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On equilibrium prices in continuous time ☆

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Abstract

State prices are the fundamental building block for dynamic asset pricing models. We provide here a general continuous-time setup that allows to derive non-trivial structural properties for state-prices from economic fundamentals. To this end, we combine general equilibrium theory and *théorie générale* of stochastic processes to characterize state prices that lead to continuous price systems on the consumption set. We also show that equilibria with such state prices exist.

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1. Introduction

Rational asset prices are the expected present value of future cash flows, properly discounted with a state price, also known as stochastic discount factor, or Arrow price. In economic terms, the state price $\psi_t(\omega)$ is the price of one unit of the numéraire consumption good (or money) delivered at time t when state ω prevails. In equilibrium, the price of a financial asset is obtained by valuing the asset's payment stream with the state price. State prices determine asset prices, interest rates, and the pricing or equivalent martingale measure. It is thus important to understand what economic theory can say about the properties of such state prices.

We study this question in a continuous-time model under uncertainty. If in equilibrium, financial markets are dynamically complete then the Arrow–Radner financial market equilibrium is also an Arrow–Debreu equilibrium. In this paper, we will characterize the structure of possible Arrow–Debreu price functionals and provide an existence result. In general, an Arrow–Debreu (efficient) equilibrium can be implemented as an Arrow–Radner financial equilibrium only if financial markets are dynamically complete. In the continuous-time setting that we analyze there are no general results that ensure that markets are dynamically complete and that existence of Arrow–Radner financial market equilibria follows from existence of Arrow–Debreu equilibria. On the other hand, existence of Arrow–Debreu equilibria certainly seems a necessary condition for the existence of financial market equilibria with endogenously complete markets. The (so far unresolved) question of financial equilibrium existence with dynamically complete (or even incomplete) financial markets is left for further research.[3]

To start with, let us consider consumption plans. We think that it is important not to exclude ex-ante any patterns that might arise from optimal behavior of rational agents. Consumption can occur in rates, as it does in the usual time-additive formulation for intertemporal consumption (Samuelson [48] and Merton [42]), but also in gulps or even in a more sophisticated, singular way, e.g. when the cumulative consumption is the running maximum of a diffusion (as it is optimal in Hindy and Huang [32], Bank and Riedel [18], or in models of durable goods, Abel and Eberly [1]). We follow the approach taken by Hindy, Huang and Kreps [33] and let cumulative consumption plans be non-decreasing processes. Under uncertainty, they are adapted to the given information filtration, of course.

Having fixed our consumption set, we can ask: what is a price? On competitive markets, it should be a linear function; and given that consumption is desirable (what we do not question here), prices should be positive. Clear as this is, linearity and positivity alone are not very helpful to derive structure for the price functional in our model.[4]

Competition yields linearity, and desirability leads to positivity. What else can we reasonably assume for rational agents? Well, some degree of continuity for preferences—small changes of plans should not affect preferences too much. We can thus ask what kind of plans are or should be considered as close by rational agents. One possible approach is the classical time-additive model: agents consume in rates $(c(t))$ that are adapted, suitably integrable processes,

[3] We believe that our result on the structure of equilibrium state prices might help to identify the right space to look for existence.

[4] In simpler models, it yields already some structure: if consumption plans are random variables in some Banach lattice, say L^p, then positive, linear functionals are continuous, and prices would be in the topological dual (see Aliprantis and Border [2, Theorem 8.6]). This approach does not work here because we will use (for economic reasons) a topology that does not turn the commodity space into a Banach lattice. Moreover, we want more than state prices being merely integrable.

and preferences are assumed to be norm—or Mackey—continuous on the consumption set. The literature shows that equilibrium state prices are non-negative, measurable, adapted processes that satisfy some degree of integrability.[5] In particular, we do not get continuous sample paths, or state prices that are diffusions (as one would like to have to justify a Samuelson–Black–Scholes-type model of asset markets). Another implausible consequence of working with this strong topology is that, when one would apply such a model to finance, bond prices need not depend continuously on maturity. A term structure would not exist.

The typical way out of this lack of structure is to impose additional assumptions on preferences and endowments. Usually,[6] it is assumed that endowments are Itô processes, and that agents have time-additive smooth expected utility functions. From the first-order conditions of utility maximization, a representative agent argument, and Itô's lemma, one obtains equilibrium state-prices that are diffusions. Here, we want to know how much one can say without imposing exogenously so much structure on endowments and preferences.

