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Unbounded exchange economies with satiation: How far can we go?☆

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ABSTRACT

We unify and generalize the existence results in Werner [Werner, J., 1987. Arbitrage and the existence of competitive equilibrium. Econometrica 55 (6), 1403–1418], Dana et al. [Dana, R.-A., Le Van, C., Magnien, F., 1999. On the different notions of arbitrage and existence of equilibrium. Journal of Economic Theory 87 (1), 169–193], Allouch et al. [Allouch, N., Le Van, C., Page Jr., F.H., 2006. Arbitrage and equilibrium in unbounded exchange economies with satiation. Journal of Mathematical Economics 42 (6), 661–674], Allouch and Le Van [Allouch, N., Le Van, C., 2008. Erratum to "Walras and dividends equilibrium with possibly satiated consumers". Journal of Mathematical Economics 45 (3–4), 320–328]. We also show that, in terms of weakening the set of assumptions, we cannot go too far.

1. Introduction

When agents trade securities, due to the possibility of short sales, the set of portfolios is not bounded from below. This implies that the set of feasible portfolios may not be bounded and the classical existence results of Arrow and Debreu (1954) and McKenzie (1959) cannot be applied. Existence results for models with unbounded action sets have been provided by Hart (1974), Hammond (1983) and Page (1987) for security markets. In these models the utility $V^i(q,\theta)$ of a portfolio θ is defined by the expected utility of its return $\mathbb{E}^{\mu^i(q)}(r\cdot\theta)$ with respect to price-dependent beliefs $\mu^i(q)$. In a different context where the utility function does not depend on prices, existence results have been provided by Werner (1987), Nielsen (1989), Page and Wooders (1996), Dana et al. (1999), Page et al. (2000), Allouch et al. (2002) and Le Van et al. (2001). To prove existence, several conditions have been proposed to limit arbitrage opportunities. In all cases the role played by those no-arbitrage conditions was to bound the economy endogenously. Dana et al. (1999) proved that all these conditions imply the compactness of the individually rational utility set,[1] which in turn is a sufficient condition for existence.[2]

In models with bounded from below consumption sets, a crucial assumption imposed (e.g. Arrow and Debreu, 1954; McKenzie, 1959) is that agents' preferences satisfy a non-satiation property (e.g. monotonicity). Actually, what is needed is non-satiation only over individually feasible actions. In security markets models (like CAPM), satiation of preferences is

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[1] That we call compactness condition (CU).

[2] Allouch (2002) introduced a weaker condition than the compactness of the individually rational utility set. In particular, an existence result was proved without assuming that preference relations are complete.

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rather a rule than an exception (see, among others, Werner, 1987; Nielsen, 1989; Allingham, 1991; Dana et al., 1999; Won et al., 2008). In his seminal paper, Werner (1987) allows for satiation but he imposes that all arbitrage opportunities are uniform among agents and that each agent has a useful portfolio. His existence result was extended by Allouch et al. (2006) where a weaker non-satiation condition (ANS) is imposed: *useless net trades are uniform among agents and each agent who is satiated has a non-empty set of useful net trades*. Recently, a weaker non-satiation condition (WNS) was proposed by Allouch and Le Van (2008): *satiation* is possible provided that each agent has satiation points available to him outside the set of individually feasible actions. They prove that this condition is sufficient for existence of a quasi-equilibrium provided that the set of individually rational and feasible allocations is compact.

The main objective of this paper is to investigate if it is possible to unify the aforementioned existence results. There are two kinds of assumptions used to prove existence: the first one deals with satiated preferences while the second one relies on a no-arbitrage condition which is the compactness of the individually rational utility set. Allouch et al. (2006) and Allouch and Le Van (2008) impose the weakest condition on satiation so far[3] while Dana et al. (1999) impose the weakest no-arbitrage condition so far, which is the compactness of the individually rational utility set.[4] Therefore, one may conjecture that existence is guaranteed under the two weakest conditions:

(a) the compactness condition (CU), i.e., the individually rational utility set is compact;
(b) the weak non-satiation condition (WNS), i.e., satiation is possible provided that each agent has satiation points available to him outside the set of individually feasible actions.

We prove that such a conjecture is not correct provided that there are more than two agents in the economy. We subsequently introduce a new condition, called *strong compactness of the individually rational utility set* (SCU) and we prove that it is sufficient for existence of a quasi-equilibrium when agents' preferences satisfy the weak non-satiation condition (WNS). We also show that, in general, condition (SCU) is stronger than the compactness of the individually rational utility set (CU)[5] but weaker than the two compactness conditions imposed in Allouch et al. (2006) and Allouch and Le Van (2008).

Our paper covers a specific case of economies with satiated agents: if satiation occurs in the set of individually feasible actions, then satiation should also occur outside. In particular we do not cover models, like the ones proposed by Allingham (1991), Nielsen (1989) and Won et al. (2008), where satiation only occurs inside the set of individually feasible actions. We also refer to Won and Yannelis (2002) where a generalization of these results is proposed.

2. The model

Consider an economy $(X^i, u^i, e^i)_{i\in I}$ where I is a finite set, for each $i\in I$, the set X^i is a subset of $\mathbb{R}^J$ with J a finite set, e^i is a vector in $\mathbb{R}^J$ and u^i is a real-valued function defined on X^i. As in Werner (1987), each $j\in J$ represents a commodity which can be a consumption good as well as a financial asset. Each $i\in I$ represents an agent, X^i his action set,[6] e^i his initial endowment and u^i his utility function. Once for all the sets $(X^i)_{i\in I}$ and the vectors $(e^i)_{i\in I}$ are fixed. The economy $(X^i, u^i, e^i)_{i\in I}$ is then denoted by $\mathcal{E}(\boldsymbol{u})$.

We denote by F the set of feasible allocations, i.e., those vectors $\boldsymbol{x} = (x^i)_{i\in I}$ in $\boldsymbol{X} \equiv \prod_{i\in I} X^i$ satisfying

$$\sum_{i\in I} x^i = \sum_{i\in I} e^i$$

and by $\mathrm{Ir}(\boldsymbol{u})$ the set of individually rational allocations, i.e., those vectors $\boldsymbol{x} = (x^i)_{i\in I}$ in $\boldsymbol{X}$ satisfying

$$\forall i\in I, \quad u^i(x^i) \geq u^i(e^i).$$

We shall denote by $\mathrm{A}(\boldsymbol{u})$ the set $\mathrm{F}\cap \mathrm{Ir}(\boldsymbol{u})$ and by $\mathrm{A}^i(\boldsymbol{u})$ the projection of $\mathrm{A}(\boldsymbol{u})$ onto X^i. An allocation in $\mathrm{A}(\boldsymbol{u})$ is said attainable and an action in $\mathrm{A}^i(\boldsymbol{u})$ is said individually attainable.[7] We denote by $\mathrm{A}^i_c(\boldsymbol{u})$ the set $X^i \setminus \mathrm{A}^i(\boldsymbol{u})$ of actions $x^i \in X^i$ that are not individually attainable. We let $\mathcal{U}(\boldsymbol{u})$ denote the utility set defined by

$$\mathcal{U}(\boldsymbol{u}) \equiv \{\boldsymbol{\lambda} \in \mathbb{R}^I : \exists \boldsymbol{x} \in \mathrm{A}(\boldsymbol{u}), \quad \forall i\in I, \quad u^i(e^i) \leq \lambda^i \leq u^i(x^i)\}.$$

From now on we assume that the economy $(X^i, u^i, e^i)_{i\in I}$ satisfies the following list of standard assumptions:

[3] They allow for satiation while Dana et al. (1999) do not.
[4] Observe that Brown and Werner (1995) and Dana et al. (1997) have the same assumption for the case of infinite dimensional economies.
[5] Except if there are at most two agents in the economy. In that case, both conditions (SCU) and (CU) are equivalent.
[6] A vector x in X^i is called an action since it can be interpreted as a consumption bundle or a portfolio.
[7] Observe that if x^i is individually attainable, then it is individually rational i.e., $u^i(x^i) \geq u^i(e^i)$ and it is individually feasible, i.e., there exists a feasible allocation $\boldsymbol{y} \in \mathrm{F}$ such that $x^i = y^i$. However, the converse is not true in general.

