Interim efficiency with MEU-preferences ☆

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Abstract

Recently Kajii and Ui (2009) [17] proposed to characterize interim efficient allocations in an exchange economy under asymmetric information when uncertainty is represented by multiple posteriors. When agents have Bewley's incomplete preferences, Kajii and Ui (2009) [17] proposed a necessary and sufficient condition on the set of posteriors. However, when agents have Gilboa–Schmeidler's MaxMin expected utility preferences, they only propose a sufficient condition. The objective of this paper is to complete Kajii and Ui's work by proposing a necessary and sufficient condition for interim efficiency for various models of ambiguity aversion and in particular MaxMin expected utility. Our proof is based on a direct application of some results proposed by Rigotti, Shannon and Stralecki (2008) [24].
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1. Introduction

We borrow from Kajii and Ui [17] the following model of an exchange economy with a single good and a finite set of possible states of nature. Finitely many agents exchange contingent contracts. There are two stages: ex-ante each agent's perception of uncertainty is represented by a family of priors; at the interim stage each agent receives a private signal about which states will

not occur and his interim perception of uncertainty is then represented by a family of posteriors. Each agent is endowed with a concave utility index function from which he derives either Bewley's incomplete preferences or Gilboa–Schmeidler's MaxMin expected utility preferences. The set of priors induces preferences at the ex-ante stage (before agents receive their private signal) and the set of posteriors induces preferences at the interim stage depending on the private signal agents receive.

It is well known that ex-ante efficiency can be characterized through a necessary and sufficient condition imposed on agents' priors. Bewley [2] and Rigotti and Shannon [23] characterized ex-ante efficiency for agents with Bewley's incomplete preferences.[1] Billot, Chateauneuf, Gilboa and Tallon [4] and Rigotti, Shannon and Stralecki [24] characterized ex-ante efficiency for agents with Gilboa–Schmeidler's MaxMin expected utility preferences.[2]

In the standard Bayesian models, Morris [21] and Feinberg [13] provided a characterization of interim efficiency in terms of agents' posteriors. Kajii and Ui [17] proposed to address the same question when agents have multiple posteriors. They identified a key concept, called the *compatible prior set*, that plays a crucial role in their analysis. The compatible prior set of an agent is the collection of all probability measures which, conditional to each private signal, coincide with a posterior.[3] When agents have Bewley's incomplete preferences, Kajii and Ui [17] succeeded to characterize interim efficiency by providing a necessary and sufficient condition in terms of compatible prior sets. For the particular case of linear utility index functions (risk neutral agents), they proved that an allocation is interim efficient if and only if the compatible prior sets of all agents have a non-empty intersection.

When agents have Gilboa–Schmeidler's MaxMin expected utility preferences, Kajii and Ui [17] proposed a condition that is only sufficient. The objective of this paper is to show that it is possible to find a necessary and sufficient condition for interim efficiency when agents have Gilboa–Schmeidler's MaxMin expected utility preferences. The condition that we propose is closely related to the necessary condition introduced by Kajii and Ui [17]. Actually we show that the concept of compatible prior set is central not only for models where agents have Bewley's incomplete preferences or Gilboa–Schmeidler's MaxMin expected utility preferences, but also for any model with general convex preferences. More precisely, we provide a general necessary and sufficient condition for interim efficiency in terms of compatible priors associated to interim subjective beliefs as introduced by Rigotti, Shannon and Stralecki [24]. All the characterization results in Kajii and Ui [17] follow as corollaries of our general characterization. In particular, having a complete characterization of ex-ante and interim efficiency, we can provide conditions under which there is no speculative trade as first studied by Milgrom and Stockey [20] for standard Bayesian models.

The paper is organized as follows. Section 2 sets up the formal framework, notation and some preliminary definitions. The characterization results proposed by Kajii and Ui [17] are presented in Section 3. Our necessary and sufficient condition for interim efficiency is stated and proved in Section 4. We illustrate in Section 5 how the results in Kajii and Ui [17] can be deduced from our general characterization. Section 6 shows how our results can be extended to encompass general convex preferences. We explore a slightly different concept of interim efficiency in Section 7 and Section 8 is devoted to no speculative trade.

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[1] See also Dana [8].

[2] See also Dana [7], Samet [25], Tallon [27], Chateauneuf, Dana and Tallon [5], Dana [9] and Dana [10].

[3] Technically, the compatible prior set of an agent is the convex hull of all sets of the agent's posteriors.

2. Set up

We consider a model of an exchange economy $\mathcal{E}$ under uncertainty with asymmetric information as presented in Kajii and Ui [17]. There is a finite set Ω of states. The set of all probability measures over Ω is denoted by $\mathrm{Prob}(\Omega)$ and we let $\mathcal{P}$ be the collection of all non-empty, convex and closed subsets of $\mathrm{Prob}(\Omega)$. The expectation of a vector $x \in \mathbb{R}^\Omega$ under a probability measure $p \in \mathrm{Prob}(\Omega)$ is denoted by $\mathbb{E}^p[x]$, i.e.,

$$\mathbb{E}^p[x] = \sum_{\omega \in \Omega} p(\omega)x(\omega).$$

If P is a set in $\mathcal{P}$ then we let

$$\mathbb{E}_P[x] = \min_{p \in P} \mathbb{E}^p[x].$$

There is a finite set I of agents. Each agent i's information is characterized by a partition Π^i of Ω. Any event $\pi \in \Pi^i$ can be interpreted as a signal received by agent i at the interim stage. Given a state $\omega \in \Omega$, the unique event $\pi \in \Pi^i$ containing ω is denoted by $\Pi^i(\omega)$. The information is assumed to be correct in the sense that if the state of nature is ω, agent i knows that the *true* state does not belong to $\Omega \setminus \Pi^i(\omega)$ but cannot discern among the states in $\Pi^i(\omega)$ which one is the true state. Each agent i has a set of priors $P^i \in \mathcal{P}$ which represents his prior beliefs, and a set of posteriors $\Phi^i(\pi) \in \mathcal{P}$ for each signal $\pi \in \Pi^i$, which represents his posterior beliefs after observing π. The collection of posteriors $\{\Phi^i(\pi)\}_{\pi \in \Pi^i}$ is denoted by Φ^i.

Assumption 2.1. For every agent i and every signal $\pi \in \Pi^i$,

(a) there exists at least one prior $p \in P^i$ such that $p(\pi) > 0$;
(b) every posterior $r \in \Phi^i(\pi)$ satisfies $r(\pi) = 1$.

For notational convenience, given a subset $\pi \subset \Omega$, we denote by $\mathcal{P}(\pi)$ the subset of $\mathcal{P}$ defined as follows: a set $P \in \mathcal{P}$ belongs to $\mathcal{P}(\pi)$ if and only if the support of any probability in P is a subset of π, i.e.,

$$\forall r \in P, \quad r(\pi) = 1.$$

Observe that for every agent i and every interim signal $\pi \in \Pi^i$, the set of posteriors $\Phi^i(\pi)$ belongs to $\mathcal{P}(\pi)$.

There is a single good in the economy, and agent i has a concave, strictly increasing and continuous differentiable utility index function $u^i : [0, \infty) \to \mathbb{R}$ which induces MaxMin expected utility preferences as defined by Gilboa and Schmeidler [14].

Definition 2.1. Agent i (strictly) prefers the contingent consumption bundle $y \in \mathbb{R}_+^\Omega$ to $x \in \mathbb{R}_+^\Omega$ at the *ex-ante* stage if

$$\mathbb{E}_{P^i}\big[u^i(y)\big] > \mathbb{E}_{P^i}\big[u^i(x)\big].$$

The set of all contingent consumption bundles that are (strictly) preferred to x at the ex-ante stage is denoted by $\mathrm{Pref}_\Omega^i(x)$.

Similarly, we can define agent i's preference relation at the interim stage.

Definition 2.2. Agent i (strictly) prefers the contingent consumption bundle $y \in \mathbb{R}_+^\Omega$ to $x \in \mathbb{R}_+^\Omega$ at the *interim* stage with private information $\pi \in \Pi^i$ if

$$\mathbb{E}_{\Phi^i(\pi)}\big[u^i(y)\big] > \mathbb{E}_{\Phi^i(\pi)}\big[u^i(x)\big].$$

The set of all contingent consumption bundles that are (strictly) preferred to x at the interim stage with private information $\pi \in \Pi^i$ is denoted by $\mathrm{Pref}_\pi^i(x)$.

An allocation $\boldsymbol{x}$ is a family $\boldsymbol{x} = (x^i)_{i \in I}$ where x^i is a vector in $\mathbb{R}_+^\Omega$ representing a contingent consumption bundle. We fix from now on an allocation $\boldsymbol{e} = (e^i)_{i \in I}$ where e^i can be interpreted as the current endowment of agent i.

Assumption 2.2. For each agent i, the contingent consumption bundle e^i is interior in the sense that

$$\forall \omega \in \Omega, \quad e^i(\omega) > 0.$$

A family $\boldsymbol{t} = (t^i)_{i \in I}$ where t^i is a vector in $\mathbb{R}^\Omega$ is called a feasible trade (from the allocation $\boldsymbol{e}$) if

$$\forall i \in I, \quad e^i + t^i \in \mathbb{R}_+^\Omega \quad \text{and} \quad \sum_{i \in I} t^i = 0.$$

Each vector t^i correspond to the net trade of agent i. We follow Kajii and Ui [17] and introduce the concepts of *ex-ante* and *weak interim* efficiency.

Definition 2.3. The allocation $\boldsymbol{e}$ is

- *ex-ante efficient* if there does not exist a feasible trade $\boldsymbol{t}$ such that each agent i prefers at the ex-ante stage the contingent consumption $e^i + t^i$ to e^i;
- *weakly interim efficient* if there does not exist a feasible trade $\boldsymbol{t}$ such that each agent i prefers at every interim stage $\pi \in \Pi^i$ the contingent consumption $e^i + t^i$ to e^i.[4]

In other words, the allocation $\boldsymbol{e}$ is not ex-ante efficient if and only if there exists a feasible trade $\boldsymbol{t}$ such that

$$\forall i \in I, \quad e^i + t^i \in \mathrm{Pref}_\Omega^i(e^i).$$

The allocation $\boldsymbol{e}$ is not weakly interim efficient if and only if there exists a feasible trade $\boldsymbol{t}$ such that

$$\forall i \in I, \forall \pi \in \Pi^i, \quad e^i + t^i \in \mathrm{Pref}_\pi^i(e^i).$$

In order to provide a characterization of efficiency in terms of primitives it is important to characterize the set of net trades t^i such that the associated contingent consumption $e^i + t^i$ is strictly preferred to the initial endowment e^i when agents have MEU-preferences. For that purpose, we need to introduce the concept of active belief.

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[4] We will introduce a concept of strong interim efficiency in Section 7.

Definition 2.4. Fix an agent i and a set $Q \in \mathcal{P}$ of beliefs. We denote by $\mathrm{Act}^i(Q)$ the set of beliefs $p \in Q$ that minimize $\mathbb{E}^p[u^i(e^i)]$ over Q, i.e.,

$$\mathrm{Act}^i(Q) = \mathrm{argmin}\{\mathbb{E}^p[u^i(e^i)]: \ p \in Q\}.$$

Any belief in $\mathrm{Act}^i(Q)$ is called an *active belief* in Q at e^i.

Since we allow for risk-averse agents, we also need to introduce the concept of risk-adjusted belief.

Definition 2.5. Fix an agent i and a belief $p \in \mathrm{Prob}(\Omega)$. The *risk-adjusted* belief $\mathrm{RA}^i(p)$ is the probability measure in $\mathrm{Prob}(\Omega)$ defined by[5]

$$\forall \omega \in \Omega, \quad \mathrm{RA}^i(p)(\omega) = \frac{p(\omega)\nabla u^i(e^i(\omega))}{\mathbb{E}^p[\nabla u^i(e^i)]}.$$

Given a set $Q \in \mathcal{P}$ of beliefs, we denote by $\mathrm{RA}^i(Q)$ the set of risk-adjusted beliefs defined by

$$\mathrm{RA}^i(Q) = \bigcup_{q \in Q}\{\mathrm{RA}^i(q)\}.$$

Adapting the arguments in Rigotti, Shannon and Stralecki [24] we can prove the following lemma. This is the crucial technical result of this paper. We provide a detailed proof for more general preferences in Section 6.