Let us consider the question of closeness of consumption plans again. This question has been discussed at length in Hindy, Huang and Kreps [33] for the deterministic, and Hindy and Huang [31] for the stochastic case. They convincingly argue that small *shifts over time* should not affect the preferences of a rational agent too harshly. After all, most agents do not care much about getting their retirement payments in twenty five years from now or twenty five years plus or minus one day.[7] In the eyes of the strong topology on rates considered above, such shifts induce a large change. The norm—or Mackey—topology on consumption rates in some L^p-space is so strong along the time dimension that an agent with continuous preferences considers consumption at nearby dates as perfect non-substitutes. This is the economic reason for the fact that equilibrium state prices can be quite discontinuous with this topology—when agents consider consumption as close dates as "independent", equilibrium theory cannot yield "similar" prices for them. This topology is too strong to yield a reasonable notion of when different consumption patterns should be "close". Hindy and Huang [31] (HH) (see also Hindy, Huang and Kreps [33] for the certainty case) thus claim that a rational agent should treat small up or downward shifts of consumption at a specific point in time as negligible deviations, of course; but both from an economic as well as a finance perspective, she should also react smoothly to small shifts of (lifetime) consumption plans over time except possibly when there is a significant change of information. In financial terms, bonds maturing at close dates should be almost perfect substitutes. In this paper, when preferences satisfy such intuitive economic properties, we should say that they exhibit *intertemporal local substitution*. Technically, the *intertemporal topology* is the weak*-topology for distribution functions on the time axis plus an L^p-topology for uncertainty. Indeed, Hindy and Huang [31] show that this topology captures exactly the economically sensible notion of intertemporal local substitution.

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[5] See the overview by Mas-Colell and Zame [41] for an account of infinite-dimensional general equilibrium theory. A classic in this regard is the first part of the existence proof in Duffie and Zame [28], see also Karatzas, Lehoczky and Shreve [36]. For further developments, consult Dana [20], Zitkovic [54], and Anderson and Raimondo [10], among others.

[6] See Duffie and Zame [28], Karatzas, Lehoczky and Shreve [35], more recently Anderson and Raimondo [10], to cite a few prominent examples.

[7] They do care, by the way, if they know in advance that there will be a huge shock to the economy exactly on that day. This is related to the question of information surprises discussed below. In most situations, it will be plausible to exclude such knowledge, at least for the far future.

In this paper we propose to address the following issue: What form would equilibrium prices take when individuals in the economy have preferences that are continuous with respect to the intertemporal topology?

Hindy and Huang [31] argue that equilibrium prices are elements of the topological dual space, i.e., they are linear functionals that are *continuous on the commodity space* for the intertemporal topology. They characterize the elements of the dual: they are given by state prices that are the sum of a martingale and some absolutely continuous process. However, they are unable to show that equilibria with such prices exist; and indeed, in general, it will be impossible to find equilibria with such prices (see our Example 1 below). What is the problem?[8] Note that the economic considerations alone yield continuity on the *consumption set*, the positive cone of the commodity space, not necessarily on the whole commodity space. Normally, this subtle difference does not play a role because a linear function Ψ that is continuous on the positive cone is also continuous on the whole space: a typical element of the commodity space X is decomposed into its positive and negative part, $X = X^+ - X^-$, and $\Psi(X) = \Psi(X^+) - \Psi(X^-)$ gives continuity on the whole space. This is not true in our model because the lattice operations $(X \mapsto X^+, X^-)$ are not continuous with respect to the intertemporal topology.

This discussion shows that we can only expect equilibrium prices to be continuous on the consumption set. When we want to know more about the structure of equilibrium prices, we thus have to answer the question: What are the positive, linear functionals that are continuous on the consumption set for the intertemporal topology?

We call a state price process defining a linear functional that is positive and continuous on the consumption set for the intertemporal topology a *compatible state price* since it is compatible with the intertemporal local substitutability of agents' preferences. The main contribution of this paper is a full characterization of compatible state prices. This result is presented in Section 2. We prove that a compatible state price process $(\psi(t))$ is the optional projection of a *continuous, but not necessarily adapted* stochastic process $(\xi(t))$. Intuitively, there is a shadow price process that has continuous sample paths. This shadow price process is not necessarily adapted to the available information. The agents thus take the conditional expectation

$$\psi(t) = \mathbb{E}\big[\xi(t)\big|\mathcal{F}_t\big]$$

as the state price. The state price inherits regularity from the shadow price, of course. We show that its sample paths are right-continuous with left hand limits; it need no longer be continuous, though. State prices can jump.

Jumps may come from the gradual release of information under uncertainty.[9] To give an example, one might have for every time $t \leqslant T$ that $\xi_t = Z$ for an $\mathcal{F}_T$-measurable random variable so that the process (ξ_t) is constant in time, but not adapted. The optional projection is then the martingale $\psi_t = \mathbb{E}[Z|\mathcal{F}_t]$ which can jump when information surprises occur. An information surprise can stem from a sudden shock to economic fundamentals (technically, like a jump from a Poisson process, e.g.). Another source of information surprises are discontinuities in the information filtration. A typical economic example are policy changes of the federal reserve: we

[8] Technically, the problem is that the dual is not a lattice which is a crucial property for the standard existence proof to work. Bank and Riedel [17] and Martins-da-Rocha and Riedel [38] have addressed this issue by proving existence in larger price spaces. Here, we are not interested in the more technical question of proving existence in some space, but rather what is the right economic space to expect prices to be in.

