi^i(e) + \beta^i(t))|\mathscr{F}^c](\omega) \leqslant \operatorname*{ess\,sup}_{\omega\in\Omega} U^i_\star(\omega, \beta^i(t))$$

which implies that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} V^i_\star(\omega, \beta^i(t)) \leqslant \operatorname*{ess\,sup}_{\omega\in\Omega} U^i_\star(\omega, \beta^i(t)). \tag{8}$$

Symmetrically we can prove

$$\operatorname*{ess\,inf}_{\omega\in\Omega} U^i_\star(\omega, t) \leqslant \operatorname*{ess\,inf}_{\omega\in\Omega} V^i_\star(\omega, t). \tag{9}$$

Since U^i satisfies Assumption S.4, we have that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} U^i_\star(\omega, \beta^i(t)) \leqslant \operatorname*{ess\,inf}_{\omega\in\Omega} U^i_\star(\omega, t). \tag{10}$$

The desired result follows from (8), (9) and (10).

Following similar arguments, it is straightforward to prove that

$$\forall t \in \mathbb{R}, \quad \operatorname*{ess\,sup}_{\omega\in\Omega} V^i_\star(\omega, t) < \operatorname*{ess\,inf}_{\omega\in\Omega} V^i_\star(\omega, \alpha^i(t)),$$

and then we get that V^i satisfies Assumptions S.1–S.4.

Claim 2 implies (see Remark 2.4) that the symmetric information economy $\mathscr{E}^\star$ satisfies Assumptions H.0–H.4 in Le Van and Truong Xuan (2001). Based on their arguments we can directly conclude that the set $\mathscr{V}$ (and therefore the set $\mathscr{U}$) is compact.[11]

4 Decentralization of Edgeworth allocations

We now recall the definition of different optimality concepts for feasible allocations.

Definition 8 *A feasible allocation $x^\bullet$ is said to be:*

1. *weakly Pareto optimal if there is no feasible allocation $y^\bullet$ satisfying $y^i \in P^i(x^i)$ for each $i \in I$,*
2. *a core allocation if it cannot be blocked by any coalition in the sense that there is no coalition $S \subseteq I$ and some $(y^i, i \in S) \in \prod_{i\in S} P^i(x^i)$ such that $\sum_{i\in S} y^i = \sum_{i\in S} e^i$,*

[11] Actually in Le Van and Truong Xuan (2001) the probability space is the continuum [0, 1] endowed with the Lebesgue measure, but the arguments can straightforwardly be adapted to abstract probability spaces. We refer to Daher, Martins-da-Rocha and Vailakis (2005) for a formal discussion.

3. *an Edgeworth allocation if there is no* $0 \neq \lambda^\bullet \in \mathbb{Q} \cap [0,1]^I$ *and some allocation* $y^\bullet$ *such that* $y^i \in P^i(x^i)$ *for each* $i \in I$ *with* $\lambda^i > 0$ *and* $\sum_{i\in I} \lambda^i y^i = \sum_{i\in I} \lambda^i e^i$,
4. *an Aubin allocation if there is no* $0 \neq \lambda^\bullet \in [0,1]^I$ *and some allocation* $y^\bullet$ *such that* $y^i \in P^i(x^i)$ *for each* $i \in I$ *with* $\lambda^i > 0$ *and* $\sum_{i\in I} \lambda^i y^i = \sum_{i\in I} \lambda^i e^i$.

Remark 12 The reader should observe that these concepts are "price free" in the sense that they are intrinsic properties of the commodity space. It is proved in (Florenzano, 2003, Proposition 4.2.6) that the set of Aubin allocations and the set of Edgeworth allocations coincide for every standard economy.

Remark 13 In our model, agent i's consumption set coincides with $L^p(\mathscr{F}^i)$ the space of contracts measurable with respect to agent i's information. Therefore the core concept introduced above is related to the private core concept introduced by Yannelis (1991) (where agent i's consumption set is $L^p_+(\mathscr{F}^i)$).

Corollary 1 *If* $\mathscr{E}$ *is a standard economy with a conditionally independent information structure then there exists an Edgeworth allocation.*

Proof Since the individually rational utility set is compact, we can apply Theorem 3.1 in Allouch and Florenzano (2004).

It is straightforward to check that every competitive allocation is an Edgeworth allocation. We prove that the converse is true.

