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Equilibria in large economies with a separable Banach commodity space and non-ordered preferences

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Abstract

The purpose of this paper is to provide an existence result of equilibria for economies with a measure space of agents, a non-trivial production sector and an infinite dimensional commodity space. The commodity space is modeled by an ordered separable Banach space whose positive cone has a non-empty interior. The *discretization* approach proposed in this paper, allows us to extend the existence results in Khan and Yannelis [Equilibrium in markets with a continuum of agents and commodities. In: Khan, M.A., Yannelis, N.C. (Eds.), Equilibrium Theory in Infinite Dimensional Spaces. Springer, Berlin, 1991] and Podczeck [Economic Theory 9 (1997) 585] to economies with a non-trivial production sector and with possibly non-ordered but convex preferences as well as partially ordered (possibly incomplete) but non-convex preferences.

JEL classification: C62; D51

Keywords: Measure space of agents; Separable Banach commodity spaces; Non-ordered but convex preferences; Partially ordered but non-convex preferences; Discretization of measurable correspondences

1. Introduction

For economies with a measure space of agents and an ordered separable[1] Banach commodity space, there exist many Walrasian equilibria existence results for exchange economies with ordered preference relations. In Khan and Yannelis (1991), the preference relations are

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[1] In Tourky and Yannelis (2001), they proved that equilibria existence results in Khan and Yannelis (1991) and Rustichini and Yannelis (1991) do not extend to non-separable commodity spaces.

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ordered and convex. In Aumann (1966), Hildenbrand (1970) and Rustichini and Yannelis (1991) or in Podczeck (1997), the preference relations are ordered but non-convex.

In both papers Khan and Yannelis (1991) and Podczeck (1997), the Gale-Nikaido–Debreu lemma is applied to the excess demand correspondence. This approach does not cover non-ordered preference relations. The *discretization* approach proposed in this paper, allows us to extend the existence results in Khan and Yannelis (1991) and Podczeck (1997) to economies with a non-trivial production sector and with possibly non-ordered but convex preference relations as well as partially ordered (possibly incomplete) but non-convex preference relations.

The *discretization* approach consists of considering an economy with a measure space of agents as the *limit* of a sequence of economies with a finite, but larger and larger, set of agents. Applying the measurability properties of the different characteristics of the economy (initial endowments, consumption sets, production sets and preference relations), we construct an increasing sequence of finite partitions of the measure space. To each partition we define a *subordinated simple* economy. Each *simple* economy will be identified as an economy with a finite set of agents. Then we apply a classical Edgeworth equilibria existence result (for which we do not need to suppose that preference relations are ordered) for economies with a finite set of agents, e.g. in Florenzano (1990). By a separation argument we obtain a sequence of allocations and prices, which will converge to a Walrasian quasi-equilibrium for the original economy.

The paper is organized as follows. In Section 2 we set out the main definitions and notations. In Section 3 we define the model of large square economies, we introduce the concepts of equilibria, we give the list of assumptions that economies will be required to satisfy and finally, we present the existence result (Theorem 3.1). Section 4 is devoted to the mathematical *discretization* of measurable correspondences. The proof of the main theorem (Theorem 3.1) is then given in Section 5. The last section is devoted to mathematical auxiliary results.

2. Notations and definitions

Consider (E, τ) a topological vector space. If $X \subset E$ is a subset, then the τ-interior of X is denoted by $\tau\text{-int}\, X$, the τ-closure of X is denoted by $\tau\text{-cl}\, X$. The convex hull of X is denoted by $\text{co}\, X$ and the τ-closed convex hull of X is denoted by $\tau\text{-}\overline{\text{co}}\, X$. We let $A(X) = \{v \in \mathbb{L} : X + \{v\} \subset X\}$ be the asymptotic cone of X. If $(C_n)_n$ is a sequence of subsets of E, the *τ-sequential upper limit* of $(C_n)_n$, is denoted by $\tau\text{-ls}\, C_n$ and is the set of all cluster point of the sequence $(C_n)_n$, i.e. x belongs to $\tau\text{-ls}\, C_n$ if there exists a sequence $(x_n)_n$ in E satisfying

$$x = \tau\text{-}\lim_n x_n \quad \text{and} \quad x_n \in C_{\varphi(n)}$$

where $\varphi : \mathbb{N} \to \mathbb{N}$ is an increasing application.

Let $(\mathbb{L}, \|\cdot\|, \geqslant)$ be an ordered separable Banach space.[2] The topology induced by the norm is denoted by s (strong). The s-dual of $\mathbb{L}$, i.e., the space of s-continuous linear functionals

[2] That is, $(\mathbb{L}, \|\cdot\|)$ is a separable Banach space and there exists a pointed ($C \cap -C = \{0\}$) closed convex cone $C \subset \mathbb{L}$ such that is the order induced by C, i.e. $x \;\; y$ whenever $x - y \in C$.

on $\mathbb{L}$, is denoted by $\mathbb{L}'$. The natural dual pairing $\langle \mathbb{L}', \mathbb{L} \rangle$ is defined by $\langle p, x \rangle := p(x)$, for every $(p, x) \in \mathbb{L}' \times \mathbb{L}$. The weak topology $\sigma(\mathbb{L}, \mathbb{L}')$ is denoted by w and the weak star topology $\sigma(\mathbb{L}', \mathbb{L})$ is denoted by w^*. The space $\mathbb{L}$ is thus endowed with two topologies s and w. Following Podczeck (1997), the Borel σ-algebra of $(\mathbb{L}, w)$ and of $(\mathbb{L}, s)$ coincide and is denoted by $\mathcal{B}(\mathbb{L})$. The positive cone of $\mathbb{L}$ is denoted by $\mathbb{L}_+ := \{x \in \mathbb{L} : x \geqslant 0\}$. We write $\mathbb{L}'_+$ for the set $\{p \in \mathbb{L}' : \forall x \in \mathbb{L}_+ p(x) \geqslant 0\}$. If $x \in \mathbb{L}$ then $x > 0$ means $x \geqslant 0$ and $x \neq 0$. If $p \in \mathbb{L}'$ then $p > 0$ means $p \geqslant 0$ and $p \neq 0$. If X is a subset of $\mathbb{L}$ and p belongs to $\mathbb{L}'$, then $\sup\{p(x) : x \in X\}$ is denoted by $\sup p(x)$.

We consider $(A, \mathcal{A}, \mu)$ a finite complete measure space, i.e. A is a set, $\mathcal{A}$ is a σ-algebra of subsets of A and μ is a finite measure on $\mathcal{A}$. The measure space $(A, \mathcal{A}, \mu)$ is complete if $\mathcal{A}$ contains all μ-negligible[3] subsets of A. A function f from A to $\mathbb{L}$ is *measurable* if for each $B \in \mathcal{B}(\mathbb{L})$, $f^{-1}(B) := \{a \in A : f(a) \in B\} \in \mathcal{A}$. A function f from A to $\mathbb{L}$ is *Bochner measurable* if there exists a sequence of simple functions $(f_n)_n$ pointwise s-converging to f, i.e.,

$$\forall a \in A, \lim_n \| f_n(a) - f(a) \| = 0.$$

Since $(\mathbb{L}, \|\cdot\|)$ is separable then following Theorem III.36 in Castaing and Valadier (1977), f is measurable if and only if f is Bochner measurable. A measurable function f from A to $\mathbb{L}$ is *Bochner integrable* if the real-valued function $\|f(\cdot)\| : a \mapsto \|f(a)\|$ is integrable. Following Diestel and Uhl (1977), a measurable function f is Bochner integrable if and only if there exists a sequence of simple functions $(f_n)_n$ such that

$$\lim_n \int_A \| f_n(a) - f(a) \| \, \mathrm{d}\mu(a) = 0.$$

For each $E \in \mathcal{A}$, the integral of f over E is defined by

$$\int_E f(a) \, \mathrm{d}\mu(a) := \lim_n \int_E f_n(a) \, \mathrm{d}\mu(a).$$

Let (D, d) be a separable metric space. A correspondence (or a multifunction) $F : A \twoheadrightarrow D$ is *graph measurable* if $\{(a, x) \in A \times D : x \in F(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(D)$. A function $f : A \to D$ is a *measurable selection* of F if f is measurable and if, for almost every $a \in A$, $f(a) \in F(a)$. If f is a measurable function from A to D, then we denote by $\{f(\cdot)\}$ the correspondence defined by $a \mapsto \{f(a)\}$. Following Theorem III.30 in Castaing and Valadier (1977), the function f is measurable if and only if $\{f(\cdot)\}$ is graph measurable. The set of measurable selections of F is denoted by $S(F)$. When $D \subset \mathbb{L}$, the set of Bochner integrable selections of F is denoted by $S^1(F)$ and we denote by F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a) \, \mathrm{d}\mu(a) := \{v \in D : \exists x \in S^1(F), v = \int_A x(a) \, \mathrm{d}\mu(a)\}$. The correspondence F is said to be *integrably bounded* if there exists an integrable function h from A to $\mathbb{R}_+$ such that for a.e. $a \in A$, for every $x \in F(a)$, $\|x\| \leqslant h(a)$.

Let X be a space and $P \subset X \times X$ be a binary relation on X. The relation P is irreflexive if $(x, x) \notin P$, for every $x \in X$. The relation P is transitive if $[(x, y) \in P$ and $(y, z) \in P]$ implies $(x, z) \in P$, for every $(x, y, z) \in X^3$. The relation P is negatively transitive if $[(x, y) \notin P$ and $(y, z) \notin P]$ implies $(x, z) \notin P$, for every $(x, y, z) \in X^3$. The

[3] A set N is μ-negligible if there exists $E \in \mathcal{A}$ such that $N \subset E$ and $\mu(E) = 0$.

relation P is a partial order it is irreflexive and transitive. The relation P is an order if it is irreflexive, transitive and negatively transitive. When P is an order, it is usually denoted by $\succ$ and $X^2 \setminus P$ is denoted by $\preceq$. Note that when P is an order, then $\preceq$ is transitive, reflexive ($x \preceq x$ for every $x \in X$) and complete (for every $(x, y) \in X^2$ either $x \preceq y$ or $y \preceq x$).

3. The model and the result

3.1. The model

We consider an ordered separable Banach space $(\mathbb{L}, \|\cdot\|, \geqslant)$ such that the positive cone $\mathbb{L}_+ := \{x \in \mathbb{L} : x \geqslant 0\}$ is closed and has a non-empty s-interior. Moreover, we consider a complete finite measure space $(A, \mathcal{A}, \mu)$, a Bochner integrable function e from A to $\mathbb{L}$, two correspondences X and Y from A into $\mathbb{L}$ and a correspondence of preference relations P in X, i.e., P is a correspondence from A into $\mathbb{L} \times \mathbb{L}$ such that for every $a \in A$, $P(a) \subset X(a) \times X(a)$ and $P(a)$ is an irreflexive relation on $X(a)$.