(A.1) the set X^i is closed convex containing e^i;
(A.2) the function u^i is upper semi-continuous and strictly quasi-concave.[8]

3. Existence under the non-satiation condition (NS)

If an action $x^i \in X^i$ is such that the set $P^i(x^i) \equiv \{y^i \in X^i : u^i(y^i) > u^i(x^i)\}$ is empty, then x^i is called a satiation point of u^i. We shall denote by $S^i(u^i)$ the set of satiation points of u^i on X^i, i.e.,

$$S^i(u^i) = \mathrm{argmax}\{u^i(x^i) : x^i \in X^i\}.$$

When there is no satiation point in X^i, the set $S^i(u^i)$ is empty. We recall now the definition of a quasi-equilibrium.

Definition 3.1. Given an economy $\mathcal{E}(\boldsymbol{u})$, a couple $(p, \boldsymbol{x})$ where $0 \neq p \in \mathbb{R}^J$ and $\boldsymbol{x} = (x^i)_{i \in I}$ is a feasible allocation in F, is a quasi-equilibrium of $\mathcal{E}(\boldsymbol{u})$ if for each i

(a) the action x^i satisfies the budget restriction $p \cdot x^i \leq p \cdot e^i$;
(b) the action x^i is weakly optimal in the budget set in the sense that for each $y^i \in P^i(x^i)$, we have $p \cdot y^i \geq p \cdot e^i$.

If the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is compact then a sufficient condition for existence of a quasi-equilibrium is the following non-satiation assumption:

(NS) the set $S^i(u^i) \cap \mathrm{A}^i(\boldsymbol{u})$ is empty for each i.

Theorem 3.1 (Dana et al., 1999). *Assume that*

(a.1) *the individually rational utility set* $\mathcal{U}(\boldsymbol{u})$ *is compact*,
(b.2) *the non-satiation condition (NS) is satisfied*,

then there exists a quasi-equilibrium.

The proof of Theorem 3.1 follows from standard arguments: see e.g. Dana et al. (1999). Observe that if the set $\mathrm{A}(\boldsymbol{u}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ of attainable allocations is compact then the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is trivially compact. However, the converse is not in general true.

4. The weak non-satiation condition (WNS)

Recently, Allouch and Le Van (2008) introduced a weaker non-satiation assumption:

(WNS) for every individually attainable action $x^i \in \mathrm{A}^i(\boldsymbol{u})$, there exists an action $y^i \in \mathrm{A}^i_c(\boldsymbol{u})$ which is not individually attainable but satisfies $u^i(y^i) \geq u^i(x^i)$.

Assumption (WNS) is obviously weaker than the non-satiation condition (NS). The weak non-satiation condition (WNS) is satisfied if and only if,

$$\forall i \in I, \quad S^i(u^i) \cap \mathrm{A}^i(\boldsymbol{u}) \neq \emptyset \Rightarrow S^i(u^i) \cap \mathrm{A}^i_c(\boldsymbol{u}) \neq \emptyset.$$

In other words, under Assumption (WNS), each agent i may have satiation points that are individually attainable, but the set of satiation points must be large enough such that there exists at least one satiation action which is not attainable.

Allouch and Le Van (2008) proved[9] that if the set $\mathrm{A}(\boldsymbol{u}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ of attainble allocations is compact, then the weak non-satiation condition (WNS) is sufficient for existence.

Theorem 4.1 (Allouch and Le Van). *Assume that*

(a.4) *the set* $\mathrm{A}(\boldsymbol{u}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ *of attainable allocations is compact*,
(b.1) *the weak non-satiation condition (WNS) is satisfied*,

then there exists a quasi-equilibrium.

[8] The function u^i is upper semi-continuous if for each $c \in \mathbb{R}$, the upper level set $\{x \in X^i : u^i(x) \geq c\}$ is closed in X^i. The function u^i is strictly quasi-concave if for every x and y in X^i, if $u^i(y) > u^i(x)$, then $u^i(\lambda y + (1-\lambda)x) > u^i(x)$ for every $\lambda \in (0, 1]$.

[9] We propose in Appendix A an alternative proof based on a very general existence result by Florenzano (2003).

This result is not comparable with the one by Dana et al. (1999). Indeed, Allouch and Le Van (2008) consider a weaker non-satiation assumption but a stronger compactness assumption. A natural question is whether an existence result generalizing both results of Dana et al. (1999) and Allouch and Le Van (2008) is possible. One may conjecture that the weakest assumptions of both results are sufficient for existence.

Conjecture 4.1. *Assume that*

(a.1) *the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is compact,*
(b.1) *the weak non-satiation condition (WNS) is satisfied,*

then there exists a quasi-equilibrium.

We first show that Conjecture 4.1 is not correct. We provide a counterexample in the following section. In Section 6 we prove that Conjecture 4.1 is correct if there are at most two agents in the economy.

5. Conjecture 4.1 is false

In this section we consider an economy with three agents and two commodities such that the individually rational utility set is compact and the weak non-satiation condition (WNS) is satisfied, but for which there is no quasi-equilibrium.

We pose $I = \{i_1, i_2, i_3\}$ and $J = \{j_1, j_2\}$. The action set of agent i_1 is given by

$$X^{i_1} \equiv [-1, \infty) \times [-1/2, \infty);$$

his utility function is given by

$$u^{i_1}(x) \equiv \begin{cases} \dfrac{x_{j_2} + x_{j_1}}{x_{j_2} + 1} & \text{if} \quad -1 \le x_{j_1} \le 1; \\ x_{j_1} & \text{if} \quad 1 \le x_{j_1}, \end{cases}$$

and he has no initial endowment, i.e., $e^{i_1} = 0$. The action set of agent i_2 is given by

$$X^{i_2} \equiv [-1, 2] \times \mathbb{R};$$

his utility function is given by

$$u^{i_2}(x) \equiv x_{j_1},$$

and he has no initial endowment, i.e., $e^{i_2} = 0$. The action set of agent i_3 is given by

$$X^{i_3} \equiv [-3, \infty) \times \{0\};$$

his utility function is given by

$$u^{i_3}(x) \equiv 0,$$

and he has no initial endowment, i.e., $e^{i_3} = 0$.

Proposition 5.1. *The economy satisfies Assumptions (A.1) and (A.2). Moreover, the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is compact and the weak non-satiation condition (WNS) is satisfied.*

Proof of Proposition 5.1. It is immediate that Assumptions (A.1) and (A.2) are satisfied. We propose to prove that the utility set $\mathcal{U}(\boldsymbol{u})$ is bounded and closed. Let $(\lambda^{i_1}, \lambda^{i_2}, \lambda^{i_3})$ in $\mathcal{U}(\boldsymbol{u})$, then there exists $\boldsymbol{x} \in \mathrm{F}$ such that

$$\forall i \in I, \quad 0 \le \lambda^i \le u^i(x^i).$$

This implies that $\lambda^{i_2} \in [0, 2]$ and $\lambda^{i_3} = 0$. Moreover, since $x^{i_1} = -x^{i_2} - x^{i_3}$, it follows that $x^{i_1}_{j_1} \le 4$ and $\lambda^{i_1} \le 4$. As a consequence

$$\mathcal{U}(\boldsymbol{u}) \subset [0, 4] \times [0, 2] \times \{0\}$$

is a bounded set. In order to prove that $\mathcal{U}(\boldsymbol{u})$ is closed, consider a sequence $\boldsymbol{\lambda}_n$ in $\mathcal{U}(\boldsymbol{u})$ converging to some $\boldsymbol{\lambda}$.

If $\lambda^{i_1} \le 1$ then

$$\forall i \in I, \quad 0 \le \lambda^i \le u^i(y^i)$$

where[10]

$$y^{i_1} \equiv \mathbf{1}_{j_1}, \quad y^{i_2} \equiv 2\mathbf{1}_{j_1} \quad \text{and} \quad y^{i_3} \equiv -3\mathbf{1}_{j_1}.$$

[10] We denote by $\mathbf{1}_{j_1}$ the vector $z \in \mathbb{R}^J$ such that $z_{j_1} = 1$ and $z_{j_2} = 0$.