[9] This was already pointed out by Hindy and Huang [31].

know in advance when they occur, but we do not know what decision will be taken. The martingale example above was already discussed in Hindy and Huang [31]. Martingales are known to jump only at surprises. Our compatible state prices can be much more complicated than just martingales plus possibly an absolutely continuous process of bounded variation. The natural question is then whether we can also have jumps even without information surprises. We show that this is not true: a compatible price may have jumps only at information surprises. When there is no surprise, the optional projection of a process coincides with its predictable projection. The intertemporal topology entails that the predictable projection is just the left-continuous version of $(\psi(t))$. Hence, at times of no surprise, sample paths are continuous. In particular, if information is revealed continuously (which is the case when the information structure is generated by a Brownian motion) then compatible prices have even continuous sample paths.

HH showed that prices in the dual space are Itô processes when the filtration is generated by a Brownian motion. Compatible prices do not satisfy this important property. To illustrate this, we propose a simple example of an economy where the Arrow–Debreu equilibrium price process (Ψ_t) is given by $\Psi_t = \exp\{W_t^+\}$ where W_t^+ is the positive part of a standard Brownian motion. Actually compatible price processes may display very unpleasant properties. We provide an example of an Arrow–Debreu exchange economy where a representative agent has *Hindy–Huang–Kreps* preferences. The felicity function is assumed to depend on an unobservable shock. Despite the fact that preferences exhibit intertemporal local substitution, we show that the economy admits an equilibrium price process which is compatible but is not a semi-martingale. Unfortunately, and contrary to what is claimed by HH, the intertemporal local substitution property of preferences is not a sufficient condition to provide an equilibrium foundation for the application of stochastic calculus in mathematical finance.

Section 3 is devoted to the relation between compatible prices and competitive equilibrium prices in a dynamically complete market. We consider a pure exchange economy with continuous-time and uncertainty populated by a finite set of heterogeneous agents that may have incentives to share risks. Following Arrow and Debreu [14] we assume a perfect model without any friction: there is no asymmetric or incomplete information (in particular no problems of moral hazard or adverse selection). Moreover, there is no transaction or entry cost, no tax system and no restriction is imposed on enforcement and monitoring technologies available to the society. In such a model, nothing prevents agents to exchange any contingent contract.[10] Since time is continuous, we have an uncountable family of possible state and time contingent contracts and all trade can be subsumed at the initial date. If we feel uncomfortable with a model where a possibly infinite number of markets open only at the first date, one may follow the approach proposed by Arrow [15] who shows that equilibrium allocations of the Arrow–Debreu model can be implemented by a more realistic structure of markets where a limited number of securities can be traded through a sequential opening of markets. This transition is not without cost, since it requires that participants of the economy have rational expectations. Duffie [25] (see also Duffie and Huang [26]) generalizes Arrow's approach to the continuous-time framework. Duffie proved that, under additional assumptions on the information structure, there exists a finite family of securities such that, at equilibrium, any contingent contract can be replicated through continuous trading of these securities. One should stress that in Duffie [25] the dividend structure of the securities that endogenously completes the market is endogenously defined, i.e., it depends on

[10] Our model should be contrasted with models that assume a particular and exogenous asset markets structure without explicitly modeling the microeconomic foundation for this restriction. We refer, among others, to Radner [44], Magill and Quinzii [37] and the literature therein.

the primitives of the economy.[11] In particular, the participants of the economy should be able to perfectly compute this security structure. We consider that this is not a restrictive assumption once we assume that agents are sufficiently skilled to have rational expectations. Indeed, we share Radner's point of view (see the discussion in Radner [45, vol. IV, p. 940]) that the rational expectation hypothesis imposes a strong rationality on the participants of the economy: agents are assumed to fully understand the economic environment (including preferences and endowments of other agents).[12]

We propose two contributions relating compatible state prices and competitive equilibria. We first show that any compatible price can be implemented as a competitive equilibrium. More precisely, we consider a representative agent with *Hindy–Huang–Kreps preferences*, i.e., the preference relation is defined by a utility function of the form

$$V(x) = \mathbb{E} \int_{[0,T]} u\big(t, Y(x)(t)\big)\, dt$$

where u is a felicity function and $Y(x)(t)$ describes the investor's level of satisfaction obtained from his consumption up to time t. It was proved by Bank and Riedel [17] that such preferences exhibit intertemporal local substitution. We show that for any compatible state price process (Ψ_t) we can construct an endowment such that (Ψ_t) is a competitive equilibrium of the associated economy. In other words, any compatible state price can be observed as an equilibrium outcome in an economy where agents have intertemporal local substitution.