An economy $\mathcal{E}$ is a list

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, Y, P, e)).$$

The commodity space is represented by $\mathbb{L}$. The natural dual pairing $\langle \mathbb{L}', \mathbb{L} \rangle$ is interpreted as the *price-commodity* pairing.

The set of agents (or consumers) is represented by A, the set $\mathcal{A}$ represents the set of admissible coalitions, and the number $\mu(E)$ represents the fraction of consumers which are in the coalition $E \in \mathcal{A}$.

For each agent $a \in A$, the consumption set is represented by $X(a) \subset \mathbb{L}$ and the preference relation by $P(a) \subset X(a) \times X(a)$. We define the correspondence[4] $P_a : X(a) \twoheadrightarrow X(a)$ by $P_a(x) = \{x' \in X(a) : (x, x') \in P(a)\}$. In particular, if $x \in X(a)$ is a consumption bundle, $P_a(x)$ is the set of consumption bundles strictly preferred to x by the agent a. The set of consumption allocations (or plans) of the economy is the set $S^1(X)$ of Bochner integrable selections of X. The aggregate consumption set X_Σ is defined by

$$X_\Sigma := \int_A X(a)\, \mathrm{d}\mu(a) := \left\{ v \in \mathbb{L} : \exists x \in S^1(X), v = \int_A x(a)\, \mathrm{d}\mu(a) \right\}.$$

The initial endowment of the consumer $a \in A$ is represented by the commodity bundle $e(a) \in \mathbb{L}$. We denote by $\omega := \int_A e(a)\, \mathrm{d}\mu(a)$ the aggregate initial endowment. The production possibilities available to the consumer $a \in A$ are represented by the set $Y(a) \subset \mathbb{L}$. The set of production allocations (or plans) of the economy is the set $S^1(Y)$ of Bochner integrable selections of Y. The aggregate production set Y_Σ is defined by

$$Y_\Sigma := \int_A Y(a)\, \mathrm{d}\mu(a) := \left\{ u \in \mathbb{L} : \exists y \in S^1(Y), \quad u = \int_A y(a)\, \mathrm{d}\mu(a) \right\}.$$

[4] Note that the binary relation $P(a)$ coincide with the graph of the correspondence P_a.

3.2. The equilibrium concepts

We present hereafter different concepts of (quasi-)equilibrium: Walrasian, free-disposal and competitive equilibrium.

Definition 3.1. A *Walrasian equilibrium* of an economy $\mathcal{E}$ is an element (x^*, y^*, p^*) of $S^1(X) \times S^1(Y) \times \mathbb{L}'$ such that $p^* \neq 0$ and satisfying the following properties.

(a) For almost every $a \in A$,

$$p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a))$$

and

$$x \in P_a(x^*(a)) \Rightarrow p^*(x) > p^*(x^*(a)).$$

(b) For almost every $a \in A$,

$$y \in Y(a) \Rightarrow p^*(y) \leqslant p^*(y^*(a)).$$

(c) $\int_A x^*(a)\,\mathrm{d}\mu(a) = \int_A e(a)\,\mathrm{d}\mu(a) + \int_A y^*(a)\,\mathrm{d}\mu(a)$.

An element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ with $p^* \neq 0$ is a *Walrasian quasi-equilibrium* of an economy $\mathcal{E}$ if the conditions (b) and (c) together with

(a′) for almost every $a \in A$,

$$p^*(x^*(a)) = p^*(e(a)) + p^*(y^*(a))$$

and

$$x \in P_a(x^*(a)) \Rightarrow p^*(x) \geqslant p^*(x^*(a)),$$

are satisfied.

Following Debreu (1982), we introduce the concept of free-disposal equilibria.

Definition 3.2. A *free-disposal equilibrium* of an economy $\mathcal{E}$ is an element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ such that $p^* > 0$ and which satisfies conditions (a) and (b) together with

(c′) $\int_A x^*(a)\,\mathrm{d}\mu(a) \leqslant \int_A e(a)\,\mathrm{d}\mu(a) + \int_A y^*(a)\,\mathrm{d}\mu(a)$.

An element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ with $p^* > 0$ is a *free-disposal quasi-equilibrium* of an economy $\mathcal{E}$ if the conditions (a′), (b) and (c′) are satisfied.

A (free-disposal) Walrasian equilibrium of a production economy $\mathcal{E}$ is clearly a (resp. free-disposal) Walrasian quasi-equilibrium of $\mathcal{E}$. We provide in the following remark, a classical condition on $\mathcal{E}$ under which a (free-disposal) Walrasian quasi-equilibrium is in fact a (resp. free-disposal) Walrasian equilibrium.

Remark 3.1. Each (free-disposal) Walrasian quasi-equilibrium (x^*, y^*, p^*) of an economy $\mathcal{E}$, is a (resp. free-disposal) Walrasian equilibrium, if we assume that, for almost every agent $a \in A$, $X(a)$ is convex, the strict-preferred set $P_a(x^*(a))$ is s-open in $X(a)$ and

$$\inf p^*(X(a)) < p^*(e(a)) + \sup p^*(Y(a)). \tag{3.1}$$

In particular, if $p^* > 0$ then the condition (3.1) is automatically valid if for almost every agent $a \in A$,

$$(\{e(a)\} + Y(a) - X(a)) \cap s\text{-int}\,\mathbb{L}_+ \neq \emptyset.$$

A Walrasian equilibrium (quasi-equilibrium) of a production economy $\mathcal{E}$ is clearly a free-disposal equilibrium (resp. quasi-equilibrium) of $\mathcal{E}$. We provide in the following remark, a classical condition on $\mathcal{E}$ under which a free-disposal equilibrium (quasi-equilibrium) is in fact an equilibrium (resp. quasi-equilibrium).

Remark 3.2. If the aggregate production set Y_Σ is free-disposal, i.e., $Y_\Sigma - \mathbb{L}_+ \subset Y_\Sigma$, then each free-disposal equilibrium (quasi-equilibrium) is in fact a Walrasian (resp. quasi-equilibrium) equilibrium.

Remark 3.3. We can find in the literature a third concept of equilibrium. In Khan and Yannelis (1991) and Rustichini and Yannelis (1991), (x^*, y^*, p^*) with $p^* > 0$, is a *competitive equilibrium* of $\mathcal{E}$ if it satisfies conditions (b), (c$'$) together with the following condition:

(a$''$) For almost every $a \in A$,

$$p^*(x^*(a)) \leqslant p^*(e(a)) + p^*(y^*(a))$$

and

$$x \in P_a(x^*(a)) \Rightarrow p^*(x) > p^*(e(a)) + p^*(y^*(a)).$$

The free-disposal property on the aggregate production set is no more sufficient to prove that a competitive equilibrium is in fact a Walrasian equilibrium. However, under a suitable *local non-satiation* property and together with the free-disposal property on the aggregate production set, we can prove that a competitive equilibrium is in fact a Walrasian equilibrium. Note moreover that if (x^*, y^*, p^*) is a free-disposal equilibrium then the value of the excess of demand is zero, i.e., $p^*(\int_A y^*(a)\,\mathrm{d}\mu(a) + \omega - \int_A x^*(a)\,\mathrm{d}\mu(a)) = 0$. This is not automatically the case if (x^*, y^*, p^*) is a competitive equilibrium.

The model of production economies defined above encompasses the model of a *private ownership economy* presented in Hildenbrand (1970). In a private ownership economy

$$\mathcal{E} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L} \rangle, (X, P, e), (Y_j, \theta_j)_{j \in J}),$$

the production sector is represented by a finite set J of firms with production sets $(Y_j)_{j \in J}$, where for every $j \in J$, $Y_j \subset \mathbb{L}$. The profit made by the firm $j \in J$ is distributed among the consumers following a share function $\theta_j : A \to \mathbb{R}_+$. The share functions are supposed to be integrable and to satisfy for each $j \in J$, $\int_A \theta_j(a)\,\mathrm{d}\mu(a) = 1$. If we let for each $a \in A$,

$$Y(a) := \sum_{j\in J} \theta_j(a)\overline{\mathrm{co}}\, Y_j$$

then we define an economy $\mathcal{E}' := ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L}\rangle, (X, Y, P, e))$. If the production sector of the private ownership economy satisfies $\sum_{j\in J} Y_j$ is closed and convex, then for every $p \in \mathbb{L}'$ and for almost every $a \in A$,

$$\int_A Y(a)\,\mathrm{d}\mu(a) = \sum_{j\in J} Y_j \quad \text{and} \quad \sup p(Y(a)) = \sum_{j\in J} \theta_j(a) \sup p(Y_j).$$

It follows that we can apply an equilibria existence result corresponding to the economy $\mathcal{E}'$, to provide a result corresponding to the private ownership economy $\mathcal{E}$.

3.3. The assumptions

We present the list of assumptions that the economy $\mathcal{E}$ will be required to satisfy. On the consumption side we consider both non-ordered but convex preference relations (Assumption C.3(i)) and partially ordered (possibly incomplete) but non-convex preference relations (Assumption C.3(ii)).

Assumption C.1 (continuity). For almost every agent $a \in A$, the consumption set $X(a)$ is closed convex and P_a is continuous, i.e., for each bundle $x \in X(a)$, $P_a(x)$ is s-open in $X(a)$ and $P_a^{-1}(x) := \{y \in X(a) : y \in P_a(x)\}$ is w-open in $X(a)$.

Assumption C.2 (atomic part). If a belongs to an atom[5] of $(A, \mathcal{A}, \mu)$ then the relation $P(a)$ is convex, i.e., for each bundle $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$.

Assumption C.3 (non-atomic part). One of the two following properties is satisfied:

(i) For almost every a on the non-atomic part of $(A, \mathcal{A}, \mu)$, the preference relation $P(a)$ is convex and $X(a) \setminus P_a^{-1}(x)$ is convex.
(ii) For almost every a on the non-atomic part of $(A, \mathcal{A}, \mu)$, the preference relation $P(a)$ is a partial order on $X(a)$.

Remark 3.4. When $X(a) \setminus P_a^{-1}(x)$ is supposed to be convex, the set $P_a^{-1}(x)$ is w-open in $X(a)$ if and only if it is s-open in $X(a)$.

Remark 3.5. Note that if $P(a)$ is partially ordered, then assuming that for every $x \in X(a)$, $X(a) \setminus P_a^{-1}(x)$ is convex, implies that for every $x \in X(a)$, $x \notin \mathrm{co}\, P_a(x)$. In particular,

[5] An element $E \in \mathcal{A}$ is an atom of $(A, \mathcal{A}, \mu)$ if $\mu(E) \neq 0$ and $[B \in \mathcal{A}$ and $B \subset E]$ implies $\mu(B) = 0$ or $\mu(E \setminus B) = 0$.

Assumptions C.1, C.2 and C.3(i) are automatically valid under Assumptions A1–A4 in Podczeck (2001) and under Assumptions 3.1 and 3.2 in Khan and Yannelis (1991).