Lemma 2.1. *Fix an agent i, a set of beliefs $Q \in \mathcal{P}$ and a net trade $t^i \in \mathbb{R}^\Omega$ such that $e^i + t^i \geqslant 0$.*

- *If $e^i + t^i$ satisfies $\mathbb{E}_Q[u^i(e^i + t^i)] > \mathbb{E}_Q[u^i(e^i)]$ then*

$$\forall p \in \mathrm{RA}^i \circ \mathrm{Act}^i(Q), \quad \mathbb{E}^p[t^i] > 0. \tag{2.1}$$

- *Reciprocally, if t^i is such that (2.1) is satisfied then there exists $\varepsilon > 0$ small enough such that $\mathbb{E}_Q[u^i(e^i + \eta t^i)] > \mathbb{E}_Q[u^i(e^i)]$ for every $\eta \in (0, \varepsilon)$.*

A direct consequence of this lemma and the fundamental theorem of welfare economics is the following characterization of ex-ante efficiency.[6]

Proposition 2.1. *The allocation e is ex-ante efficient if and only if*

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{Act}^i(P^i) \neq \emptyset.$$

It is natural to investigate whether a similar characterization is possible for weak interim efficiency.

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[5] If $f : [0, \infty) \to \mathbb{R}$ is differentiable on $(0, \infty)$, we denote by $\nabla f(\alpha)$ the differential of f at $\alpha > 0$.

[6] See Proposition 2 and Proposition 7 in Rigotti, Shannon and Stralecki [24] or Proposition 5 in Kajii and Ui [17]. We also refer to Dana [7], Samet [25], Tallon [27], Chateauneuf, Dana and Tallon [5], Billot, Chateauneuf, Gilboa and Tallon [4], Dana [9] and Dana [10] for related results.

3. The characterization proposed by Kajii and Ui

Kajii and Ui [17] introduced the key concept of compatible priors which is a natural way to construct a family of priors when starting from a family of posteriors.

Definition 3.1. Fix an agent i and a family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs where for each interim signal $\pi \in \Pi^i$, the set $Q(\pi)$ belongs to $\mathcal{P}(\pi)$. A probability $p \in \text{Prob}(\Omega)$ is said to be a *Q-compatible prior* if for every interim signal $\pi \in \Pi^i$, the conditional probability $p(\cdot \mid \pi)$, when it exists, belongs to the set $Q(\pi)$. The set of all Q-compatible priors is denoted by $\text{CP}^i(Q)$.

Kajii and Ui [17] showed that the set of Q-compatible priors is actually the convex hull of the union of all sets $Q(\pi)$, i.e.,

$$\text{CP}^i(Q) = \text{co} \bigcup_{\pi \in \Pi^i} Q(\pi).$$

In particular the set $\text{CP}^i(Q)$ is non-empty, convex and closed, i.e., it belongs to $\mathcal{P}$.

Unfortunately, Kajii and Ui [17] did not propose a "general" characterization of weak interim efficiency similar to the characterization given in Proposition 2.1 for ex-ante efficiency. They propose a sufficient condition and prove that this condition is necessary provided that the interim utility of the initial endowment is independent of the signal received, i.e., the following mapping

$$\pi \mapsto \mathbb{E}_{\Phi^i(\pi)}\big[u^i(e^i)\big]$$

is constant over Π^i for every agent i.

Definition 3.2. The allocation e is said to have *constant interim utility* if for every agent i, the utility at an interim stage of the contingent consumption e^i is independent of the signal received, i.e.,

$$\forall i \in I, \ \forall \pi, \sigma \in \Pi^i, \quad \mathbb{E}_{\Phi^i(\pi)}\big[u^i(e^i)\big] = \mathbb{E}_{\Phi^i(\sigma)}\big[u^i(e^i)\big].$$

Kajii and Ui [17] proved the following (partial) characterization.

Proposition 3.1.

(a) *The allocation e is weakly interim efficient if* [7]

$$\bigcap_{i \in I} \text{RA}^i \circ \text{Act}^i \circ \text{CP}^i(\Phi^i) \neq \emptyset. \tag{3.1}$$

(b) *If the allocation e has constant interim utility then condition* (3.1) *is also necessary.*

In order to provide a necessary and sufficient condition for weak interim efficiency, Kajii and Ui [17] introduced the concept of full insurance at the interim stage.

Definition 3.3. A contingent consumption bundle x has the *full-insurance property at the interim stage* for agent i if it is *privately measurable* in the sense that the restriction of x to each signal $\pi \in \Pi^i$ is constant.

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[7] We abuse notations by writing $\text{RA}^i \circ \text{Act}^i \circ \text{CP}^i(\Phi^i)$ instead of $\text{RA}^i[\text{Act}^i[\text{CP}^i(\Phi^i)]]$.

Kajii and Ui [17] proposed the following necessary and sufficient condition when the allocation *e* is privately measurable.

Proposition 3.2. *Assume that the allocation **e** is privately measurable in the sense that for each agent i, the contingent consumption bundle e^i has the full-insurance property at the interim stage. Then the allocation **e** is weakly interim efficient if and only if*

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{CP}^i\left(\Phi^i\right) \neq \emptyset. \tag{3.2}$$

4. A necessary and sufficient condition for weak interim efficiency

We propose to improve the latter results by exhibiting a necessary and sufficient condition. For expositional reasons we abuse notations by writing $\mathrm{CP}^i \circ \mathrm{RA}^i \circ \mathrm{Act}^i(\Phi^i)$ instead of

$$\mathrm{CP}^i\left[\left(\mathrm{RA}^i\left[\mathrm{Act}^i\left(\Phi^i(\pi)\right)\right]\right)_{\pi \in \Pi^i}\right].$$

Theorem 4.1. *The allocation **e** is weakly interim efficient if and only if*

$$\bigcap_{i \in I} \mathrm{CP}^i \circ \mathrm{RA}^i \circ \mathrm{Act}^i\left(\Phi^i\right) \neq \emptyset. \tag{4.1}$$

*In other words, the allocation **e** is weakly interim efficient if and only if there exists a probability measure $q \in \mathrm{Prob}(\Omega)$ and for each i, a family $r^i = (r^i_\pi)_{\pi \in \Pi^i}$ with r^i_π an active belief of $\Phi^i(\pi)$ at e^i such that*

$$\forall i \in I, \quad q = \sum_{\pi \in \Pi^i} q(\pi)\,\mathrm{RA}^i\left(r^i_\pi\right).$$

The proof will be a very simple consequence of Proposition 2.1. To see this, we need the following intermediate result.

Lemma 4.1. *The allocation **e** is weakly interim efficient if and only if there does not exist a feasible trade **t** such that*

$$\forall i \in I, \ \forall p \in \bigcup_{\pi \in \Pi^i} \mathrm{RA}^i \circ \mathrm{Act}^i\left(\Phi^i(\pi)\right), \quad \mathbb{E}^p\left[t^i\right] > 0. \tag{4.2}$$

Proof of Lemma 4.1. We first prove the "if" part. Assume that there does not exist a feasible trade satisfying (4.2) but the allocation *e* is not weakly interim efficient. Then there exists a feasible trade *t* such that

$$\forall i \in I, \ \forall \pi \in \Pi^i, \quad \mathbb{E}_{\Phi^i(\pi)}\left[u^i\left(e^i + t^i\right)\right] > \mathbb{E}_{\Phi^i(\pi)}\left[u^i\left(e^i\right)\right].$$

Following Lemma 2.1 we must have

$$\forall i \in I, \ \forall \pi \in \Pi^i, \ \forall p \in \mathrm{RA}^i \circ \mathrm{Act}^i\left(\Phi^i(\pi)\right), \quad \mathbb{E}^p\left[t^i\right] > 0.$$

This contradicts the assumption that there does not exist a feasible trade satisfying (4.2).

We now prove the "only if" part. Assume that the allocation *e* is weakly interim efficient but there exists a feasible trade τ such that

$$\forall i \in I, \ \forall p \in \bigcup_{\pi \in \Pi^i} \mathrm{RA}^i \circ \mathrm{Act}^i\left(\Phi^i(\pi)\right), \quad \mathbb{E}^p\left[\tau^i\right] > 0.$$

Fix an agent i and a signal $\pi \in \Pi^i$. We have

$$\forall p \in \mathrm{RA}^i \circ \mathrm{Act}^i\big(\Phi^i(\pi)\big), \quad \mathbb{E}^p\big[\tau^i\big] > 0.$$

It follows from Lemma 2.1 that there exists $\varepsilon_\pi^i > 0$ such that

$$\forall \eta \in \big(0, \varepsilon_\pi^i\big), \quad \mathbb{E}_{\Phi^i(\pi)}\big[u^i\big(e^i + \eta\tau^i\big)\big] > \mathbb{E}_{\Phi^i(\pi)}\big[u^i\big(e^i\big)\big].$$

We let $\varepsilon > 0$ be defined by

$$\varepsilon = \min\big\{\varepsilon_\pi^i \colon i \in I \text{ and } \pi \in \Pi^i\big\}$$

and for each i, we pose $t^i = \varepsilon\tau^i$. The allocation $\boldsymbol{t} = (t^i)_{i \in I}$ is a feasible trade such that

$$\forall i \in I, \ \forall \pi \in \Pi^i, \quad e^i + t^i \in \mathrm{Pref}_\pi^i\big(e^i\big)$$

which leads to a contradiction. $\square$

The proof of Theorem 4.1 follows from Lemma 4.1 and Proposition 2.1. We provide the straightforward details hereafter.

Proof of Theorem 4.1. Consider the *modified* economy $\widehat{\mathcal{E}}$ where

- each agent i is risk-neutral in the sense that his utility index $\widehat{u}^i$ is linear, i.e., $\widehat{u}^i(c) = c$;
- each agent i's set of priors $\widehat{P}^i$ is defined by

$$\widehat{P}^i = \mathrm{co} \bigcup_{\pi \in \Pi^i} \mathrm{RA}^i \circ \mathrm{Act}^i\big(\Phi^i(\pi)\big).$$

According to Lemma 4.1, the allocation $\boldsymbol{e}$ is weakly interim efficient for the economy $\mathcal{E}$ if and only if it is ex-ante efficient for the economy $\widehat{\mathcal{E}}$. Observe that $\widehat{P}^i$ coincides with $\mathrm{CP}^i \circ \mathrm{RA}^i \circ \mathrm{Act}^i(\Phi^i)$. Applying Proposition 2.1, we obtain the desired result. $\square$

We provide hereafter an example where Theorem 4.1 can be applied but not the results in Kajii and Ui [17].

Example 4.1. Consider an exchange economy with two risk-neutral agents $I = \{i_1, i_2\}$ and four states of nature $\Omega = \{\omega_1, \ldots, \omega_4\}$.[8] Agent i_1 may receive two signals $\Pi^{i_1} = \{a, b\}$ with $a = \{\omega_1, \omega_2\}$ and $b = \{\omega_3, \omega_4\}$. His posterior beliefs are given by

$$\Phi^{i_1}(a) = \big\{p \in \mathrm{Prob}(\Omega) \colon p(\omega_1) + p(\omega_2) = 1 \text{ and } 1/4 \leqslant p(\omega_1) \leqslant 1/2\big\}$$

and

$$\Phi^{i_1}(b) = \big\{p \in \mathrm{Prob}(\Omega) \colon p(\omega_3) + p(\omega_4) = 1 \text{ and } 1/4 \leqslant p(\omega_3) \leqslant 1/2\big\}.$$

Agent i_1's contingent bundle is $e^{i_1} = (1, 3, 3, 1)$. Agent i_2 has no private information, i.e., $\Pi^{i_2} = \{\Omega\}$ and his posterior beliefs are represented by a single probability

$$\Phi^{i_2}(\Omega) = \big\{(1/4, 1/4, 1/8, 3/8)\big\}.$$

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[8] The utility index function u^i of each agent i is defined by $u^i(c) = c$ for each $c \geqslant 0$.

Agent i_2's contingent bundle is any interior vector $e^{i_2} \in \mathbb{R}_{++}^{\Omega}$. We can compute the set of active priors for agent i_1:

$$\mathrm{Act}^{i_1}\big(\Phi^{i_1}(a)\big) = \big\{(1/2, 1/2, 0, 0)\big\} \quad \text{and} \quad \mathrm{Act}^{i_1}\big(\Phi^{i_1}(b)\big) = \big\{(0, 0, 1/4, 3/4)\big\}.$$

It follows that

$$\mathrm{CP}^{i_1} \circ \mathrm{Act}^{i_1}\big(\Phi^{i_1}\big) = \big\{ p \in \mathrm{Prob}(\Omega) \colon \ p(\omega_1) = p(\omega_2) \text{ and } p(\omega_4) = 3p(\omega_3) \big\}.$$

Since the unique posterior belief of agent i_2 belongs to $\mathrm{CP}^{i_1} \circ \mathrm{Act}^{i_1}(\Phi^{i_1})$, we can apply Theorem 4.1 to conclude that the allocation e is weakly interim efficient. Since the interim expected utility of agent i_1 is not constant, we cannot apply Proposition 6 in Kajii and Ui [17].[9] Neither can we apply Proposition 7 in Kajii and Ui [17] since e^{i_1} does not have the full-insurance property at the interim stage.

