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RESEARCH ARTICLE

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Stochastic equilibria for economies under uncertainty with intertemporal substitution

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Abstract We consider the model of a stochastic pure exchange economy with a finite set of agents whose preferences exhibit local substitution in the sense of Hindy and Huang (1992). In order to prove the existence of Arrow–Debreu equilibria, it is assumed in Bank and Riedel (2001) that smooth subgradients exist (Assumption 1 in Bank and Riedel (2001)) and that they are uniformly bounded from above and away from zero (Assumption 2 in Bank and Riedel 2001).

In this paper, we prove that the existence of smooth subgradients implies local properness of preferences. By a slight improvement of classical existence results of the literature, we prove that the local properness of preferences is a sufficient condition for the existence of equilibria, rendering Assumption 2 in Bank and Riedel (2001) superfluous.

Keywords Stochastic pure exchange economies · Intertemporal substitutability of consumption · Arrow–Debreu equilibrium · Local properness · Optional random measures

JEL Classification Numbers D51 · D91

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1 Introduction

Continuous trading of long lived securities allows rational agents to finance rich varieties of consumption plans: from continuous and smooth pension plans to more risk–loving plans that prescribe gulps in good times and pauses in bad times to even fancier plans that might be related to all–time highs of some indices. Given all these possibilities, one naturally asks what are the economically sensible restrictions for preferences over these consumption plans? To what kind of state prices in the sense of Arrow and Debreu do these preferences lead in equilibrium? More fundamentally, does an equilibrium exist at all? If it exists, can it be implemented by trading in securities?

An intriguing approach to all these questions has been put forward by Hindy and Huang (1992) (see also Hindy, Huang, and Kreps 1992, for the deterministic case). In addition to the usual rationality axioms for preferences, they convincingly argue that *continuity* of preferences has an eminent role to play in continuous time models. If you get your chocolate bar now or an instant from now does somehow matter (and the younger you are the more you might suffer), but before you can start crying, the instant has passed and you get the chocolate bar — so your overall sufferings from the instant delay remain negligible after all. In a broader and more adult perspective, it seems most natural to assume that a rational agent is almost indifferent if his plans for the next year are shifted slightly by, say, a second or a minute, or even a day. Economically, consumption of a good at one point in time is a close substitute for consumption of the same good an instant later or earlier. In other words, preferences exhibit local substitution in time. Hindy and Huang (1992) thus impose the strong continuity requirement that preferences be continuous with respect to small time shifts. As Hindy and Huang identify the commodity space with the space of all finite signed random measures on the time axis, mathematically, a small shift in time corresponds to a small variation with respect to the weak topology for random measures. This approach, economically natural as it is, has led to deep and difficult problems for General Equilibrium Theory. This paper studies the existence problem and aims to identify the weakest assumptions on the primitives that allow to prove existence of an equilibrium.

We consider the model of a stochastic pure exchange economy with a finite set of agents whose preferences exhibit local substitution in the sense of Hindy and Huang (1992) (see also Hindy, Huang, and Kreps 1992, for the case of certainty). In the case of certainty, the problem of existence of an Arrow–Debreu equilibrium is solved by Mas-Colell and Richard (1991) (see Aliprantis 1997, which corrects two critical steps in the proof of Mas-Colell and Richard 1991) for norm-proper economies. The Mas-Colell–Richard theorem does not apply to the uncertain framework since the space of norm-continuous price functionals in the Hindy–Huang model (which consists of semimartingales with absolutely continuous compensator) is not a lattice. The existence of an Arrow-equilibrium with a norm-continuous price functional for norm-proper economies, is left as an open question in Hindy and Huang (1992).

In the case with uncertainty Bank and Riedel (2001) proposed a solution. Their approach consists on first proving the existence of a *weak equilibrium* in the sense that the price functional is a bounded and optional process, which is not necessarily continuous for the norm-topology introduced in Hindy and Huang (1992).

Then, under additional assumptions on the information flow and the utility subgradients, they prove that any price functional of a weak equilibrium is in fact continuous on the consumption set (but not necessarily on the commodity space) for the Hindy–Huang norm.

To prove the existence of a weak equilibrium, Bank and Riedel follow the abstract Mas-Colell–Richard approach by using a disaggregated version of the Negishi approach. However they do not assume that the economy is norm-proper, but they assume that smooth subgradients exist for the utility functions and that these subgradients are bounded above and away from zero on the set of feasible allocations. Since the properness assumption is a condition on the marginal rates of substitution, the smoothness assumption on utility functions in Bank–Riedel seems to be related to the uniform properness assumption on preferences used in Mas-Colell and Richard (1991). Moreover, it was proved in Araujo and Monteiro (1989) (for topological vector lattices) and Podczeck (1996) (for more general commodity-price pairings) that, in order to prove the existence of an equilibrium, the uniform properness assumption on preferences can be weakened into a local properness assumption. Regarding to these existence results, there are two questions: What is the exact relation between the existence of smooth subgradients and the norm-properness assumption? Is it possible to prove the existence of stochastic equilibria for economies with locally norm-proper preferences as in the classical literature?

We prove (Propositions 3 and 5) that the existence of smooth subgradients (Assumptions 1 and 4 in Bank and Riedel 2001) implies that preferences are locally proper for suitable topologies. Moreover we slightly improve Mas-Colell–Richard's existence result to prove (Theorem 2) the existence of equilibria for economies with locally norm-proper preferences. Therefore both existence results Theorems 1 and 2 in Bank and Riedel (2001) follow from a slight improvement of the Mas-Colell–Richard's existence result. In particular the uniform boundedness assumption on subgradients introduced in Bank and Riedel (2001, Assumption 2) is superfluous. We provide in the last section an example of Hindy–Huang–Kreps utility functions that are locally proper but for which smooth subgradients may not exist. Some proofs are referred to the Appendix.

2 The model

We consider a stochastic pure exchange economy where a finite set I of agents live in a world of uncertainty from time 0 to time T. There is a single consumption good available for consumption at any time $t \in [0, T]$. Uncertainty is modelled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each $\omega \in \Omega$ is a state of nature which is a complete description of one possible realization of all exogenous sources of uncertainty from time 0 to time T. The sigma-field $\mathcal{F}$ is the collection of events which are distinguishable at time T and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with a filtration $\mathbb{F} = \{\mathcal{F}(t) : t \in [0, T]\}$ which represents the time evolution of the agents' knowledge about the states of nature. We assume that $\mathcal{F}(0)$ is $\mathbb{P}$-almost surely trivial and that $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness.

The set of positive, increasing and right-continuous functions from $[0, T]$ to $\mathbb{R}_+$ is denoted by M_+. We represent the consumption bundle of an agent by a process

$x : (\omega, t) \mapsto x(\omega, t)$, where $x(\omega, t)$ represents the cumulative consumption from time 0 to time T and satisfies

(a) for each $\omega \in \Omega$, $x(\omega) : t \mapsto x(\omega, t)$ belongs to M_+,
(b) for each $t \in [0, T]$, $x(t) : \omega \mapsto x(\omega, t)$ is $\mathcal{F}(t)$-measurable and $x(T)$ belongs to $L^1(\mathbb{P})$.

The set of ($\mathbb{P}$-equivalent classes of) mappings $x : \Omega \to M_+$ such that the process $(\omega, t) \mapsto x(\omega, t)$ satisfies (a) and (b) is denoted by E_+ and the linear span of E_+ will be denoted by E and is called the commodity space. If z belongs to E then there exists x, y in E_+ such that $z = x - y$. In particular z is an $\mathbb{F}$-adapted process having right-continuous and bounded variation sample paths. We can endow E with the linear order $\geqslant$ defined by the cone E_+ in the sense that $y \geqslant x$ if $y - x$ belongs to E_+. If y belongs to E_+ then the order interval $[0, y]$ is defined by $[0, y] := \{x \in E : x \in E_+ \text{ and } y - x \in E_+\}$.