Remark 3.6. Following the notations of Section 2, when preference relations are ordered, we have

$$X(a) \setminus P_a^{-1}(x) = \{y \in X(a) | y \succeq_a x\}.$$

If $\{y \in X(a) : y \succeq_a x\}$ is supposed to be convex then the relation $P(a)$ is automatically convex. In particular, Assumptions C.1, C.2 and C.3(ii) are implied by Assumptions E1–E3 and B1–B2 in Podczeck (1997), by Assumptions a2 and a3 in Rustichini and Yannelis (1991) and by Assumptions 3.1 and 3.2 in Khan and Yannelis (1991). In these three papers, preference relations are supposed to be ordered, but in Assumption C.3(ii), preference relations are only required to be partially ordered.

We say that two agents a and b are equivalent, denoted by $a \sim b$, if $\mu(\{a\}) = \mu(\{b\})$, $X(a) = X(b)$, $e(a) = e(b)$ and $P(a) = P(b)$. Two equivalent agents play the same role in the economy. The binary relation $\sim$ is an equivalence. Each equivalence class represents a *type* of consumers. We let A^{na} be the non-atomic part of A. To deal with partially ordered but non-convex preference relations, we need the following assumption.

Assumption (A). If $F : A^{\mathrm{na}} \twoheadrightarrow \mathbb{L}$ is a graph measurable and integrably bounded correspondence with non-empty and w-compact values, such that for every $(b, c) \in A^{\mathrm{na}}$, $b \sim c$ implies $F(b) = F(c)$, then

$$\int_{A^{\mathrm{na}}} \overline{\mathrm{co}}\, F(a)\, \mathrm{d}\mu(a) = \int_{A^{\mathrm{na}}} F(a)\, \mathrm{d}\mu(a).$$

Remark 3.7. Following Theorem 3.1 in Podczeck (1997), Assumption A is implied by Assumptions A1–A2 in Podczeck (1997) which formulate that there are many agents of (almost) every type. If there exists a fixed w-compact set K such that for every $a \in A^{\mathrm{na}}$, $F(a) \subset K$ then Assumption A1 (many more agents than commodities) in Rustichini and Yannelis (1991) implies Assumption A. For several refinements of the Lyapunov Theorem, we refer to Tourky and Yannelis (2001).

Assumption (C) (consumption side). Assumptions C.1 and C.2 are valid and either Assumptions C.3(ii) and A are valid or Assumption C.3 (i) is valid.

Assumption (M) (measurability). The correspondences X and Y are graph measurable, i.e.,

$$\{(a, x) \in A \times \mathbb{L} : x \in X(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L})$$

and

$$\{(a, y) \in A \times \mathbb{L} : y \in Y(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}),$$

and the correspondence of preference relations P is lower graph measurable, i.e.,

$$\forall y \in S(X), \{(a, x) \in A \times \mathbb{L} : (x, y(a)) \in P(a)\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{L}).$$

Remark 3.8. In Khan and Yannelis (1991) and Podczeck (1997), the correspondences X and P are supposed to be graph measurable. It can be proved (see Martins-da-Rocha, 2002) that in the framework of Khan and Yannelis (1991) and Podczeck (1997), graph measurability of the correspondence of preference relations implies lower graph measurability, in particular Assumption M is valid.

Remark 3.9. In Podczeck (2001), it is assumed that preference relations are Aumann measurable. It can be proved (see Martins-da-Rocha, 2002) that in the framework of Podczeck (2001), Aumann measurability of the correspondence of preference relations implies lower graph measurability, in particular Assumption M is valid.

Assumption (P) (production side). The aggregate production set Y_Σ and the set $Y_\Sigma - \mathbb{L}_+$ are closed convex subsets of $\mathbb{L}$.

Assumption (S) (survival). For almost every $a \in A$,

$$0 \in (\{e(a)\} + X(a) - Y(a)) \neq \emptyset.$$

Remark 3.10. Assumption S means that we have compatibility between individual needs and resources. In Khan and Yannelis (1991) and Podczeck (2001), the initial endowment is supposed to lie in the consumption set, i.e., for almost every $a \in A$, $e(a) \in X(a)$ and inaction is a possible production plan.

Assumption (B) (bounded). The correspondence X is integrably bounded with w-compact values.

Remark 3.11. We can find Assumption B in Khan and Yannelis (1991), Podczeck (1997, 2001) and Rustichini and Yannelis (1991). In order to apply Theorem A.1, this assumption is the natural framework to deal with general Banach commodity spaces. Note that under Assumptions M, S and B, the aggregate consumption set X_Σ is non-empty.

Assumption (LNS) (local non-satiation). For almost every agent $a \in A$, for every bundle $x \in X(a)$:

(i) if x is a satiation point, i.e. $P_a(x) = \emptyset$, then for every $y \in Y(a)$, $x \geqslant e(a) + y$;
(ii) if x is not a satiation point, then $x \in \overline{\text{co}}\, P_a(x)$.

Remark 3.12. In Podczeck (1997, 2001), economies in consideration are free-disposal exchange economies, i.e., for every $a \in A$, $Y(a) = -\mathbb{L}_+$. It follows that Assumptions B4–B5 in Podczeck (1997) and C5–C6 in Podczeck (2001) imply Assumption LNS.

Assumption (SS) (strong survival). For almost every agent $a \in A$,

$$(\{e(a)\} + Y(a) - X(a)) \cap s\text{−int}\,\mathbb{L}_+ \neq \emptyset.$$

Remark 3.13. In the framework of exchange economies ($Y(a) = \{0\}$ or $Y(a) = -\mathbb{L}_+$), Podczeck (1997, 2001) and Khan and Yannelis (1991) supposed that for almost every agent $a \in A$, $[\{e(a)\} - X(a)] \cap s\text{−int}\,\mathbb{L}_+ \neq \emptyset$. This obviously implies that Assumption SS is valid.

Assumption (FD) (free disposal). The aggregate production set is free-disposal, i.e., $Y_\Sigma - \mathbb{L}_+ \subset Y_\Sigma$.

3.4. Existence result

We shall now state the main result of the paper.

Theorem 3.1. *If $\mathcal{E}$ is an economy satisfying* Assumptions **C**, **M**, **P**, **S**, **B** *and* **LNS**, *then there exists a free-disposal quasi-equilibrium* (x^*, y^*, p^*). *If moreover $\mathcal{E}$ satisfies* **SS**, *then* (x^*, y^*, p^*) *is a free-disposal Walrasian equilibrium. If moreover $\mathcal{E}$ satisfies* **SS** *and* **FD**, *then* (x^*, y^*, p^*) *is a Walrasian equilibrium.*

Remark 3.14. Under Assumption LNS, Theorem 3.1 extends Theorem 5.1 in Podczeck (1997), Theorem 6.1 in Rustichini and Yannelis (1991) and the main theorem in Khan and Yannelis (1991), to economies with a non-trivial production sector. Moreover, for economies with convex preference relations Theorem 3.1 extends Theorem 5.1 in Podczeck (1997) and the main theorem in Khan and Yannelis (1991), to economies with non-ordered preference relations. And for economies with possibly non-convex preference relations, Theorem 3.1 extends Theorem 5.1 in Podczeck (1997) and Theorem 6.1 in Rustichini and Yannelis (1991) to economies with possibly incomplete preference relations.

Although Khan–Yannelis and Rustichini–Yannelis succeed in proving the existence of a competitive equilibrium without Assumption LNS, but they assume that for some s-compact subset of the commodity space, say K, the endowment of each agent belongs to K.

Remark 3.15. If we let $\tilde{Y} : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined for every $a \in A$ by

$$\tilde{Y}(a) := \mathrm{cl}(\overline{\mathrm{co}}\, Y(a) + A(Y_\Sigma)),$$

then following Theorem III.40 in Castaing and Valadier (1977), Assumption M is valid. Moreover, following Proposition A.2, $\tilde{Y}$ satisfies Assumption P and $\mathcal{E}$ has a free-disposal satiation quasi-equilibrium if and only if

$$\tilde{\mathcal{E}} = ((A, \mathcal{A}, \mu), \langle \mathbb{L}, \mathbb{L}' \rangle, (X, \tilde{Y}, P, e))$$

has a free-disposal satiation quasi-equilibrium.

It follows that in Theorem 3.1, we can replace Assumptions S and SS by the weaker Assumptions S′ and SS′ defined by

Assumption (S′). For almost every $a \in A$,

$$e(a) \in X(a) - \tilde{Y}(a).$$

Assumption (SS′). For almost every agent $a \in A$,

$$(\{e(a)\} + \tilde{Y}(a) - X(a)) \cap s\text{−int}\,\mathbb{L}_+ \neq \emptyset.$$

4. Discretization of measurable correspondences

4.1. Notations and definitions

We consider $(A, \mathcal{A}, \mu)$ a complete finite measure space and (D, d) a separable metric space.

Definition 4.1. A partition $\sigma = (A_i)_{i\in I}$ of A is a *measurable partition* if I is a finite set and for every $i \in I$, the set A_i is non-empty and belongs to $\mathcal{A}$. A finite subset A^σ of A is *subordinated to the partition* σ if there exists a family $(a_i)_{i\in I} \in \prod_{i\in I} A_i$ such that $A^\sigma = \{a_i : i \in I\}$.

Given a couple (σ, A^σ), where $\sigma = (A_i)_{i\in I}$ is a measurable partition of A, and $A^\sigma = \{a_i : i \in I\}$ is a finite set subordinated to σ, we consider $\phi(\sigma, A^\sigma)$ the mapping which maps each measurable function f to a simple measurable function $\phi(\sigma, A^\sigma)(f)$, defined by

$$\phi(\sigma, A^\sigma)(f) := \sum_{i\in I} f(a_i)\chi_{A_i},$$

where χ_{A_i} is the characteristic[6] function associated with A_i. Note that the sum is well defined since there exists at most one non-zero factor.

Definition 4.2. A function $s : A \to D$ is called *a simple function subordinated* to f if there exists a couple (σ, A^σ), where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $s = \phi(\sigma, A^\sigma)(f)$.

Given a couple (σ, A^σ), where $\sigma = (A_i)_{i\in I}$ is a measurable partition of A, and $A^\sigma = \{a_i : i \in I\}$ is a finite set subordinated to σ, we consider $\psi(\sigma, A^\sigma)$, the mapping which maps each measurable correspondence $F : A \twoheadrightarrow D$ to a simple measurable correspondence $\psi(\sigma, A^\sigma)(F)$, defined by

$$\psi(\sigma, A^\sigma)(F) := \sum_{i\in I} F(a_i)\chi_{A_i}.$$

[6] That is, for every $a \in A$, $\chi_{A_i}(a) = 1$ if $a \in A_i$ and $\chi_{A_i}(a) = 0$ elsewhere.

Definition 4.3. A correspondence $S : A \to D$ is called a *simple correspondence subordinated* to a correspondence F if there exists a couple (σ, A^σ), where σ is a measurable partition of A, and A^σ is a finite set subordinated to σ, such that $S = \psi(\sigma, A^\sigma)(F)$.