5. Relation with the literature

In order to simplify the comparison between our characterization result and those presented in Kajii and Ui [17], we propose to state some properties satisfied by the operators Act^i, CP^i and RA^i. The details of the proofs are postponed to appendices.

Lemma 5.1. *For every agent i and every family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs $Q(\pi) \in \mathcal{P}(\pi)$, we have*

$$\mathrm{CP}^i \circ \mathrm{RA}^i(Q) = \mathrm{RA}^i \circ \mathrm{CP}^i(Q).$$

Remark 5.1. As a consequence of Theorem 4.1 and Lemma 5.2 we obtain the following equivalent characterization: The allocation e is weakly interim efficient if and only if

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{CP}^i \circ \mathrm{Act}^i\big(\Phi^i\big) \neq \emptyset. \tag{5.1}$$

Lemma 5.2. *For every agent i and every family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs $Q(\pi) \in \mathcal{P}(\pi)$, we always have*

$$\mathrm{Act}^i \circ \mathrm{CP}^i(Q) \subset \mathrm{CP}^i \circ \mathrm{Act}^i(Q).$$

The converse inclusion is true if e has constant interim utility.

The partial characterization proved by Kajii and Ui [17] (and presented in Proposition 3.1) follows from our main characterization result (Theorem 4.1) and the two preceding lemmas. One may want to compare Proposition 3.2 with our general necessary and sufficient condition. The following characterization is a straightforward consequence of Theorem 4.1.

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[9] We can check that

$$\mathbb{E}_{\Phi^{i_1}(a)}\big[u^{i_1}(e^{i_1})\big] = 2 \quad \text{and} \quad \mathbb{E}_{\Phi^{i_1}(b)}\big[u^{i_1}(e^{i_1})\big] = 3/2.$$

Observe moreover that

$$\mathrm{Act}^{i_1} \circ \mathrm{CP}^{i_1}\big(\Phi^{i_1}\big) = \big\{(0, 0, 1/4, 3/4)\big\}.$$

Since the unique posterior belief of agent i_2 does not belong to $\mathrm{Act}^{i_1} \circ \mathrm{CP}^{i_1}(\Phi^{i_1})$, condition (7) in Kajii and Ui [17] cannot be necessary.

Proposition 5.1. *Assume that the allocation e is privately measurable. Then the allocation e is weakly interim efficient if and only if*

$$\bigcap_{i \in I} \mathrm{CP}^i(\Phi^i) \neq \emptyset. \tag{5.2}$$

Proof of Proposition 5.1. Assume that the allocation e is privately measurable. It is sufficient to prove that for each agent i and each private signal $\pi \in \Pi^i$, we have $\mathrm{RA}^i \circ \mathrm{Act}^i(\Phi^i(\pi)) = \Phi^i(\pi)$. Since e^i is constant on π, we denote by $e^i(\pi)$ its value. Observe that for each $r_\pi \in \Phi^i(\pi)$ we have $\mathbb{E}^{r_\pi}[u^i(e^i)] = u^i(e^i(\pi))$. In particular, any posterior belief is active, i.e., $\mathrm{Act}^i[\Phi^i(\pi)] = \Phi^i(\pi)$.

We propose now to prove that there is no need to adjust for risk. This is very intuitive since there is no risk. More precisely, let κ_π be a probability measure in $\mathrm{RA}^i(\Phi^i(\pi))$. There exists a posterior belief $r_\pi \in \Phi^i(\pi)$ such that

$$\forall \omega \in \Omega, \quad \kappa_\pi(\omega) = \frac{1}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} \nabla u^i(e^i(\omega)) r_\pi(\omega).$$

Since e^i is constant on π, the function $\omega \mapsto \nabla u^i(e^i(\omega))$ is also constant on π. We denote by $\nabla u^i(e^i(\pi))$ its value. It follows that $\mathbb{E}^{r_\pi}[\nabla u^i(e^i)] = \nabla u^i(e^i(\pi))$ implying that $\kappa_\pi = r_\pi$. $\square$

Our necessary and sufficient condition (5.2) seems to be different from (3.2) the condition proposed by Kajii and Ui [17]. Actually, when the allocation e has the full-insurance property at the interim stage, there is no need to adjust Φ-compatible priors to risk.

Lemma 5.3. *If the allocation e is privately measurable then*

$$\forall i \in I, \quad \mathrm{RA}^i \circ \mathrm{CP}^i(\Phi^i) = \mathrm{CP}^i(\Phi^i).$$

The proofs of Lemma 5.1, Lemma 5.2 and Lemma 5.3 can be found in Appendices A, B and C respectively.

6. General convex preferences and subjective beliefs

Until now we assumed that agents have MaxMin expected utility preferences. When uncertainty is represented by multiple priors and posteriors, there are other modelings of preferences: the incomplete preferences model of Bewley [3], the convex Choquet model of Schmeidler [26], the smooth second-order prior models of Klibanoff, Marinacci and Mukerji [18] and Nau [22], the second-order expected utility model of Ergin and Gul [12], the confidence preferences model of Chateauneuf and Faro [6], the multiplier model of Hansen and Sargent [16], and the variational preferences model of Maccheroni, Marinacci and Rustichini [19]. We propose to follow the approach initiated by Rigotti, Shannon and Stralecki [24] by considering a broad class of convex preferences which encompasses as special cases all the aforementioned models.

Each agent i has an ex-ante preference relation $\succ_\Omega^i$ on contingent consumption bundles in $\mathbb{R}_+^\Omega$ defining the correspondence $\mathrm{Pref}_\Omega^i : \mathbb{R}_+^\Omega \to \mathbb{R}_+^\Omega$ of strictly preferred contingent consumption bundles:

$$\forall x \in \mathbb{R}_+^\Omega, \quad \mathrm{Pref}_\Omega^i(x) = \{y \in \mathbb{R}_+^\Omega : y \succ_\Omega^i x\}.$$

Similarly, for each possible interim signal $\pi \in \Pi^i$, agent i is endowed with an interim preference relation $\succ_\pi^i$ on consumption bundles contingent to the information π, defining the correspon-

dence $\mathrm{Pref}_\pi^i : \mathbb{R}_+^\pi \to \mathbb{R}_+^\pi$ of strictly preferred contingent consumption bundles at the interim stage.

If x is a vector in $\mathbb{R}^\Omega$ and σ is a subset of Ω, we denote by $x \mid \sigma$ the restriction of x to σ, i.e., $x \mid \sigma$ is the vector in $\mathbb{R}^\sigma$ defined by $(x \mid \sigma)(\omega) = x(\omega)$ for each $\omega \in \sigma$. Preference relations are assumed to satisfy the following properties

Assumption 6.1. For each agent i, for each $\sigma \in \{\Omega\} \cup \Pi^i$, the binary relation $\succ_\sigma^i$ is

(a) irreflexive, i.e., $x \notin \mathrm{Pref}_\sigma^i(x)$ for all $x \in \mathbb{R}_+^\sigma$;
(b) convex, i.e., the set $\mathrm{Pref}_\sigma^i(x)$ is convex for all $x \in \mathbb{R}_+^\sigma$;
(c) monotone, i.e., $x + h \in \mathrm{Pref}_\sigma^i(x)$ for all $x, h \in \mathbb{R}_+^\sigma$ and h interior[10];
(d) continuous, i.e., the set $\mathrm{Pref}_\sigma^i(x)$ is open in $\mathbb{R}_+^\sigma$.

Following Rigotti, Shannon and Stralecki [24] we introduce the concepts of ex-ante and interim subjective beliefs.

Definition 6.1. The set Sub_Ω^i of *ex-ante subjective beliefs* (or *subjective priors*) of agent i at e^i is

$$\mathrm{Sub}_\Omega^i = \left\{ p \in \mathrm{Prob}(\Omega) \colon \mathbb{E}^p[x] \geqslant \mathbb{E}^p[e^i], \ \forall x \in \mathrm{Pref}_\Omega^i(e^i) \right\}.$$

The set Sub_π^i of *interim subjective beliefs* (or *subjective posteriors*) of agent i at e^i with private information $\pi \in \Pi^i$ is

$$\mathrm{Sub}_\pi^i = \left\{ r \in \mathrm{Prob}(\pi) \colon \mathbb{E}^r[y] \geqslant \mathbb{E}^r[e^i \mid \pi], \ \forall y \in \mathrm{Pref}_\Omega^i(e^i \mid \pi) \right\}.$$

For any $\sigma \in \{\Omega\} \cup \Pi^i$, the set Sub_σ^i is non-empty. Indeed, the vector $e^i \mid \sigma$ does not belong to the convex set $\mathrm{Pref}_\Omega^i(e^i \mid \sigma)$. Applying the Separating Hyperplane Theorem there exists a non-zero vector $\xi \in \mathbb{R}_+^\sigma$ supporting the set $\mathrm{Pref}_\Omega^i(e^i \mid \sigma)$ at $e^i \mid \sigma$, i.e.,

$$\forall y \in \mathrm{Pref}_\sigma^i(e^i \mid \sigma), \quad \xi \cdot y = \sum_{\omega \in \sigma} \xi(\omega) y(\omega) \geqslant \sum_{\omega \in \sigma} \xi(\omega) e^i(\omega) = \xi \cdot (e^i \mid \sigma).$$

Fix a state $\omega \in \sigma$, $\varepsilon > 0$ and let h_ε be the vector in $\mathbb{R}_{++}^\sigma$ defined by $h_\varepsilon(\omega') = \varepsilon$ for every $\omega' \neq \omega$ and $h_\varepsilon(\omega) = 1$. Since the binary relation $\succ_\sigma^i$ is monotone we get $\xi \cdot h_\varepsilon \geqslant 0$. Passing to the limit when ε tends to 0 we can conclude that the vector ξ is a non-zero vector in $\mathbb{R}_+^\sigma$. Since ξ is not zero we can normalize ξ such that r defined by

$$\forall \omega \in \sigma, \quad r(\omega) = \frac{\xi(\omega)}{\sum_{\omega' \in \sigma} \xi(\omega')}$$

can be assimilated to a probability measure in $\mathrm{Prob}(\sigma)$. For any vector $y \in \mathbb{R}^\sigma$, the inner product $r \cdot y$ is then denoted by $\mathbb{E}^r[y]$. Abusing notations we will assimilate the probability measure $r \in \mathrm{Prob}(\sigma)$ to its natural extension in $\mathrm{Prob}(\Omega)$ by posing $r(\omega) = 0$ if ω does not belong to σ.

The set Sub_σ^i is also compact and convex, therefore it belongs to $\mathcal{P}(\sigma)$. We can adapt the arguments in Rigotti, Shannon and Stralecki [24] to prove the following intermediary result which is the counterpart of Lemma 6.1 for general convex preferences.

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[10] The vector $h \in \mathbb{R}_+^\sigma$ is interior if it is strictly positive, i.e., $h(\omega) > 0$ for every $\omega \in \sigma$.

Lemma 6.1. *Fix an agent i, a signal $\sigma \in \{\Omega\} \cup \Pi^i$ and a vector $t^i \in \mathbb{R}^\sigma$ such that $(e^i \mid \sigma) + t^i$ belongs to $\mathbb{R}_+^\sigma$.*

- *If $(e^i \mid \sigma) + t^i$ belongs to $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$ then*

$$\forall p \in \mathrm{Sub}_\sigma^i, \quad \mathbb{E}^p[t^i] > 0. \tag{6.1}$$

- *Reciprocally, if t^i is such that (6.1) is satisfied then there exists $\varepsilon > 0$ small enough such that*

$$\forall \eta \in (0, \varepsilon), \quad (e^i \mid \sigma) + \eta t^i \in \mathrm{Pref}_\sigma^i(e^i \mid \sigma).$$

For the sake of completeness, we provide a detailed proof.