Remark 1 Observe that if x, y are vectors in E such that $y \geqslant x$ then there exists $\Omega^* \in \mathcal{F}$ with $\mathbb{P}\Omega^* = 1$ and such that for each $\omega \in \Omega^*$, the function $t \longmapsto y(\omega, t) - x(\omega, t)$ is increasing with $y(\omega, 0) - x(\omega, 0) \geqslant 0$. In particular we have for each $\omega \in \Omega^*$,

$$y(\omega, t) \geqslant x(\omega, t) \quad \forall t \in [0, T].$$

If $B(T)$ denotes the space of bounded functions defined on $[0, T]$ and $\mathcal{B}$ denotes the Borelian sigma-algebra on $[0, T]$, then we let $L^\infty(\mathbb{P}, B(T))$ denote the space (up to $\mathbb{P}$-indistinguishability) of all bounded and $\mathcal{F} \otimes \mathcal{B}$-measurable processes $\psi : \Omega \times [0, T] \to \mathbb{R}$ in the sense that the function

$$\omega \mapsto \sup_{t \in [0,T]} |\psi(\omega, t)|$$

is essentially bounded, i.e. belongs to $L^\infty(\mathbb{P})$. There is a natural duality $\langle ., . \rangle$ on $L^\infty(\mathbb{P}, B(T)) \times E$ defined by

$$\forall (\psi, x) \in L^\infty(\mathbb{P}, B(T)) \times E, \quad \langle \psi, x \rangle = \mathbb{E} \int_{[0,T]} \psi(t) dx(t).$$

A process is said optional if it is $\mathcal{O}$-measurable where $\mathcal{O}$ is the sigma-field on $\Omega \times [0, T]$ generated by right-continuous $\mathbb{F}$-adapted processes with left-limits. The space of bounded processes $\psi \in L^\infty(\mathbb{P}, B(T))$ that are optional is denoted by F and we denote by F_+ the order dual cone, i.e.

$$F_+ := \{\psi \in F : \langle \psi, x \rangle \geqslant 0, \quad \forall x \in E_+\}.$$

If $\psi \in F_+$ then ψ is called a price process. The duality product $\langle \psi, x \rangle$ is the value of the consumption bundle $x \in E_+$ under the price ψ where $\psi(\omega, t)$ is interpreted to be the time 0 price of one unit of consumption at time t in state ω, per unit of probability.

We prove (see Appendix A.1 for the proof) in the following proposition that the framework of stochastic pure exchange economies with intertemporal substitution fits the structural conditions widely used in the literature dealing with the lattice theoretical framework (see Florenzano 2003, Section 5.3.2).

Proposition 1 *The pair $\langle F, E\rangle$ is a Riesz dual pair, i.e.*

(i) $\langle F, E\rangle$ *is a dual pair,*[1]
(ii) the space E endowed with the partial order defined by E_+ is a linear vector lattice,
(iii) the space F endowed with the partial order defined by F_+ is a linear vector lattice.

Moreover if $\psi^\bullet = (\psi^i, i \in I)$ is a finite family in F_+ then the Riesz-Kantorovich functional $\mathcal{R}_{\psi^\bullet}$ defined by

$$\forall x \in E_+, \quad \mathcal{R}_{\psi^\bullet}(x) := \sup \left\{ \sum_{i\in I} \langle \psi^i, x^i\rangle \ : \ \sum_{i\in I} x^i = x \quad \text{and} \quad x^i \in E_+ \right\}$$

satisfies

$$\forall x \in E_+, \quad \mathcal{R}_{\psi^\bullet}(x) = \langle \sup \psi^\bullet, x\rangle$$

where $\sup \psi^\bullet$ *is the bounded optional process in F defined by*

$$\forall t \in [0, T], \quad [\sup \psi^\bullet](t) := \max\{\psi^i(t) \ : \ i \in I\}.$$

In particular a bounded optional process ψ belongs to F_+ if and only if for every $t \in [0, T]$, we have $\psi(t) \geqslant 0$.

Definition 1 *An economy $\mathcal{E}$ is a family $\mathcal{E} = (V^\bullet, e^\bullet)$ where $V^\bullet = (V^i, i \in I)$ with $V^i : E_+ \longrightarrow \mathbb{R}$ and $e^\bullet = (e^i, i \in I)$ with $e^i \in E_+$.*

For each $i \in I$, the functional $V^i : E_+ \longrightarrow \mathbb{R}$ represents the utility function of agent i, and e^i represents the cumulative income stream (initial endowment) of agent i. We let $e = \sum_{i\in I} e^i$ denote the aggregate endowment and if $x \in E_+$ the set $\{y \in E_+ \ : \ V^i(y) > V^i(x)\}$ is denoted by $P^i(x)$. An allocation is a vector $x^\bullet = (x^i, i \in I)$ where $x^i \in E_+$. It is said feasible or attainable if $\sum_{i\in I} x^i = e$. The set of attainable allocations is denoted by $\mathcal{A}$. We define hereafter the standard notion of Arrow–Debreu equilibrium.

Definition 2 *The pair $(\psi, x^\bullet)$ of a price process $\psi \in F_+$ and an allocation $x^\bullet$ is called an Arrow–Debreu equilibrium if*

(a) the price process ψ belongs to F_+ and $\langle \psi, e\rangle > 0$,
(b) the allocation $x^\bullet$ is feasible, and
(c) for each $i \in I$, the consumption plan x^i maximizes agent i's utility over all consumption plans y satisfying the budget constraint $\langle \psi, y\rangle \leqslant \langle \psi, e^i\rangle$, i.e.

$$x^i \in \operatorname{argmax}\{V^i(y) \ : \ y \in E_+ \quad \textit{and} \quad \langle \psi, y\rangle \leqslant \langle \psi, e^i\rangle\}.$$

Remark 2 Observe that if $(\psi, x^\bullet)$ is an equilibrium then the budget constraints are binding, i.e. for each $i \in I$, $\langle \psi, x^i\rangle = \langle \psi, e^i\rangle$.

[1] That is, if $\langle \psi, x\rangle = 0$ for each $x \in E$ then $\psi = 0$, and if $\langle \psi, x\rangle = 0$ for each $\psi \in F$, then $x = 0$.

3 Assumptions on primitives

Since $0 \leqslant x(t) \leqslant x(T)$ and $x(T) \in L^1(\mathbb{P})$ for every $x \in E_+$, the space E is a subspace of $L^1(\mathcal{O}, \mathbb{P} \otimes \kappa)$ where $\kappa = \lambda + \delta_T$ with λ the Lebesgue measure on $\mathcal{B}$. Following Hindy and Huang (1992) (see also Hindy, Huang, and Kreps 1992) we consider on E the restriction of the $L^1(\mathcal{O}, P \otimes \kappa)$-norm, i.e. we consider the norm $\|.\|$ defined by

$$\forall x \in E, \quad \|x\| = \mathbb{E} \int_{[0,T]} |x(t)| \kappa(dt) = \mathbb{E} \int_{[0,T]} |x(t)| dt + \mathbb{E}|x(T)|.$$

It is argued in Hindy and Huang (1992) that this norm induces a topology on the set of consumption bundles that exhibits intuitive economic properties, in particular it captures the notion that consumption at adjacent dates are almost perfect substitutes. If $(x_n, n \in \mathbb{N})$ is a sequence in E_+ norm-converging to $x \in E_+$, then there exists a subsequence $(x_{n_k}, k \in \mathbb{N})$ and $\Omega^* \in \mathcal{F}$ such that $\mathbb{P}\Omega^* = 1$ and

$$\forall \omega \in \Omega^*, \quad \lim_{k \to +\infty} \int_{[0,T]} |x(\omega, t) - x_{n_k}(\omega, t)| dt + |x(\omega, T) - x_{n_k}(\omega, T)| = 0. \tag{1}$$

On the linear span M of M_+ we may consider the weak-star topology defined by the family of semi-norms $(p_f, f \in C)$ where C is the space of continuous functions from $[0, T]$ to $\mathbb{R}$ and

$$p_f(z) = \int_{[0,T]} f(t) dz(t), \quad \forall z \in M.$$

Following Hindy, Huang, and Kreps (1992, Proposition 5), (1) implies the weak-star convergence of $(x_{n_k}(\omega), k \in \mathbb{N})$ to $x(\omega)$, for each $\omega \in \Omega^*$. The weak-star topology restricted to M_+ is metrizable by the Prohorov distance d_P. Following Bank and Riedel (2001), we let d be the distance defined on E_+ by

$$\forall (x, y) \in E_+^2, \quad d(x, y) = \mathbb{E} \min\{d_P(x, y), 1\} + \mathbb{E}|x(T) - y(T)|.$$

Therefore, we have the following result.