Theorem 2 *If* $\mathscr{E}$ *is a standard economy with a conditionally independent information structure then for every Edgeworth allocation* $x^\bullet$ *there exists a price* ψ *such that* $(x^\bullet, \psi)$ *is a competitive equilibrium. Moreover, for every* $i \in I$ *there exists* $\lambda^i > 0$ *such that*

$$\mathbb{E}[\psi|\mathscr{F}^i] = \lambda^i U^i_\star(x^i).$$

In particular, we have

$$\forall (i,j) \in I \times I, \quad \mathbb{E}[\lambda^i U^i_\star(x^i)|\mathscr{F}^c] = \mathbb{E}[\lambda^j U^j_\star(x^j)|\mathscr{F}^c].$$

Proof Let $x^\bullet$ be an Edgeworth allocation. From Remark 12 it is actually an Aubin allocation, in particular

$$0 \notin G(x^\bullet) := \text{co} \bigcup_{i\in I} \left(P^i(x^i) - \{e^i\}\right). \tag{11}$$

The set $L^p(\mathscr{F}^i)$ is denoted by E^i, the set $L^p(\mathscr{F}^c)$ is denoted by E^c and we let $E = \sum_{i\in I} E^i$. Observe that E may be a strict subspace of $L^p(\mathscr{F})$. We endow E with the topology τ for which a base of 0-neighborhoods is

$$\left\{ \sum_{i\in I} \alpha^i B \cap E^i \ : \ \alpha^i > 0, \quad \forall i \in I \right\}$$

where B is the closed unit ball in $L^p(\mathscr{F})$ for the p-norm topology.[12] From the Structure Theorem in (Aliprantis and Border, 1999, p. 168) this topology is well defined, Hausdorff and locally convex. Observe moreover that the restriction of the τ-topology to each subspace E^i coincide with the restriction of the p-norm topology. Following standard arguments,[13] each utility function u^i is p-norm continuous. It follows that $G(x^\bullet)$ has a τ-interior point. Applying a convex separation theorem, there exists a τ-continuous linear functional $\pi \in (E, \tau)'$ such that $\pi \neq 0$ and for each $g \in G(x^\bullet)$, we have $\pi(g) \geqslant 0$. In particular it follows that

$$\forall i \in I, \ \forall y^i \in P^i(x^i), \quad \pi(y^i) \geqslant \pi(x^i). \tag{12}$$

Since preference relations are strictly increasing, we have that x^i belongs to the closure of $P^i(x^i)$. Therefore $\pi(x^i) \geqslant \pi(e^i)$. But since $x^\bullet$ is attainable, we must have $\pi(x^i) = \pi(e^i)$ for each $i \in I$. The linear functional π is not zero, therefore there exists $j \in I$ and $z^j \in E^j$ such that $\pi(z^j) < \pi(e^j)$. We now claim that if $y^j \in P^j(x^j)$ then $\pi(y^j) > \pi(e^j)$. Indeed, assume by way of contradiction that $\pi(y^j) = \pi(e^j)$. Since the set $P^j(x^j)$ is p-norm open in E^j, there exists $\alpha \in (0, 1)$ such that $\alpha y^j + (1-\alpha)z^j \in P^j(x^j)$. From (12) we get $\alpha\pi(y^j) + (1-\alpha)\pi(z^j) \geqslant \pi(e^j)$: contradiction. We have thus proved that

$$\forall y^j \in P^j(x^j), \quad \pi(y^j) > \pi(x^j).$$

By strict-monotonicity of preference relations, we obtain that $\pi(\mathbf{1}) > 0$ where $\mathbf{1}$ is the vector in E^c defined by $\mathbf{1}(\omega) = 1$ for any $\omega \in \Omega$. Choosing $z^i = e^i - \mathbf{1}$ for every $i \neq j$, we follow the previous argument to get that

$$\forall i \in I, \ \forall y^i \in P^i(x^i), \quad \pi(y^i) > \pi(e^i),$$

or equivalently

$$\forall i \in I, \quad x^i \in \operatorname{argmax}\left\{u^i(z^i) : z^i \in E^i \quad \text{and} \quad \pi(z^i) \leqslant \pi(e^i)\right\}. \tag{13}$$

In order to prove that $x^\bullet$ is an equilibrium allocation, it is sufficient to prove that there exists a price $\psi \in L^q(\mathscr{F})$ which represents π in the sense that $\pi = \langle \psi, . \rangle$.

Claim 3 For every $i \in I$, there exists $\psi^i \in L^q(\mathscr{F}^i)$ such that

$$\forall z^i \in E^i, \quad \pi(z^i) = \langle \psi^i, z^i \rangle.$$

Moreover, there exists $\lambda^i > 0$ such that $\psi^i = \lambda^i U^i_\star(x^i)$.