Proof of Lemma 6.1. Assume that $(e^i \mid \sigma) + t^i$ belongs to $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$. We propose to prove that (6.1) is satisfied. Since $(e^i \mid \sigma)$ is strictly positive and $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$ is open in $\mathbb{R}_+^\sigma$, there exists $\alpha \in (0, 1]$ close enough to 1 such that $(e^i \mid \sigma) + \alpha t^i$ is strictly positive and still belongs to $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$. Since the set $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$ is open in $\mathbb{R}^\sigma$, there exists $\varepsilon > 0$ small enough such that[11]

$$(e^i \mid \sigma) + \alpha t^i - \varepsilon \mathbf{1}_\sigma \in \mathrm{Pref}_\sigma^i(e^i \mid \sigma).$$

It follows from the definition of Sub_σ^i that

$$\forall p \in \mathrm{Sub}_\sigma^i, \quad \mathbb{E}^p[t^i] \geqslant \frac{\varepsilon}{\alpha} > 0.$$

Assume now that $t^i \in \mathbb{R}^\sigma$ is such that $(e^i \mid \sigma) + t^i$ belongs to $\mathbb{R}_+^\sigma$ and (6.1) is satisfied. Since $e^i \mid \sigma$ is strictly positive, there exists $\bar{\varepsilon}$ such that for every $\varepsilon \in (0, \bar{\varepsilon}]$ the vector $(e^i \mid \sigma) + \varepsilon t^i$ belongs to $\mathbb{R}_+^\sigma$. We claim that there exists at least one $\varepsilon \in (0, \bar{\varepsilon}]$ such that $(e^i \mid \sigma) + \varepsilon t^i$ belongs to $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$. Assume by way of contradiction that

$$\big\{(e^i \mid \sigma) + \varepsilon t^i : \varepsilon \in (0, \bar{\varepsilon}]\big\} \cap \mathrm{Pref}_\sigma^i(e^i \mid \sigma) = \emptyset.$$

Applying the Separating Hyperplane Theorem there exists a non-zero vector $\xi \in \mathbb{R}^\sigma$ such that

$$\forall \varepsilon \in (0, \bar{\varepsilon}], \ \forall x \in \mathrm{Pref}_\sigma^i(e^i \mid \sigma), \quad \xi \cdot x \geqslant \xi \cdot \big[(e^i \mid \sigma) + \varepsilon t^i\big].$$

Letting ε tend to 0, we obtain that ξ supports $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$ at $(e^i \mid \sigma)$. Following a previous discussion we can prove that ξ belongs $\mathbb{R}_+^\sigma$. Normalizing if necessary, we can assume that ξ is a subjective belief, i.e., belongs to Sub_σ^i. For each $n \in \mathbb{N}$ we let

$$x_n = (e^i \mid \sigma) + \big(1/(n+1)\big)\mathbf{1}_\sigma.$$

Since the preference relation $\succ_\sigma^i$ is monotone we have $x_n \in \mathrm{Pref}_\sigma^i(e^i \mid \sigma)$ implying that[12]

$$\frac{1}{n+1} \geqslant \bar{\varepsilon}\,\mathbb{E}^\xi[t^i].$$

Passing to the limit this leads to the contradiction: $0 \geqslant \mathbb{E}^\xi[t^i]$. We have thus proved that there exists $\varepsilon \in (0, \bar{\varepsilon}]$ such that $(e^i \mid \sigma) + \varepsilon t^i$ belongs to $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$. We claim that actually $(e^i \mid \sigma) + \bar{\eta}t^i$ belongs to $\mathrm{Pref}_\sigma^i(e^i \mid \sigma)$ for every $\bar{\eta} \in (0, \varepsilon]$. Fix $\bar{\eta} \in (0, \varepsilon]$. Following the previous

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[11] The vector $\mathbf{1}_\sigma$ in $\mathbb{R}^\sigma$ is defined by $\mathbf{1}_\sigma(\omega) = 1$ for every $\omega \in \sigma$.

[12] As usual, for any vector $z \in \mathbb{R}^\sigma$ the notation $\xi \cdot z$ is replaced by $\mathbb{E}^\xi[z]$ since ξ is a probability measure defined on σ.

argument we can prove that there exists $\eta \in (0, \bar{\eta})$ such that $(e^i \mid \sigma) + \eta t^i$ belongs to $\mathrm{Pref}^i_\sigma (e^i \mid \sigma)$. Observe that $(e^i \mid \sigma) + \bar{\eta} t^i$ is a convex combination of $(e^i \mid \sigma) + \eta t^i$ and $(e^i \mid \sigma) + \varepsilon t^i$. Since the set $\mathrm{Pref}^i_\sigma (e^i \mid \sigma)$ is convex, we get the desired result. $\square$

We adapt the concepts of efficiency to this general framework.

Definition 6.2. The allocation e is

- *ex-ante efficient* if there does not exist a feasible trade t such that each agent i prefers at the ex-ante stage the contingent consumption $e^i + t^i$ to e^i in the sense that $e^i + t^i \succ^i_\Omega e^i$;
- *weakly interim efficient* if there does not exist a feasible trade t such that each agent i prefers at every interim stage $\pi \in \Pi^i$ the contingent consumption $(e^i \mid \pi) + (t^i \mid \pi)$ to $(e^i \mid \pi)$ in the sense that $(e^i \mid \pi) + (t^i \mid \pi) \succ^i_\pi (e^i \mid \pi)$.

As a direct consequence of Lemma 6.1 and the fundamental theorem of welfare economics, we obtain the following characterization of ex-ante efficiency due to Rigotti, Shannon and Stralecki [24].

Theorem 6.1. *The allocation e is ex-ante efficient if and only if*

$$\bigcap_{i \in I} \mathrm{Sub}^i_\Omega \neq \emptyset.$$

Following almost verbatim the arguments in the proof of Theorem 4.1 we obtain the following characterization.[13]

Theorem 6.2. *The allocation e is weakly interim efficient if and only if*

$$\bigcap_{i \in I} \mathrm{CP}^i \big[(\mathrm{Sub}^i_\pi)_{\pi \in \Pi^i} \big] \neq \emptyset.$$

Rigotti, Shannon and Stralecki [24] studied the relationships between the notion of subjective belief and those arising in several common models of ambiguity. We propose to interpret the two previous characterization results for the two models of ambiguity studied in Kajii and Ui [17]. One could also do the same for the other models studied in Rigotti, Shannon and Stralecki [24].[14]

6.1. Bewley's incomplete preferences

In this section we consider that each agent i's preferences are defined as follows:

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[13] For every interim signal $\pi \in \Pi^i$, a subjective belief $r_\pi \in \mathrm{Sub}^i_\pi$ can be interpreted as a probability measure in $\mathrm{Prob}(\Omega)$ by posing $r_\pi(\omega) = 0$ for every $\omega \notin \pi$. Therefore, we abuse notations and consider that Sub^i_π is a subset of $\mathrm{Prob}(\Omega)$ implying that the formula

$$\mathrm{CP}^i \big[(\mathrm{Sub}^i_\pi)_{\pi \in \Pi^i} \big]$$

is well defined.

[14] The Bewley's incomplete preference model is not studied in Rigotti, Shannon and Stralecki [24] since they restrict their attention to complete and transitive binary relations.

- ex-ante,

$$\forall x \in \mathbb{R}_+^\Omega, \quad \mathrm{Pref}_\Omega^i(x) = \left\{ y \in \mathbb{R}_+^\Omega \colon \forall p \in P^i, \ \mathbb{E}^p\big[u^i(y)\big] > \mathbb{E}^p\big[u^i(x)\big] \right\};$$

- for every interim signal $\pi \in \Pi^i$,[15]

$$\forall x \in \mathbb{R}_+^\pi, \quad \mathrm{Pref}_\pi^i(x) = \left\{ y \in \mathbb{R}_+^\pi \colon \forall r \in \Phi^i(\pi), \ \mathbb{E}^r\big[u^i(y)\big] > \mathbb{E}^r\big[u^i(x)\big] \right\}.$$

For these specific convex preferences we can compute explicitly the set of subjective beliefs.

Lemma 6.2. *For each agent i we have*

$$\mathrm{Sub}_\Omega^i = \mathrm{RA}^i\big(P^i\big) \quad and \quad \mathrm{Sub}_\pi^i = \mathrm{RA}^i\big(\Phi^i(\pi)\big), \quad \forall \pi \in \Pi^i.$$

The arguments of the proof are standard: the result follows from the concavity of the utility index u^i. As a direct consequence of Theorem 6.1, Theorem 6.2 and the previous lemma, we obtain the following necessary and sufficient conditions for ex-ante and weak interim efficiency.

Proposition 6.1. *Assume that all agents have Bewley's incomplete preferences. The allocation **e** is ex-ante efficient if and only if*

$$\bigcap_{i \in I} \mathrm{RA}^i\big(P^i\big) \neq \emptyset.$$

Proposition 6.1 corresponds to Proposition 1 in Kajii and Ui [17] which is due to Bewley [2] and Rigotti and Shannon [23].

Proposition 6.2. *Assume that all agents have Bewley's incomplete preferences. The allocation **e** is weakly interim efficient if and only if*

$$\bigcap_{i \in I} \mathrm{CP}^i \circ \mathrm{RA}^i\big(\Phi^i\big) \neq \emptyset.$$

Proposition 6.2 corresponds to Proposition 2 in Kajii and Ui [17].[16]

6.2. Gilboa–Schmeidler's MaxMin expected utility preferences

In this section we consider that each agent i's preferences are defined as follows:

- ex-ante,

$$\forall x \in \mathbb{R}_+^\Omega, \quad \mathrm{Pref}_\Omega^i(x) = \left\{ y \in \mathbb{R}_+^\Omega \colon \min_{p \in P^i} \mathbb{E}^p\big[u^i(y)\big] > \min_{p \in P^i} \mathbb{E}^p\big[u^i(x)\big] \right\};$$

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[15] If x is a vector in $\mathbb{R}_+^\pi$ and r is a probability in $\mathrm{Prob}(\pi)$ we let $\mathbb{E}^r[u^i(x)] = \sum_{\omega \in \pi} r(\omega)u^i(x(\omega))$.

[16] Actually in Kajii and Ui [17] the necessary and sufficient condition for weak interim efficiency is

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{CP}^i\big(\Phi^i\big) \neq \emptyset.$$

We proved (see Lemma 5.1) that this condition is equivalent to the one stated in Proposition 6.2.

- for every interim signal $\pi \in \Pi^i$,

$$\forall x \in \mathbb{R}_+^\pi, \quad \mathrm{Pref}_\pi^i(x) = \left\{ y \in \mathbb{R}_+^\pi : \min_{r \in \Phi^i(\pi)} \mathbb{E}^r[u^i(y)] > \min_{r \in \Phi^i(\pi)} \mathbb{E}^r[u^i(x)] \right\}.$$

For these specific convex preferences Rigotti, Shannon and Stralecki [24] have computed explicitly the set of subjective beliefs.[17]

Lemma 6.3. *For each agent i we have*

$$\mathrm{Sub}_\Omega^i = \mathrm{RA}^i \circ \mathrm{Act}^i(P^i) \quad and \quad \mathrm{Sub}_\pi^i = \mathrm{RA}^i \circ \mathrm{Act}^i(\Phi^i(\pi)), \quad \forall \pi \in \Pi^i.$$

As a direct consequence of Theorem 6.1, Theorem 6.2 and the previous lemma, we obtain the necessary and sufficient conditions for ex-ante and weak interim efficiency presented in Proposition 2.1 and Theorem 4.1.

7. Interim efficiency and common knowledge

In this section we consider the framework of the previous section where each agent i is endowed with an ex-ante preference relation $\succ_\Omega^i$ and an interim preference relation $\succ_\pi^i$ for every private signal $\pi \in \Pi^i$. We assume that Assumption 6.1 is satisfied, i.e., for each $\sigma \in \{\Omega\} \cup \Pi^i$, the preference relation $\succ_\sigma^i$ is irreflexive, convex, monotone and continuous.

The no-trade theorem of Milgrom and Stockey [20] says that, upon the new arrival of information to the agents, it cannot be commonly known among them that there are some trading opportunities which can make them mutually beneficial, provided that the previous allocation (before the arrival of the new information) is ex-ante efficient. To illustrate this we consider that the allocation e is the outcome of an ex-ante trade process and assume it is ex-ante efficient. If the state of nature is $s \in \Omega$, each agent i knows (and only knows) at the interim stage that the true state belongs to $\pi^i = \Pi^i(s)$. One should first define which objects agents may have incentives to trade. If agent i proposes to trade a consumption bundle $x_{\pi^i} : \pi^i \to \mathbb{R}_+$, he is revealing to the other agents his private signal π^i. We follow Wilson [29] by considering that agents do not want to communicate their private information. If we assume that the family $(\Pi^i)_{i \in I}$ of private partitions is common knowledge then agents can trade consumption bundles contingent to the common knowledge event $E = \Pi^c(s)$ where $\Pi^c(s)$ is the unique atom of the common knowledge partition Π^c containing s.[18] Let $(y_E^i)_{i \in I}$ be an allocation of consumption bundles $y_E^i \in \mathbb{R}_+^E$ contingent to the event E and feasible, i.e.,

$$\forall \omega \in E, \quad \sum_{i \in I} y_E^i(\omega) = \sum_{i \in I} e^i(\omega).$$

When would agents accept to exchange the allocation $(e^i \mid E)_{i \in I}$ with the allocation $(y_E^i)_{i \in I}$? Agent i should accept if the restriction $(y_E^i \mid \pi^i)$ is strictly preferred to $(e^i \mid \pi^i)$, i.e.,

$$(y_E^i \mid \pi^i) \succ_{\pi^i}^i (e^i \mid \pi^i) \quad \text{or equivalently} \quad (y_E^i \mid \pi^i) \in \mathrm{Pref}_{\pi^i}^i(e^i \mid \pi^i),$$

or if there is status-quo, i.e.,

$$(y_E^i \mid \pi^i) = (e^i \mid \pi^i).$$

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[17] See Lemma 1 in Rigotti, Shannon and Stralecki [24] and Appendix A in Kajii and Ui [17].