Lemma 1 *If $(x_n, n \in N)$ is a sequence in E_+ which is norm-converging to $x \in E_+$, then there exists a subsequence $(x_{n_k}, k \in \mathbb{N})$ which is d-convergent to x.*

In order to prove the existence of an Arrow–Debreu equilibrium, we consider the following list of assumptions that an economy can satisfy.

Assumption (C) *For each $i \in I$,*

(C.1) the initial endowment e^i belongs to E_+ and is not zero, i.e. $e^i > 0$,
(C.2) the utility function $V^i : E_+ \longrightarrow \mathbb{R}$ is concave,
(C.3) the utility function V^i is norm-upper semicontinuous on the order interval $[0, e]$.[2]

[2] That is, if $(x_n, n \in \mathbb{N})$ is a sequence in $[0, e]$ which norm-converges to x in $[0, e]$, then

$$\limsup_{n \to \infty} V^i(x_n) \leqslant V^i(x).$$

Assumption (U) *For each $i \in I$, for every $x \in E_+$, there exists a positive bounded optional process $\nabla V^i(x) \in F_+$ with*

(U.1) for each $j \in I$, we have $\langle \nabla V^i(x), e^j \rangle > 0$,
(U.2) the vector $\nabla V^i(x)$ satisfies the subgradient property

$$\forall y \in E_+, \quad V^i(y) - V^i(x) \leqslant \langle \nabla V^i(x), y - x \rangle$$

(U.3) this subgradient is continuous in the sense that,

$$\forall y \in E_+, \quad \lim_{\varepsilon \downarrow 0} \langle \nabla V^i(\varepsilon y + (1-\varepsilon)x), y - x \rangle = \langle \nabla V^i(x), y - x \rangle.$$

We will prove in Subsection 4.3 that Assumption U is stronger than the usual local properness assumption on preferences widely used in the general equilibrium literature.

Remark 3 Let $\mathcal{E} = (V^\bullet, e^\bullet)$ be an economy. Preferences of agent i are said increasing if $V^i(x+y) \geqslant V^i(x)$ for every x, y in E_+; strictly increasing if $V^i(x+y) > V^i(x)$ for every x, y in E_+ with $y \neq 0$. Note that if $\mathcal{E}$ satisfies Assumption U, then preferences of agent i are increasing; they are strictly increasing if and only if $\nabla V^i(x)$ is strictly positive for every $x \in E_+$.

Remark 4 Let $(V^\bullet, e^\bullet)$ be an economy satisfying Assumption U, then for each i, j in I, the initial endowment e^j is strongly desirable for agent i in the sense that

$$\forall x \in E_+, \quad \forall t > 0, \quad V^i(x + te^j) > V^i(x).$$

Remark 5 Assume that Assumption U.2 is satisfied,

(a) if preferences of agent i are strictly increasing and $e^j > 0$ for each $j \in J$, then Assumption U.1 is satisfied,
(b) if preferences of agent i are increasing and for each $j \in I$, there exists a strictly positive integrable adapted process ξ^j such that $de^j(t) = \xi^j(t)dt$, then Assumption U.1 is satisfied.

The following theorem is the main result of the paper for utility functions with smooth subgradients.

Theorem 1 *If an economy $\mathcal{E} = (V^\bullet, e^\bullet)$ satisfies Assumptions C and U then there exists an Arrow–Debreu equilibrium $(\psi, x^\bullet)$ such that for each $i \in I$, there exists $\lambda^i > 0$ satisfying*

$$\psi = \sup\{\lambda^i \nabla V^i(x^i) \; : \; i \in I\}.$$

We prove in Subsection 4.3 that Theorem 1 is a corollary of an existence result (Corollary 1) for economies with proper preferences.

Remark 6 Theorem 1 generalizes the main result in Bank and Riedel (2001, Theorem 1). Assumption 1 in Bank and Riedel (2001) implies Assumptions C and U. Indeed, it is assumed in Bank and Riedel (2001) that

- utility functions V^i are strictly concave: we only assume concavity,

- utility functions V^i are d-continuous on the whole set E_+: we only assume norm-upper semicontinuity on the order interval $[0, e]$,[3]
- utility functions are strictly increasing: we only assume that utility functions are increasing and that every initial endowment is strongly desirable for each agent.

But the main improvement of this theorem is to prove that we do not need to assume neither that subgradients are uniformly bounded from above nor that they are uniformly bounded away from zero. More precisely, we prove that the following assumption (Bank and Riedel 2001, Assumption 2) is superfluous:

(U.4) there exists positive optional processes b and B in $L^1_+(\mathbb{P} \otimes de) \setminus \{0\}$ such that

$$b \leqslant \nabla V^i(z) \leqslant B \quad [\mathbb{P} \otimes de]\text{−a.e.}$$

for all $z \in [0, e]$.

4 Existence of equilibria under properness

To prove existence of an Arrow–Debreu equilibrium we follow the approach introduced by Peleg and Yaari (1970). First we prove the existence of an Edgeworth equilibrium $x^\bullet$ and then we decentralize this allocation by a price process ψ to get an Arrow–Debreu equilibrium $(\psi, x^\bullet)$.

4.1 Existence of Edgeworth equilibria

We recall well-known properties of optimality for allocations.

Definition 3 *An attainable allocation $x^\bullet \in \mathcal{A}$ is said to be:*

1. *an Edgeworth equilibrium if there is no $0 \neq \lambda^\bullet \in (\mathbb{Q} \cap [0, 1])^I$ and some allocation $y^\bullet$ such that $V^i(y^i) > V^i(x^i)$ for each $i \in I$ with $\lambda^i > 0$ and satisfying $\sum_{i \in I} \lambda^i y^i = \sum_{i \in I} \lambda^i e^i$,*
2. *an Aubin (or fuzzy) equilibrium if there is no $0 \neq \lambda^\bullet \in [0, 1]^I$ and some allocation $y^\bullet$ such that $V^i(y^i) > V^i(x^i)$ for each $i \in I$ with $\lambda^i > 0$ and satisfying $\sum_{i \in I} \lambda^i y^i = \sum_{i \in I} \lambda^i e^i$.*

The reader should observe that these concepts are "price free" in the sense that they are intrinsic properties of the commodity space. Recall[4] that if utility functions are concave then the set of Aubin equilibria and the set of Edgeworth equilibria coincide.

Proposition 2 *If an economy satisfies Assumption C then the set of Edgeworth equilibria is non-empty.*

[3] If V^i is d-continuous on E_+, then it is d-upper semicontinuous on E_+ and then, from Lemma 1, it is norm-upper semicontinuous on E_+.

[4] See Proposition 4.2.6 in Florenzano (2003).