Since $\boldsymbol{y}$ belongs to $\mathrm{A}(\boldsymbol{u})$ it follows that $\boldsymbol{\lambda}$ belongs to $\mathcal{U}(\boldsymbol{u})$.

Assume now that $\lambda^{i_1} > 1$. Since $\boldsymbol{\lambda}_n$ belongs to $\mathcal{U}(\boldsymbol{u})$, there exists a sequence $\boldsymbol{x}_n$ in $\mathrm{A}(\boldsymbol{u})$ such that

$$\forall i \in I, \quad 0 \le \lambda_n^i \le u^i(x_n^i).$$

For each n, we have

$$x_n^{i_1} + x_n^{i_2} + x_n^{i_3} = 0 \quad \text{and} \quad (x_{j_1,n}^{i_1}, x_{j_1,n}^{i_2}, x_{j_1,n}^{i_3}) \ge (-1, -1, -3).$$

Passing to a subsequence if necessary, we can suppose that the sequence

$$(x_{j_1,n}^{i_1}, x_{j_1,n}^{i_2}, x_{j_1,n}^{i_3})_n$$

converges to a vector $(x_{j_1}^{i_1}, x_{j_1}^{i_2}, x_{j_1}^{i_3})$ in $\mathbb{R}^I$ satisfying

$$x_{j_1}^{i_1} + x_{j_1}^{i_2} + x_{j_1}^{i_3} = 0.$$

Since $\lambda^{i_1} > 1$, for n large enough we have $x_{j_1,n}^{i_1} > 1$, implying that $u^{i_1}(x_n^{i_1}) = x_{j_1,n}^{i_1}$. It follows that for n large enough

$$\lambda_n^{i_1} \le x_{j_1,n}^{i_1}, \quad \lambda_n^{i_2} \le x_{j_1,n}^{i_2} \quad \text{and} \quad \lambda_n^{i_3} = 0.$$

Passing to the limit, we get

$$\lambda^{i_1} \le x_{j_1}^{i_1}, \quad \lambda^{i_2} \le x_{j_1}^{i_2} \quad \text{and} \quad \lambda^{i_3} = 0.$$

This implies that

$$\forall i \in I, \quad 0 \le \lambda^i \le u^i(z^i)$$

where

$$z^{i_1} \equiv x_{j_1}^{i_1} \mathbf{1}_{j_1}, \quad z^{i_2} \equiv x_{j_1}^{i_2} \mathbf{1}_{j_1} \quad \text{and} \quad z^{i_3} \equiv x_{j_1}^{i_3} \mathbf{1}_{j_1}.$$

Since $\boldsymbol{z}$ belongs to $\mathrm{A}(\boldsymbol{u})$ it follows that $\boldsymbol{\lambda}$ belongs to $\mathcal{U}(\boldsymbol{u})$.

Agent i_1 satisfies the non-satiation condition (NS). Agents i_2 and i_3 satisfy Assumption (WNS). Indeed, if $\boldsymbol{x}$ belongs to F then

$$(x_{j_1}^{i_1}, x_{j_1}^{i_2}, x_{j_1}^{i_3}) \in [-1, 4] \times [-1, 2] \times [-3, 2].$$

implying that the action $3\mathbf{1}_{j_1}$ belongs to $S^{i_3}(u^{i_3}) \cap \mathrm{A}_c^{i_3}(\boldsymbol{u})$. We have thus proved that agent i_3 satisfies Assumption (WNS). Since $\boldsymbol{x} \in \boldsymbol{X}$, we have

$$x_{j_1}^{i_1} \ge -1 \quad \text{and} \quad x_{j_1}^{i_3} = 0$$

implying by feasibility that $x_{j_1}^{i_2} \le 1$. It follows that the vector $2\mathbf{1}_{j_1} + 2\mathbf{1}_{j_2}$ belongs to $S^{i_2}(u^{i_2}) \cap \mathrm{A}_c^{i_2}(\boldsymbol{u})$ and agent i_2 satisfies Assumption (WNS). □

Remark 5.1. Consider the sequence $(\boldsymbol{x}_n)_n$ of feasible allocations defined by

$$x_n^{i_1} \equiv -\mathbf{1}_{j_1} + n\mathbf{1}_{j_2}, \quad x_n^{i_2} \equiv \mathbf{1}_{j_1} - n\mathbf{1}_{j_2} \quad \text{and} \quad x_n^{i_3} \equiv 0.$$

For each n, we let $\lambda_n^i \equiv u^i(x_n^i)$. The sequence $(\boldsymbol{\lambda}_n)_n$ belongs to the utility set and

$$\lim_{n\to\infty} (\lambda_n^{i_1}, \lambda_n^{i_2}, \lambda_n^{i_3}) = (1, 1, 0).$$

The limit $\boldsymbol{\lambda} \equiv (1, 1, 0)$ belongs to the utility set since $u^i(x^i) = \lambda^i$ with

$$x^{i_1} \equiv \mathbf{1}_{j_1}, \quad x^{i_2} \equiv \mathbf{1}_{j_1} \quad \text{and} \quad x^{i_3} \equiv -2\mathbf{1}_{j_1}.$$

The role of third agent to ensure the compactness of the utility set is crucial in this example. Indeed, if we consider the same economy but with only agents i_1 and i_2, then for each n the pair $(\lambda_n^{i_1}, \lambda_n^{i_2})$ belongs to the utility set but the limit $(1, 1)$ does not.

Proposition 5.2. *There is no quasi-equilibrium.*

Proof of Proposition 5.2. Assume that $(p, \boldsymbol{x})$ is a quasi-equilibrium. For each i, we have $p \cdot x^i \le p \cdot e^i = 0$. Since the allocation $\boldsymbol{x}$ is feasible, it follows that $p \cdot x^i = 0$, in other words the vector x^i belongs to the budget line $L(p)$ defined by

$$L(p) \equiv \{x \in \mathbb{R}^J : p \cdot x = p_{j_1} x_{j_1} + p_{j_2} x_{j_2} = 0\}.$$

We consider the three following cases:

(i) Assume $x^{i_2}_{j_1} < 2$. For every $y \in X^{i_2}$ with $y_{j_1} > 0$, we have $x^{i_2} + \alpha y \in P^{i_2}(x^{i_2})$ for $\alpha > 0$ small enough. This implies that $p \cdot y \geq 0$. Then necessarily $p_{j_2} = 0$ and $p_{j_1} > 0$. This implies that $L(p) = \{x \in \mathbb{R}^J : x_{j_1} = 0\}$. Therefore $x^{i_1}_{j_1} = 0$. But, for $\alpha > 0$ small enough, we have $x^{i_1} + \mathbf{1}_{j_2} - \alpha \mathbf{1}_{j_1} \in P^{i_1}(x^{i_1})$ implying by weak optimality that $p_{j_1}(-\alpha) + p_{j_2} \geq 0$. This contradicts the fact that $p_{j_2} = 0$ and $p_{j_1} > 0$.
(ii) Assume $x^{i_2}_{j_1} = 2$ and $x^{i_2}_{j_2} \neq 0$. Since x^{i_2} belongs to the budget line $L(p)$ it follows that $p_{j_1} \neq 0$. Then the only possibility for x^{i_3} to belong to $L(p) \cap X^{i_3}$ is that $x^{i_3} = 0$. It then follows that $x^{i_1}_{j_1} = -2$ which yields a contradiction with the fact that x^{i_1} belongs to X^{i_1}.
(iii) Assume $x^{i_2}_{j_1} = 2$ and $x^{i_2}_{j_2} = 0$. Since x^{i_2} belongs to the budget line $L(p)$ it follows that $p_{j_1} = 0$. As a consequence we must have $x^{i_1}_{j_2} = 0$. But, for $\alpha > 0$ small enough, we have $x^{i_1} + \mathbf{1}_{j_1} - \alpha \mathbf{1}_{j_2} \in P^{i_1}(x^{i_1})$ implying by weak optimality that $p_{j_1} + p_{j_2}(-\alpha) \geq 0$. Since $p_{j_1} = 0$ this implies $p_{j_2} < 0$. Similarly, for $\alpha > 0$ small enough, we have $x^{i_1} - \mathbf{1}_{j_1} + \alpha \mathbf{1}_{j_2} \in P^{i_1}(x^{i_1})$ implying by weak optimality that $-p_{j_1} + p_{j_2}(\alpha) \geq 0$. This contradicts the fact that $p_{j_1} = 0$ and $p_{j_2} < 0$. □

6. General existence result under Assumption (WNS)

Keeping the generality of the non-satiation assumption introduced by Allouch and Le Van (2008), we propose to investigate under which additional assumptions Conjecture 4.1 is correct. We introduce an assumption called strong compactness of the utility set (SCU).