[12] The space $L^p(\mathscr{F})$ is endowed with the p-norm topology defined by the sup-norm $\|.\|_\infty$ if $p = \infty$, i.e.

$$\forall x \in L^\infty(\mathscr{F}), \quad \|x\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega} |x(\omega)|$$

and the p-norm $\|.\|_p$ if $p \in [1, +\infty)$, i.e.

$$\forall x \in L^p(\mathscr{F}), \quad \|x\|_p = \left[\int_\Omega |x(\omega)|^p \mathbb{P}(d\omega)\right]^{1/p}.$$

[13] We refer to Appendix A in Daher, Martins-da-Rocha and Vailakis (2005) for formal arguments.

Proof Fix $i \in I$ and let $z^i \in E^i$ such that $\langle U^i_\star(x^i), z^i\rangle > 0$. Following standard arguments,[14] we can prove that

$$\langle U^i_\star(x^i), z^i\rangle = \lim_{t\downarrow 0} \frac{1}{t}\{u^i(x^i + tz^i) - u^i(x^i)\}.$$

Therefore there exists $t > 0$ such that $x^i + tz^i \in P^i(x^i)$. Applying (13) we get that $\pi(z^i) > 0$. We have thus proved that

$$\forall z^i \in E^i, \quad \langle U^i_\star(x^i), z^i\rangle > 0 \Longrightarrow \pi(z^i) > 0.$$

It then follows[15] that there exists $\lambda^i > 0$ such that the restriction $\pi_{|E^i}$ of π to E^i coincides with the linear functional $\langle \lambda^i U^i_\star(x^i), .\rangle$.

Since $E^c \subset E^i \cap E^j$ the family $\psi^\bullet = (\psi^i, i \in I)$ is compatible in the sense that

$$\forall (i, j) \in I \times I, \quad \mathbb{E}[\psi^i|\mathscr{F}^c] = \mathbb{E}[\psi^j|\mathscr{F}^c].$$

We denote by ψ^c the vector in $L^q(\mathscr{F}^c)$ defined by $\psi^c = \mathbb{E}[\psi^i|\mathscr{F}^c]$ and we let ψ be the vector in $L^q(\mathscr{F})$ defined by

$$\psi = \psi^c + \sum_{i\in I}(\psi^i - \psi^c). \tag{14}$$

Claim 4 The vector ψ represents the linear functional π.

Proof Since the information structure is conditionally independent, we can easily check that

$$\forall i \in I, \quad \mathbb{E}[\psi|\mathscr{F}^i] = \psi^i.$$

If z is a vector in E, then there exists $z^\bullet = (z^i, i \in I)$ with $z^i \in E^i$ such that $z = \sum_{i\in I} z^i$. Therefore

$$\langle \psi, z\rangle = \sum_{i\in I}\langle \psi, z^i\rangle = \sum_{i\in I}\mathbb{E}[\psi z^i] = \sum_{i\in I}\mathbb{E}[\psi^i z^i] = \sum_{i\in I}\pi(z^i) = \pi(z),$$

which implies that $\pi = \langle \psi, .\rangle$.

It follows from (13) and Claim 4 that $(x^\bullet, \psi)$ is a competitive equilibrium.

[14] We refer to Appendix A in Daher, Martins-da-Rocha and Vailakis (2005) for formal arguments.

[15] Let $\langle X, X^\star\rangle$ be a dual pair. Assume that there exists $x^\star \in X^\star$ and $0 \neq \pi$ a linear functional from X to $\mathbb{R}$ such that for every $x \in X$, if $\langle x, x^\star\rangle > 0$ then $\pi(x) > 0$. Let $\Delta = \{(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+ : a + b = 1\}$ be the simplex in $\mathbb{R}^2$, we then have

$$\Delta \cap \{(\langle x, x^\star\rangle, -\pi(x)) : x \in X\} = \emptyset.$$

Applying a convex separation argument, we get the existence of $\lambda > 0$ such that $\langle \lambda x^\star, .\rangle = \pi$.

Remark 14 Observe that in the proof of Theorem 2, the conditional independence assumption on the information structure is only used in Claim 4 to prove that the linear functional π is representable by a function ψ in $L^q(\mathscr{F})$. In particular if the set Ω is finite, then Theorem 2 is valid without assuming that the information structure is conditionally independent.

Due to Corollary 1 and Theorem 2 we are able to generalize Theorem 1 in Cheng (1991), Theorem 1 in Dana and Le Van (1996) and Theorem 1 in Le Van and Truong Xuan (2001) to economies with differential information.