Proof In order to apply Theorem 3.1 in Allouch and Florenzano (2004), it is sufficient to prove that the following set

$$\mathcal{V} := \left\{ v^\bullet = (v^i, i \in I) \in \mathbb{R}^I \;:\; \exists x^\bullet \in \mathcal{A}, \quad \forall i \in I, \quad V^i(e^i) \leqslant v^i \leqslant V^i(x^i) \right\}$$

is a compact subset of $\mathbb{R}^I$. Let $(v_n^\bullet, n \in \mathbb{N})$ be a sequence in $\mathcal{V}$, then there exists $(x_n^\bullet, n \in \mathbb{N})$ a sequence of attainable allocations $x_n^\bullet \in \mathcal{A}$ such that

$$\forall n \in \mathbb{N}, \quad \forall i \in I, \quad V^i(e^i) \leqslant v_n^i \leqslant V^i(x_n^i). \tag{2}$$

Fix $i \in I$, we claim that the sequence $(v_n^i, n \in \mathbb{N})$ is bounded. Observe that if each utility function V^i is increasing then $V^i(x_n^i) \leqslant V^i(e)$ which implies that the sequence $(v_n^i, n \in \mathbb{N})$ is bounded. Now for the general case, assume by way of contradiction that there exists a subsequence, still denoted by $(v_n^i, n \in \mathbb{N})$, converging to $+\infty$. For each $n \in \mathbb{N}$, the vector x_n^i belongs to $[0, e]$. It then follows that

$$\sup_{n \in \mathbb{N}} \int_{\Omega \times [0,T]} |x_n^i(\omega, t)| [P \otimes \kappa](d\omega, dt) \leqslant \|e\| < +\infty.$$

Applying Komlós (1967) there exists a subsequence, still denoted by $(x_n^i, n \in \mathbb{N})$, such that the sequence $(y_n^i, n \in \mathbb{N})$ defined by

$$y_n^i = \frac{1}{n+1} \sum_{k=0}^{n} x_k^i$$

is convergent to $y^i \in L^1(\mathcal{O}, \mathbb{P} \otimes \kappa)$ for $[\mathbb{P} \otimes \kappa]$-almost every (ω, t). Therefore y^i is positive, increasing and right-continuous, i.e. $y^i \in E_+$. Moreover since y_n^i belongs to $[0, e]$, it follows that y^i belongs to $[0, e]$. In particular,

$$\forall n \in N, \quad |y_n^i(\omega, t) - y^i(\omega, t)| \leqslant 2e(\omega, t) \quad [\mathbb{P} \otimes \lambda]\text{−a.e.}$$

and applying the Lebesgue Dominated Convergence Theorem we get that the sequence $(y_n^i, n \in \mathbb{N})$ is norm convergent to y^i. Since V^i is upper semicontinuous and concave we get

$$\begin{aligned} +\infty = \limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} v_k^i &\leqslant \limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} V^i(x_k^i) \\ &\leqslant \limsup_{n \to \infty} V^i(y_n^i) \leqslant V^i(y^i), \end{aligned}$$

which yields a contradiction. Passing to a subsequence if necessary we get that the sequence $(v_n^i, n \in \mathbb{N})$ is convergent to some $v^i \in \mathbb{R}$ satisfying

$$\begin{aligned} v^i = \limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} v_n^i &\leqslant \limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} V^i(x_k^i) \\ &\leqslant \limsup_{n \to \infty} V^i(y_n^i) \leqslant V^i(y^i). \end{aligned}$$

Moreover the allocation $y^\bullet = (y^i, i \in I)$ belongs to $\mathcal{A}$. It then follows that the vector $(v^i, i \in I)$ belongs to $\mathcal{V}$. □

4.2 Decentralizing Edgeworth equilibria

It is straightforward to check that every Arrow–Debreu equilibrium is an Edgeworth equilibrium. The main difficulty consists in proving the converse. We propose to follow the classical literature[5] dealing with infinite dimensional commodity-price spaces by introducing the concept of proper economies.

Definition 4 **(τ-properness)** *Let τ be a Hausdorff locally convex linear topology on E. An economy $(V^\bullet, e^\bullet)$ is τ-proper if for every Edgeworth equilibrium $x^\bullet$, for each $i \in I$, there is a set $\widehat{P}^i(x^i)$ such that*

(i) the vector $x^i + e$ is a τ-interior point of $\widehat{P}^i(x^i)$,
(ii) the set $\widehat{P}^i(x^i)$ is convex and satisfies the following additional convexity property

$$\forall z \in \widehat{P}^i(x^i) \cap E_+, \quad \forall t \in (0,1), \quad tz + (1-t)x^i \in \widehat{P}^i(x^i) \cap E_+$$

(iii) we can extend preferences in the following way

$$\widehat{P}^i(x^i) \cap E_+ \cap A_{x^i} \subset P^i(x^i) \subset \widehat{P}^i(x^i) \cap E_+$$

where $A_{x^i} \subset E$ is a radial set at x^i.[6]

Remark 7 We say that an economy is *strongly* τ-proper if condition (iii) in Definition 4 is replaced by the following condition (iii'):

(iii') we can extend preferences in the following way

$$\widehat{P}^i(x^i) \cap E_+ = P^i(x^i).$$

Strong τ-properness was introduced by Tourky (1999) and is used, among others, by Aliprantis, Tourky, and Yannelis (2001) and Aliprantis, Florenzano, and Tourky (2004, 2005). We refer to Aliprantis, Tourky, and Yannelis (2000) for a comparison of the different notions of properness used in the literature. Observe that if $\mathcal{E} = (V^\bullet, e^\bullet)$ is an economy (satisfying the following monotonicity Assumption M) such that for each $i \in I$, it is possible to extend V^i to a τ-continuous and concave function $\widehat{V}^i : E \longrightarrow \mathbb{R}$, then the economy is τ-proper.[7] Moreover, τ-properness is slightly weaker than strong τ-properness. However this slight difference is crucial in order to compare properness with the existence of smooth subgradients (see Proposition 3).

Definition 5 **(H-properness)** *Given a subspace H of F, an economy $(V^\bullet, e^\bullet)$ is H-proper if there exists a Hausdorff locally convex linear topology τ on E such that*

(a) the economy $(V^\bullet, e^\bullet)$ is τ-proper,
(b) the τ-topological dual $(E, \tau)'$ is a subspace of H,

[5] We refer, among others, to Deghdak and Florenzano (1999), Florenzano (2003), Mas-Colell and Richard (1991), Podczeck (1996), Tourky (1999), Aliprantis, Tourky, and Yannelis (2001), and Aliprantis, Florenzano, and Tourky (2004, 2005).

[6] A subset A of E is radial at $x \in A$ if for each $y \in E$, there exists $\bar{\alpha} \in (0, 1]$ such that $(1-\alpha)x + \alpha y$ belongs to A for every $\alpha \in [0, \bar{\alpha}]$.

[7] Take $\widehat{P}^i(x) := \{y \in E : \widehat{V}^i(y) > \widehat{V}^i(x)\}$.

(c) *for every family* $\pi^\bullet = (\pi^i, i \in I)$ *of* τ*-continuous linear functionals* $\pi^i \in (E, \tau)'$ *the supremum* $\sup \psi^\bullet$ *still belongs to* H.

If H is a linear subspace of F then the cone $H \cap F_+$ is denoted by H_+.

Remark 8 Given a dual pair $\langle H, E\rangle$ and an economy $\mathcal{E}$, we consider the following requirements:

(i) the economy $\mathcal{E}$ is $\sigma(E, H)$-proper,
(ii) the economy $\mathcal{E}$ is $\tau(E, H)$-proper,
(iii) the economy $\mathcal{E}$ is H-proper,

where $\tau(E, H)$ is the Mackey topology on E relatively to the dual pair $\langle H, E\rangle$. The topology $\sigma(E, H)$ is the weakest Hausdorff locally convex topology compatible with the dual pair $\langle H, E\rangle$ and $\tau(E, H)$ is the strongest Hausdorff locally convex topology compatible with the dual pair $\langle H, E\rangle$. Therefore condition (i) implies condition (ii). If the space H is stable under the supremum operation[8] then condition (ii) implies condition (iii).

We consider the following monotonicity assumption.

Assumption (M) *For every Edgeworth equilibrium* $x^\bullet$*, for each* $i \in I$*, the following property is satisfied:*

$$\forall j \in I, \quad \forall t > 0, \quad x^i + te^j + E_+ \subset P^i(x^i).$$

Remark 9 From Remarks 3-4, Assumptions C and U imply Assumption M.

For decentralizing Edgeworth equilibria when the commodity space is infinite dimensional, Mas-Colell (1986) introduces the properness assumption on preferences in order to compensate for the fact that the consumption sets may have empty interior. Mas-Colell's work was extended in various important directions; one of them is the following existence result which is closely related to existence results in Podczeck (1996) and Deghdak and Florenzano (1999).