Definition 6.1. The individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is *strongly* compact if

(SCU) for every sequence $(\boldsymbol{x}_n)_n$ in $\mathrm{A}(\boldsymbol{u})$ of attainable rational allocations there exist a feasible allocation $\boldsymbol{y}$ and a subsequence $(\boldsymbol{x}_{n_k})_k$ satisfying

$$\forall i \in I, \quad u^i(y^i) \geq \lim_{k\to\infty} u^i(x^i_{n_k}) \tag{6.1}$$

together with[11]

$$\forall i \in I, \quad \lim_{k\to\infty} \frac{\mathbf{1}_{S^i(u^i)}(x^i_{n_k})}{1 + ||x^i_{n_k}||^2}(y^i - x^i_{n_k}) = 0. \tag{6.2}$$

Remark 6.1. The previous expression of (SCU) was chosen for its concision. Passing to subsequences if necessary, it is possible to prove that the strong compactness of individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is equivalent to the following statement: for every sequence $(\boldsymbol{x}_n)_n$ in $\mathrm{A}(\boldsymbol{u})$ of attainable allocations there exist a feasible allocation $\boldsymbol{y}$ and a subsequence $(\boldsymbol{x}_{n_k})_k$ satisfying

$$\forall i \in I, \quad u^i(y^i) \geq \lim_{k\to\infty} u^i(x^i_{n_k}) \tag{6.3}$$

and such that for each i one of the following conditions is satisfied:

(a) the subsequence $(x^i_{n_k})_k$ is unbounded,[12]
(b) the subsequence $(x^i_{n_k})_k$ converges to y^i,
(c) for k large enough, the action $x^i_{n_k}$ is not a satiation point.

In other words, condition (6.2) can be replaced by

$$I = I_{\mathrm{S}} \cup I_{\mathrm{C}} \cup I_{\mathrm{U}}$$

where

$$I_{\mathrm{S}} \equiv \{i \in I : x^i_{n_k} \notin S^i(u^i), \forall k \in \mathbb{N}\}, \quad I_{\mathrm{C}} \equiv \{i \in I \setminus I_{\mathrm{S}} : \lim_{k\to\infty} x^i_{n_k} = y^i\}$$

and

$$I_{\mathrm{U}} \equiv \{i \in I \setminus I_{\mathrm{S}} : \lim_{k\to\infty} ||x^i_{n_k}|| = \infty\}.$$

[11] If A is a subset of $\mathbb{R}^J$ then $\mathbf{1}_A$ is the function from $\mathbb{R}^J$ to $\{0, 1\}$ defined by $\mathbf{1}_A(x) \equiv 1$ if $x \in A$ and $\mathbf{1}_A(x) \equiv 0$ elsewhere. The space $\mathbb{R}^J$ is endowed with the norm defined by $||x|| = \sum_{j\in J}|x(j)|$ for every vector $x = (x(j))_{j\in J}$.

[12] Since we can always pass to a subsequence, on can replace this condition by the following one: the subsequence $(x^i_{n_k})_k$ converges to ∞.

Remark 6.2. It is straightforward to check that the strong compactness of the individually rational utility implies its compactness. Indeed, assume that $\mathcal{U}(\boldsymbol{u})$ is strongly compact and let $(\boldsymbol{\lambda}_n)_n$ be sequence in $\mathcal{U}(\boldsymbol{u})$. There exists a sequence $(\boldsymbol{x}_n)_n$ in $\mathrm{A}(\boldsymbol{u})$ of attainable allocations satisfying $u^i(x_n^i) \geq \lambda_n^i \geq u^i(e^i)$. From (SCU), there exists a feasible allocation $\boldsymbol{y}$ and a subsequence $(\boldsymbol{x}_{n_k})_k$ satisfying

$$\forall i \in I, \quad u^i(y^i) \geq \lim_{k\to\infty} u^i(x_{n_k}^i). \tag{6.4}$$

This implies that for each i the subsequence $(\lambda_{n_k}^i)_k$ is bounded. Passing to another subsequence if necessary, we can assume without any loss of generality that the sequence $(\lambda_{n_k}^i)_k$ converges to some λ^i. From (6.4) it follows that $\boldsymbol{\lambda}$ belongs to $\mathcal{U}(\boldsymbol{u})$.

Remark 6.3. Observe that if $(\boldsymbol{x}_n)_n$ is a sequence in $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ then for each i we have

$$\mathbf{1}_{S^i(u^i)}(x_n^i) = \mathbf{1}_{S^i(u^i)\cap A^i(\boldsymbol{u})}(x_n^i).$$

This implies that (6.2) is trivially fulfilled under the usual non-satiation condition (NS). In particular, under Assumption (NS), the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is strongly compact if and only if it is compact.

The main existence result of this paper is the following generalization of Theorem 3 in Dana et al. (1999) (see Theorem 3.1) and Theorem 2 in Allouch and Le Van (2008).

Theorem 6.1. *Assume that*

(a.2) *the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is strongly compact,*
(b.1) *the weak non-satiation condition (WNS) is satisfied,*

then there exists a quasi-equilibrium.

Proof of Theorem 6.1. Let $\mathcal{E}(\boldsymbol{u})$ be an economy satisfying the weak non-satiation condition (WNS) and such that the individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is strongly compact. It follows immediately that for each agent i, the set

$$\operatorname{argmax}\{u^i(x) : x \in \mathrm{A}^i(\boldsymbol{u})\}$$

is non-empty. Let ξ^i be an element of $\operatorname{argmax}\{u^i(x) : x \in \mathrm{A}^i(\boldsymbol{u})\}$. Applying Assumption (WNS), there exists $\zeta^i \in \mathrm{A}_c^i(\boldsymbol{u})$ such that $u^i(\zeta^i) \geq u^i(\xi^i)$.

For each i, we let v^i be the function defined on X^i by[13]

$$\forall x \in X^i, \quad v^i(x) \equiv u^i(x) + \mathbf{1}_{S^i(u^i)}(x)\exp\{-d(x, \zeta^i)\} \tag{6.5}$$

where $\mathbf{1}_{S^i(u^i)}$ is the indicator function of the set $S^i(u^i)$. Observe that if the economy $\mathcal{E}(u)$ satisfies the non-satiation assumption (NS) then, for each agent i, we have $v^i = u^i$. We claim that the economy $\mathcal{E}(\boldsymbol{v})$ satisfies all the assumptions required to apply Theorem 3.1. □

Claim 6.1. *The economy $\mathcal{E}(\boldsymbol{v})$ satisfies Assumptions (A.1) and (A.2).*

Proof of Claim 6.1. We only have to prove that Assumption (A.2) is satisfied. We denote by M^i the extended real number defined by

$$M^i \equiv \sup\{u^i(x) : x \in X^i\}.$$

Let $c \in \mathbb{R} \cup \{\infty\}$ then

$$\{x \in X^i : v^i(x) \geq c\} = \begin{cases} S^i(u^i) \cap B(\zeta^i, -\ln(c - M^i)) & \text{if} \quad c \geq M^i \\ \{x \in X^i : u^i(x) \geq c\} & \text{if} \quad c < M^i \end{cases}$$

where $B(\zeta^i, r) \equiv \{x \in \mathbb{R}^J : d(\zeta^i, x) \leq r\}$ if $r \geq 0$ and $B(\zeta^i, r) \equiv \emptyset$ if $r < 0$. It follows that the set $\{x \in X^i : v^i(x) \geq c\}$ is closed convex, implying that the economy $\mathcal{E}(\boldsymbol{v})$ satisfies Assumption (A.2). □

Claim 6.2. *The utility set $\mathcal{U}(\boldsymbol{v})$ is compact.*

Proof of Claim 6.2. Let $(\boldsymbol{\lambda}_n)_n$ be a sequence in $\mathcal{U}(\boldsymbol{v})$, there exists a sequence $(\boldsymbol{x}_n)_n$ of allocations in $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{v})$ such that for each n

$$\forall i \in I, \quad v^i(e^i) \leq \lambda_n^i \leq v^i(x_n^i). \tag{6.6}$$

[13] We endow the space $\mathbb{R}^J$ with the distance d associated with the norm $\|\cdot\|$ and defined by $d(x, y) = \|x - y\|$ for every x, y in $\mathbb{R}^J$.