Remark 4.1. If f is a function from A to D, let $\{f(\cdot)\}$ be the correspondence from A into D, defined for every $a \in A$ by $\{f(\cdot)\}(a) := \{f(a)\}$. We check that

$$\psi(\sigma, A^\sigma)(\{f(\cdot)\}) = \{\phi(\sigma, A^\sigma)(f)\}.$$

The space of all non-empty subsets of D is denoted by $\mathcal{P}^*(D)$. We let τ_{W_d} be the Wisjman topology on $\mathcal{P}^*(D)$, i.e., the weak topology on $\mathcal{P}^*(D)$ generated by the family of distance functions $(d(x, \cdot))_{x \in D}$.

4.2. Approximation of measurable correspondences

Hereafter we assert that for a countable set of graph measurable correspondences, there exists a sequence of measurable partitions *approximating* each correspondence. The proof of the following theorem is given in Martins-da-Rocha (2002).

Theorem 4.1. *We consider $(A, \mathcal{A}, \mu)$ a complete finite measure space and (D, d) a separable metric space. Let $\mathcal{F}$ be a countable set of graph measurable correspondences with non-empty values from A into D and let $\mathcal{G}$ be a finite set of integrable functions from A into $\mathbb{R}$. There exists a sequence $(\sigma^n)_n$ of finer and finer measurable partitions $\sigma^n = (A_i^n)_{i \in I^n}$ of A, satisfying the following properties.*

(a) *Let $(A^n)_n$ be a sequence of finite sets A^n subordinated to the measurable partition σ^n and let $F \in \mathcal{F}$. For every $n \in \mathbb{N}$, we define the simple correspondence $F^n := \psi(\sigma^n, A^n)(F)$ subordinated to F. Then for each $a \in A$, the set $F(a)$ is the Wijsman limit of the sequence $(F^n(a))_n$, i.e.*

$$\forall a \in A \ \ \forall x \in A, \lim_n d(x, F^n(a)) = d(x, F(a)).$$

(b) *There exists a sequence $(A^n)_n$ of finite sets A^n subordinated to the measurable partition σ^n, such that for each n, if we let $f^n := \phi(\sigma^n, A^n)(f)$ be the simple function subordinated to each $f \in \mathcal{G}$, then*

$$\forall f \in \mathcal{G} \ \ \forall a \in A, |f^n(a)| \leqslant 1 + \sum_{g \in \mathcal{G}} |g(a)|.$$

In particular, for each $f \in \mathcal{G}$,

$$\lim_{n \to \infty} \int_A |f^n(a) - f(a)| \, \mathrm{d}\mu(a) = 0.$$

Remark 4.2. The property (a) implies in particular that, if $(x^n)_n$ is a sequence of D, d-converging to $x \in D$, then

$$\forall a \in A, \lim_n d(x^n, F^n(a)) = d(x, F(a)).$$

It follows that if F is non-empty closed valued, then property (a) implies that

$$\forall a \in A, \operatorname{ls} F^n(a) \subset F(a).$$

Moreover, if $F = \{f(\cdot)\}$, where $f : A \to D$ is a function, then if $a \in A$, $F(a) = \{f(a)\}$ is the Wisjman limit of the sequence[7] $(F^n(a) = \{f^n(a)\})_n$ if and only if $f(a)$ is the d-limit of the sequence $(f^n(a))_n$.

5. Proof of main theorem

5.1. Free-disposal satiation equilibria

Hereafter, we introduce an auxiliary concept of quasi-equilibrium.

Definition 5.1. An element $(x^*, y^*, p^*) \in S^1(X) \times S^1(Y) \times \mathbb{L}'$ is a *free-disposal satiation quasi-equilibrium* of the economy $\mathcal{E}$ if $p^* > 0$ and if the following properties are satisfied.

(i) For almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Rightarrow p^*(x) \geqslant p^*(y) + p^*(e(a)).$$

(ii) $\int_A x^*(a)\,\mathrm{d}\mu(a) \leqslant \int_A e(a)\,\mathrm{d}\mu(a) + \int_A y^*(a)\,\mathrm{d}\mu(a)$.

If (x^*, y^*, p^*) is a free-disposal quasi-equilibrium of an economy $\mathcal{E}$, then (x^*, y^*, p^*) is clearly a free-disposal satiation quasi-equilibrium of $\mathcal{E}$.

Remark 5.1. Under Assumption LNS, each free-disposal satiation quasi-equilibrium (x^*, y^*, p^*) of an economy $\mathcal{E}$, is in fact a free-disposal quasi-equilibrium of $\mathcal{E}$.

Following Remarks 3.1, 3.2 and 5.1, to prove the existence of a Walrasian equilibrium, it is sufficient (under Assumptions C, SS, LNS and FD) to prove the following lemma.

Lemma 1. *If $\mathcal{E}$ is an economy satisfying* Assumptions **C**, **M**, **P**, **S** *and* **B**, *then a free-disposal satiation quasi-equilibrium of $\mathcal{E}$ exists.*

5.2. Existence of free-disposal satiation equilibria for polytope economies

We propose first to prove an auxiliary existence result (the following Lemma 2) for *polytope* economies, i.e., economies satisfying the following Assumption K. This first step allows us to isolate the crucial aspect of the new approach, which is the approximation of economies with a measure space of agents (measurable correspondences) by a sequence of economies with a finite set of agents (resp. simple correspondences). Moreover, the framework of *polytope* economies allows us to deal with non-ordered but convex preference relations, as well as, ordered but non-convex preference relations.

[7] Where $f^n := \varphi(\sigma^n, A^n)(f)$.

Assumption (K). There exist a finite set $K = \{0, \dots, r\}$ and Bochner integrable functions $(x_k)_{k\in K}$, $(y_k)_{k\in K}$ from A to $\mathbb{L}$ such that for almost every agent $a \in A$,

$$X(a) = \mathrm{co}\,\{x_0(a), \dots, x_r(a)\} \quad \text{and} \quad Y(a) = \mathrm{co}\,\{y_0(a), \dots, y_r(a)\}.$$

Lemma 2. *If $\mathcal{E}$ is an economy satisfying* Assumptions C, M, P, S *and* K, *then a free-disposal satiation quasi-equilibrium of $\mathcal{E}$ exists.*

Proof. We can suppose (considering a translation if necessary) that for almost every $a \in A$, $e(a) = 0$. Following Proposition A.1, there exist a sequence $(f_k)_k$ of measurable selections of X and a sequence $(g_k)_k$ of measurable selections of Y such that for every $a \in A$,

$$X(a) = s\text{-cl}\,\{f_k(a) : k \in \mathbb{N}\} \quad \text{and} \quad Y(a) = s\text{-cl}\,\{g_k(a) : k \in \mathbb{N}\}.$$

Assumption S implies that the correspondence $X \cap Y$ defined by $a \mapsto X(a) \cap Y(a)$ has non-empty values. Since X and Y are graph measurable, then we can check that $X \cap Y$ is graph measurable. It follows (appealing once again to Proposition A.1) that without any loss of generality, we can suppose for every $a \in A$, $x_0(a) = f_0(a) = g_0(a) = y_0(a)$. We let for every k, $R_k : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined by $R_k(a) := \{x \in X(a) : f_k(a) \notin P_a(x)\}$. Then for almost every agent $a \in A$, for every $x \in \mathbb{L}$,

$$d(x, X(a)) = 0 \Leftrightarrow x \in X(a) \quad \text{and} \quad d(x, Y(a)) = 0 \Leftrightarrow x \in Y(a),$$

and for every $x \in X(a)$,

$$\forall k \in \mathbb{N}, d(x, R_k(a)) > 0 \Leftrightarrow f_k(a) \in P_a(x).$$

Following Assumption K, we let for each $a \in A$,

$$h(a) := \max\{\|x_k(a)\|, \|y_k(a)\| : 0 \leqslant k \leqslant r\}.$$

It follows that the correspondences X and Y are integrably bounded by h. Applying Theorem 4.1 and Remark 4.1 to

$$\mathcal{F} := \{\{f_k(\cdot)\}, \{g_k(\cdot)\}, \{x_j(\cdot)\}, \{y_j(\cdot)\}, X, Y, R_k : (k, j) \in \mathbb{N} \times K\}$$

and $\mathcal{G} := \{h(\cdot)\}$, there exists a sequence $(\sigma^n)_n$ of measurable partitions $\sigma^n = (A_i^n)_{i\in S^n}$ of $(A, \mathcal{A})$, and a sequence $(A^n)_n$ of finite sets $A^n = \{a_i^n : i \in S^n\}$ subordinated to the measurable partition σ^n, satisfying the following properties.

Fact 5.1. *For every $a \in A$,*

(i) *for every n, $h^n(a) \leqslant 1 + h(a)$ and for every $(k, j) \in \mathbb{N} \times K$,*

$$s\text{-}\lim_n (f_k^n(a), g_k^n(a)) = (f_k(a), g_k(a))$$

and

$$s\text{-}\lim_n (x_j^n(a), y_j^n(a)) = (x_j(a), y_j(a));$$

(ii) *for every sequence* $(x^n)_n$ *of* $\mathbb{L}$, *s-converging to* $x \in \mathbb{L}$,

$$\lim_n d(x^n, X^n(a)) = d(x, X(a)), \qquad \lim_n d(x^n, Y^n(a)) = d(x, Y(a))$$

and

$$\lim_n d(x^n, R^n_k(a)) = d(x, R_k(a)),$$

where d *is the distance function associated to the norm* $\| \cdot \|$.

We let, for each $a \in A$,

$$K_1(a) := \overline{\mathrm{co}} \bigcup_{k \in K} \{x^n_k(a) : n \in \mathbb{N}\} \quad \text{and} \quad K_2(a) := \overline{\mathrm{co}} \bigcup_{k \in K} \{y^n_k(a) : n \in \mathbb{N}\}.$$

A direct consequence of Fact 5.1 together with Theorem III.40 in Castaing and Valadier (1977) is the following result.

Fact 5.2. *The correspondences* K_1 *and* K_2 *are graph measurable, integrably bounded with non-empty, s-compact and convex values.*

We construct now a sequence of economies with a finite set of consumers. We distinguish two cases. In the first case (Claim 5.1) preference relations are possibly non-ordered but convex, in the second case (Claim 5.2) preference relations are ordered but possibly non-convex.

Claim 5.1. *If* $\mathcal{E}$ *satisfies* Assumption C.3(i), *then a free-disposal satiation quasi-equilibrium exists.*

Proof. For every n, we denote by $\mathcal{G}^n$ the following *finite* production economy:

$$\mathcal{G}^n = (\langle \mathbb{L}', \mathbb{L} \rangle, (X^n_i, Y^n_i - \mathbb{L}_+, P^n_i)_{i \in I^n})$$

where $I^n := \{i \in S^n : \mu(A^n_i) \neq 0\}$ is the finite set of consumers. The consumption set of consumer $i \in I^n$ is given by $X^n_i := \mu(A^n_i) X(a^n_i)$[8] and the production set is given by $Y^n_i - \mathbb{L}_+$, where $Y^n_i := \mu(A^n_i) Y(a^n_i)$. The preference relations are given by $P^n_i := \mu(A^n_i) P(a^n_i)$.