The second contribution of Section 3 is an existence result. We show that any economy exhibiting intertemporal local substitution and a suitable (local) properness condition[13] admits a competitive equilibrium state price that is compatible. The complete characterization of compatible prices that we propose plays a crucial role in the proof. It allows us to substantially improve the existence results in Bank and Riedel [17] and Martins-da-Rocha and Riedel [38]. Indeed our properness assumption is less restrictive and more importantly, we do not need to assume that the information structure is quasi-left-continuous. This is an assumption on the way new information is revealed to the agents. Economically, an information flow corresponds to a quasi-left-continuous filtration if information surprises occur only at times which cannot be predicted.[14] However, from an economic perspective it is important not to restrict the information structure to be quasi-left-continuous. Indeed, many economic announcements (the policy change

[11] It is also clear that financial markets need not be complete if we start with a fixed set of securities. This is obvious in the diffusion case when the number of securities is smaller than the number of independent Brownian motions generating the information flow. In a general semimartingale setting, one would already need an infinity of assets to complete the market because there is usually no finite martingale basis then (for example, if we have a jump-diffusion with exponentially distributed jumps).

[12] Anderson and Raimondo [10] follow the different approach proposed by Radner [44] and Magill and Quinzii [37]. Although they do not model explicitly the reasons why agents cannot design the appropriate securities exhibited by Duffie, they assume that the set of contracts agents can arrange is restricted to an exogenously given family. They provide an exogenous non-degeneracy condition on the security dividends to ensure that markets are dynamically complete. Anderson and Raimondo do not assume intertemporal local substitution of agents preferences. It is not the issue that we address in this paper but we consider that it would be interesting to know whether the structure for state prices that we obtain under intertemporal local substitutability gives useful information for decentralizing the Arrow–Debreu equilibrium allocation through trading in an exogenously given security structure.

[13] Such properness is necessary in infinite-dimensional models, see Mas-Colell and Zame [41] for a discussion.

[14] See Hindy and Huang [31] for a precise definition. An information flow generated by a Brownian motion or a Poisson process is quasi-left-continuous.

of the Federal reserve or profit announcements in stock exchange markets) are examples of an information surprise which occurs at a time known in advance.

Before we conclude the introduction, we would like to mention a technical contribution of the paper. The method to derive the characterization relies on the *théorie générale* of stochastic processes as developed in Dellacherie and Meyer [24]. It turns out that much in the same way as Itô's theory of stochastic integration is taylor-made for the Samuelson–Black–Scholes theory of asset markets, the *théorie générale* suits our general theory for equilibrium state prices. This is the first connection of general equilibrium theory and *théorie générale*, and we hope that more interesting results can spring from this relation in the future.

The paper is organized as follows. The next section sets up the intertemporal model and presents the main contribution of the paper: the characterization of compatible prices. In particular we show which properties of the state prices are or are not inherited from the intertemporal local substitution of preferences. In Section 3 we show that any compatible price can be implemented as an equilibrium and we establish existence of equilibria with such prices. The last section contains the detailed proof of the characterization result.

2. Prices compatible with local substitutability

We consider a stochastic pure exchange economy where a finite set I of agents live in a world of uncertainty from time 0 to time T. Uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each $\omega \in \Omega$ is a state of nature which is a complete description of one possible realization of all exogenous sources of uncertainty from time 0 to time T. The sigma-field $\mathcal{F}$ is the collection of events which are distinguishable at time T and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is endowed with a filtration $\mathbb{F} = \{\mathcal{F}(t)\colon t \in [0, T]\}$ which represents the time evolution of the agents' knowledge about the states of nature. We assume that $\mathcal{F}(0)$ is $\mathbb{P}$-almost surely trivial and that $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness. A process is said optional if it is $\mathcal{O}$-measurable where $\mathcal{O}$ is the sigma-field on $\Omega \times [0, T]$ generated by right-continuous $\mathbb{F}$-adapted processes with left-limits.

2.1. Consumption set

There is a single consumption good available for consumption at any time $t \in [0, T]$. The set of positive, non-decreasing and right-continuous functions from $[0, T]$ to $\mathbb{R}_+$ is denoted by M_+. We represent the consumption bundle of an agent by a process $x : (\omega, t) \mapsto x(\omega, t)$, where $x(\omega, t)$ (sometimes denoted by $x_t(\omega)$) represents the cumulative consumption from time 0 to time t and satisfies

(a) for each $\omega \in \Omega$, the function $x(\omega)$ belongs to M_+,
(b) for each $t \in [0, T]$, the random variable x_t is $\mathcal{F}(t)$-measurable and x_T belongs to $L^p(\mathbb{P})$

where $1 \leqslant p < +\infty$.

The set of ($\mathbb{P}$-equivalent classes of) mappings $x : \Omega \to M_+$ such that the process $(\omega, t) \mapsto x(\omega, t)$ satisfies (a) and (b) is denoted by E_+ and the linear span of E_+ will be denoted by E. The space E_+ is called the consumption set and E is called the commodity space. Observe that any consumption bundle x in E_+ is an $\mathbb{F}$-adapted process having right-continuous and bounded variation sample paths and therefore can be identified with an optional random measure denoted by dx. If z belongs to E then there exist x, y in E_+ such that $z = x - y$. We can endow E with the linear order $\geqslant$ defined by the cone E_+ in the sense that $y \geqslant x$ if $y - x$ belongs to E_+. If y belongs to E_+ then the order interval $[0, y]$ is defined by $[0, y] := \{x \in E\colon x \in E_+ \text{ and } y - x \in E_+\}$. The space E endowed with the partial order defined by E_+ is a linear vector lattice (see Martins-da-Rocha and Riedel [38, Proposition 1]).