Theorem 2 *Let* $\mathcal{E}$ *be an economy satisfying Assumptions C and M and let* H *be a subspace of* F*. If the economy* $\mathcal{E}$ *is* H*-proper then for every Edgeworth equilibrium* $x^\bullet$ *there exists a price* $\psi \in H_+$ *such that* $(\psi, x^\bullet)$ *is an Arrow–Debreu equilibrium.*

The proof of Theorem 2 is postponed to Appendix A.2. The approach is based on a convexity result due to Podczeck (1996) and the proof follows almost verbatim Deghdak and Florenzano (1999). Following Propositions 1 and 2 and Remark 8, we have the following corollary to Theorem 2.

Corollary 1 *Consider an economy* $\mathcal{E} = (V^\bullet, e^\bullet)$ *satisfying Assumptions C and M. If* $\mathcal{E}$ *is* $\tau(E, F)$*-proper then there exists an Arrow-Debreu equilibrium* $(\psi, x^\bullet)$ *with a price process in* F_+.

[8] Or equivalently if the ordered vector space (H, H_+) is a vector lattice.

4.3 Properness vs existence of subgradients

In order to prove that the main result in Bank and Riedel (2001) is a special case of Corollary 1, we have to check that Assumptions C.2 and U imply $\sigma(E, F)$-properness.

Proposition 3 *If* $\mathcal{E} = (V^\bullet, e^\bullet)$ *is an economy satisfying Assumptions C.2 and U then* $\mathcal{E}$ *is* $\sigma(E, F)$*-proper.*

Proof Let $\mathcal{E} = (V^\bullet, e^\bullet)$ be an economy satisfying Assumptions C.2 and U. We denote by τ the weak topology $\sigma(E, F)$ on E. Let $x \in E_+$, we pose

$$\widehat{P^i}(x) = \left\{y \in E \,:\, \langle \nabla V^i(x), y - x\rangle > 0\right\}.$$

This set is τ-open, convex and from Assumption U.1, we have $x + e \in \widehat{P^i}(x)$. Let $y \in P^i(x)$ then $y \in E_+$ and from Assumption U.2,

$$0 < V^i(y) - V^i(x) \leqslant \langle \nabla V^i(x), y - x\rangle$$

which implies that $y \in \widehat{P^i}(x)$. We have thus proved that $P^i(x) \subset \widehat{P^i}(x) \cap E_+$. Now let $y \in \widehat{P^i}(x) \cap E_+$ and let $t \in (0, 1)$. From Assumption U.2

$$\frac{1}{t}\left[V^i(ty + (1-t)x) - V^i(x)\right] \geqslant \langle \nabla V^i(ty + (1-t)x), y - x\rangle.$$

From Assumption U.3

$$\lim_{t \downarrow 0}\langle \nabla V^i(ty + (1-t)x), y - x\rangle = \langle \nabla V^i(x), y - x\rangle > 0.$$

It then follows that there exists $t_y \in (0, 1)$ such that

$$\forall t \in (0, t_y], \quad V^i(x + t(y - x)) > V^i(x).$$

Let $A_x = \{x + t(y - x) : y \in E, \quad t \in (0, t_y]\}$ where $t_y = 1$ if $y \notin E_+$. The set A_x is radial at x and $\widehat{P^i}(x) \cap A_x \cap E_+ \subset P^i(x)$. □

Remark 10 Following the previous proof, the existence of smooth subgradients (Assumptions C.2 and U) implies that each set $P^i(x)$ can be extended by a $\sigma(E, F)$-open half space $\widehat{P^i}(x)$. This is a more restrictive assumption than $\sigma(E, F)$-properness and a much more restrictive assumption than $\tau(E, F)$-properness. In particular Corollary 1 is a substantial generalization of Theorem 1 (and thus of Theorem 1 in Bank and Riedel 2001).

5 Structure of equilibrium prices

Following Hindy and Huang (1992) and Bank and Riedel (2001), we propose now to provide sufficient conditions in order to prove the existence of an Arrow–Debreu equilibrium $(\psi, x^\bullet)$ such that given the price process ψ, the linear value functional $\langle \psi, .\rangle : E \to \mathbb{R}$ is norm continuous on the consumption set E_+.

Let G be the subspace of F of all bounded semimartingales with a compensator of bounded variation with almost surely continuous paths, i.e. a process $\psi \in F$

belongs to G if there exists a local martingale M and a process A of bounded variation with almost surely continuous paths such that $\psi = A + M$. In order to prove the existence of a price equilibrium in G we need to assume that the filtration $\mathbb{F}$ is quasi left-continuous. This is an assumption on the way new information is revealed to the agents. Economically, an information flow corresponds to a quasi left-continuous filtration[9] if information surprises (in the sense of Hindy and Huang 1992) occur only at times which cannot be predicted. The announcement of a policy change of the Federal reserve is an example for an information surprise which occurs at a time known in advance.

Proposition 4 *If the filtration $\mathbb{F}$ is quasi left-continuous then the price-commodity pairing $\langle G, E\rangle$ is a Riesz dual pair (in particular the subspace G is stable under the supremum operation). Moreover for every $\psi \in G$, the linear value functional $\langle \psi, .\rangle : E \to \mathbb{R}$ is norm continuous on the consumption set E_+.*

Proof Since the norm dual E' is a subspace of G, it is straightforward to check that $\langle G, E\rangle$ is a dual pair. The lattice property of G follows from points 1 and 2 in the proof of Theorem 2 in Bank and Riedel (2001), where it is proved that the supremum of two processes in G still lies in G. The norm continuity of the linear functional $\langle \psi, .\rangle$ follows from Lemma 1 and point 3 in the proof of Theorem 2 in Bank and Riedel (2001).

It is proved in Bank and Riedel (2001, Theorem 2) that under quasi left-continuity of the filtration, there exists an equilibrium with a value functional norm-continuous on E_+ if every utility gradient $\nabla V^i(x)$ in Assumption U belongs to G. We propose to generalize this result to G-proper economies. According to Proposition 4, we have the following corollary to Theorem 2.

Corollary 2 *Assume that the filtration $\mathbb{F}$ is quasi left-continuous. Consider an economy $\mathcal{E} = (V^\bullet, e^\bullet)$ satisfying Assumption C. If $\mathcal{E}$ is G-proper then there exists an Arrow-Debreu equilibrium $(\psi, x^\bullet)$ with $\psi \in G_+$. In particular the value functional $\langle \psi, .\rangle$ is norm-continuous on E_+.*

Assumption (U') *Assumption U is satisfied and for each $i \in I$, for every $x \in E_+$ the subgradient $\nabla V^i(x)$ belongs to G.*

The main result of this section is the following proposition. We prove that Assumptions 1 and 4 in Bank and Riedel (2001) imply $\sigma(E, G)$-properness. In particular it follows that Corollary 2 is a strict improvement of Theorem 2 in Bank and Riedel (2001).

Proposition 5 *If $\mathcal{E} = (V^\bullet, e^\bullet)$ is an economy satisfying Assumptions C.2 and U' then $\mathcal{E}$ is $\sigma(E, G)$-proper.*

Proof The proof follows almost verbatim the proof of Proposition 3 since

$$\widehat{P}^i(x) = \{y \in E : \langle \nabla V^i(x), y - x\rangle > 0\}$$

is $\sigma(E, G)$-open when $\nabla V^i(x)$ belongs to G. □

[9] See Hindy and Huang (1992) for a precise definition. An information flow generated by a Brownian motion or a Poisson process is quasi left-continuous.

6 Example

We consider *Hindy–Huang–Kreps preferences*, i.e. preferences given by utility functionals of the form

$$V^i(x) = \mathbb{E}\int_{[0,T]} u^i(t, Y(x)(t))\kappa(dt)$$

where $u^i : [0, T]\times\mathbb{R}_+ \to \mathbb{R}$ denotes a felicity function for agent i, and the quantity

$$Y(x)(t) = \int_{[0,t]} \beta e^{-\beta(t-s)} dx(s)$$

describes the investor's level of satisfaction obtained from his consumption up to time $t \in [0, T]$. The constant $\beta > 0$ measures how fast satisfaction decays.