[18] Π^c is the finest partition among those which are coarser than all partitions $\{\Pi^i : i \in I\}$. See Aumann [1].

We denote by $\Pi^i(E)$ the set of all atoms $\pi \in \Pi^i$ contained in E. The set $\Pi^i(E)$ is the collection of all signals agent i may have received according to the common knowledge information. Assuming that each agents' characteristics are common knowledge, agent i can accept to trade only if for every signal $\pi \in \Pi^c(E)$ we also have

$$(y_E^i \mid \pi) \in \left\{ (e^i \mid \pi) \right\} \cup \mathrm{Pref}_\pi^i(e^i \mid \pi). \tag{7.1}$$

Indeed, if agent i accepts to trade despite the fact that there exists $\pi \in \Pi^i(E)$ such that (7.1) is not satisfied then the other agents will infer that the true state does not belong to π. Since we assume that each agent does not want to reveal information to the others, we are lead to consider the following concept of interim efficiency.

Definition 7.1. The allocation e is *status-quo interim efficient* if there does not exist a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and for every signal $\pi \in \Pi^i$, we have

$$(y^i \mid \pi) \in \left\{ (e^i \mid \pi) \right\} \cup \mathrm{Pref}_\pi^i(e^i \mid \pi)$$

and for at least one agent j and one signal $\pi^j \in \Pi^j$, we have

$$(y^j \mid \pi^j) \in \mathrm{Pref}_{\pi^j}^j(e^j \mid \pi^j).$$

Remark 7.1. One may define the concept of status-quo ex-ante efficiency as follows: the allocation e is status-quo ex-ante efficient if there does not exist a feasible allocation $(y^i)_{i \in I}$ such that for each agent i, we have

$$y^i \in \left\{ e^i \right\} \cup \mathrm{Pref}_\Omega^i(e^i)$$

and for at least one agent j we have

$$y^j \in \mathrm{Pref}_\Omega^j(e^j).$$

If an allocation is status-quo ex-ante efficient then it is ex-ante efficient. The converse is also true since ex-ante preferences are monotone and continuous.[19]

If the allocation e is status-quo interim efficient then it is weakly interim efficient. The converse is not true as illustrated by the following example.

Example 7.1. Consider an exchange economy with four states of nature $\Omega = \{\omega_1, \ldots, \omega_4\}$ and two risk-neutral agents $I = \{i_1, i_2\}$ with MEU-preferences. Each agent i may receive two signals $\Pi^i = \Pi$ with $\Pi = \{\alpha, \beta\}$, $\alpha = \{\omega_1, \omega_2\}$ and $\beta = \{\omega_3, \omega_4\}$. Contingent to every signal π, every agent i has a unique posterior, i.e., $\Phi^i(\pi)$ is a singleton. Moreover, we assume that agents have the same posterior contingent to the signal β, i.e., $\Phi^{i_1}(\beta) = \Phi^{i_2}(\beta)$, but have different posteriors contingent to the signal α, i.e., $\Phi^{i_1}(\alpha) \neq \Phi^{i_2}(\alpha)$. We propose to show that the allocation $(e^i)_{i \in I}$ is weakly interim efficient (as defined by Kajii and Ui [17]). Assume by way of contradiction

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[19] Let $(y^i)_{i \in I}$ be a feasible allocation such that $y^i = e^i$ or $y^i \succ_\Omega^i e^i$ for each agent i and $y^j \succ_\Omega^j e^j$ for some agent j. Since preferences are continuous, there exists α close enough to 1 such that the allocation $(z^i)_{i \in I}$ defined by $z^i = \alpha y^i + (1 - \alpha)e^i$ is feasible, interior and satisfies $z^i = e^i$ or $z^i \succ_\Omega^i e^i$ for each agent i and $z^j \succ_\Omega^j e^j$. Since the binary relation $\succ_\Omega^j$ is continuous there exists $\varepsilon > 0$ small enough such that $x^j = z^j - \varepsilon \mathbf{1}_\Omega \succ_\Omega^j e^j$. For every agent $i \neq j$ we pose $x^i = z^i + \beta \mathbf{1}_\Omega$ where $\beta(\#I - 1) = \varepsilon$. Observe that the allocation $(x^i)_{i \in I}$ is feasible and satisfies $x^i \succ_\Omega^i e^i$ for each agent i.

that there exists a feasible allocation $(y^i)_{i \in I}$ such that $(y^i \mid \pi) \succ^i_\pi (e^i \mid \pi)$ for every agent i and every signal $\pi \in \Pi^i$. If we denote by r_β the common posterior contingent to the signal β, one must have $\mathbb{E}^{r_\beta}[y^i \mid \beta] > \mathbb{E}^{r_\beta}[e^i \mid \beta]$ for each i: this contradicts the feasibility of $(y^i)_{i \in I}$. We have thus proved that the allocation $(e^i)_{i \in I}$ is weakly interim efficient but it is not status-quo interim efficient. Indeed, since posteriors contingent to the signal α are different, agents have incentives to trade at the interim stage if the state of nature is ω_1 or ω_2.

We will show later on (see Corollary 7.1) that when the unique common knowledge event is the whole space, weak interim efficiency and status-quo interim efficiency are equivalent provided that preferences are strictly monotone as defined hereafter.

Definition 7.2. Interim preferences are said to be *strictly monotone* if for every agent i and every signal $\pi \in \Pi^i$ we have $x + h \succ^i_\pi x$ for every consumption bundles $x, h \in \mathbb{R}^\pi_+$ where $h \neq 0$.

Observe that interim MEU-preferences are strictly monotone if every posterior belief has full support in the sense that for every agent i, for every signal $\pi \in \Pi^i$, every posterior $r \in \Phi^i(\pi)$ satisfies $\operatorname{supp} r = \pi$.[20]

In most models with ambiguity aversion, the strict preference relation $\succ^i_\pi$ of agent i contingent to signal π is the strict part of a reflexive, complete and transitive binary relation.

Definition 7.3. Preferences are said *complete, transitive and convex* if for every agent i and every signal $\sigma \in \{\Omega\} \cup \Pi^i$, the binary relation $\succ^i_\sigma$ satisfies Assumption 6.1 and is the strict part of a reflexive, complete, transitive and convex binary relation $\succeq^i_\sigma$.[21]

When preferences are complete transitive and convex one may consider the concept of interim efficiency used by Milgrom and Stockey [20] and Morris [21].

Definition 7.4. Assume that preferences are complete transitive and convex. The allocation e is *strongly interim efficient* if there does not exist a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and for every signal $\pi \in \Pi^i$, we have

$$(y^i \mid \pi) \succeq^i_\pi (e^i \mid \pi)$$

and for at least one agent j and one signal $\pi^j \in \Pi^j$, we have

$$(y^j \mid \pi^j) \succ^j_{\pi^j} (e^j \mid \pi^j).$$

Remark 7.2. Assume that preferences are complete transitive and convex. If the allocation e is strongly interim efficient then it is status-quo interim efficient. The converse is true if interim preferences are strictly convex[22] or strictly increasing (see Remark 7.5).

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[20] If $r \in \operatorname{Prob}(\Omega)$ we denote by $\operatorname{supp} r$ the support of r defined by $\operatorname{supp} r = \{\omega \in \Omega : r(\omega) > 0\}$.

[21] We omit the standard definition of a reflexive, complete and transitive binary relation $\succeq^i_\sigma$. The binary relation $\succeq^i_\sigma$ is said convex when $\{y \in \mathbb{R}^\sigma_+ : y \succeq^i_\sigma x\}$ is convex for every $x \in \mathbb{R}^\sigma_+$.

[22] In the sense that for every agent $i \in I$, for every signal $\pi \in \Pi^i$ and every consumption bundle $z_\pi \in \mathbb{R}^\pi_+$,

$$\left[z_\pi \neq (e^i \mid \pi) \text{ and } z_\pi \succeq^i_\pi (e^i \mid \pi) \right] \implies \alpha z_\pi + (1 - \alpha)(e^i \mid \pi) \succ^i_\pi (e^i \mid \pi), \quad \forall \alpha \in (0, 1).$$

Remark 7.3. If preferences are complete transitive and convex then the set of subjective beliefs coincide with the set of probabilities (or prices) supporting upper contour sets as defined by Yaari [30] and Rigotti, Shannon and Stralecki [24]. More precisely, for every agent i and every signal $\sigma \in \{\Omega\} \cup \Pi^i$, the vector $p \in \mathrm{Prob}(\sigma)$ is a subjective belief if and only if $\mathbb{E}^p[y] \geqslant \mathbb{E}^p[x]$ for every $y \succeq^i_\sigma x$.

We proved that the existence of a common prior compatible with subjective posteriors is a necessary and sufficient condition for weak interim efficiency. To characterize status-quo and strong interim efficiency, we need to strengthen this condition.

Definition 7.5. Fix an agent i and a family $Q = (Q(\pi))_{\pi \in \Pi^i}$ of posterior beliefs where for each signal $\pi \in \Pi^i$, the set $Q(\pi)$ belongs to $\mathcal{P}(\pi)$. A probability $p \in \mathrm{Prob}(\Omega)$ is said to be *fully compatible* with posteriors of Q if for every signal $\pi \in \Pi^i$ we have $p(\pi) > 0$ and the conditional probability $p(\cdot \mid \pi)$ belongs to the set $Q(\pi)$. The set of all priors fully compatible with posteriors of Q is denoted by

$$\mathrm{CP}^i_+(Q).$$

Remark 7.4. A prior p is compatible with posteriors of Q if there exists $(\lambda_\pi, r_\pi)_{\pi \in \Pi^i}$ with $r_\pi \in Q(\pi)$ and λ a vector in $\mathrm{Prob}(\Pi^i)$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi.$$

The prior p is fully compatible with posteriors of Q if λ has a full support, i.e., $\lambda_\pi > 0$ for each possible signal $\pi \in \Pi^i$.

It is straightforward to check that existence of a common prior that is fully compatible with subjective posteriors is a sufficient condition for status-quo interim efficiency and strong interim efficiency (when defined).[23] This sufficient condition turns out to be necessary if interim preferences are strictly monotone. To provide a proof we will use the following characterization result.

Proposition 7.1. *Assume that interim preferences are strictly monotone. If there is a unique common knowledge event, i.e., $\Pi^c = \{\Omega\}$, then weak interim efficiency implies the existence of a common prior that is fully compatible with subjective posteriors.*

Proof of Proposition 7.1. Assume that e is weakly interim efficient. Applying Theorem 6.2 there exists a probability $p \in \mathrm{Prob}(\Omega)$ such that for each agent i and each signal $\pi \in \Pi^i$, there exists a subjective belief $r^i_\pi \in \mathrm{Sub}^i_\pi$ satisfying

$$\forall i \in I, \quad p = \sum_{\pi \in \Pi^i} p(\pi) r^i_\pi. \tag{7.2}$$

Fix an agent $k \in I$. Since p belongs to $\mathrm{Prob}(\Omega)$ there exists at least one signal $\sigma \in \Pi^k$ such that $p(\sigma) > 0$. We have to prove that $p(\pi) > 0$ for every agent i and every signal $\pi \in \Pi^i$. Using the strict monotonicity of preferences we obtain the following intermediary step.

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[23] See Appendix D for a detailed proof.

Claim 7.1. *Fix an agent i and a signal $\pi \in \Pi^i$ consistent with σ in the sense that $\pi \cap \sigma \neq \emptyset$. Then $p(\pi) > 0$.*

Proof of Claim 7.1. Since the preference relation $\succ_{\sigma}^k$ is strictly increasing the subjective belief r_{σ}^k has full support, i.e., $\operatorname{supp} r_{\sigma}^k = \sigma$. Since $p(\sigma) > 0$ we obtain that $p(\{\omega\}) > 0$ for every $\omega \in \sigma$, implying the desired result. □

Now fix an agent j and a signal $\mu \in \Pi^j$. Since Ω is an element of the common knowledge partition Π^c, there exists a finite chain

$$\big((i_1, \pi_1), \ldots, (i_n, \pi_n)\big)$$

such that $(i_1, \pi_1) = (k, \sigma)$, $(i_n, \pi_n) = (j, \mu)$ and

$$\forall s \in \{1, 2, \ldots, n-1\}, \quad \pi_s \cap \pi_{s+1} \neq \emptyset.$$

Applying recursively Claim 7.1 we get that $p(\mu) = p(\pi_n) > 0$. The probability p is a common prior that is fully compatible with subjective posteriors. □

The following equivalence result is a direct corollary of Proposition 7.1 and Appendix D.