We assert that the economy $\mathcal{G}^n$ satisfies all the assumptions[9] of Proposition 4 in Florenzano (1990) and thus there exist $(x^n_i)_{i \in I^n} \in \prod_{i \in I^n} X^n_i$ and $(y^n_i)_{i \in I^n} \in \prod_{i \in I^n} Y^n_i$ such that $\sum_i x^n_i \leqslant \sum_i y^n_i$ and $0 \notin G$, where[10]

$$G := \mathbb{Q}\text{-co} \bigcup_{i \in I^n} (\mathrm{co}\, P^n_i(x^n_i) - \mathrm{co}\, Y^n_i - \mathbb{L}_+).$$

Applying Proposition A.3 there exists $p^n \in \mathbb{L}' \setminus \{0\}$ satisfying $p^n > 0$ and such that for every $i \in I^n$, if $(x, y) \in P^n_i(x^n_i) \times Y^n_i$ then $p^n(x - y) \geqslant 0$.

8 The consumer a^n_i *represents* the coalition A^n_i.

9 In particular Assumption S is valid, since for almost every $a \in A$, $f_0(a) = g_0(a)$.

10 We refer to Proposition A.3 for the definition of the $\mathbb{Q}$-convex hull.

Let, for every n,

$$x^n := \sum_{i\in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i\in I^n} \frac{y_i^n}{\mu(A_i^n)} \chi_{A_i^n}.$$

For each n, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying

$$\int_A x^n(a)\,\mathrm{d}\mu(a) \leqslant \int_A y^n(a)\,\mathrm{d}\mu(a) \tag{5.1}$$

$$\forall a \in \bigcup_{i\in I^n} A_i^n,\ (x, y) \in P_a^n(x^n(a)) \times Y^n(a) \Rightarrow p^n(x) \geqslant p^n(y), \tag{5.2}$$

where X^n and Y^n are defined by Fact 5.1 and similarly $P^n := \sum_i P(a_i^n)\chi_{A_i^n}$. Note that for almost every $a \in A$, for each n, $x^n(a) \in K_1(a)$ and $y^n(a) \in K_2(a)$. Applying Fact 5.2 and Theorem A.1,[11] there exist Bochner integrable functions $x^*, y^* : A \to \mathbb{L}$ such that

$$\int_A x^*\,\mathrm{d}\mu = \lim_n \int_A x^n\,\mathrm{d}\mu \quad \text{and} \quad \int_A y^*\,\mathrm{d}\mu = \lim_n \int_A y^n\,\mathrm{d}\mu, \tag{5.3}$$

for almost every $a \in A^{\mathrm{na}}$,

$$x^*(a) \in \overline{\mathrm{co}}\, s\text{-ls}\,\{x^n(a)\} \quad \text{and} \quad y^*(a) \in \overline{\mathrm{co}}\, s\text{-ls}\,\{y^n(a)\}, \tag{5.4}$$

for every $a \in A^{\mathrm{pa}}$,

$$x^*(a) \in s\text{-ls}\,\{x^n(a)\} \quad \text{and} \quad y^*(a) \in s\text{-ls}\,\{y^n(a)\}, \tag{5.5}$$

where A^{na} is the non-atomic part of $(A, \mathcal{A}, \mu)$ and A^{pa} is the purely atomic part of $(A, \mathcal{A}, \mu)$. Since, for every n, $p^n \in \mathbb{L}'_+ \setminus \{0\}$, we may suppose[12] that $(p^n)_n$ w^*-converging to p^*, with $p^* \in \mathbb{L}'_+ \setminus \{0\}$.

We propose to prove that (x^*, y^*, p^*) is a free-disposal satiation quasi-equilibrium of $\mathcal{E}$. We let

$$A_0 := \bigcup_n \bigcup_{i\in S^n\setminus I^n} A_i^n,$$

then we easily check that $\mu(A_0) = 0$. Let now A' be a measurable subset of $A \setminus A_0$ with $\mu(A \setminus A') = 0$ and such that all *almost every where* assumptions and properties are satisfied for every $a \in A'$.

To prove condition (ii) of Definition 5.1, we need to prove that $(x^*, y^*) \in S^1(X) \times S^1(Y)$. Let $a \in A'$, by construction, we have that for every n, $x^n(a) \in X^n(a)$, and thus, for every n, $d(x^n(a), X^n(a)) = 0$. We apply Fact 5.1 to conclude that for every $\xi \in s\text{-ls}\,\{x^n(a)\}$, $d(\xi, X(a)) = 0$. It follows that $s\text{-ls}\,\{x^n(a)\} \subset X(a)$. Since $x^*(a) \in \overline{\mathrm{co}}\, s\text{-ls}\,\{x^n(a)\}$, applying Assumption K, we get that $x^*(a) \in X(a)$. We prove similarly that $y^* \in S^1(Y)$. Following (5.1) and (5.3), condition (ii) is thus valid.

[11] Since the correspondences K_1 and K_2 have s-compact values, we have that w-ls $= s$-ls.

[12] Indeed, since there exists $u \in \mathbb{L}_+$ and V a $\|\cdot\|$-open symmetric neighborhood of zero such that $\{u\} + V \subset \mathbb{L}_+$, then the sequence $(p^n)_n$ may be chosen such that, for each $n \in \mathbb{N}$, for each $v \in V$, $|p^n(v)| \quad p^n(u) = 1$. Now since $(\mathbb{L}, \|\cdot\|)$ is separable, applying Alaoglu Compactness Theorem, we can extract a w^*-converging subsequence.

We will now prove that (x^*, y^*, p^*) satisfies condition (i) of Definition 5.1. Let $a \in A'$ and $(x, y) \in P_a(x^*(a)) \times Y(a)$. We let $\mathcal{I}$ be the set of strictly increasing functions from $\mathbb{N}$ into $\mathbb{N}$. We can suppose that there exists $(\phi, \psi) \in \mathcal{I}^2$ such that $(f_{\phi(k)}(a))_k$ s-converges to x and that $(g_{\psi(k)}(a))_k$ s-converges to y. To prove that $p^*(x - y) \geqslant 0$, it is sufficient to prove that for every k large enough, $p^*(f_{\phi(k)}(a)) \geqslant p^*(g_{\psi(k)}(a))$. Following Assumption C, there exist $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$, $f_{\phi(k)}(a) \in P_a(x^*(a))$. Consider $k \geq k_0$ and let $i := \phi(k)$ and $j := \psi(k)$.

We assert that there exists $\alpha \in \mathcal{I}$ such that

$$\forall n \in \mathbb{N}, \left(f_i^{\alpha(n)}(a), g_j^{\alpha(n)}(a)\right) \in P_a^{\alpha(n)}\left(x^{\alpha(n)}(a)\right) \times Y^{\alpha(n)}(a). \tag{5.6}$$

Indeed, by definition of $Y^n(a)$, we have that $g_j^n(a) \in Y^n(a)$. Suppose now that for every $\alpha \in \mathcal{I}$, there exist $\beta \in \mathcal{I}$ such that

$$\forall n \in \mathbb{N}, d\left(x^{\alpha\circ\beta(n)}(a), R_i^{\alpha\circ\beta(n)}(a)\right) = 0.$$

Applying (ii) of Fact 5.1, it follows that for every $\xi \in s\text{-ls}\{x^n(a)\}$, $d(\xi, R_i(a)) = 0$, i.e., $\xi \in R_i(a)$. Following Assumption C, $R_i(a)$ is closed convex if a belongs to the non-atomic part of $(A, \mathcal{A}, \mu)$. Applying (5.4) and (5.5), we conclude that $x^*(a) \in R_i(a)$, i.e., $f_i(a) \notin P_a(x^*(a))$: contradiction.

Applying (5.6) together with (5.2), we obtain that, for every n,

$$p^{\alpha(n)}(f_i^{\alpha(n)}(a) - g_j^{\alpha(n)}(a)) \geqslant 0.$$

Applying Fact 5.1, we have that $(f_i^n(a) - g_j^n(a))_{n\in\mathbb{N}}$ s-converges to $f_i(a) - g_j(a)$. Since $(p^n)_n$ w^*-converges to p^*, we get that $p^*(f_i(a)) \geqslant p^*(g_j(a))$. □

We consider now the case of ordered but possibly non-convex preference relations.

Claim 5.2. *If $\mathcal{E}$ satisfies* Assumptions C.3(ii) *and* A, *then a free-disposal satiation quasi-equilibrium exists.*

Proof. Following Theorem 2 in Sondermann (1980), for almost every $a \in A$, there exists an s-upper semi-continuous utility function u_a representing the binary relation $P(a)$ on $X(a)$, in the sense that

$$(x, x') \in P(a) \Rightarrow u_a(x) < u_a(x').$$

We denote by $A^{\mathrm{na}} \subset A$ the non-atomic part of $(A, \mathcal{A}, \mu)$ and A^{pa} the purely atomic part of $(A, \mathcal{A}, \mu)$. We let, for almost every $a \in A^{\mathrm{na}}$,

$$\tilde{P}(a) := \{(x, x') \in X(a) \times X(a) : u_a(x) < u_a(x')\}$$

and for each $a \in A^{\mathrm{pa}}$, $\tilde{P}(a) := P(a)$. Note that for almost every $a \in A$, $P(a) \subset \tilde{P}(a)$. We define the correspondence $\tilde{R}$ from A into $\mathbb{L} \times \mathbb{L}$ by, for almost every $a \in A^{\mathrm{na}}$, $\tilde{R}(a) := \{(z, z') \in X(a) \times X(a) : u_a(z) \leqslant u_a(z')\}$; and for every $a \in A^{\mathrm{pa}}$, $\tilde{R}(a) := R(a)$.

In order to use the same *limit* argument as in Claim 5.1, we define *convex* preference relations. This construction is borrowed from Hildenbrand (1974) (problem 7, p. 94). We define $\hat{P}: A \twoheadrightarrow \mathbb{L} \times \mathbb{L}$ by, for every a in the non-atomic part A^{na} of $(A, \mathcal{A}, \mu)$,

$$\hat{P}(a) := \{(x, y) \in X(a) \times X(a) : x \notin \overline{\mathrm{co}}\, \tilde{R}_a(y)\}$$

where $\tilde{R}_a(y) := \{x \in X(a) : u_a(x) \geqslant u_a(y)\}$;[13] and we define for every a in the purely atomic part A^{pa}, $\hat{P}(a) = P(a)$. For almost every $a \in A$, for each $y \in X(a)$, $X(a) \setminus \hat{P}_a^{-1}(y) := \overline{\mathrm{co}}\, \tilde{R}_a(y)$ is closed convex.