We consider the linear mapping $\phi : E \to E$ defined by

$$\forall t \in [0, T], \quad \phi(x)(t) = \int_{[0,t]} \exp\{\beta s\} dx(s).$$

For each $x \in E$, the vector $\phi(x)$ is defined by the optional random measure $d[\phi(x)](t) = \exp\{\beta t\} dx(t)$. The linear mapping ϕ is bijective and the inverse mapping ϕ^{-1} is given by

$$\forall t \in [0, T], \quad \phi^{-1}(x)(t) = \int_{[0,t]} \exp\{-\beta s\} dx(s).$$

We introduce on E the following norm ρ:

$$\forall x \in E, \quad \rho(x) := \|\phi(x)\| = \mathbb{E}\int_{[0,T]} |\phi(x)(t)|\kappa(dt).$$

The proof of the following lemma is postponed to Appendix A.3.

Lemma 2 *The following properties are satisfied:*

(a) the norm-topology and the ρ-topology coincide on E_+,
(b) the ρ-topological dual $(E, \rho)'$ consists of functionals $\langle \psi, .\rangle$ with ψ an optional process of the form

$$\psi(t) = [A(t) + M(t)], \quad \forall t \in [0, T]$$

where M is a bounded martingale and A is an absolutely continuous process (with $A(0) = 0$) whose derivative A' is bounded. In particular the ρ-topological dual $(E, \rho)'$ coincides with the norm-topological dual $(E, \|.\|)'$.

In order to apply Theorem 2 we consider the following list of assumptions.

Assumption (V) *For each $i \in I$,*

$$V^i(x) = \mathbb{E}\int_{[0,T]} u^i(t, Y(x)(t))\kappa(dt), \quad \forall x \in E_+$$

where $u^i : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

(V.1) for each $t \in [0, T]$, the function $u^i(t, .) : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing and concave,
(V.2) for each $y \in \mathbb{R}_+$, the function $u^i(., y) : [0, T] \to \mathbb{R}$ is $\mathcal{B}$-measurable and the function $u^i(., 0)$ belongs to $L^1(\mathcal{B}, \kappa)$,
(V.3) for each $t \in [0, T]$ the right-derivative $\partial_y u^i(t, 0+)$ exists and the function $\partial_y u^i(., 0+)$ belongs to $L^\infty_+(\mathcal{B}, \kappa)$.

Theorem 3 *If $\mathcal{E} = (V^\bullet, e^\bullet)$ is an economy satisfying Assumption V then*

(a) for each $i \in I$, the utility function V^i is concave, strictly increasing and norm-continuous on E_+,
(b) if moreover $e \neq 0$ then the economy $\mathcal{E}$ is ρ-proper.

Proof Let $\mathcal{E} = (V^\bullet, e^\bullet)$ be an economy satisfying Assumption V. Concavity and strict-monotonicity of V^i is inherited from concavity and strict-monotonicity of u^i. We let $\widehat{u}^i$ be the function from $[0, T] \times \mathbb{R}$ to $\mathbb{R}$ defined by

$$\widehat{u}^i(t, y) = \begin{cases} u^i(t, y) & \text{if } y \geqslant 0 \\ \\ u^i(t, 0) + y\theta^i(t) & \text{if } y < 0 \end{cases}$$

where $\theta^i(t) = \partial_y u^i(t, 0+)$. Note that

$$\forall (t, y) \in [0, T] \times \mathbb{R}, \quad |\widehat{u}^i(t, y)| \leqslant u^i(t, 0) + |y|\theta^i(t). \tag{3}$$

For each $i \in I$, we let $\widehat{V}^i : E \to \mathbb{R}$ be defined by

$$\widehat{V}^i(x) = \mathbb{E} \int_{[0,T]} \widehat{u}^i(t, Y(x)(t))\kappa(dt).$$

The function $\widehat{V}^i$ is concave and coincide with V^i on E_+. In particular

$$\forall x \in E_+, \quad P^i(x) = \widehat{P}^i(x) \cap E_+ \quad \text{where} \quad \widehat{P}^i(x) = \{y \in E : \widehat{V}^i(y) > \widehat{V}^i(x)\}.$$

Claim 1 *For each $i \in I$, the function $\widehat{V}^i$ is ρ-continuous on E.*

Proof Let $(x_n, n \in \mathbb{N})$ be a sequence in E which is ρ-converging to $x \in E$. Since

$$\forall t \in [0, T], \quad |Y(x_n - x)(t)| = \beta e^{-\beta t}|\phi(x_n - x)(t)| \leqslant \beta|\phi(x_n - x)(t)|$$

the sequence $(Y(x_n), n \in \mathbb{N})$ is norm-converging to $Y(x)$. Then there exists a subsequence $(Y(x_{n_k}), k \in \mathbb{N})$ which converges $[\mathbb{P} \otimes \kappa]$-a.e. to $Y(x)$. Since $\widehat{u}^i(t, .)$ is continuous, we have that $[\mathbb{P} \otimes \kappa]$-a.e.

$$\lim_{k\to\infty} \widehat{u}^i(t, Y(x_{n_k})(t)) = \widehat{u}^i(t, Y(x)(t)). \tag{4}$$

Moreover, applying (3) we have that

$$|\widehat{u}^i(t, Y(x_{n_k})(t))| \leqslant u^i(t, 0) + \theta^i(t)|Y(x_{n_k})(t)|, \quad [\mathbb{P} \otimes \kappa]\text{−a.e.} \tag{5}$$

Since the sequence $(Y(x_{n_k}), k \in \mathbb{N})$ is norm-convergent, it is $[P \otimes \kappa]$-uniformly integrable. It then follows from (5) and (4) that the sequence $(\eta_k, k \in \mathbb{N})$ defined by

$$\forall k \in \mathbb{N}, \quad \forall t \in [0, T], \quad \eta_k(t) = \widehat{u}^i(t, Y(x_{n_k})(t))$$

is $[P\otimes\kappa]$-uniformly integrable and converges $[P\otimes\kappa]$-a.e. to $\eta(t) = \widehat{u}^i(t, Y(x)(t))$. Therefore the sequence $(\eta_k, k \in \mathbb{N})$ is norm-convergent to η and

$$\lim_{k\to\infty} \widehat{V}^i(x_{n_k}) = \lim_{k\to\infty} \mathbb{E}\int_{[0,T]} \eta_k(t)\kappa(dt) = \mathbb{E}\int_{[0,T]} \eta(t)\kappa(dt) = \widehat{V}^i(x).$$

Following the standard subsequence argument, we get the ρ-continuity of $\widehat{V}^i$ on E. □

Part (a) of the theorem follows from Claim 1 and Lemma 2. Let $x \in E_+$. Since $\widehat{V}^i$ is concave and ρ-continuous on E, the set $\widehat{P}^i(x)$ is convex and ρ-open. Since $\widehat{V}^i$ is strictly-increasing, $\widehat{V}^i(x + e) > \widehat{V}^i(x)$ and $x^i + e$ belongs to $\widehat{P}^i(x)$. We have thus proved part (b). □

Applying Theorems 2 and 3, we get the following corollary.

Corollary 3 *Assume that the filtration $\mathbb{F}$ is quasi-left continuous and let $\mathcal{E} = (V^\bullet, e^\bullet)$ be an economy satisfying assumption V. If for each $i \in I$ the initial endowment $e^i \neq 0$ then there exists an Arrow–Debreu equilibrium $(\psi, x^\bullet)$ such that the value functional $\langle\psi, .\rangle$ is norm-continuous on E_+.*

Proof Let $\mathcal{E} = (V^\bullet, e^\bullet)$ be an economy satisfying assumption V. We claim that the economy $\mathcal{E}$ is G-proper.[10] Indeed, we know from Theorem 3 that the economy is ρ-proper. From Lemma 2 the ρ-topological dual $(E, \rho)'$ coincides with the norm-topological dual $(E, \|.\|)'$. In particular, $(E, \rho)'$ is a subspace of G. Now let $\psi^\bullet = (\psi^i, i \in I)$ be a family of processes in $(E, \rho)'$. From Lemma 2, each ψ^i belongs to G. From Proposition 4, the pair $\langle G, E\rangle$ is a Riesz dual pair. In particular, if we let $\psi = \sup\{\psi^i : i \in I\}$ then ψ still belongs to G. We have thus proved that the economy is G-proper. Applying Theorem 2 there exists $(\psi, x^\bullet)$ an Arrow–Debreu equilibrium with $\psi \in G_+$. Applying Proposition 4 the functional $\langle\psi, .\rangle$ is norm-continuous on E_+. □

Remark 11 In Bank and Riedel (2001) (see also Duffie and Skiadas 1994; and Bank and Riedel 2001b), it is assumed that the felicity function u^i is twice continuously differentiable in its second argument. Since we do not need to require the existence of smooth subgradients, we only need to assume right-differentiability of the felicity function at the origin (Assumption V.3).