Since $v^i(e^i) \geq u^i(e^i)$ the allocation $\boldsymbol{x}_n$ also belongs to $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$. The individually rational utility set $\mathcal{U}(\boldsymbol{u})$ is strongly compact, therefore, passing to a subsequence if necessary, we may assume that there exists an allocation $\boldsymbol{y}$ in $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ satisfying

$$\forall i \in I, \quad u^i(y^i) \geq \lim_n u^i(x_n^i) \tag{6.7}$$

and

$$I = I_\mathrm{S} \cup I_\mathrm{C} \cup I_\mathrm{U}$$

where

$$I_\mathrm{S} \equiv \{i \in I : x_{n_k}^i \notin S^i(u^i), \forall k \in \mathbb{N}\}, \quad I_\mathrm{C} \equiv \{i \in I \setminus I_\mathrm{S} : \lim_{k \to \infty} x_{n_k}^i = y^i\}$$

and

$$I_\mathrm{U} \equiv \{i \in I \setminus I_\mathrm{S} : \lim_{k \to \infty} ||x_{n_k}^i|| = \infty\}.$$

□

Claim 6.3. *For each i we have*

$$\lim_{n \to \infty} v^i(x_n^i) \leq v^i(y^i).$$

Proof of Claim 6.3. Let $i \in I_\mathrm{S}$, by construction of v^i we have $v^i(x_n^i) = u^i(x_n^i)$ for all n. It follows that

$$\lim_{n \to \infty} v^i(x_n^i) = \lim_{n \to \infty} u^i(x_n^i) \leq u^i(y^i).$$

The desired result follows from the fact that $u^i(y^i) \leq v^i(y^i)$.

For each $i \in I_\mathrm{C}$ the result follows from convergence of the sequence $(x_n^i)_n$ to y^i and upper semicontinuity of v^i.

Now let $i \in I_\mathrm{U}$, since

$$\lim_{n \to \infty} d(x_n^i, \zeta^i) = \infty$$

we have

$$\lim_{n \to \infty} v^i(x_n^i) = \lim_{n \to \infty} u^i(x_n^i).$$

The desired result follows from

$$\lim_{n \to \infty} u^i(x_n^i) \leq u^i(y^i) \leq v^i(y^i).$$

Combining (6.6) and Claim 6.3 we prove that the set $\mathcal{U}(\boldsymbol{v})$ is compact. □

In order to apply Theorem 3.1, it is sufficient to prove that the non-satiation condition (NS) is satisfied. This follows from the construction of the function v^i. Indeed, let x^i be a vector in $A^i(\boldsymbol{v})$. If there exists $y^i \in X^i$ satisfying $u^i(y^i) > u^i(x^i)$ then x^i does not belong to $S^i(u^i)$ and

$$v^i(y^i) \geq u^i(y^i) > u^i(x^i) = v^i(x^i)$$

implying that x^i does not belong to $S^i(v^i)$. Now assume that x^i belongs to $S^i(u^i)$. There exists $(x^k)_{k \neq i}$ such that $(x^k)_{k \in I}$ belongs to $A(\boldsymbol{v}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{v})$. Since for each $v^i(e^i) \geq u^i(e^i)$, we have that $(x^k)_{k \in I}$ also belongs to $A(\boldsymbol{u}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ and therefore x^i belongs to $A^i(\boldsymbol{u})$. By construction the vector ζ^i does not belong to $A^i(\boldsymbol{u})$, implying that

$$v^i(\zeta^i) = u^i(\zeta^i) + \mathbf{1}_{S^i(u^i)}(\zeta^i)\exp\{0\} \geq u^i(x^i) + 1 > u^i(x^i) + \exp\{-d(x^i, \zeta^i)\} = v^i(x^i). \tag{6.8}$$

Therefore the action x^i does not belong to $S^i(v^i)$. We have thus proved that

$$A^i(\boldsymbol{v}) \cap S^i(v^i) = \emptyset$$

i.e., the economy $\mathcal{E}(\boldsymbol{v})$ satisfies Assumption (NS).

Applying Theorem 3.1 to the economy $\mathcal{E}(\boldsymbol{v})$ there exists a quasi-equilibrium $(p, \boldsymbol{x})$ of $\mathcal{E}(\boldsymbol{v})$. Therefore the allocation $\boldsymbol{x}$ is feasible and for each i

(a) the action x^i satisfies the budget restriction $p \cdot x^i \leq p \cdot e^i$;
(b) the action x^i is weakly optimal for the utility function v^i in the budget set, i.e., for each $y^i \in X^i$,

$$v^i(y^i) > v^i(x^i) \Rightarrow p \cdot y^i \geq p \cdot e^i. \tag{6.9}$$

We claim that $(p, \boldsymbol{x})$ is a quasi-equilibrium of the economy $\mathcal{E}(\boldsymbol{u})$. To see this it is sufficient to prove that x^i is weakly optimal for the utility function u^i in the budget set. Let $y^i \in X^i$, if $u^i(y^i) > u^i(x^i)$ then x^i does not belong to $S^i(u^i)$ and $u^i(x^i) = v^i(x^i)$. Since $v^i(y^i) \geq u^i(y^i)$, we obtain that $v^i(y^i) > v^i(x^i)$, implying from (6.9) that $p \cdot y^i \geq p \cdot e^i$.

7. When is the utility set strongly compact?

Following Remark 6.3, Assumption (SCU) appears as a generalization of Assumption (CU) suitably adapted to a framework with satiation points. We provide in this section three natural situations where the strong compactness of the individually rational utility set is either automatically satisfied or follows from its compactness.

7.1. Economies with two agents

If there are two agents then Conjecture 4.1 is valid.

Proposition 7.1. *If there are at most two agents, then the strong compactness of the individually rational utility set follows from its compactness.*

Proof of Proposition 7.1. Let $\mathcal{E}(\boldsymbol{u})$ be an economy with two agents,[14] i.e., $I = \{i_1, i_2\}$. Assume that the individually rational utility set is compact. We have to prove that Assumption (SCU) is satisfied. Let $(\boldsymbol{x}_n)_n$ be a sequence in $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$. Since the individually rational utility set is compact, there exist an attainable allocation $\boldsymbol{y} \in \mathrm{A}(\boldsymbol{u})$ and a subsequence $(\boldsymbol{x}_{n_k})_k$ satisfying

$$\forall i \in I, \quad u^i(y^i) \geq \lim_{k\to\infty} u^i(x^i_{n_k}).$$

If the sequence $(x^{i_1}_{n_k})_k$ is bounded then, by feasibility, the sequence $(x^{i_2}_{n_k})_k$ is also bounded. Passing to a subsequence if necessary, we can assume that there exists $\boldsymbol{x}$ such that

$$\forall i \in I, \quad \lim_{k\to\infty} (x^i - x^i_{n_k}) = 0.$$

Since utility functions are upper semi-continuous, passing to a subsequence if necessary, we have

$$\forall i \in I, \quad \lim_{k\to\infty} u^i(x^i_{n_k}) \leq u^i(x^i).$$

Assumption (SCU) is thus satisfied.

Assume now that the sequence $(x^{i_1}_{n_k})_k$ is not bounded. By feasibility, it follows that the sequence $(x^{i_2}_{n_k})_k$ is also unbounded. Now, Assumption (SCU) follows from Remark 6.1. □

As a direct consequence of Theorem 6.1 we obtain the following existence result.