Corollary 7.1. *Assume that interim preferences are strictly monotone. If there is a unique common knowledge event, i.e., $\Pi^c = \{\Omega\}$, then weak interim efficiency, status-quo interim efficiency and strong interim efficiency (when defined) are equivalent.*

Another consequence of Proposition 7.1 is the following characterization.

Proposition 7.2. *Assume that interim preferences are strictly monotone. The allocation e is status-quo interim efficient (or strongly interim efficient when preferences are complete transitive and convex) if and only if for every common knowledge atom $E \in \Pi^c$ the allocation $(e^i \mid E)_{i \in I}$ is weakly interim efficient when the state space is restricted to E.*

Remark 7.5. Assume that preferences are complete transitive and convex. If interim preferences are strictly increasing then we can apply Proposition 7.2 to show that the allocation e is status-quo interim efficient if and only if it is strongly interim efficient.

We can now state our main characterization result.

Theorem 7.1. *Assume that interim preferences are strictly monotone. Existence of a common prior that is fully compatible with subjective posteriors is a necessary and sufficient condition for status-quo interim efficiency (or strong interim efficiency when preferences are complete transitive and convex).*

In other words, when interim preferences are strictly monotone, the allocation e is status-quo interim efficient (or strongly interim efficient when defined) if and only if

$$\bigcap_{i \in I} \mathrm{CP}_+^i\big[(\mathrm{Sub}_{\pi}^i)_{\pi \in \Pi^i}\big] \neq \emptyset.$$

Proof of Theorem 7.1. The sufficiency part follows from Appendix D. Now assume that the allocation e is status-quo interim efficient (or strongly interim efficient when defined). Fix a common knowledge event $E \in \Pi^c$. It follows from Proposition 7.2 that the allocation $(e^i \mid E)_{i \in I}$ is weakly interim efficient when the state space is restricted to E. Applying Proposition 7.1 we get the existence of a probability $p_E \in \mathrm{Prob}(E)$ and for each agent i, a family $(\lambda_\pi^i, r_\pi^i)_{\pi \in \Pi^i(E)}$ with $r_\pi^i \in \mathrm{Sub}_\pi^i$ and $(\lambda_\pi^i)_{\pi \in \Pi^i(E)}$ a probability in $\mathrm{Prob}(E)$ with full support such that

$$\forall i \in I, \quad p_E = \sum_{\pi \in \Pi^i(E)} \lambda_\pi^i r_\pi^i.$$

We let p be the probability in $\mathrm{Prob}(\Omega)$ defined by

$$\forall \omega \in \Omega, \quad p(\omega) = p_{\Pi^c(\omega)}(\omega).^{24}$$

It is straightforward to check that p is a common prior fully compatible with subjective posteriors. $\square$

8. No speculative trade

For expositional reasons, we assume in this section that agents have MEU-preferences satisfying the assumptions of Section 2. Following Kajii and Ui [17], we propose to investigate under which conditions speculative trade is impossible.[25]

Definition 8.1. We say that there is no *weak speculative trade* if ex-ante efficiency of the allocation e implies that it is also weakly interim efficient.

As a direct consequence of Proposition 2.1 and Theorem 4.1 we get the following general characterization.

Proposition 8.1. *There is no weak speculative trade if and only if*

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{Act}^i(P^i) \neq \emptyset \quad \Longrightarrow \quad \bigcap_{i \in I} \mathrm{CP}^i \circ \mathrm{RA}^i \circ \mathrm{Act}^i(\Phi^i) \neq \emptyset. \tag{8.1}$$

The two general characterization results (Corollary 8 and Corollary 9) proposed by Kajii and Ui [17] follow from the previous result together with Lemmas 5.1–5.3.

Proposition 8.2. *Assume that the allocation e has constant interim utility. Then there is no weak speculative trade if and only if*

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{Act}^i(P^i) \neq \emptyset \quad \Longrightarrow \quad \bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{Act}^i \circ \mathrm{CP}^i(\Phi^i) \neq \emptyset. \tag{8.2}$$

Assume that the allocation e is privately measurable. Then there is no weak speculative trade if and only if

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{Act}^i(P^i) \neq \emptyset \quad \Longrightarrow \quad \bigcap_{i \in I} \mathrm{CP}^i(\Phi^i) \neq \emptyset. \tag{8.3}$$

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[24] Recall that $\Pi^c(\omega)$ is the unique atom of the common knowledge partition containing ω.

[25] Most of the results stated in this section have straightforward proofs. We propose in Appendix E the details of the arguments for Proposition 8.3, Corollaries 8.1 and 8.2.

Remark 8.1. In Kajii and Ui [17, Corollary 9], the condition (8.3) is replaced by

$$\bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{Act}^i(P^i) \neq \emptyset \quad \Longrightarrow \quad \bigcap_{i \in I} \mathrm{RA}^i \circ \mathrm{CP}^i(\Phi^i) \neq \emptyset. \tag{8.4}$$

We proved (see Lemma 5.3) that when the allocation e is privately measurable then for every i, we have $\mathrm{RA}^i \circ \mathrm{CP}^i(\Phi^i) = \mathrm{CP}^i(\Phi^i)$. This implies that both conditions (8.4) and (8.3) are equivalent.

The necessary and sufficient condition (8.1) imposes an abstract relation between the set of priors P^i and the family $\Phi^i = (\Phi^i(\pi))_{\pi \in \Pi^i}$ of posteriors. We propose to investigate sufficient conditions relating priors and posteriors to preclude weak speculative trade. A straightforward sufficient condition is proposed hereafter.

Proposition 8.3. *Assume that for each agent i, for every active prior $p^i \in \mathrm{Act}^i(P^i)$ and for every signal $\pi \in \Pi^i$ the conditional probability $p^i(\cdot \mid \pi)$, when it exists, is an active posterior, i.e.,*

$$\forall p^i \in \mathrm{Act}^i(P^i), \ \forall \pi \in \Pi^i, \quad p^i(\pi) > 0 \quad \Longrightarrow \quad p^i(\cdot \mid \pi) \in \mathrm{Act}^i(\Phi^i(\pi)). \tag{8.5}$$

Then there is no weak speculative trade.

Condition (8.5) is still an abstract relation between the set of priors and the family of posteriors. However, it is now simple to provide explicit conditions on the way agents "up-date" their beliefs to guarantee that weak speculative trade is impossible. Kajii and Ui [17] used the concept of *full Bayesian updating*. We consider a weaker concept.

Definition 8.2. We say that the set of posteriors Φ^i is *Bayesian consistent* with P^i if for every prior belief $p \in P^i$ and for every private signal $\pi \in \Pi^i$ plausible according to p, i.e., $p(\pi) > 0$, the conditional probability $p(\cdot \mid \pi)$ is a possible posterior, i.e.,

$$\forall p \in P^i, \ \forall \pi \in \Pi^i, \quad p(\pi) > 0 \quad \Longrightarrow \quad p(\cdot \mid \pi) \in \Phi^i(\pi). \tag{8.6}$$

As a simple corollary of Proposition 8.3 we obtain the following no-trade result.

Corollary 8.1. *Assume that for each agent i the set of posteriors is Bayesian consistent with the set of priors. If the allocation e is privately measurable (or equivalently satisfies the full insurance property at the interim stage) then there is no weak speculative trade.*

This is a slight generalization of Proposition 10 in Kajii and Ui [17] since we only assume that the set of posteriors is Bayesian consistent with priors, while Kajii and Ui [17] assumed that posteriors are the full Bayesian updating of priors in the sense that for every agent i,

$$\forall \pi \in \Pi^i, \quad \Phi^i(\pi) = \mathrm{cl}\{p(\cdot \mid \pi): \ p \in P^i \text{ and } p(\pi) > 0\}.$$

We can also obtain the no-trade result of Epstein and Schneider [11] and Wakai [28] as a simple corollary of Proposition 8.3. These authors proved that if priors are rectangular sets with respect to posteriors, then MEU-preferences are dynamically consistent which in turn implies that there is no weak speculative trade.[26] In order to define the concept of a rectangular prior set,

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[26] For the readers interested in dynamically consistent update rules for decision making under ambiguity, we refer to Hanany and Klibanoff [15] and the literature therein.

we introduce some notations. Fix a probability $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ in $\text{Prob}(\Pi^i)$ representing beliefs about the private signals of agent i. Let $r = (r_\pi)_{\pi \in \Pi^i}$ be a family of posteriors. The probability in $\text{Prob}(\Omega)$ defined by

$$\sum_{\pi \in \Pi^i} \lambda_\pi r_\pi$$

is denoted by $\lambda \otimes r$. If Λ^i is a set of probabilities in $\text{Prob}(\Pi^i)$, we denote by $\Lambda^i \otimes \Phi^i$ the set of all probabilities $\lambda \otimes r$ where $\lambda \in \Lambda^i$ and $r_\pi \in \Phi^i(\pi)$ for each $\pi \in \Pi^i$. Observe that the set $\text{CP}^i(\Phi^i)$ of Φ^i-compatible priors coincides with the set $\text{Prob}(\Pi^i) \otimes \Phi^i$.

Definition 8.3. The set of priors is *rectangular with respect to posteriors* (or equivalently P^i is Φ^i-rectangular) if there exists a closed and convex set Λ^i of beliefs in $\text{Prob}(\Pi^i)$ about the realization of private signals such that

$$P^i = \Lambda^i \otimes \Phi^i$$

and such that for every private signal $\pi \in \Pi^i$, there exists $\lambda \in \Lambda^i$ such that $\lambda_\pi > 0$.

If $P^i = \Lambda^i \otimes \Phi^i$ then Λ^i must coincide with $P^i(\Pi^i)$ the set of all probabilities $(p(\pi))_{\pi \in \Pi^i}$ defined by all priors $p \in P^i$. Observe that if Φ^i is Bayesian consistent with P^i then we have

$$P^i \subset P^i(\Pi^i) \otimes \Phi^i.$$

When the inclusion is replaced by an equality, we obtain that P^i is Φ^i-rectangular. It is straightforward to check if the set of priors is rectangular with respect to posteriors then posteriors are the full Bayesian updating of priors.[27]

Remark 8.2. One may want to investigate sets of priors that are *weakly* rectangular in the following sense: there exists a closed set Λ^i (possibly non-convex) of beliefs in $\text{Prob}(\Pi^i)$ such that[28]

$$P^i = \text{co}\big[\Lambda^i \otimes \Phi^i\big].$$

Since each set $\Phi^i(\pi)$ is convex, it is straightforward to check that $P^i = [\text{co}\,\Lambda^i] \otimes \Phi^i$, implying that any set of priors that is weakly rectangular is automatically rectangular.

As a simple corollary of Proposition 8.3 we obtain the following no-trade result which corresponds to Proposition 11 in Kajii and Ui [17].

Corollary 8.2. *If for each agent the set of priors is rectangular with respect to posteriors, then there is no weak speculative trade.*

We propose hereafter an example where Proposition 8.3 can be applied but the results in Kajii and Ui [17] cannot.

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[27] Actually we also have that Φ^i is the maximum likelihood updating of P^i in the sense that

$$\forall \pi \in \Pi^i, \quad \Phi^i(\pi) = \Big\{ p(\cdot \mid \pi) \colon\ p \in \underset{q \in P^i}{\arg\max}\, q(\pi) \Big\}.$$

[28] Given Definition 2.1, the set of priors $\Lambda^i \otimes \Phi^i$ and the set $\text{co}[\Lambda^i \otimes \Phi^i]$ define the same ex-ante preference relation.