Corollary 2 *Every standard economy with a conditionally independent information structure has a competitive allocation.*

Remark 15 Observe that when $p = +\infty$, contrary to Cheng (1991) and Le Van and Truong Xuan (2001), we do not need to establish Mackey-continuity of the utility functions to guarantee that equilibrium prices belong to $L^1(\mathscr{F})$. As in Dana and Le Van (1996), our equilibrium prices are related to individual marginal utilities. The continuity requirement (i.e. prices belong to $L^1(\mathscr{F})$) follows then directly from the standard assumptions on the utility functions.

5 Finitely many states of nature and no-arbitrage prices

In this section we assume that the set Ω is finite. Without any loss of generality, we may assume that $\mathbb{P}\{\omega\} > 0$ for every $\omega \in \Omega$. For every $1 \leqslant p \leqslant +\infty$, the space $L^p(\mathscr{F}^i)$ coincides with $L^0(\mathscr{F}^i)$ the space of $\mathscr{F}^i$ measurable functions from Ω to $\mathbb{R}$.

In Theorem 2 we proved an equilibrium existence result for economies in which the information structure satisfies the conditional independence property. We show below that, when Ω is finite, the existence of equilibrium prices can be established without assuming conditional independence of the information structure. Indeed, in this case one can show that for standard economies a no-arbitrage price always exists. It is well-known in the literature (see Allouch et al. (2002) and the references therein) that in the finite dimensional case, the existence of a no-arbitrage price is sufficient for the existence of an equilibrium price system. However, it should be stressed that dropping out conditional independence leaves open the question of execution of contracts. In that respect, we believe that conditional independence is a relevant restriction even in the finite dimensional case. We impose hereafter the following assumption which is a weaker version of Assumption S.4.

(wS.4) There exist three numbers $\alpha^i < \tau^i < \beta^i$ such that,

$$\sup_{\omega\in\Omega} U^i_\star(\omega, \beta^i) < \inf_{\omega\in\Omega} U^i_\star(\omega, \tau^i) \quad \text{and} \quad \sup_{\omega\in\Omega} U^i_\star(\omega, \tau^i) < \inf_{\omega\in\Omega} U^i_\star(\omega, \alpha^i).$$

For each $i \in I$, we let R^i be the subset of $L^0(\mathscr{F}^i)$ defined by

$$R^i = \left\{ y^i \in L^0(\mathscr{F}^i) : \forall t \geqslant 0, \quad u^i(e^i + t y^i) \geqslant u^i(e^i) \right\}.$$

Since the function u^i is concave, it is standard (see (Rockafellar, 1970, Theorem 8.7)) that the set R^i satisfies the following property:

$$\forall x^i \in L^0(\mathscr{F}^i), \quad R^i = \{y^i \in L^0(\mathscr{F}^i) : \forall t \geqslant 0, \quad u^i(x^i + ty^i) \geqslant u^i(x^i)\}. \tag{15}$$

In our framework, we can prove that $\mathbf{1}$ is a no-arbitrage price, as defined in Werner (1987).

Proposition 2 *For each $i \in I$, if y^i is a non-zero vector in R^i then $\langle \mathbf{1}, y^i \rangle > 0$.*

Proof Let $x^i \in L^p(\mathscr{F}^i)$ be defined as follows

$$x^i = \beta^i \mathbf{1}_{\{y^i > 0\}} + \alpha^i \mathbf{1}_{\{y^i < 0\}} + \tau^i \mathbf{1}_{\{y^i = 0\}}.$$

Since $y^i \in R^i$, one has $u^i(x^i + y^i) \geqslant u^i(x^i)$. Following standard arguments, one has

$$\langle U^i_\star(x^i), y^i \rangle \geqslant u^i(x^i + y^i) - u^i(x^i) \geqslant 0.$$

Let $\gamma^i \in \mathbb{R}$ be such that

$$\inf_{\omega \in \Omega} U^i_\star(\omega, \tau^i) \leqslant \gamma^i \leqslant \sup_{\omega \in \Omega} U^i_\star(\omega, \tau^i).$$

Observe that $\gamma^i > 0$ and $\gamma^i y^i(\omega) \geqslant U^i_\star(\omega, x^i(\omega)) y^i(\omega)$ for almost every $\omega \in \Omega$. From Assumption wS.4, $\gamma^i > U^i_\star(\omega, x^i(\omega))$ for every $\omega \in \{y^i > 0\}$ and $\gamma^i < U^i_\star(\omega, x^i(\omega))$ for every $\omega \in \{y^i < 0\}$.

Since $y^i \neq 0$, it follows that

$$\gamma^i \langle \mathbf{1}, y^i \rangle > \langle U^i_\star(x^i), y^i \rangle \geqslant 0.$$

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