Corollary 7.1. *Consider an economy with at most two agents. If*

(a.1) *the individually rational utility set* $\mathcal{U}(\boldsymbol{u})$ *is compact,*
(b.1) *the weak non-satiation condition (WNS) is satisfied,*

then there exists a quasi-equilibrium.

7.2. Compactness of attainable allocations

We propose to prove that our main existence result Theorem 6.1 generalizes Theorem 2 in Allouch and Le Van (2008).

Proposition 7.2. *If the set* $\mathrm{A}(\boldsymbol{u})$ *of attainable allocations is compact then the individually rational utility set* $\mathcal{U}(\boldsymbol{u})$ *is strongly compact.*

Proof of Proposition 7.2. Let $\mathcal{E}(\boldsymbol{u})$ be an economy such that the set $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ is compact. We have to prove that Assumption (SCU) is satisfied. Let $(\boldsymbol{x}_n)_n$ be a sequence in $\mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$. By compactness, there exist a subsequence $(\boldsymbol{x}_{n_k})_k$ and an allocation $\boldsymbol{x}$ satisfying

$$\forall i \in I, \quad \lim_{k\to\infty} (x^i - x^i_{n_k}) = 0.$$

Since utility functions are upper semi-continuous, passing to a subsequence if necessary we have

$$\forall i \in I, \quad \lim_{k\to\infty} u^i(x^i_{n_k}) \leq u^i(x^i).$$

[14] We do not consider the trivial case of an economy with only one agent.

Assumption (SCU) is thus satisfied. □

As a direct consequence of Theorem 6.1 we obtain the main existence result in Allouch and Le Van (2008).

Corollary 7.2. *Assume that*

(a.4) *the set* $\mathrm{A}(\boldsymbol{u}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ *of attainable allocations is compact,*
(b.1) *the weak non-satiation condition (WNS) is satisfied,*

then there exists a quasi-equilibrium.

7.3. Economies satisfying a no-arbitrage assumption

In order to introduce a no-arbitrage assumption, we recall some definitions. For each action $x^i \in X^i$, we denote by $\mathrm{As}^i(x^i)$ the asymptotic (or recession) cone of the set $\hat{P}^i(x^i) = \{y^i \in X^i : u^i(y^i) \geq u^i(x^i)\}$, i.e.,

$$\mathrm{As}^i(x^i) \equiv \{v \in \mathbb{R}^J : \forall t \geq 0, \forall y^i \in \hat{P}^i(x^i), y^i + tv \in \hat{P}^i(x^i)\}.$$

We denote by $\mathrm{L}^i(x^i)$ the lineality space defined by

$$\mathrm{L}^i(x^i) \equiv \mathrm{As}^i(x^i) \cap -\mathrm{As}^i(x^i).$$

Let $\mathrm{Span}(X^i)$ be the smallest linear subspace of $\mathbb{R}^J$ containing X^i. The lineality space $\mathrm{L}^i(x^i)$ is the largest linear subspace of $\mathrm{Span}(X^i)$ contained in $\mathrm{As}^i(x^i)$. We borrow from Allouch et al. (2002) the following assumptions:

(WU) for each agent i and each individually rational action $x^i \in \hat{P}^i(e^i)$ we have $\mathrm{L}^i(x^i) = \mathrm{L}^i(e^i)$;
(WNMA) for every family $(y^i)_{i \in I}$ with $y^i \in \mathrm{As}^i(e^i)$,

$$\sum_{i \in I} y^i = 0 \Rightarrow y^i \in \mathrm{L}^i(e^i), \quad \forall i \in I.$$

In Allouch et al. (2002) Assumption (WU) is called weak uniformity and Assumption (WNMA) is called weak no market arbitrage. It was proved by Allouch et al. (2002) (see Theorem 1) that if Assumptions (WU) and (WNMA) are satisfied then the individually rational utility set is compact. We claim that the individually rational utility set is actually strongly compact.

Proposition 7.3. *If an economy satisfies the weak uniformity condition (WU) and the weak no market arbitrage condition (WNMA), then the individually rational utility set is strongly compact.*

Before providing the arguments of the proof, we introduce some notations. Let $[\mathrm{L}^i(e^i)]^\perp$ be the subspace of $\mathrm{Span}(X^i)$ orthogonal[15] to $\mathrm{L}(e^i)$. Every vector $x^i \in X^i$ can be uniquely decomposed as a sum $\Psi^i(x^i) + \Theta^i(x^i)$, where $\Psi^i(x^i) \in [\mathrm{L}^i(e^i)]^\perp$ and $\Theta^i(x^i) \in \mathrm{L}^i(e^i)$. Observe that if x^i belongs to $\hat{P}^i(e^i)$ then the vector $\Psi^i(x^i)$ also belongs to $\hat{P}^i(e^i)$ since $\Psi^i(x^i) \in \hat{P}^i(e^i) - \mathrm{L}(e^i)$. Under Assumption (WU) we also have that $\Psi^i(x^i)$ belongs to $\hat{P}^i(x^i)$. We denote by $\mathrm{A}(\boldsymbol{u})^\perp$ the orthogonal projection of $\mathrm{A}(\boldsymbol{u})$ on $\prod_{i \in I}[\mathrm{L}^i(e^i)]^\perp$, i.e.,

$$\mathrm{A}(\boldsymbol{u})^\perp \equiv \{(\Psi^i(x^i))_{i \in I} : (x^i)_{i \in I} \in \mathrm{A}(\boldsymbol{u})\}.$$

It was proved by Allouch et al. (2002) (see Theorem 1) that the weak no market arbitrage condition (WNMA) is equivalent to the compactness of the set $\mathrm{A}(\boldsymbol{u})^\perp$. This will play a crucial role on the arguments of the following proof.[16]

Proof of Proposition 7.3. Consider an economy satisfying Assumptions (WU) and (WNMA). Let $(\boldsymbol{x}_n)_n$ be a sequence of attainable allocations. Passing to a subsequence if necessary we can assume that $I = I_\mathrm{c} \cup I_\mathrm{U}$ where

(a) for each $i \in I_\mathrm{c}$, there exists $x^i \in X^i$ such that the sequence $(x^i_n)_n$ converges to x^i,
(b) for each $i \in I_\mathrm{U}$, the sequence $(||x^i_n||)_n$ converges to ∞.

Observe that $(\Psi^i(x^i_n))_{i \in I}$ belongs to $\mathrm{A}(\boldsymbol{u})^\perp$. Assumption (WNMA) implies that $\mathrm{A}(\boldsymbol{u})^\perp$ is compact. Therefore, there exists $\boldsymbol{z} \in \mathrm{A}(\boldsymbol{u})$ such that, passing to a subsequence if necessary,

$$\forall i \in I, \quad \lim_{n \to \infty} \Psi^i(x^i_n) = \Psi^i(z^i).$$

[15] For the usual scalar product in $\mathbb{R}^J$.
[16] We would like to thank an anonymous referee for pointing this to us.

In particular one has

$$\forall i \in I_C, \quad \Psi^i(z^i) = \Psi^i(x^i).$$

The feasibility of $(z^i)_{i \in I}$ implies that

$$\sum_{i \in I_U} z^i + \sum_{i \in I_C} x^i = \sum_{i \in I} e^i + \sum_{i \in I_C} \eta^i$$

where

$$\forall i \in I_C, \quad \eta^i = \Theta^i(x^i) - \Theta^i(z^i).$$

For each $i \in I_C$ we let $y^i \equiv x^i - \eta^i$ and for each $i \in I_U$ we let $y^i = z^i$. Since $x^i \in \hat{P}^i(e^i)$ and $\eta^i \in \mathrm{L}^i(e^i)$, it follows that $y^i \in \hat{P}^i(e^i)$. Therefore the allocation $\boldsymbol{y} = (y_i)_{i \in I}$ is attainable, i.e., $\boldsymbol{y} \in \mathrm{A}(\boldsymbol{u})$.