Claim 5.3. *For almost every $a \in A$, $\hat{P}(a)$ satisfies the following convex property,*

$$\forall x \in X(a), x \notin \mathrm{co}\, \hat{P}_a(x).$$

Proof. If $a \in A^{\mathrm{pa}}$, the claim is trivial, let thus $a \in A^{\mathrm{na}}$. Suppose that there exists $x \in X(a)$ such that $x \in \mathrm{co}\, \hat{P}_a(x)$. Then there exists a finite set $\mathcal{K}$ such that $x = \sum_{k \in \mathcal{K}} x_k$, where $x_k \in \hat{P}_a(x)$. The binary relation $\tilde{R}(a)$ is a complete pre-order, then there exists $k_0 \in \mathcal{K}$, such that for every $k \in \mathcal{K}$, $x_k \in \tilde{R}_a(x_{k_0})$. It follows that $x \in \mathrm{co}\, \tilde{R}_a(x_{k_0})$, in particular this leads to $x_{k_0} \notin \hat{P}_a(x)$, a contradiction. □

We are now ready to construct the sequence of economies with a finite set of consumers. Following the notations of Fact 5.1, for each n, we denote by $\mathcal{E}^n$ the following *finite* economy $\mathcal{E}^n = (\langle \mathbb{L}', \mathbb{L} \rangle, (X_i^n, \hat{Y}_i^n, \hat{P}_i^n)_{i \in I^n})$, where $I^n := \{i \in S^n : \mu(A_i^n) \neq 0\}$ is the finite set of consumers. The consumption set of the consumer $i \in I^n$ is given by $X_i^n := \mu(A_i^n) X(a_i^n)$ and the production set is given by $\hat{Y}_i^n - \mathbb{L}_+$, where $\hat{Y}_i^n := \mu(A_i^n)[Y(a_i^n) + (1/n)\{u\}]$ and u is any vector in s-int $\mathbb{L}_+$. The preference relations are given by $\hat{P}_i^n := \mu(A_i^n) \hat{P}(a_i^n)$.

We assert that the economy $\mathcal{E}^n$ satisfies all the assumptions of Proposition 4 in Florenzano (1990). It follows that there exists $(x_i^n)_{i \in I^n} \in \prod_{i \in I^n} X_i^n$, $(y_i^n)_{i \in I^n} \in \prod_{i \in I^n} \hat{Y}_i^n$ such that $\sum_{i \in I^n} x_i^n \leqslant \sum_{i \in I^n} y_i^n$ and such that $0 \notin G$, where[14]

$$G := \mathbb{Q}\text{-co} \bigcup_{i \in I^n} (\mathrm{co}\, \hat{P}_i^n(x_i^n) - \mathrm{co}\, \hat{Y}_i^n - \mathbb{L}_+).$$

Applying Proposition A.3 there exists $p^n \in \mathbb{L}' \setminus \{0\}$ satisfying $p^n > 0$,[15] and such that for every $i \in I^n$, if $(x, y) \in \hat{P}_i^n(x_i^n) \times \hat{Y}_i^n$ then $p^n(x - y) \geqslant 0$.

We let for every n,

$$x^n := \sum_{i \in I^n} \frac{x_i^n}{\mu(A_i^n)} \chi_{A_i^n} \quad \text{and} \quad y^n := \sum_{i \in I^n} \left(\frac{y_i^n}{\mu(A_i^n)} - \frac{1}{n} u \right) \chi_{A_i^n}.$$

[13] Since u_a is s-upper semi-continuous, then $\tilde{R}_a(y)$ is s-closed. Moreover, if $\tilde{R}_a(y)$ is convex, then $\hat{P}(a) = \tilde{P}(a)$.

[14] We refer to Proposition A.3 for the definition of the $\mathbb{Q}$-convex hull.

[15] Since u is an s-interior point of $\mathbb{L}_+$, there exists V a symmetric s-open neighborhood of 0 such that $\{u\} + V \subset \mathbb{L}_+$. In particular, without any loss of generality, we can choose the price p^n such that for each $v \in V$, $|p^n(v)|$ $p^n(u) = 1$.

For each n, we have defined integrable selections $x^n \in S^1(X^n)$ and $y^n \in S^1(Y^n)$ satisfying

$$\int_A x^n(a)\,\mathrm{d}\mu(a) \leqslant \int_A y^n(a)\,\mathrm{d}\mu(a) + \frac{1}{n}u \tag{5.7}$$

$$\forall a \in \bigcup_{i\in I^n} A_i^n,\ (x,y) \in \hat{P}_a^n(x^n(a)) \times Y^n(a) \Rightarrow p^n(x) > p^n(y), \tag{5.8}$$

where X^n, Y^n are defined by Fact 5.1 and similarly $Q^n := \sum_i Q(a_i^n)\chi_{A_i^n}$, where $Q \in \{P, \tilde{P}, \hat{P}, \tilde{R}\}$. Since, for every n, $\sup_{v\in V}|p^n(v)| \leqslant p^n(u) = 1$, there exists a subsequence of $(p^n)_n$ w^*-converging to p^*, with $p^*(u) = 1$.

For each $p \in \mathbb{L}'$, for each correspondences $Z, W: A \to \mathbb{L}$ and for each $a \in A$, we let

$$B(a, p, Z, W) = \{z \in Z(a): p(z) \leqslant \sup p(W(a))\}$$

and

$$\beta(a, p, Z, W) = \{z \in Z(a): p(z) < \sup p(W(a))\}.$$

We let $B(a) := B(a, p^*, X, Y)$ and $\beta(a) = \beta(a, p^*, X, Y)$. Moreover, for each $n \in \mathbb{N}$, we let $B^n(a) := B(a, p^n, X^n, Y^n)$ and $\beta^n(a) := \beta(a, p^n, X^n, Y^n)$.
Now we define the correspondence D, G and H by, for each correspondence of preference relations Q in Z and for each $a \in A$,

$$D(a, p, Z, W, Q) := \{z \in B(a, p, Z, W): Q_a(z) \cap B(a, p, Z, W) = \emptyset\},$$

$$G(a, p, Z, W, Q) := \{z \in Z(a): Q_a(z) \cap B(a, p, Z, W) = \emptyset\},$$

and

$$H(a, p, Z, W, Q) := \{z \in Z(a): Q_a(z) \cap \beta(a, p, Z, W) = \emptyset\}.$$

We define $D^*(a, Q) := D(a, p^*, X, Y, Q)$, $G^*(a, Q) := G(A, p^*, X, Y, Q)$ and $H^*(a, Q) := H(a, p^*, X, Y, Q)$. Moreover, for each $n \in \mathbb{N}$, we define $D^n(a, Q) := D(a, P^n, X^n, Y^n, Q)$, $G^N(a, Q) := G(a, p^n, X^n, Y^n, Q)$ and $H^N(a, Q) := H(a, p^n, X^n, Y^n, Q)$.

Claim 5.4. *For each $n \in \mathbb{N}$ and for each $a \in A^{\mathrm{na}}$,*

$$G^n(a, \hat{P}^n) \subset \overline{\mathrm{co}}\, G^n(a, \tilde{P}^n) \subset \overline{\mathrm{co}}\, G^n(a, P^n) \tag{5.9}$$

and for every $a \in A^{\mathrm{pa}}$, $G^n(a, \hat{P}^n) = G^n(a, \tilde{P}^n) = G^n(a, P^n)$.

Proof. Indeed, if $a \in A^{\mathrm{pa}}$ then $\hat{P}^n(a) = \tilde{P}^n(a) = P^n(a)$ and the result follows. Now let $a \in A^{\mathrm{na}}$ and $x \in G^n(a, \hat{P}^n)$. The set $X^n(a)$ is s-compact, the strict-preference relation $\tilde{P}^n(a)$ is irreflexive, transitive with s-open lower sections. Hence, following a classical maximal argument, the set $D^n(a, \tilde{P}^n)$ is non-empty. Let $\tilde{x} \in D^n(a, \tilde{P}^n)$, then $\tilde{x} \in B(a, p^n)$ and since $x \in G^n(a, \hat{P}^n)$, we have that $(x, \tilde{x}) \notin \hat{P}^n(a)$, i.e., $x \in \overline{\mathrm{co}}\, \tilde{R}_a^n(\tilde{x})$. Since $\tilde{R}^n(a)$ is transitive and complete, it is straightforward to verify that $\tilde{R}_a^n(\tilde{x}) \subset G^n(a, \tilde{P}^n) \subset G^n(a, P^n)$, and thus $x \in \overline{\mathrm{co}}\, G^n(a, P^n)$. □

Since (x^n, p^n) satisfies (5.8), it follows[16] that for a.e. $a \in A$, $x^n(a) \in G^n(a, \hat{P}^n)$. Applying the previous claim, it follows that $x^n(a) \in \overline{\text{co}}\, G^n(a)$. Note that for almost every $a \in A$, for each n, $x^n(a) \in K_1(a)$ and $y^n(a) \in K_2(a)$. Applying Fact 5.2 and Theorem A.1, there exist Bochner integrable functions $x^*, y^* : A \to \mathbb{L}$ such that

$$\int_A x^* \, d\mu = \lim_n \int_A x^n \, d\mu \quad \text{and} \quad \int_A y^* \, d\mu = \lim_n \int_A y^n \, d\mu, \tag{5.10}$$

for almost every $a \in A^{\text{na}}$,

$$x^*(a) \in \overline{\text{co}}\, s\text{-ls}\, \{x^n(a)\} \quad \text{and} \quad y^*(a) \in \overline{\text{co}}\, s\text{-ls}\, \{y^n(a)\}, \tag{5.11}$$

for every $a \in A^{\text{pa}}$,

$$x^*(a) \in s\text{-ls}\, \{x^n(a)\} \quad \text{and} \quad y^*(a) \in s\text{-ls}\, \{y^n(a)\}. \tag{5.12}$$

Following verbatim the arguments of Claim 5.1,

$$s\text{-ls}\, X^n(a) \subset X(a) \quad \text{and} \quad s\text{-ls}\, Y^n(a) \subset Y(a).$$

With Assumption K, $\overline{\text{co}}\, X(a) = X(a)$ and $\overline{\text{co}}\, Y(a) = Y(a)$, it follows that $x^* \in S^1(X)$ and $y^* \in S^1(Y)$. Applying (5.7),

$$\int_A x^*(a) \, d\mu(a) \leqslant \int_A y^*(a) \, d\mu(a). \tag{5.13}$$

Once again, following verbatim the arguments of Claim 5.1, we prove that for almost every $a \in A$,

$$s\text{-ls}(H^n(a, P^n)) \subset H^*(a, P).$$

Applying Carathéodory Convexity Theorem, for almost every $a \in A$,

$$s\text{-ls}(\text{co}\, H^n(a, P^n)) \subset \text{co}\, s\text{-ls}(H^n(a, P^n)) \subset \text{co}\, H^*(a, P).$$

It follows[17] that for almost every $a \in A$,

$$a \in A^{\text{na}} \Rightarrow x^*(a) \in \overline{\text{co}}\, H^*(a, P) \quad \text{and} \quad a \in A^{\text{pa}} \Rightarrow x^*(a) \in H^*(a, P).$$

Claim 5.5. *The correspondence $H^*(\cdot, P) : a \mapsto H^*(a, P)$ is graph measurable.*

Proof. In this proof, we denote $\beta(a) := \beta(a, p^*, P)$ and $H(a) := H^*(a, P)$. Indeed let $A^\beta := \{a \in A : \beta(a) \neq \emptyset\}$. Since X and Y are graph measurable, then β is graph measurable and $A^\beta \in \mathcal{A}$. Applying Proposition A.1, there exists a sequence $(h^k)_k$ of measurable selections of $\beta_{|A^\beta}$ satisfying, for every $a \in A^\beta$, $(h^k(a))_k$ is dense in $\beta(a)$. We let, for each k, $z^k(a) = h^k(a)$ if $a \in A^\beta$ and $z^k(a) = X_0(a)$ elsewhere. It follows that

$$\forall a \in A^\beta,\ H(a) = \bigcap_\nu R_{z^\nu}(a) \quad \text{and} \quad \forall a \in A \setminus A^\beta,\ H(a) = X(a),$$

[16] This is the reason why we introduce u in the construction of $\hat{Y}_i^n$.