A Appendix

A.1 Proof of Proposition 1

When the space E is endowed with the norm

$$\forall x \in E, \quad \|x\| := \mathbb{E}\int_{[0,T]} |x(t)|\kappa(dt)$$

[10] Recall that G is the subspace of F of all bounded semimartingales with a compensator of bounded variation with almost surely continuous paths.

then following Hindy and Huang (1992), for every $\|.\|$-continuous linear functional $\pi \in (E, \|.\|)'$ there exists a bounded semimartingale ψ with an absolutely continuous compensator such that $\pi = \langle \psi, . \rangle$. In particular $(E, \|.\|)'$ is a subset of F.

When the space E is endowed with the total variation norm

$$\forall x \in E, \quad \|x\|_V := \mathbb{E} \int_{[0,T]} d|x|(t)$$

then F is a subset of $(E, \|.\|_V)'$, i.e. for every $\psi \in F$, the linear function $\langle \psi, . \rangle$ is $\|.\|_V$-continuous. Indeed for every $\psi \in F$,

$$|\langle \psi, x \rangle| \leqslant \mathbb{E} \int_{[0,T]} |\psi(t)| d|x|(t) \leqslant \|\psi\|_\infty \, \|x\|_V$$

where

$$\|\psi\|_\infty := \mathbb{P}\text{−ess sup} \sup_{t \in [0,T]} |\psi(t)|.$$

Since the space F satisfies

$$(E, \|.\|)' \subset F \subset (E, \|.\|_V)'$$

it is now straightforward to prove that $\langle F, E \rangle$ is a dual pair.

Condition (ii) of Proposition 1 follows from the fact that (M, M_+) is a linear vector lattice.

Claim 2 *If $\psi^\bullet = (\psi^i, i \in I)$ is a finite family in F_+ then*

$$\forall x \in E_+, \quad \mathcal{R}_{\psi^\bullet}(x) = \langle \sup \psi^\bullet, x \rangle.$$

Proof The vector $\sup \psi^\bullet$ in F_+ is denoted by ψ. Fix $x \in E_+$. Since for every $z \in E_+$

$$\langle \psi, z \rangle = \mathbb{E} \int_{[0,T]} \psi(t) dz(t),$$

we have that $\langle \psi, z \rangle \geqslant \langle \psi^i, z \rangle$. It then follows that $\mathcal{R}_{\psi^\bullet}(x) \leqslant \langle \psi, x \rangle$. We propose now to prove that $\langle \psi, x \rangle \leqslant \mathcal{R}_{\psi^\bullet}(x)$. For each $i \in I$, we consider the bounded measurable process $\xi^i := \mathbf{1}_{\{\psi^i = \psi\}}$ and we let $n := \sum_i \xi^i$. Note that $n \geqslant 1$. We let x^i be the optional random measure in E_+ defined by

$$dx^i(t) = \chi^i(t) dx(t) \quad \text{where} \quad \chi^i = \xi^i n^{-1}.$$

It is straightforward to check that $\langle \psi^i, x^i \rangle = \langle \psi, x^i \rangle$. Since $\sum_i x^i = x$ we have that

$$\mathcal{R}_{\psi^\bullet}(x) \geqslant \sum_{i \in I} \langle \psi^i, x^i \rangle = \sum_{i \in I} \langle \psi, x^i \rangle = \langle \psi, x \rangle,$$

which implies that $\mathcal{R}_{\psi^\bullet}(x) = \langle \psi, x \rangle$. □

Condition (iii) of Proposition 1 follows from Claim 2.

A.2 Proof of Theorem 2

Let H be a linear subspace of F and let $(V^\bullet, e^\bullet)$ be a H-proper economy satisfying Assumptions C and M. Then there exists a Hausdorff locally convex topology τ such that

(a) the economy $(V^\bullet, e^\bullet)$ is τ-proper,
(b) the τ-topological dual $(E, \tau)'$ is a subspace of H,
(c) for every family $(\pi^i, i \in I)$ of τ-continuous linear functionals $\pi^i \in (E, \tau)'$ the supremum $\sup \psi^\bullet$ still belongs to H.

Let $x^\bullet$ be an Edgeworth equilibrium of the economy $\mathcal{E}$. Since utility functions are concave, the allocation $x^\bullet$ is in fact an Aubin equilibrium. In particular

$$0 \notin \mathcal{G} := \operatorname{co} \bigcup_{i \in I} \left[P^i(x^i) - \{e^i\} \right].$$

We introduce the order ideal $E(e)$ defined by

$$E(e) = \bigcup_{\lambda > 0} \lambda[-e, e] = \{x \in E : \exists \lambda > 0, \quad -\lambda e \leqslant x \leqslant +\lambda e\}.$$

Notice that when $E(e)$ is equipped with the Riesz norm

$$\|x\|_e := \inf\{\lambda > 0 : \ -\lambda e \leqslant x \leqslant +\lambda e\},$$

the order intervals $\lambda[-e, e]$, for each $\lambda > 0$ form a basis for the 0-neighborhoods of this topology, thus e is $\|.\|_e$-interior point of the positive cone $E_+(e) := E(e) \cap E_+$. Note that for each $i \in I$, $\{e^i, x^i\}$ is a subset of $E_+(e)$.

Claim 3 *There exists a linear functional $p : E(e) \to \mathbb{R}$ such that*

$$\forall i \in I, \quad p(x^i) = p(e^i) > 0 \quad \textit{and} \quad p(y) > p(x^i), \quad \forall y \in E_+(e) \cap P^i(x^i). \tag{6}$$

Proof From Assumption M we have that $x^i + e^i + E_+$ is a subset of $P^i(x^i)$. Hence $e + E_+(e)$ is a subset of $\mathcal{G} \cap E(e)$ and $(2/\#I)e$ is $\|.\|_e$-interior point of $\mathcal{G} \cap E(e)$. Applying a convex separation theorem, there exists a non-zero linear functional[11] $p : E(e) \to \mathbb{R}$ such that for every $y \in \mathcal{G} \cap E(e)$, $p(y) \geqslant 0$ and such that $p(e) > 0$. In particular,

$$\forall i \in I, \quad \forall y \in E(e) \cap P^i(x^i), \quad p(y) \geqslant p(e^i). \tag{7}$$

From Assumption M we have that for every $t > 0$, $x^i + te^i \in E(e) \cap P^i(x^i)$. Applying (7) we get that $p(e^i) \geqslant 0$ and $p(x^i) \geqslant p(e^i)$. But $\sum_{i \in I} x^i = \sum_{i \in I} e^i$, which implies that

$$\forall i \in I, \quad p(x^i) = p(e^i). \tag{8}$$

Since $p(e) = \sum_{i \in I} p(e^i)$ there exists $j \in I$ such that $p(e^j) > 0$. Now let $i \in I$. We know that $p(e^i) \geqslant 0$ and we claim that $p(e^i) > 0$. Assume by way of contradiction that $p(e^i) = 0$. Then for any $\alpha \in (0, 1)$, $p(\alpha[x^j + e^i]) < p(x^j) = p(e^j)$.

[11] In fact p is $\|.\|_e$-continuous.