Since u^i is upper semi-continuous, passing to a subsequence if necessary we have

$$\forall i \in I_C, \quad u^i(x^i) \geq \lim_{n \to \infty} u^i(x_n^i)$$

and

$$\forall i \in I_C, \quad u^i(\Psi^i(z^i)) \geq \lim_{n \to \infty} u^i(\Psi^i(x_n^i)).$$

Since the economy satisfies the weak uniformity assumption, we have

$$\forall i \in I_C, \quad u^i(y^i) = u^i(x^i) \geq \lim_{n \to \infty} u^i(x_n^i)$$

and

$$\forall i \in I_U, \quad u^i(y^i) = u^i(\Psi^i(z^i)) \geq \lim_{n \to \infty} u^i(\Psi^i(x_n^i)) \geq \lim_{n \to \infty} u^i(x_n^i).$$

We have thus proved that (6.1) holds. Assumption (SCU) follows then from Remark 6.1. □

As a direct consequence of Theorem 6.1 we obtain the following existence result.

Corollary 7.3. *Assume that*

(a.3) *the weak uniformity condition (WU) and the weak no market arbitrage condition (WNMA) are satisfied,*
(b.1) *the weak non-satiation condition (WNS) is satisfied,*

then there exists a quasi-equilibrium.

Recently, Allouch et al. (2006) proved that every economy satisfying (a.3) admits a quasi-equilibrium provided that the following non-satiation assumption is satisfied:

(ANS) For every $x^i \in A^i(\boldsymbol{u})$, if $x^i \in S^i(u^i)$ then $\mathrm{As}^i(x^i) \setminus \mathrm{L}^i(x^i) \neq \emptyset$.

Actually, for economies satisfying the weak uniformity condition (WU) and the weak no market arbitrage condition (WNMA), the non-satiation condition (ANS) implies the weak non-satiation condition (WNS).

Proposition 7.4. *Consider an economy satisfying the weak uniformity condition (WU) and the weak no market arbitrage condition (WNMA). If it satisfies Assumption (ANS) then it also satisfies Assumption (WNS).*

As a consequence of Proposition 7.4, the existence result in Allouch et al. (2006) is a particular case of Corollary 7.3.

Proof of Proposition 7.4. Consider an economy satisfying the weak uniformity condition (WU), the weak no market arbitrage condition (WNMA) and the non-satiation condition (ANS). Fix an agent $k \in I$. If $S^k(u^k) \cap A^k(\boldsymbol{u})$ is empty then the claim is true. If not, we let $x^k \in S^k(u^k) \cap A^k(\boldsymbol{u})$. Since Assumption (ANS) is satisfied we can choose a vector $\xi^k \in \mathrm{As}^k(x^k) \setminus \mathrm{L}^k(x^k)$. In particular, one has

$$\forall \lambda > 0, \quad x^k + \lambda \xi^k \in S^k(u^k).$$

We propose to prove that there exists $\lambda > 0$ such that $x^k + \lambda \xi^k \notin \mathrm{A}^k(\boldsymbol{u})$. Assume by way of contradiction that

$$\forall \lambda > 0, \quad x^k + \lambda \xi^k \in \mathrm{A}^k(\boldsymbol{u}).$$

This implies that for each $\lambda > 0$, there exists $(y_\lambda^i)_{i \in I}$ in $\mathrm{A}(\boldsymbol{u})$ such that $y_\lambda^k = x^k + \lambda \xi^k$. Observe that

$$\Psi^k(y_\lambda^k) = \Psi^k(x^k) + \lambda \Psi^k(\xi^k).$$

Since the weak uniformity condition (WU) is satisfied, the vector $\Psi^k(\xi^k)$ belongs to $\mathrm{As}^k(x^k)\setminus\{0\}$. This implies that the allocation $(\Psi^i(y^i))_{i\in I}$ is unbounded and belongs to $\mathrm{A}(\boldsymbol{u})^\perp$. This contradicts Assumption (WNMA). $\square$

Remark 7.1. The proof of Proposition 7.4 was suggested by an anonymous referee. See also Allouch and Le Van (2008).

7.4. An example

We propose to illustrate the generality of Assumption (SCU) by providing an example of an economy with a strongly compact individually rational utility set but for which neither the set of attainable allocations is compact, nor the weak no market arbitrage assumption is satisfied. In particular, existence of a quasi-equilibrium for this economy does not follow from any of the existence results proposed so far in the literature.

We pose $I=\{i_1,i_2,i_3\}$ and $J=\{j_1,j_2\}$.

- The action set of agent i_1 is $X^{i_1}\equiv\mathbb{R}^J$, his utility function is $u^{i_1}(x)\equiv\min\{x_{j_1},x_{j_2}\}$ and his initial endowment is $e^{i_1}\equiv 0$.
- The action set of agent i_2 is $X^{i_2}\equiv\mathbb{R}^J$, his utility function is $u^{i_2}(x)\equiv x_{j_2}$ and his initial endowment is $e^{i_2}\equiv 0$.
- The action set of agent i_3 is $X^{i_3}\equiv[-1,1]\times[-1,1]$, his utility function is $u^{i_3}(x)\equiv x_{j_1}$ and his initial endowment is $e^{i_3}\equiv 0$.

Observe that this economy satisfies Assumptions (A.1) and (A.2). Agents i_1 and i_2 satisfy the non-satiation condition (NS). We propose to prove that agent i_3 satisfies Assumption (WNS). The set of satiation actions is $S^{i_3}(u^{i_3})=\{x\in X^{i_3}: x_{j_1}=1\}$. The action $x^{i_3}=\mathbf{1}_{j_1}+\mathbf{1}_{j_2}$ belongs to $S^{i_3}(u^{i_3})$ but does not belong to $\mathrm{A}^{i_3}(\boldsymbol{u})$. Indeed, assume by way of contradiction that x^{i_3} belongs $\mathrm{A}^{i_3}(\boldsymbol{u})$. Then there exist x^{i_1} and x^{i_2} such that $(x^{i_1},x^{i_2},x^{i_3})$ is attainable, i.e., belongs to $\mathrm{A}(\boldsymbol{u})$. This implies that

$$x^{i_1}_{j_2}\geq 0 \quad\text{and}\quad x^{i_2}_{j_2}\geq 0$$

and contradicts the feasibility condition $x^{i_1}_{j_2}+x^{i_2}_{j_2}+x^{i_3}_{j_2}=0$.

The weak uniformity Assumption (WU) is satisfied since

$$\forall x\in X^{i_1},\quad \mathrm{As}^{i_1}(x)=\mathbb{R}^J_+ \quad\text{and}\quad L^{i_1}(x)=\{0\},$$

$$\forall x\in X^{i_2},\quad \mathrm{As}^{i_2}(x)=\{y\in\mathbb{R}^J: y_{j_2}\geq 0\} \quad\text{and}\quad L^{i_2}(x)=\{(x_{j_1},0): x_{j_1}\in\mathbb{R}\},$$

and

$$\forall x\in X^{i_3},\quad \mathrm{As}^{i_3}(x)=\{0\}=L^{i_3}(x).$$

Observe that if we let $y=\mathbf{1}_{j_1}$ then for every $\lambda>0$, the allocation

$$(e^{i_1}+\lambda y, e^{i_2}-\lambda y, e^{i_3})$$

is attainable, i.e., belongs to $\mathrm{A}(\boldsymbol{u})$. This implies that the set $\mathrm{A}(\boldsymbol{u})$ of attainable allocations is not compact and the weak no market arbitrage condition (WNMA) is not satisfied. In particular, none of the existence results in the literature can be applied. However, this economy satisfies the assumptions of our main theorem since the individually rational utility set is strongly compact. Indeed, let $(\boldsymbol{x}_n)_n$ be a sequence in $\mathrm{A}(\boldsymbol{u})$ of attainable allocations. Since X^{i_3} is compact, passing to a subsequence if necessary, we can assume that there exists $x^{i_3}\in X^{i_3}$ such that

$$\lim_{n\to\infty} x^{i_3}_n = x^{i_3}.$$

By feasibility, one has

$$(x^{i_1}_{j_2,n}, x^{i_2}_{j_2,n}, x^{i_3}_{j_2,n})\in\mathbb{R}_+\times\mathbb{R}_+\times[-1,1] \quad\text{and}\quad \sum_{i\in I} x^i_{j_2,n}=0.$$