[17] Recall that for every n, $x^n(a) \in \overline{\text{co}}\, G^n(a, P^n) \subset \overline{\text{co}}\, H^n(a, P^n)$. But $H^n(a, P^n)$ is a s-closed subset of $X^n(a) = \text{co}\, \{x_0^n(a), \ldots, x_r^n(a)\}$, hence $\overline{\text{co}}\, H^n(a, P^n) = \text{co}\, H^n(a, P^n)$.

where $R_{z^\nu}(a) = \{x \in X(a) : (x, z^\nu) \notin P(a)\}$. Applying Assumption M, for each k, R_{zk} is graph measurable and H is then graph measurable. □

We apply now Assumption A,

$$\int_A x^*(a)\,\mathrm{d}\mu(a) \in \int_{A^{\mathrm{na}}} \overline{\mathrm{co}}\,[H^*(a, P)]\,\mathrm{d}\mu(a) + \int_{A^{\mathrm{pa}}} H^*(a, P)\,\mathrm{d}\mu(a)$$
$$= \int_A H^*(a, P)\,\mathrm{d}\mu(a).$$

That is, there exists $\bar{x} \in S^1(X)$ such that for almost every agent $a \in A$, $\bar{x}(a)$ belongs to $H^*(a, P)$ and following (5.13), $\int_A \bar{x} \leqslant \int_A e(a)\,\mathrm{d}\mu(a) + \int_A y^*(a)\,\mathrm{d}\mu(a)$. It follows that $(\bar{x}, y^*, p^*)$ is a free-disposal satiation quasi-equilibrium of the economy $\mathcal{E}$. □

The proof of Lemma 2 is a direct consequence of Claims 5.1 and 5.2. □

5.3. *Proof of Lemma 1*

We now apply Lemma 2 to prove Lemma 1.

Proof. Let $\mathcal{E}$ be an economy satisfying Assumptions C, M, P, S and B. Following Remark 3.15 and Proposition A.2, we can suppose without any loss of generality that for almost every $a \in A$, $Y(a)$ is a closed convex subset of $\mathbb{L}$ and that for almost every $a \in A$, $e(a) = 0$. From Assumption S, the correspondence $X \cap Y$ defined by $a \mapsto X(a) \cap Y(a)$ has non-empty values. Applying Theorem III.40 in Castaing and Valadier (1977), the correspondence $X \cap Y$ is graph measurable. Now applying Proposition A.1, there exist $f_0 \in S^1(X)$ and $g_0 \in S^1(Y)$ such that for almost every $a \in A$, $f_0(a) = g_0(a)$. Once again applying Proposition A.1, there exist a sequence $(f_k)_k$ of measurable selections of X and a sequence $(g_k)_k$ of measurable selections of Y such that for every $a \in A$,

$$X(a) = s\text{-cl}\,\{f_k(a) : k \in \mathbb{N}\} \quad \text{and} \quad Y(a) = s\text{-cl}\,\{g_k(a) : k \in \mathbb{N}\}.$$

For each $\nu \in \mathbb{N}$, let $\mathcal{E}^\nu = ((A, \mathcal{A}, \mu), \langle \mathbb{L}', \mathbb{L}\rangle, (X^\nu, Y^\nu, P^\nu))$, where for each agent $a \in A$, the consumption and production sets are defined by

$$X^\nu(a) := \mathrm{co}\,\{f_0(a), \ldots, f_\nu(a)\} \subset X(a)$$

and

$$Y^\nu(a) := \mathrm{co}\,\{g_0(a), g_1^\nu(a), \ldots, g_\nu^\nu(a)\} \subset Y(a),$$

where for each $1 \leqslant k \leqslant \nu$, $g_k^\nu(a) = g_k(a)$ if $\|g_k(a)\| \leqslant \nu$ and $g_k^\nu(a) = g_0(a)$ either. The preference relations are defined by $P^\nu(a) := P(a) \cap (X^\nu(a) \times X^\nu(a))$. For each ν, the economy $\mathcal{E}^\nu$ satisfies Assumptions C, M, P, S and K. Applying Lemma 2, we obtain the following fact.

Fact 5.3. *For each* ν, *there exists*[18] $(x^\nu, p^\nu) \in S^1(X^\nu) \times \mathbb{L}'_+$ *with* $p^\nu \neq 0$ *and such that there exists* $A^\nu \in \mathcal{A}$, *with* $\mu(A \setminus A^\nu) = 0$ *and satisfying the following properties*.

(i) *For every* $a \in A^\nu$, $(x, y) \in P^\nu_a(x^\nu(a)) \times Y^\nu(a) \Rightarrow p^\nu(x - y) \geqslant 0$.
(ii) $\int_A x^\nu(a)\,\mathrm{d}\mu(a) \in Y_\Sigma - \mathbb{L}_+$.

Applying Theorem A.1, there exists a Bochner integrable function $x^* \in S^1(X)$ such that

$$\int_A x^*(a)\,\mathrm{d}\mu(a) = \lim_\nu \int_A x^\nu(a)\,\mathrm{d}\mu(a). \tag{5.14}$$

$$\text{for a.e. } a \in A^{\mathrm{na}}, \quad x^*(a) \in \overline{\mathrm{co}}\, w\text{-ls}\,\{x^\nu(a)\} \tag{5.15}$$

$$\text{for all } a \in A^{\mathrm{pa}}, \quad x^*(a) \in w\text{-ls}\,\{x^\nu(a)\}. \tag{5.16}$$

For every ν, $p^\nu > 0$. Since s-int $\mathbb{L}_+ \neq \emptyset$, we can choose the sequence $(p^\nu)_\nu$ such that $(p^\nu)_\nu$ w^*-converges to p^*, with $p^* > 0$. Following Assumption P, (5.14) and (ii) of Fact 5.3, there exists $y^* \in S^1(Y)$ such that

$$\int_A x^*(a)\,\mathrm{d}\mu(a) \leqslant \int_A y^*(a)\,\mathrm{d}\mu(a). \tag{5.17}$$

For the rest of the proof, we distinguish two cases. In the first case (Claim 5.6) preference relations are possibly non-ordered but convex, in the second case (Claim 5.7) preference relations are ordered but possibly non-convex.

Claim 5.6. *If* $\mathcal{E}$ *satisfies* Assumption C.3(i), *then a free-disposal satiation quasi-equilibrium exists*.

Proof. We propose to prove that (x^*, y^*, p^*) is a free-disposal satiation quasi-equilibrium of $\mathcal{E}$. Following (5.17) it suffices to prove that for almost every $a \in A$,

$$(x, y) \in P_a(x^*(a)) \times Y(a) \Rightarrow p^*(x) \geqslant p^*(y).$$

Let $a \in A \setminus (\cup_\nu A^\nu)$ and let $(x, y) \in P_a(x^*(a)) \times Y(a)$. We let $\mathcal{I}$ be the set of strictly increasing functions from $\mathbb{N}$ into $\mathbb{N}$. We can suppose that there exists $(\phi, \psi) \in \mathcal{I}^2$ such that $(f_{\phi(k)}(a))_k$ s-converges to x and that $(g_{\psi(k)}(a))_k$ s-converges to y. Moreover, we can suppose that for every k large enough,

$$g_{\psi(k)}(a) = g^{\psi(k)}_{\psi(k)}(a) \in Y^k(a).$$

To prove that $p^*(x - y) \geqslant 0$, it is sufficient to prove that for every k large enough, $p^*(f_{\phi(k)}(a)) \geqslant p^*(g_{\psi(k)}(a))$. Following Assumption C, there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$, $f_{\phi(k)}(a) \in P_a(x^*(a))$. Let $k \geqslant k_0$ and let $i := \phi(k)$ and $j := \psi(k)$.
We can suppose that there exists $\alpha \in \mathcal{I}$ such that for every ν,

$$(f_i(a), g_j(a)) \in P^{\alpha(\nu)}_a(x^{\alpha(\nu)}(a)) \times Y^{\alpha(\nu)}(a).$$

[18] Recall that for every ν, $S^1(X^\nu) \subset S^1(X)$.

Indeed, for every $\nu \geqslant k$, $(f_i(a), g_j(a)) \in X^\nu(a) \times Y^\nu(a)$. Suppose that for every $\alpha \in \mathcal{I}$, there exists $\beta \in \mathcal{I}$ such that

$$\forall \nu \in \mathbb{N},\ x^{\alpha\circ\beta(\nu)}(a) \in R_i(a) := X(a) \setminus P_a^{-1}(f_i(a)).$$

Applying Assumption C, it follows that w-ls $\{x^\nu(a)\} \subset R_i(a)$. But $R_i(a)$ is closed convex if $a \in A^{\mathrm{na}}$. Applying (5.15) and (5.16), we conclude that $x^*(a) \in R_i(a)$, i.e., $f_i(a) \notin P_a(x^*(a))$: Contradiction. It follows that there exists $\alpha \in \mathcal{I}$ such that for every ν, $(f_i(a), g_j(a)) \in P_a^{\alpha(\nu)}(x^{\alpha(\nu)}(a)) \times Y^{\alpha(\nu)}(a)$.
Thus applying (i) of Fact 5.3, we obtain that, for every ν,

$$p^{\alpha(\nu)}(f_i(a) - g_j(a)) \geqslant 0.$$

Since $(p^\nu)_\nu$ w^*-converges to p^*, it follows that $p^*(f_i(a)) \geqslant p^*(g_j(a))$. □

We consider now the case of ordered but possibly non-convex preference relations.