Remark 1. Observe that if x, y are vectors in E such that $y \geqslant x$ then there exists $\Omega^* \in \mathcal{F}$ with $\mathbb{P}\Omega^* = 1$ and such that for each $\omega \in \Omega^*$, the function $t \mapsto y(\omega, t) - x(\omega, t)$ is non-decreasing with $y(\omega, 0) - x(\omega, 0) \geqslant 0$. In particular we have for each $\omega \in \Omega^*$,

$$y(\omega, t) \geqslant x(\omega, t), \quad \forall t \in [0, T].$$

2.2. Topologies

Since $0 \leqslant x_t \leqslant x_T$ and $x_T \in L^1(\mathbb{P})$ for every $x \in E_+$, the space E is a subspace of $L^1(\mathcal{O}, \mathbb{P} \otimes \kappa)$ where $\kappa = \lambda + \delta_T$ with λ the Lebesgue measure on $\mathcal{B}$ the Borelian sigma-algebra on $[0, T]$ and δ_T the Dirac measure on T. Following Hindy and Huang [31] we consider on E the restriction of the $L^p(\mathcal{O}, P \otimes \kappa)$-norm, i.e., we consider the norm $\|\cdot\|$ defined by

$$\forall x \in E, \quad \|x\| = \left[\mathbb{E} \int_{[0,T]} \left|x(t)\right|^p \kappa(dt)\right]^{\frac{1}{p}} = \left[\mathbb{E} \int_{[0,T]} \left|x(t)\right|^p dt + \mathbb{E}\left|x(T)\right|^p\right]^{\frac{1}{p}}.$$

It is argued in Hindy and Huang [31] that this norm, called intertemporal norm, induces a topology on the set of consumption bundles that exhibits intuitive economic properties, in particular it captures the notion that consumption at adjacent dates are almost perfect substitutes except possibly at information surprises. Usually, we refer to the topology generated by the intertemporal norm when we speak about continuity, open sets, etc. Occasionally, we will use other topologies as well, though. For $z \in E$ and fixed $\omega \in \Omega$, the function $z(\omega)$ can be associated with a signed measure $d[z(\omega)]$ on the time interval $[0, T]$. We denote by $\|z(\omega)\|_{\mathrm{tot}}$ the total variation of the measure $d[z(\omega)]$ (and we will drop the ω frequently, as usual). The expectation of the total variation of z leads to the strong topology on E as given by the norm

$$\|z\|_{\mathrm{s}} := \mathbb{E}\|z\|_{\mathrm{tot}}.$$

Note that convergence in the strong topology entails convergence in the intertemporal topology. Moreover, E is a topological vector lattice when endowed with the order generated by E_+ and the strong topology.

Denote by $\mathcal{T}$ the set of all stopping times $\tau \leqslant T$. If h is a random variable and τ a stopping time in $\mathcal{T}$, we denote by $\delta_\tau h$ the simple random measure that delivers $h(\omega)$ units of the consumption good at time $\tau(\omega)$ and nothing elsewhere. In particular δ_τ is the Dirac measure on τ.

2.3. Prices

The weakest notion of a price is that of a non-negative linear functional on E. The algebraic dual (the space of linear functionals from E to $\mathbb{R}$) is denoted by $E^\star$ and $E^\star_+$ denotes the cone of non-negative linear functionals, i.e., $\pi \in E^\star$ is non-negative if $\pi(x) \geqslant 0$ for every $x \in E_+$. If $B(T)$ denotes the space of bounded functions defined on $[0, T]$ then we let $L^q(\mathbb{P}, B(T))$ denote the space (up to $\mathbb{P}$-indistinguishability) of all $\mathcal{F} \otimes \mathcal{B}$-measurable processes $\psi : \Omega \times [0, T] \to \mathbb{R}$ such that the function

$$\omega \to \sup_{t\in[0,T]} \left|\psi(\omega,t)\right|$$

belongs to $L^q(\mathbb{P})$ where $q \in (1,+\infty]$ is the conjugate of p. There is a natural duality $\langle\cdot,\cdot\rangle$ on $L^q(\mathbb{P}, B(T)) \times E$ defined by

$$\langle\psi, z\rangle = \mathbb{E} \int_{[0,T]} \psi(t)\, dz(t).$$

The space of processes in $L^q(\mathbb{P}, B(T))$ that are optional is denoted by F and we denote by F_+ the order dual cone, i.e.,

$$F_+ := \bigl\{\psi \in F\colon \langle\psi, x\rangle \geqslant 0,\ \forall x \in E_+\bigr\}.$$