This implies that there exists $(x^{i_1}_{j_2},x^{i_2}_{j_2})\in\mathbb{R}_+\times\mathbb{R}_+$ such that, passing to a subsequence if necessary,

$$\lim_{n\to\infty}(x^{i_1}_{j_2,n},x^{i_2}_{j_2,n})=(x^{i_1}_{j_2},x^{i_2}_{j_2}).$$

In particular, the market for commodity j_2 clears, i.e.,

$$\sum_{i\in I} x^i_{j_2}=0=\sum_{i\in I} e^i_{j_2}. \tag{7.1}$$

Moreover, one has

$$\lim_{n\to\infty} u^{i_2}(x^{i_2}_n)=x^{i_2}_{j_2} \quad\text{and}\quad \lim_{n\to\infty} u^{i_3}(x^{i_3}_n)=x^{i_3}_{j_1}. \tag{7.2}$$

Since $x_n^{i_1}$ is individually rational, we must have

$$\forall n \in \mathbb{N}, \quad x_{j_1,n}^{i_1} \geq 0.$$

We propose to split the study in two cases:

(i) Assume that the sequence $(x_{j_1,n}^{i_1})_n$ is not bounded. Passing to a subsequence if necessary, we can assume that $\lim_n x_{j_1,n}^{i_1} = \infty$. In that case, for n large enough, one has

$$u^{i_1}(x_n^{i_1}) = x_{j_2,n}^{i_1}$$

implying that

$$\lim_{n\to\infty} u^{i_1}(x_n^{i_1}) = x_{j_2}^{i_1}.$$

We let $\boldsymbol{y}$ be the allocation defined by

$$y^{i_1} \equiv (x_{j_2}^{i_1} + 1)\mathbf{1}_{j_1} + x_{j_2}^{i_1}\mathbf{1}_{j_2}, \quad y^{i_2} \equiv -(x_{j_2}^{i_1} + 1 - x_{j_1}^{i_3})\mathbf{1}_{j_1} + x_{j_2}^{i_2}\mathbf{1}_{j_2} \quad \text{and} \quad y^{i_3} \equiv x^{i_3}.$$

We can check that the allocation $\boldsymbol{y} = (y^i)_{i\in I}$ is feasible and satisfies

$$\forall i \in I, \quad u^i(y^i) \geq \lim_{n\to\infty} u^i(x_n^i).$$

Therefore (6.1) is satisfied. Moreover, we can decompose I as follows

$$I = I_s \cup I_c \quad \text{where} \quad I_s = \{i_1, i_2\} \quad \text{and} \quad I_c = \{i_3\}.$$

For every $i \in I_s$, the action x_n^i is not a satiation point. For $i = i_3 \in I_c$, we have $\lim_{n\to\infty} x_n^{i_3} = y^{i_3}$. We have thus proved that (6.2) is satisfied.

(ii) Assume that the sequence $(x_{j_1,n}^{i_1})_n$ is bounded. Passing to a subsequence if necessary, we can assume that there exists $x_{j_1}^{i_1}$ such that $\lim_{n\to\infty} x_{j_1,n}^{i_1} = x_{j_1}^{i_1}$. By feasibility and convergence of $(x_n^{i_3})_n$, we can conclude that there exists $x_{j_1}^{i_2}$ such that $\lim_{n\to\infty} x_{j_1,n}^{i_2} = x_{j_1}^{i_2}$. We have thus proved that

$$\forall i \in I, \quad \lim_{n\to\infty} x_n^i = x^i.$$

It is now straightforward to prove that (6.1) and (6.2) are both satisfied.

Appendix A. An alternative proof of Theorem 2 in Allouch and Le Van (2008)

Allouch and Le Van (2008) proved that if

(a.4) the set $\mathrm{A}(\boldsymbol{u}) = \mathrm{F} \cap \mathrm{Ir}(\boldsymbol{u})$ of attainable allocations is compact,
(b.1) the weak non-satiation condition (WNS) is satisfied,

then there exists a quasi-equilibrium. In this section we propose an alternative proof of this result based on a very general existence result proposed by Florenzano (2003).

Proof of Theorem 4.1. Let $\mathcal{E}(\boldsymbol{u}) = (X^i, u^i, e^i)_{i\in I}$ be an economy with a compact set $\mathrm{A}(\boldsymbol{u})$ of attainable allocations and satisfying the weak non-satiation condition (WNS). We split the set of agents in two parts: we let $I_{NS} \equiv \{i \in I : S^i(u^i) = \emptyset\}$ be the set of agents that are never satiated and we let $I_s \equiv \{i \in I : S^i(u^i) \neq \emptyset\}$ be the set of agents that may be satiated. We propose to modify the characteristics of the agents that may be satiated. Fix $i \in I_s$. The set $\mathrm{A}^i(\boldsymbol{u})$ is compact and since u^i is upper semi-continuous, there exists $\xi^i \in \operatorname{argmax}\{u^i(x) : x \in \mathrm{A}^i(\boldsymbol{u})\}$. Applying Assumption (WNS),

$$\exists \zeta^i \in S^i(u^i) \setminus A^i(\boldsymbol{u}), \quad u^i(\zeta^i) \geq u^i(\xi^i) = \sup\{u^i(x) : x \in \mathrm{A}^i(\boldsymbol{u})\}.$$

We consider another economy $\mathcal{G} = (Y^i, Q^i, e^i)_{i \in I}$ with non-ordered preferences and such that for each i the consumption set Y^i is defined by

$$Y^i \equiv \{x^i \in X^i : u^i(x^i) \geq u^i(e^i)\}$$

and for each bundle $y^i \in Y^i$ the set $Q^i(y^i)$ of strictly preferred bundles is defined by[17]

$$Q^i(y^i) \equiv \begin{cases} P^i(y^i) \cup \{\zeta^i\} & \text{if} \quad i \in I_S \\ P^i(y^i) & \text{if} \quad i \in I_{\text{NS}} \end{cases}$$

where $P^i(y^i) \equiv \{z^i \in X^i : u^i(z^i) > u^i(y^i)\}$.

Applying Assumptions (A.1) and (A.2), the consumption set Y^i is closed convex and contains the initial endowment e^i. Observe that the set of feasible allocations of the economy $\mathcal{G}$ coincides with the set $\mathrm{A}(\boldsymbol{u})$ of attainable allocations of the economy $\mathcal{E}(\boldsymbol{u})$. In particular it is compact. By construction, for each feasible allocation $\boldsymbol{y} = (y^i)_{i \in I}$ the strictly preferred set $Q^i(y^i)$ is non-empty. Moreover, since $\zeta^i \notin \mathrm{A}^i(\boldsymbol{u})$ we have $y^i \notin Q^i(y^i)$ for each individually feasible bundle y^i. In order to apply Proposition 3.2.3 in Florenzano (2003) it is sufficient to prove the following claim. $\square$

Claim Appendix A.1. For each i the correspondence Q^i has convex upper sections[18] and open lower sections.[19]

Proof of Claim A.1. The claim is obvious if $i \in I_{\text{NS}}$. Let $i \in I_S$. For each bundle $y^i \in Y^i$, we have

$$Q^i(y^i) = \begin{cases} P^i(y^i) & \text{if} \quad y^i \notin S^i \\ \{\zeta^i\} & \text{if} \quad y^i \in S^i. \end{cases}$$

It follows that Q^i has convex values. For each $z^i \in Y^i$, we have

$$(Q^i)^{-1}(z^i) = \begin{cases} (P^i)^{-1}(z^i) \cap Y^i & \text{if} \quad z^i \neq \zeta^i \\ Y^i & \text{if} \quad z^i = \zeta^i. \end{cases}$$

It follows that Q^i has open lower sections. $\square$

We can now apply Proposition 3.2.3 in Florenzano (2003) to obtain the existence of a quasi-equilibrium of $\mathcal{G}$ which is obviously a quasi-equilibrium of $\mathcal{E}(\boldsymbol{u})$.

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[17] Observed that $\zeta^i \in Y^i$ for each i.

[18] I.e. for each bundle $y^i \in Y^i$, the set $Q^i(y^i)$ is convex.

[19] I.e. for each $z^i \in Y^i$, the set $(Q^i)^{-1}(z^i) \equiv \{y^i \in Y^i : Q^i(y^i) \ni z^i\}$ is open in Y^i.