Example 8.1. Consider the economy described in Example 4.1. We now fix prior beliefs for both agents. Since agent i_2 has no private information, we assume that he has a unique prior belief which coincides with his posterior belief, i.e., $P^{i_2} = \{(1/4, 1/4, 1/8, 3/8)\}$. To describe the priors of agent i_1, we propose the following parametrization of his posterior beliefs:

$$\forall \pi \in \Pi^{i_1} = \{a, b\}, \quad \Phi^{i_1}(\pi) = \{r_\pi^\theta \colon \theta \in \Theta\}$$

where $\Theta = [0, 1/4]$ and

$$r_a^\theta = (1/2 - \theta, 1/2 + \theta, 0, 0) \quad \text{and} \quad r_b^\theta = (0, 0, 1/4 + \theta, 3/4 - \theta).$$

The parameter θ can be interpreted as a scenario that agent i_1 considers plausible. This parameter describes agent i_1's ambiguity in the sense that θ cannot be observed and agent i_1 has no beliefs about its realization. We assume that contingent to a scenario θ, agent i_1 believes he will receive the signal a with probability $\lambda_a^\theta \in (0, 1)$. We pose $\lambda_b^\theta = 1 - \lambda_a^\theta$ such that λ^θ belongs to $\mathrm{Prob}(\Pi^{i_1})$. The prior beliefs of agent i_1 are constructed in the following way:

$$P^{i_1} = \mathrm{co}\{\lambda^\theta \otimes r^\theta \colon \theta \in \Theta\}.$$

For each plausible scenario θ, agent i_1's prior belief is represented by the probability measure

$$p^\theta = \lambda^\theta \otimes r^\theta = \left(\lambda_a^\theta((1/2) - \theta), \lambda_a^\theta(1/2 + \theta), \lambda_b^\theta(1/4 + \theta), \lambda_b^\theta(3/4 - \theta)\right).$$

Observe that Φ^{i_1} is the full Bayesian updating of P^{i_1} in the sense that $\Phi^{i_1}(\pi) = \{p(\cdot \mid \pi) \colon p \in P^{i_1}\}$. However, P^{i_1} is not Φ^{i_1}-rectangular.

We assume that $\theta \mapsto \lambda_a^\theta$ is strictly increasing with $\lambda_a^0 = 1/2$.[29] Since

$$\mathbb{E}^{p^\theta}\left[u^{i_1}\left(e^{i_1}\right)\right] = 2\theta + \frac{3}{2} + \frac{\lambda_a^\theta}{2}$$

it follows that the only active prior belief is p^0, i.e., $\mathrm{Act}^{i_1}(P^{i_1}) = \{\lambda^0 \otimes r^0\}$. Since p^0 coincides with the unique prior of agent i_2, the allocation e is ex-ante efficient. Observe that

$$p^0(\cdot \mid a) = r_a^0 \in \mathrm{Act}^{i_1}\left(\Phi^{i_1}(a)\right) \quad \text{and} \quad p^0(\cdot \mid b) = r_b^0 \in \mathrm{Act}^{i_1}\left(\Phi^{i_1}(b)\right)$$

implying that we can apply Proposition 8.3 to conclude that there is no weak speculative trade. Since the interim expected utility of agent i_1 is not constant, we cannot apply Corollary 8 in Kajii and Ui [17]. Neither can we apply Corollary 9 or Corollary 10 since e^{i_1} does not have the full-insurance property at the interim stage.

One may consider another concept of speculative trade where weak interim efficiency is replaced by strong interim efficiency.

Definition 8.4. We say that there is no *strong speculative trade* if ex-ante efficiency of the allocation e implies that it is also strongly interim efficient.

The concept of strong speculative trade corresponds to the one used in Milgrom and Stockey [20] and Morris [21]. The following example illustrates the differences with the concept of weak speculative trade used by Kajii and Ui [17].

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[29] Take for example $\lambda^\theta = (1/2 + \theta, 1/2 - \theta)$ for every $\theta \in [0, 1/4]$.

Example 8.2. We consider the economy defined in Example 7.1. We recall that contingent to each signal $\pi = \Pi = \{\alpha, \beta\}$, each agent i has a single posterior r_π^i, and that agents share the same posterior contingent to β, i.e., there exists r_β such that $r_\beta^i = r_\beta$. We also assume that endowments are privately measurable and priors are rectangular with respect to posteriors. Priors are defined as follows:

$$P^i = \text{Prob}(\Pi) \otimes \Phi^i = \text{co}\{r_\alpha^i, r_\beta\}.$$

There is no speculative trade since Corollary 8.1 and Corollary 8.2 apply. Observe that if $e^i(\omega_3) < e^i(\omega_2)$ then the allocation e is ex-ante efficient.[30] However, if the posteriors r^{i1} and r^{i2} are distinct, then agents can agree to trade after observing the signal α.

The previous example shows that Corollaries 8.1 and 8.2 are not valid if we replace "weak speculative trade" by "strong speculative trade". However, it is straightforward to adapt Proposition 8.3 in order to obtain a sufficient condition for no strong speculative trade.

Proposition 8.4. *Assume that for each agent i and each active prior $p^i \in \text{Act}^i(P^i)$, every signal $\pi \in \Pi^i$ is plausible and the conditional probability $p^i(\cdot \mid \pi)$ is an active posterior, i.e.,*

$$\forall p^i \in \text{Act}^i(P^i), \ \forall \pi \in \Pi^i, \quad p^i(\pi) > 0 \quad and \quad p^i(\cdot \mid \pi) \in \text{Act}^i(\Phi^i(\pi)). \tag{8.7}$$

Then there is no strong speculative trade.

Remark 8.3. We do not need to assume that interim preferences are strictly monotone for the above proposition to be valid. Indeed, the existence of a common prior fully compatible with subjective posteriors is a sufficient condition for strong interim efficiency even if interim preferences are not strictly monotone (see Appendix D).

Observe that if ex-ante preferences are strictly monotone then any active prior $p^i \in \text{Act}^i(P^i)$ will assign positive probability to any signal $\pi \in \Pi^i$. Actually we do not need to assume that preferences are strictly monotone.

Definition 8.5. Ex-ante preferences are said *signal-monotone* if for any contingent consumption bundles x and h in $\mathbb{R}_+^\Omega$ we have $x + h \succ_\Omega^i x$ when there exists at least one signal $\pi \in \Pi^i$ such that h is strictly positive on π, i.e., $h(\omega) > 0$ for every $\omega \in \pi$.

A straightforward consequence of Proposition 8.4 is the following counter-part of Corollary 8.1.

Corollary 8.3. *Assume that ex-ante preferences are signal-monotone and the set of posteriors is Bayesian consistent with the set of priors. If the allocation e is privately measurable then there is no strong speculative trade.*

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[30] Since e^i is privately measurable, for each signal π, we denote by $e^i(\pi)$ the constant value of e^i in π. Observe that since $P^i = \text{co}\{r_\alpha^i, r_\beta\}$, it follows that

$$\mathbb{E}^{r_\alpha^i}[e^i] = e^i(\alpha) = e^i(\omega_2) > e^i(\omega_3) = e^i(\beta) = \mathbb{E}^{r_\beta}[e^i].$$

We then get that $\text{Act}^i(P^i) = \{r_\beta\}$ and the allocation e is ex-ante efficient.

Signal-monotonicity is automatically satisfied when priors are fully rectangular with respect to posteriors as defined below.

Definition 8.6. The set of priors is *fully rectangular with respect to posteriors* if for each agent i we have $P^i = \Lambda^i \otimes \Phi^i$ where Λ^i is a closed convex set of beliefs in $\text{Prob}(\Pi^i)$ satisfying that every $\lambda \in \Lambda^i$ assigns positive probability $\lambda_\pi > 0$ to every signal $\pi \in \Pi^i$.

A straightforward consequence of Proposition 8.4 is the following counter-part of Corollary 8.2.

Corollary 8.4. *If for each agent the set of priors is fully rectangular with respect to posteriors, then there is no strong speculative trade.*

Observe that the set of priors in Example 8.2 are rectangular with respect to posteriors but not fully rectangular.

Appendix A. Proof of Lemma 5.1

We first prove that $\text{CP}^i \circ \text{RA}^i(Q) \subset \text{RA}^i \circ \text{CP}^i(Q)$.

Proof. Let $q \in \text{CP}^i \circ \text{RA}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\text{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$q = \sum_{\pi \in \Pi^i} \lambda_\pi \, \text{RA}^i(r_\pi) \quad \text{and} \quad r_\pi \in Q(\pi), \quad \forall \pi \in \Pi^i.$$

For each state $\omega \in \Omega$, we have

$$q(\omega) = \sum_{\pi \in \Pi^i} \frac{\lambda_\pi}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} r_\pi(\omega) \nabla u^i\big(e^i(\omega)\big) = \alpha \sum_{\pi \in \Pi^i} \mu_\pi r_\pi(\omega) \nabla u^i\big(e^i(\omega)\big)$$

where

$$\alpha = \sum_{\pi \in \Pi^i} \frac{\lambda_\pi}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} \quad \text{and} \quad \mu_\pi = \frac{1}{\alpha} \frac{\lambda_\pi}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]}, \quad \forall \pi \in \Pi^i.$$

We let p be defined by

$$p = \sum_{\pi \in \Pi^i} \mu_\pi r_\pi.$$

Observe that p belongs to $\text{CP}^i(Q)$ and $q = \beta \, \text{RA}^i(p)$ where $\beta = \alpha \mathbb{E}^p[\nabla u^i(e^i)]$. To finish the proof it is sufficient to show that β is equal to 1. Since for each signal $\pi \in \Pi^i$, the support of r_π is a subset of π, we have

$$\mathbb{E}^p[\nabla u^i(e^i)] = \sum_{\omega \in \Omega} \left[\sum_{\pi \in \Pi^i} \mu_\pi r_\pi(\omega) \right] \nabla u^i\big(e^i(\omega)\big)$$

$$= \sum_{\pi \in \Pi^i} \mu_\pi \sum_{\omega \in \pi} r_\pi(\omega) \nabla u^i\big(e^i(\omega)\big)$$

$$= \sum_{\pi \in \Pi^i} \mu_\pi \mathbb{E}^{r_\pi} \big[\nabla u^i \big(e^i\big) \big]$$

$$= \sum_{\pi \in \Pi^i} \frac{\lambda_\pi}{\alpha} = \frac{1}{\alpha}.$$

We have thus proved that $q = \mathrm{RA}^i(p)$ where p belongs to $\mathrm{CP}^i(Q)$. $\square$

Now, we prove that $\mathrm{RA}^i \circ \mathrm{CP}^i(Q) \subset \mathrm{CP}^i \circ \mathrm{RA}^i(Q)$.

Proof. Fix $p \in \mathrm{CP}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\mathrm{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi \quad \text{and} \quad r_\pi \in Q(\pi), \quad \forall \pi \in \Pi^i.$$

Now let q be the risk-adjusted prior $\mathrm{RA}^i(p)$. For each state $\omega \in \Omega$, we have

$$q(\omega) = \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi(\omega) \nabla u^i\big(e^i(\omega)\big)$$

$$= \sum_{\pi \in \Pi^i} \mu_\pi \frac{1}{\mathbb{E}^{r_\pi}[\nabla u^i(e^i)]} r_\pi(\omega) \nabla u^i\big(e^i(\omega)\big)$$

where

$$\mu_\pi = \frac{\lambda_\pi \times \mathbb{E}^{r_\pi}[\nabla u^i(e^i)]}{\mathbb{E}^p[\nabla u^i(e^i)]}.$$

Since the vector $(\mu_\pi)_{\pi \in \Pi^i}$ belongs to $\mathrm{Prob}(\Pi^i)$, we obtain that

$$q = \sum_{\pi \in \Pi^i} \mu_\pi \, \mathrm{RA}^i(r_\pi)$$

and therefore q belongs to $\mathrm{CP}^i \circ \mathrm{RA}^i(Q)$. $\square$

Appendix B. Proof of Lemma 5.2

We first prove that $\mathrm{Act}^i \circ \mathrm{CP}^i(Q) \subset \mathrm{CP}^i \circ \mathrm{Act}^i(Q)$.

Proof. Fix $p \in \mathrm{Act}^i \circ \mathrm{CP}^i(\Phi^i)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\mathrm{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi \quad \text{and} \quad r_\pi \in Q(\pi), \quad \forall \pi \in \Pi^i.$$

It is sufficient to show that each r_σ actually belongs to $\mathrm{Act}^i(Q(\sigma))$ for every signal $\sigma \in \Pi^i$ satisfying $\lambda_\sigma > 0$. Fix a signal σ with $\lambda_\sigma > 0$ and any posterior $q_\sigma \in Q(\sigma)$. We let p^σ be the probability in $\mathrm{Prob}(\Omega)$ defined by

$$p^\sigma = \lambda_\sigma q_\sigma + \sum_{\pi \neq \sigma} \lambda_\pi r_\pi.$$

The probability measure p^σ belongs to $\mathrm{CP}^i(Q)$ by construction. It follows that

$$\mathbb{E}^p\big[u^i(e^i)\big] \leqslant \mathbb{E}^{p^\sigma}\big[u^i(e^i)\big]$$

implying that

$$\mathbb{E}^{r_\sigma}\big[u^i(e^i)\big] \leqslant \mathbb{E}^{q_\sigma}\big[u^i(e^i)\big].$$

We have thus proved that r_σ belongs to $\mathrm{Act}^i(Q(\sigma))$. $\quad\square$

Now we prove that if the interim utility of e^i is independent of the signal received then $\mathrm{CP}^i \circ \mathrm{Act}^i(Q) \subset \mathrm{Act}^i \circ \mathrm{CP}^i(Q)$.