The pair $\langle F, E\rangle$ is a Riesz dual pair (see Martins-da-Rocha and Riedel [38, Proposition 1]) and a process $\psi \in F$ belongs to F_+ if and only if $\psi(t) \geqslant 0$ for every $t \in [0,T]$. To each non-negative process $\psi \in F_+$ we can consider the non-negative linear functional $\langle\psi,\cdot\rangle$ in $E^\star_+$ defined by

$$\forall z \in E, \quad \langle\psi, z\rangle = \mathbb{E}\int \psi\, dz.$$

By abuse of notations, we still denote F_+ (and F) the space of linear functionals associated to processes in F_+ (resp. F). If a price $\pi \in E^\star_+$ is represented by an optional process $\psi \in F_+$ in the sense that $\pi = \langle\psi,\cdot\rangle$, then the process ψ is called a state price. In that case the duality product $\langle\psi, x\rangle$ is the value of the consumption bundle $x \in E_+$ under the price ψ where $\psi(\omega,t)$ is interpreted to be the time 0 price of one unit of consumption at time t in state ω, per unit of probability.

2.4. Compatible prices: characterization

In general, prices in F_+ will not be compatible with the notion of intertemporal substitution as they might assign very different prices to consumption plans that are close in the intertemporal topology. One might therefore aim to find prices in the topological dual $(E, \|\cdot\|)'$ of E. As shown by Hindy and Huang [31], every linear functional $\pi \in (E, \|\cdot\|)'$ continuous for the intertemporal norm can be represented[15] by a semi-martingale ψ satisfying

$$\psi_t = A_t + M_t$$

where A is an adapted process with absolutely continuous sample path, i.e.,

$$A_t = A_0 + \int_{[0,t]} a_s\, d\lambda(s)$$

with

$$a \in L^q(\mathcal{O}, \mathbb{P}\otimes\kappa) \quad \text{and} \quad a_T \in L^q(\mathbb{P})$$

and M is the martingale defined by

$$M_t = \mathbb{E}[-a_T - A_T|\mathcal{F}_t], \quad \forall t \in [0,T].$$

[15] In the sense that $\pi = \langle\psi,\cdot\rangle$.

We denote by K the space of processes ψ representing $\|\cdot\|$-continuous linear functionals in $(E, \|\cdot\|)'$.[16] However, there is in general no hope to obtain equilibrium prices in K as it is not a lattice. On the other hand, all that counts for equilibrium theory are linear functionals restricted to the *consumption set* E_+, the positive cone of the commodity space. So we relax the requirement of continuity on the whole space and aim only for continuity on the consumption set E_+. A price is called *compatible* if it is continuous with respect to the intertemporal topology on E_+. Denote the set of compatible prices by H_+.

This leads to several important questions: What is the form of compatible prices? Which properties of such prices are or are not inherited from the intertemporal local substitution of preferences? What is the relation between compatible prices and competitive equilibrium prices in a dynamically complete market? Let us answer the first question.

Theorem 1. *A non-negative linear functional $\pi \in E_+^\star$ is a compatible price if and only if it can be written as*

$$\pi(x) = \mathbb{E}\int \psi\, dx, \quad \forall x \in E_+,$$

for a non-negative, right-continuous processes ψ with left-limits that satisfies the following conditions:

- *The process ψ is the optional projection*[17] *of a* (*not necessarily adapted*) *continuous process ξ with*

$$\mathbb{E} \sup_{t\in[0,T]} |\xi_t| < \infty.$$

- *The process*

$$\psi^* = \sup_{t\in[0,T]} \psi_t$$

belongs to $L^q(\mathcal{F}, \mathbb{P})$.

Proof. For the connoisseurs, we sketch parts of our representation theorem here. The complete and detailed proof of Theorem 1 is postponed to Section 4. We take a non-negative, linear price functional on the space of all optional random measures with total variation in L^p. Fixing τ in $\mathcal{T}$, we consider only the restriction on payment streams that pay off at time τ. This gives us a family $(\pi^\tau)_{\tau\in\mathcal{T}}$ of linear mappings from $L^p(\mathcal{F}_\tau)$ to the real numbers. With a fixed maturity, there are no issues of shifting etc., so that this mapping is norm-continuous; Riesz' theorem gives us a random variable $z^\tau \in L^q(\mathcal{F}_\tau)$ that represents this mapping. So we obtain a family of random variables $(z^\tau)_{\tau\in\mathcal{T}}$ where every z^τ is $\mathcal{F}_\tau$-measurable. We show that this large family is consistent in the sense that we have

$$z^\tau = z^\sigma$$

on the event $\{\tau = \sigma\}$ for two stopping times σ and τ. Such families are called $\mathcal{T}$-systems in the *théorie générale*. The question is if we can find an adapted stochastic process (ψ_t) such that

[16] Observe that K is a subset of F.
[17] That is $\psi_\tau = \mathbb{E}[\xi_\tau | \mathcal{F}_\tau]$ for every stopping time $\tau \leqslant T$.