But from strong desirability of e^i, $V^j(x^j + e^i) > V^j(x^j)$. Hence by concavity of V^j, there exists $\alpha \in (0, 1)$ such that $V^j(\alpha[x^j + e^i]) > V^j(x^j)$, which yields a contradiction. Therefore we have proved that

$$\forall i \in I, \quad \forall y \in E(e) \cap P^i(x^i), \quad p(y) \geqslant p(x^i) \quad \text{and} \quad p(x^i) = p(e^i) > 0. \tag{9}$$

Let $i \in I$ and $y \in E(e) \cap P^i(x)$. In order to get (6) we have to prove that $p(y) > p(x^i)$. Assume by way of contradiction that $p(y) = p(x^i)$, then

$$\forall \alpha \in (0, 1), \quad p(\alpha y) < p(x^i). \tag{10}$$

Since $V^i(y) > V^i(x^i)$ and V^i is concave, there exists $\beta \in (0, 1)$ such that

$$\forall \alpha \in [\beta, 1), \quad V^i(\alpha y) > V^i(x^i). \tag{11}$$

Relations (10) and (11) are in contradiction with (9). □

Let us recall a convexity result due to Podczeck (1996). For a detailed proof, see Aliprantis, Florenzano, and Tourky (2004).

Lemma 3 *Let (L, τ) be an ordered topological vector space, let M be a vector subspace of L (endowed with the induced order), let Y be an open and convex subset of L such that $Y \cap M_+ \neq \emptyset$ and let $y \in M_+ \cap \operatorname{cl} Y$. If p is a linear functional on M satisfying*

$$p(y) \leqslant p(z), \quad \forall z \in Y \cap M_+,$$

then there exists some $\pi \in (L, \tau)'$ such that

$$\forall z \in M_+, \quad \pi(z) \leqslant p(z) \quad \textit{and} \quad \forall z \in Y, \quad p(y) = \pi(y) \leqslant \pi(z).$$

Applying Lemma 3 with $L = E$, $M = E(e)$, Y the τ-interior of $\widehat{P}^i(x^i)$, and $y = x^i$, we obtain[12] for each $i \in I$, a τ-continuous linear functional $\pi^i = \langle \psi^i, . \rangle$ in $(E, \tau)'$, where $\psi^i \in H$ and such that

$$\forall z \in E_+(e), \quad \langle \psi^i, z \rangle \leqslant p(z) \tag{12}$$

and

$$\forall y \in \widehat{P}^i(x^i), \quad \langle \psi^i, y \rangle \geqslant \langle \psi^i, x^i \rangle = p(x^i). \tag{13}$$

From Assumption M and (13) we get that for each $i \in I$, the individual price ψ^i belongs to H_+. Let $\psi = \sup \psi^\bullet$, then from condition (c) of the H-properness property, the price ψ belongs to H_+.

[12] If $z \in \widehat{P}^i(x^i) \cap E_+(e)$ then, since A_{x^i} is radial at x^i, there exists $t > 0$ such that $tz + (1-t)x^i$ belongs to $P^i(x^i)$. Therefore $tp(z) + (1-t)p(x^i) \geqslant p(x^i)$, i.e. $p(z) \geqslant p(x^i)$.

Claim 4 *The pair $(\psi, x^\bullet)$ is an Arrow–Debreu equilibrium of $\mathcal{E}$.*

Proof Following Proposition 1 we have the following relation

$$\forall z \in E_+, \quad \langle \psi, z\rangle = \sup\left\{\sum_{i\in I}\langle \psi^i, z^i\rangle : \sum_{i\in I} z^i = z \quad \text{and} \quad z^i \in E_+\right\}. \tag{14}$$

From (12) we have that for every $z \in E_+(e)$, $\langle \psi, z\rangle \leqslant p(z)$. In particular we have $\langle \psi, x^i\rangle \leqslant p(x^i)$. But from (13) we have that $\langle \psi, x^i\rangle \geqslant \langle \psi^i, x^i\rangle = p(x^i)$. Therefore we have

$$\forall i \in I, \quad \langle \psi, x^i\rangle = p(x^i).$$

Moreover since $e = \sum_{i\in I} e^i = \sum_{i\in I} x^i$ and $p(x^i) = p(e^i)$, we get from (14) that

$$\forall i \in I, \quad \langle \psi, e^i\rangle = p(e^i) = p(x^i) = \langle \psi, x^i\rangle. \tag{15}$$

Now fix $y \in P^i(x^i)$. From property (iii) of τ-properness the vector y belongs to $\widehat{P}^i(x^i)$. Applying (13) we have

$$\langle \psi, y\rangle \geqslant \langle \psi^i, y\rangle \geqslant p(x^i) = \langle \psi, e^i\rangle.$$

Since $\langle \psi, e^i\rangle = p(e^i) > 0$, following the same argument as in the proof of Claim 3 we actually have that

$$\forall y \in P^i(x^i), \quad \langle \psi, y\rangle > \langle \psi, e^i\rangle. \tag{16}$$

Claim 4 follows from (15) and (16). □

A.3 Proof of Lemma 2

Proof The proof of part (a) follows almost verbatim Bank and Riedel (2000). Let $(x_n, n \in \mathbb{N})$ be a sequence in E_+ norm-converging to $x \in E_+$. From Lemma 1 there exists a subsequence $(x_{n_k}, k \in \mathbb{N})$ and $\Omega^* \in \mathcal{F}$ with $\mathbb{P}\Omega^* = 1$ such that for every $\omega \in \Omega^*$, the sequence $(x_{n_k}(\omega), k \in \mathbb{N})$ converges weakly to $x(\omega)$ in M_+. Note that, for any fixed $t \in \{\Delta x = 0\}$, the function

$$s \longmapsto e^{\beta s}\mathbf{1}_{[0,t]}(s)$$

is continuous dx-a.e. Hence, the Portemanteau Theorem yields

$$\lim_{k\to\infty} \phi(x_{n_k})(t) = \phi(x)(t)$$

for all such t. In particular, we have that the sequence $(\eta_k, k \in \mathbb{N})$ converges $[\mathbb{P} \otimes \kappa]$-a.e. to η where

$$\forall t \in [0, T], \quad \eta_k(t) = \phi(x_{n_k})(t) \quad \text{and} \quad \eta(t) = \phi(x)(t).$$

Moreover

$$\forall t \in [0, T], \quad |\eta_k(t)| = \int_{[0,t]} e^{\beta s} dx_{n_k}(s) \leqslant e^{\beta T} x_{n_k}(t) = e^{\beta T}|x_{n_k}(t)|$$

which implies that the sequence $(\eta_k, k \in \mathbb{N})$ is $[P \otimes \kappa]$-uniformly integrable. Therefore

$$\lim_{k\to\infty} \rho(x_{n_k} - x) = \lim_{k\to\infty} \mathbb{E} \int_{[0,T]} |\eta_k(t) - \eta(t)| \kappa(dt) = 0.$$

Similarly we can prove that if $(x_n, n \in \mathbb{N})$ is a sequence in $E_+ \rho$-converging to $x \in E_+$ then there exists a subsequence $(x_{n_k}, k \in \mathbb{N})$ which is norm-converging to x.

We propose now to prove part (b). Let $p \in (E, \rho)'$. Since $\rho(x) = \|\phi(x)\|$ the linear mapping $\pi := p \circ \phi^{-1}$ belongs to $(E, \|.\|)'$. From Proposition 5 in Hindy and Huang (1992), there exists an optional process $\Phi = A + M$ where M is a bounded martingale and A is an absolutely continuous process whose derivative A' is bounded and such that $\pi = \langle \Phi, . \rangle$. Therefore for every $x \in E$,

$$p(x) = \pi\,[\phi(x)] = \langle \Phi, \phi(x) \rangle = \mathbb{E} \int_{[0,T]} \Phi(t) e^{\beta t} dx(t).$$

It then follows that $p = \langle \psi, . \rangle$ where $\psi(t) = [A(t) + M(t)] \exp\{\beta t\}$. Apply Itô's Lemma to $M(t)\exp(\beta t)$ to get

$$M(t)\exp(\beta t) = M(0) + \int_0^t \beta \exp(\beta s) M(s) ds + \int_0^t \exp(\beta s) dM(s).$$

Note that the quadratic covariation between a martingale and a smooth process is always zero. It follows that $M(t)\exp(\beta t)$ is the sum of the bounded martingale $N(t) = M(0) + \int_0^t \exp(\beta s) dM(s)$ and the absolutely continuous process $B(t) = \int_0^t \beta \exp(\beta s) M(s) ds$ with bounded derivative. □

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