Proof. Let $q \in \mathrm{CP}^i \circ \mathrm{Act}^i(Q)$. There exist $\lambda = (\lambda_\pi)_{\pi \in \Pi^i}$ a vector in $\mathrm{Prob}(\Pi^i)$ and a family $(r_\pi)_{\pi \in \Pi^i}$ such that

$$q = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi \quad \text{and} \quad r_\pi \in \mathrm{Act}^i\big[Q(\pi)\big], \quad \forall \pi \in \Pi^i, .$$

Since the function $\pi \mapsto \mathbb{E}_{\Phi^i(\pi)}[u^i(e^i)]$ is constant on Π^i, we denote its value by $u^i(\Pi^i)$. Observe that

$$\mathbb{E}^q\big[u^i(e^i)\big] = \sum_{\pi \in \Pi^i} \lambda_\pi \mathbb{E}_{\Phi^i(\pi)}\big[u^i(e^i)\big] = u^i(\Pi^i).$$

Now, fix $\eta \in \mathrm{CP}^i(Q)$. There exist $\gamma = (\gamma_\pi)_{\pi \in \Pi^i}$ a vector in $\mathrm{Prob}(\Pi^i)$ and a family $(\kappa_\pi)_{\pi \in \Pi^i}$ such that

$$\eta = \sum_{\pi \in \Pi^i} \gamma_\pi \kappa_\pi \quad \text{and} \quad \kappa_\pi \in Q(\pi), \quad \forall \pi \in \Pi^i.$$

Observe that

$$\begin{aligned}
\mathbb{E}^q\big[u^i(e^i)\big] = u^i(\Pi^i) &= \sum_{\pi \in \Pi^i} \gamma_\pi \mathbb{E}_{Q(\pi)}\big[u^i(e^i)\big] \\
&\leqslant \sum_{\pi \in \Pi^i} \gamma_\pi \mathbb{E}^{\kappa_\pi}\big[u^i(e^i)\big] \\
&\leqslant \mathbb{E}^\eta\big[u^i(e^i)\big].
\end{aligned}$$

We have thus proved that q belongs to $\mathrm{Act}^i \circ \mathrm{CP}^i(Q)$. $\quad\square$

Appendix C. Proof of Lemma 5.3

Let p be a probability measure in $\mathrm{CP}^i(\Phi^i)$. There exist a vector $(\lambda_\pi)_{\pi \in \Pi^i}$ in $\mathrm{Prob}(\Pi^i)$ and a posterior belief $r_\pi \in \Phi^i(\pi)$ for each signal $\pi \in \Pi^i$ such that

$$p = \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi.$$

We denote by q the risk adjusted probability $\mathrm{RA}^i(p)$. It follows that for every state $\omega \in \Omega$,

$$q(\omega) = \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \sum_{\pi \in \Pi^i} \lambda_\pi r_\pi(\omega) \nabla u^i(e^i(\omega))$$

$$= \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \sum_{\pi \in \Pi^i} \nabla u^i(e^i(\pi)) \lambda_\pi r_\pi(\omega)$$

$$= \sum_{\pi \in \Pi^i} \gamma_\pi r_\pi(\omega)$$

where $\gamma_\pi = \nabla u^i(e^i(\pi)) \lambda_\pi / \mathbb{E}^p[\nabla u^i(e^i)]$. Observe that

$$\mathbb{E}^p[\nabla u^i(e^i)] = \sum_{\pi \in \Pi^i} \lambda_\pi \mathbb{E}^{r_\pi}[u^i(e^i)] = \sum_{\pi \in \Pi^i} \lambda_\pi \nabla u^i(e^i(\pi)).$$

This implies that the vector $(\gamma_\pi)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$ and therefore the risk-adjusted Φ^i-compatible prior q is also a Φ^i-compatible prior, i.e., $q \in \text{CP}^i(\Phi^i)$. We have thus proved that $\text{RA}^i \circ \text{CP}^i(\Phi^i) \subset \text{CP}^i(\Phi^i)$.

Conversely, we can always write p in the following form

$$p = \sum_{\pi \in \Pi^i} \frac{1}{\mathbb{E}^p[\nabla u^i(e^i)]} \eta_\pi \nabla u^i(e^i(\pi)) r_\pi$$

where $\eta_\pi = \lambda_\pi \mathbb{E}^p[\nabla u^i(e^i)] / \nabla u^i(e^i(\pi))$. Since the vector $(\eta_\pi)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$ we get that p is also a risk-adjusted Φ^i-compatible prior. We have thus proved that $\text{CP}^i(\Phi^i) \subset \text{RA}^i \circ \text{CP}^i(\Phi^i)$.

Appendix D. Sufficient condition for status-quo and strong interim efficiency

Assume that there exists a probability $p \in \text{Prob}(\Omega)$ such that for each agent i and every signal $\pi \in \Pi^i$, we have $p(\pi) > 0$ and there exists a subjective belief $r_\pi^i \in \text{Sub}_\pi^i$ satisfying

$$\forall i \in I, \quad p = \sum_{\pi \in \Pi^i} p(\pi) r_\pi^i.$$

We first prove that the allocation e is status-quo interim efficient. Assume by way of contradiction that e is not status-quo interim efficient. Then there exists a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and every signal $\pi \in \Pi^i$, we have

$$(y^i \mid \pi) \in \{(e^i \mid \pi)\} \cup \text{Pref}_\pi^i(e^i \mid \pi)$$

and for at least one agent k and one signal $\sigma \in \Pi^k$, we have

$$(y^k \mid \sigma) \in \text{Pref}_\sigma^k(e^k \mid \sigma).$$

Since for each agent i and signal $\pi \in \Pi^i$ the probability r_π^i is a subjective belief, we have

$$\begin{cases} \mathbb{E}^{r_\pi^i}[y^i] = \mathbb{E}^{r_\pi^i}[e^i] & \text{if } (y^i \mid \pi) = (e^i \mid \pi), \\ \mathbb{E}^{r_\pi^i}[y^i] > \mathbb{E}^{r_\pi^i}[e^i] & \text{if } (y^i \mid \pi) \in \text{Pref}_\pi^i(e^i \mid \pi). \end{cases}$$

In particular, we have

$$\mathbb{E}^{r_\sigma^k}[y^k] > \mathbb{E}^{r_\sigma^k}[e^k].$$

Summing over agents and signals, we get the following contradiction:

$$\sum_{i \in I} \mathbb{E}^p[y^i] = \sum_{i \in I} \sum_{\pi \in \Pi^i} p(\pi) \mathbb{E}^{r^i_\pi}[y^i] > \sum_{i \in I} \sum_{\pi \in \Pi^i} p(\pi) \mathbb{E}^{r^i_\pi}[e^i] = \sum_{i \in I} \mathbb{E}^p[e^i].$$

Assume that preferences are complete, transitive and convex. We now prove that the allocation e is strongly interim efficient. Assume by way of contradiction that e is not strongly interim efficient. Then there exists a feasible allocation $(y^i)_{i \in I}$ such that for each agent i and every signal $\pi \in \Pi^i$, we have

$$(y^i \mid \pi) \succeq^i_\pi (e^i \mid \pi)$$

and for at least one agent k and one signal $\sigma \in \Pi^k$, we have

$$(y^k \mid \sigma) \succ^k_\sigma (e^k \mid \sigma).$$

Since r^k_σ is a subjective belief we have

$$\mathbb{E}^{r^k_\sigma}[y^k] > \mathbb{E}^{r^k_\sigma}[e^k]. \tag{D.1}$$

If follows from Remark 7.3 that every probability r^i_π is supporting the upper contour of $(e^i \mid \pi)$, i.e.,

$$\mathbb{E}^{r^i_\pi}[y^i] \geqslant \mathbb{E}^{r^i_\pi}[e^i]. \tag{D.2}$$

Combining (D.1) and (D.2) we get the following contradiction

$$\sum_{i \in I} \mathbb{E}^p[y^i] = \sum_{i \in I} \sum_{\pi \in \Pi^i} p(\pi) \mathbb{E}^{r^i_\pi}[y^i] > \sum_{i \in I} \sum_{\pi \in \Pi^i} p(\pi) \mathbb{E}^{r^i_\pi}[e^i] = \sum_{i \in I} \mathbb{E}^p[e^i].$$

Appendix E. No speculative trade: Proofs

In this section we propose the detailed proofs of some results presented in Section 8.

Proof of Proposition 8.3. We only have to check that (8.1) is satisfied. Assume that there exist a probability $q \in \text{Prob}(\Omega)$ and for each agent i, an active prior $p^i \in \text{Act}^i(P^i)$ such that

$$\forall i \in I, \quad q = \text{RA}^i(p^i).$$

Fix a state $\omega \in \Omega$ and an agent i. We denote by π the associated signal $\Pi^i(\omega)$. If $p^i(\pi) > 0$ then

$$
\begin{aligned}
q(\omega) &= \frac{1}{\mathbb{E}^{p^i}[\nabla u^i(e^i)]} \nabla u^i\big(e^i(\omega)\big) p^i(\omega) \\
&= \frac{p^i(\pi)}{\mathbb{E}^{p^i}[\nabla u^i(e^i)]} \nabla u^i\big(e^i(\omega)\big) p^i(\omega \mid \pi) \\
&= \lambda^i_\pi \, \text{RA}^i\big(p^i(\cdot \mid \pi)\big)(\omega)
\end{aligned}
$$

where

$$\lambda^i_\pi = \frac{p^i(\pi) \mathbb{E}^{p^i(\cdot \mid \pi)}[\nabla u^i(e^i)]}{\mathbb{E}^{p^i}[\nabla u^i(e^i)]}.$$

Since the family $(\lambda^i_\pi)_{\pi \in \Pi^i}$ belongs to $\text{Prob}(\Pi^i)$, it follows that for each agent i, the probability q belongs to the set $\text{CP}^i \circ \text{RA}^i \circ \text{Act}^i(\Phi^i)$. $\quad\square$

Proof of Corollary 8.1. We only have to check that (8.5) is satisfied. Fix an agent i, an active prior $p^i \in \text{Act}^i(P^i)$ and a private signal $\pi \in \Pi^i$ with $p^i(\pi) > 0$. Since P^i is Bayesian consistent with Φ^i, the conditional belief $p^i(\cdot \mid \pi)$ belongs to $\Phi^i(\pi)$. We should now prove that $p^i(\cdot \mid \pi)$ is active. Actually, a direct consequence of the measurability of e^i is that any prior is active, i.e., $\text{Act}^i(\Phi^i(\pi)) = \Phi^i(\pi)$. Indeed, for every posterior $r_\pi \in \Phi^i(\pi)$, we have

$$\mathbb{E}^{r_\pi}\big[u^i\big(e^i\big)\big] = u^i\big(e^i(\pi)\big). \qquad \square$$

Proof of Corollary 8.2. We only have to check that (8.5) is satisfied. Fix an agent i, an active prior $p^i \in \text{Act}^i(P^i)$ and a private signal $\pi \in \Pi^i$ with $p^i(\pi) > 0$. Since the set of priors P^i is Φ^i-rectangular, it is Bayesian consistent with Φ^i. Therefore, the conditional belief $p^i(\cdot \mid \pi)$ belongs to $\Phi^i(\pi)$. We should now prove that the posterior $p^i(\cdot \mid \pi)$ is active, i.e., belongs to $\text{Act}^i(\Phi^i(\pi))$. We know that p^i is an active prior, i.e.,

$$\forall q \in P^i, \quad \mathbb{E}^{p^i}\big[u^i\big(e^i\big)\big] \leqslant \mathbb{E}^q\big[u^i\big(e^i\big)\big]. \tag{E.1}$$

Fix an arbitrary posterior $r_\pi \in \Phi^i(\pi)$ and let q be the probability in $\text{Prob}(\Omega)$ defined by

$$q = p^i(\pi)r_\pi + \sum_{\sigma \neq \pi} p^i(\sigma)p(\cdot \mid \sigma).$$

Since the set of priors is Φ^i-rectangular, the probability q also belongs to P^i. Applying (E.1) we get

$$\mathbb{E}^{p^i(\cdot \mid \pi)}\big[u^i\big(e^i\big)\big] \leqslant \mathbb{E}^{r_\pi}\big[u^i\big(e^i\big)\big].$$

We have proved that $p^i(\cdot \mid \pi)$ is an active posterior. $\quad \square$

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