$\psi_\tau = z^\tau$ for all stopping times τ. In this case, one says that (ψ_t) recollects the family $(z^\tau)_{\tau\in\mathcal{T}}$. It has been shown by Dellacherie and Lenglart [22] that a $\mathcal{T}$-system can be recollected if it is left-continuous in expectation[18] (and this kind of continuity is necessary, in general). Fortunately, the intertemporal topology we use here gives us even continuity in expectation. Note that we use here the shifting property of the intertemporal topology proposed by HH: as small shifts over time are considered as close in that topology, the price of a unit of the consumption good delivered at τ_n has to approach the price of one unit of the consumption good at τ if $\tau_n \to \tau$. The recollecting process (ψ_t) is our desired state price. From continuity in expectation, it is even cadlag.[19] And here comes the final clue: the cadlag process (ψ_t) is the optional projection of a *continuous, but not necessarily adapted* stochastic process (ξ_t); this is a result by Bismut [19] and Emery [29]. $\square$

2.5. Compatible prices: properties

We denote by H the space of right-continuous processes with left-limits that are bounded in L^q and are the optional projection of a continuous process bounded in L^1. We have proved that a non-negative linear functional π is compatible if and only if it can be represented by a process ψ in $H_+ = H \cap F_+$, i.e., $\pi = \langle \psi, \cdot\rangle$.

In general, compatible state prices can have jumps although they are the conditional expectation of a continuous, but maybe not adapted process. The occurrence of jumps can be traced back to the way information is revealed. Let us study this phenomenon in more detail. We start with our continuous, not adapted process ξ, let ψ be its optional projection, and let $\phi(t) = \psi(t-)$ be its left-continuous variant (with right-limits). As the price functional is continuous with respect to the intertemporal topology, it is continuous in expectation; in particular we have $\lim \mathbb{E}\psi(\tau_n) = \mathbb{E}\psi(\tau)$ for any increasing sequence τ_n of stopping times that converge to the stopping time τ. In the language of the *théorie générale*, ψ is regular, and hence, ϕ is the predictable projection of ψ and ξ, i.e.,

$$\mathbb{E}\big[\xi(t)\big|\mathcal{F}(t-)\big] = \mathbb{E}\big[\psi(t)\big|\mathcal{F}(t-)\big] = \phi(t),$$

see Dellacherie and Meyer [23, Chapter VI, 50(d)]. The jumps of ψ are the points in time when $\phi(t) \neq \psi(t)$. These jumps can be described by a countable set of stopping times (see Dellacherie and Meyer [23, Theorem 46]). We claim that jumps can only occur at *information surprises* τ.[20] To see this, let τ be a predictable stopping time announced by a sequence of stopping times (τ_n) and suppose that $\mathcal{F}(\tau-) = \bigvee_{n\in\mathbb{N}} \mathcal{F}(\tau_n) = \mathcal{F}(\tau)$. Then we have

$$\phi(\tau) = \psi(\tau-) = \mathbb{E}\big[\xi(\tau)\big|\mathcal{F}(\tau-)\big] = \mathbb{E}\big[\xi(\tau)\big|\mathcal{F}(\tau)\big] = \psi(\tau),$$

and ψ is continuous in τ.

---

[18] This is not sample-path left-continuity. A $\mathcal{T}$-system $(w^\tau)_{\tau\in\mathcal{T}}$ is left-continuous in expectation if $\mathbb{E}w^{\tau_n} \to \mathbb{E}w^\tau$ whenever $\tau_n \uparrow \tau$ a.s.

[19] This is again a classic result from the *théorie générale*, see Dellacherie and Meyer [24, Theorem 48].

[20] We follow Hindy and Huang [31] and call a stopping time τ a time of surprise if it is non-predictable or if it is predictable by a sequence of stopping times $\tau_n \uparrow \tau$ but the information filtration has a discontinuity in τ in the sense that $\mathcal{F}(\tau-) = \bigvee_{n\in\mathbb{N}} \mathcal{F}(\tau_n) \neq \mathcal{F}(\tau)$.

Let us compare the state prices that are identified in the above theorem with the state prices that Hindy and Huang [31] characterize and those that Bank and Riedel [17] use. In both papers, state prices are semi-martingales of the form

$$\psi_t = A_t + M_t,$$

where $M = (M_t)$ is a martingale and $A = (A_t)$ a continuous (Bank–Riedel) or even absolutely continuous (Hindy–Huang) process. Both classes of processes belong to H: if we let

$$\xi_t = M_T + A_t$$

then ξ has continuous sample paths, but is in general no longer adapted (because M_T is usually not known at time t). The optional projection of ξ satisfies

$$^o\xi_t = \mathbb{E}[M_T + A_t|\mathcal{F}_t] = M_t + A_t = \psi_t.$$

In particular, $K \subset H$.