Claim 5.7. *If $\mathcal{E}$ satisfies* Assumptions C.3(ii) *and* A, *then a free-disposal satiation quasi-equilibrium exists.*

Proof. Following notations introduced in the proof of Lemma 2, we let $H^\nu(a) := H(a, p^\nu, X^\nu, Y^\nu, P^\nu)$ and $H^*(a) := H(a, p^*, X, Y, P)$. Fact 5.3 implies that for almost every $a \in A$, for every ν, $x^\nu(a) \in H^\nu(a)$.

Claim 5.8. *We assert that for every $a \in A \setminus (\cup_\nu A^\nu)$, w-ls $H^\nu(a) \subset H^*(a)$.*

Proof. Indeed, let $a \in A \setminus (\cup_n A^\nu)$ and $z^*(a) \in w$-ls $H^\nu(a)$. Since $X(a)$ is w-closed, $z^*(a) \in w$-ls $X^\nu(a) \subset X(a)$. To prove that $z^*(a) \in H^*(a)$, it is sufficient to prove that

$$(z, y) \in P_a(z^*(a)) \times Y(a) \Rightarrow p^*(z) \geqslant p^*(y).$$

We let $\mathcal{I}$ be the set of strictly increasing function from $\mathbb{N}$ into $\mathbb{N}$. We can suppose that there exists $(\phi, \psi) \in \mathcal{I}^2$ such that $(f_{\phi(k)}(a))_k$ s-converges to z and that $(g_{\psi(k)}(a))_k$ s-converges to y. Moreover, we can suppose that for every k large enough,

$$g_{\psi(k)}(a) = g_{\psi(k)}^{\psi(k)}(a) \in Y^k(a).$$

To prove that $p^*(z - y) \geqslant 0$, it is sufficient to prove that for every k large enough, $p^*(f_{\phi(k)}(a)) \geqslant p^*(g_{\psi(k)}(a))$. Following Assumption C.3(ii), there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$, $f_{\phi(k)}(a) \in P_a(z^*(a))$. Consider $k \geqslant k_0$ and let $i := \phi(k)$ and $j := \psi(k)$. Since $z^*(a) \in w$-ls $H^\nu(a)$, for each ν, there exists $z^\nu \in H^\nu(a)$ such that $z^*(a) \in w$-ls $\{z^\nu\}$. We assert that there exists $\alpha \in \mathcal{I}$, such that for every ν,

$$(f_i(a), g_j(a)) \in P_a^{\alpha(\nu)}(z^{\alpha(\nu)}) \times Y^{\alpha(\nu)}(a).$$

Indeed, for every $\nu \geqslant k$, $(f_i(a), g_j(a)) \in X^\nu(a) \times Y^\nu(a)$. Suppose that for every $\alpha \in \mathcal{I}$, there exist $\beta \in \mathcal{I}$ such that

$$\forall \nu \in \mathbb{N},\ z^{\alpha\circ\beta(\nu)} \in R_i(a).$$

Applying Assumption C.3(ii), it follows that $w\text{-ls}\{z^\nu\} \subset R_i(a)$ and then $z^*(a) \in R_i(a)$, i.e., $f_i(a) \notin P_a(z^*(a))$: contradiction. It follows that there exists $\alpha \in \mathcal{I}$, such that for every ν, $(f_i(a), g_j(a)) \in P_a^{\alpha(\nu)}(z^{\alpha(\nu)}) \times Y^{\alpha(\nu)}(a)$.
Thus applying (i) of Fact 5.3, we obtain that, for every ν large enough,

$$p^{\alpha(\nu)}(f_i(a) - g_j(a)) \geqslant 0.$$

Since $(p^\nu)_\nu$ w^*-converges to p^*, it follows that $p^*(f_i(a)) \geqslant p^*(g_j(a))$. □

We proved in Lemma 2 that H^* is graph measurable. With Assumption A we get that

$$\int_A x^*(a)\,\mathrm{d}\mu(a) \in \int_{A^{\mathrm{pa}}} \overline{\mathrm{co}}\, H^*(a)\,\mathrm{d}\mu(a) + \int_{A^{\mathrm{na}}} H^*(a)\,\mathrm{d}\mu(a) = \int_A H^*(a)\,\mathrm{d}\mu(a).$$

It follows that there exists an integrable selection $\bar{x}$ of H^* such that $\int_A \bar{x} = \int_A x^*$, i.e., $(\bar{x}, y^*, p^*)$ is a free-disposal satiation quasi-equilibrium of $\mathcal{E}$. □

The proof of Lemma 1 is a direct consequence of Claims 5.6 and 5.7. □

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Appendix A. Mathematical auxiliary results

A.1. Integration of correspondences

We consider $(A, \mathcal{A}, \mu)$ a finite complete positive measure space and (D, d) a complete separable metric space.
Following Aumann (1969), graph measurable correspondences have measurable selections.

Proposition A.1. *Consider F a graph measurable correspondence from A into D with non-empty values. Then there exists a sequence $(z_n)_n$ of measurable selections of F, such that for every $a \in A$, $(z_n(a))_n$ is dense in $F(a)$.*

If $F : A \twoheadrightarrow \mathbb{L}$ is a correspondence from A into $\mathbb{L}$, the set of integrable selections of F is denoted by $S^1(F)$. We denote by F_Σ the following (possibly empty) set $F_\Sigma := \int_A F(a)\,\mathrm{d}\mu(a) := \{v \in \mathbb{L} : \exists x \in S^1(F)\ v = \int_A x(a)\,\mathrm{d}\mu(a)\}$.

Proposition A.2. *Consider $F : A \twoheadrightarrow \mathbb{L}$ a graph measurable correspondence. If F_Σ is non-empty, we let $G : A \twoheadrightarrow \mathbb{L}$ be the correspondence defined by*

$$\forall a \in A,\ G(a) := s\text{-cl}[\overline{\mathrm{co}}\, F(a) + A(F_\Sigma)].$$

If F_Σ is non-empty and closed convex then $G_\Sigma = F_\Sigma$, and for every $p \in \mathbb{L}'$, if there exists an integrable selection g^* of G such that for a.e. $a \in A$, $p(g^*(a)) = \sup p(G(a))$, then there exists an integrable selection f^* of F satisfying for a.e. $a \in A$, $p(f^*(a)) = \sup p(F(a))$ and $\int_A f^* = \int_A g^*$.

Proof. Following Theorem III.40 in Castaing and Valadier (1977), the correspondence G is graph measurable and $F_\Sigma \subset G_\Sigma$. Moreover if $p \in \mathbb{L}'$ then for every $a \in A$, $\sup p(G(a)) = \sup p(F(a)) + \sup p(A(F_\Sigma))$. Note that, since $A(F_\Sigma)$ is a cone containing zero,

$$\sup p(A(F_\Sigma)) \in \{0, \infty\}.$$

Suppose now that F_Σ is closed convex and that there exists $v \in G_\Sigma$ such that $v \notin F_\Sigma$. Since F_Σ is closed convex, by a separation argument there exists $p \in \mathbb{L}'$ with $p \neq 0$ such that $p(v) > \sup p(F_\Sigma)$. It follows that $\sup p(A(F_\Sigma)) = 0$ and following Proposition 6[19] in Hildenbrand (1970),

$$\sup p(F_\Sigma) = \int_A \sup p(F(a))\, \mathrm{d}\mu(a) = \int_A \sup p(G(a))\, \mathrm{d}\mu(a) = \sup p(G_\Sigma).$$

Thus $p(v) > \sup p(G_\Sigma)$ and this contradicts the fact that $v \in G_\Sigma$. The second part of Proposition A.2 is a direct consequence of the previous result. □

Theorem A.1. *Suppose F is an integrably bounded correspondence, with non-empty, w-compact and convex values. If $(f^n)_n$ is a sequence of integrable selections of F, then there exist an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and $f^* \in S^1(F)$ an integrable selection of F, such that*

$$\int_A f^*(a)\, \mathrm{d}\mu(a) = \lim_n \int_A f^{\phi(n)}(a)\, \mathrm{d}\mu(a),$$

and

$$\text{for a.e. } a \in A^{\mathrm{na}}, \quad f^*(a) \in \overline{\mathrm{co}}\, w\text{-ls}\, \{f^{\phi(n)}(a)\}$$

$$\text{for all } a \in A^{\mathrm{pa}}, \quad f^*(a) \in w\text{-ls}\, \{f^{\phi(n)}(a)\},$$

where A^{na} is the non-atomic part of $(A, \mathcal{A}, \mu)$ and A^{pa} is the purely atomic part of $(A, \mathcal{A}, \mu)$.

Proof. For each n, we let $v^n := \int_A f^n \,\mathrm{d}\mu$. Following Corollary 2.6 in Diestel et al. (1993) and Theorem 15 (p. 422) in Dunford and Schwartz (1966), the sequence $(v^n)_n$ is relatively compact. Applying Lemma 6.6 in Podczeck (1997) or Corollary 4.4 in Balder and Hess (1995), we get the desired result. Note that a more general result is given in Yannelis (1988). □

For more precisions about measurability and integration of correspondences, we refer to papers Yannelis (1991a,b).

[19] This latter result is stated in terms of $\mathbb{R}^n$-valued correspondences. However, as can be seen from its proof, it generalizes directly to the context of a separable Banach space.

A.2. Separation of $\mathbb{Q}$-convex sets

Let $(\mathbb{L}, \tau)$ be a topological vector space. A set G is called $\mathbb{Q}$-convex if for every $x, y \in G$, for every $t \in [0, 1] \cap \mathbb{Q}$, $tx + (1 - t)y \in G$. The $\mathbb{Q}$-convex hull of a set G is the smallest $\mathbb{Q}$-convex set containing G. We present hereafter a result of decentralization for a $\mathbb{Q}$-convex set.

Proposition A.3. *Let $(\mathbb{L}, \tau)$ be a topological vector space and G be a $\mathbb{Q}$-convex subset with a τ-interior point and such that $0 \notin G$. Then there exists a non-zero continuous linear functional $p \in (L, \tau)'$ such that*

$$\forall x \in G,\ p(x) \geqslant 0.$$

Proof. The interior $\operatorname{int} G$ of G is a non-empty and $\mathbb{Q}$-convex subset of $\mathbb{L}$. Let $x \in G$, for each $\lambda \in [0, 1] \cap \mathbb{Q}$, $\lambda x + (1 - \lambda)u \in \operatorname{int} G$, if $u \in \operatorname{int} G$. It follows that

$$\operatorname{int} G \subset G \subset \operatorname{cl} \operatorname{int} G.$$

Since $\operatorname{int} G$ is τ-open, it is in fact convex. Now $0 \notin \operatorname{int} G$ and we can apply a Convex Separation Theorem to provide the existence of a non-zero continuous linear functional $p \in (\mathbb{L}, \tau)'$ such that for every $x \in \operatorname{int} G$, $p(x) \geqslant 0$. With a limit argument, we prove that for every $x \in G$, $p(x) \geqslant 0$. □

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