Having identified the structure of compatible prices, the next obvious question is whether we really have equilibria with such prices. The next section will provide the positive answer to the equilibrium existence problem. As the space of compatible prices is quite large compared to the dual space identified by Hindy and Huang, one might hope that one might be able to find equilibrium prices in that space.[21] The following example shows that this is impossible in general.

Example 1. In general, equilibrium prices do not lie in the space K identified by Hindy and Huang [31]. To see this, let us take a concrete model where $(\mathcal{F}_t)$ is the (augmented) filtration of a Brownian motion $(W(t))$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two agents $i = 1, 2$ with linear preferences of the form

$$U^i(x^i) = \mathbb{E} \int_{[0,T]} \psi^i(t)\, dx^i(t)$$

for $\psi^1(t) = \exp(W(t))$ and $\psi^2(t) = 1$. Clearly, we have $\psi^i \in K$ for $i = 1, 2$. Let endowments be

$$e^1(t) = \int_0^t 1_{\{W_s > 0\}}\, ds$$

and

$$e^2(t) = \int_0^t 1_{\{W_s \leqslant 0\}}\, ds.$$

Let $\psi(t) = \max\{\psi^1(t), \psi^2(t)\} = \exp(W(t)^+)$. We claim that (ψ, e^1, e^2) is an Arrow–Debreu equilibrium (see Definition 1 below). Indeed, the gradient of agent i's utility is ψ^i. We have $\psi^1 \leqslant \psi$ everywhere and equality if $W(t) > 0$ which coincides with $de^1(t) > 0$. By Lemma 3.4

[21] Hindy and Huang [31, p. 797], express the hope that "further research produces more powerful existence theorems which can accommodate our model".

in Bank and Riedel [18], e^1 is optimal for agent 1. The same argument establishes optimality for agent 2. Hence, equilibrium is proved.[22] By Tanaka's formula, $W(t)^+ = M(t) + L(t)$ for a martingale M and local time L. The local time of Brownian motion is a non-decreasing, yet nowhere differentiable process. As a consequence,[23] $\psi \notin K$.

3. Equilibria with compatible prices

Each agent i is characterized by a utility function $V^i : E_+ \to \mathbb{R}$ which represents his preference relation on the space E_+ of consumption patterns and by a vector $e^i \in E_+$ which represents the cumulative income stream (initial endowment). An economy is a pair

$$\mathcal{E} = (\boldsymbol{V}, \boldsymbol{e})$$

where $\boldsymbol{V} = (V^i)_{i\in I}$ and $\boldsymbol{e} = (e^i)_{i\in I}$. We let $e = \sum_{i\in I} e^i$ denote the aggregate endowment and if $x \in E_+$ the set $\{y \in E_+ \colon\ V^i(y) > V^i(x)\}$ is denoted by $P^i(x)$. An allocation is a vector $\boldsymbol{x} = (x^i)_{i\in I}$ where $x^i \in E_+$. It is said feasible or attainable if $\sum_{i\in I} x^i = e$. The set of attainable allocations is denoted by $\mathcal{A}$.

3.1. Equilibrium concepts

We define hereafter the standard notion of Arrow–Debreu equilibrium.

Definition 1. The pair $(\psi, \boldsymbol{x})$ of a price process ψ and an allocation $\boldsymbol{x}$ is called an Arrow–Debreu equilibrium if

(a) the price process ψ belongs to F_+ and $\langle \psi, e\rangle > 0$;
(b) the allocation $\boldsymbol{x}$ is attainable, i.e., $\boldsymbol{x} \in \mathcal{A}$; and
(c) for each agent i, the consumption plan x^i maximizes agent i's utility over all consumption plans y satisfying the budget constraint $\langle \psi, y\rangle \leqslant \langle \psi, e^i\rangle$, i.e.,

$$x^i \in \operatorname{argmax}\bigl\{V^i(y) \colon\ y \in E_+ \text{ and } \langle \psi, y\rangle \leqslant \bigl\langle \psi, e^i\bigr\rangle\bigr\}.$$

A possible interpretation is that a complete set of markets open at the initial date $t = 0$ for consumption good delivery at any date in any state of nature. Markets are assumed to be competitive in the sense that agents take the price functional $\langle \psi, \cdot\rangle$ as given. Each agent can sell his initial endowment e^i and buy a consumption plan $x \in E_+$ as far as he can afford it, i.e., $\langle \psi, x\rangle \leqslant \langle \psi, e^i\rangle$. The real number $\langle \psi, x\rangle$ is interpreted as the price at time $t = 0$ of the consumption claim x, and therefore the real number $\psi(\omega, t)$ is interpreted as the time $t = 0$ price (per unit of probability) of the contract that promises to deliver one unit of the unique good at time t in state ω.

Remark 2. Observe that if $(\psi, \boldsymbol{x})$ is an equilibrium then the budget constraints are binding, i.e., for each i, we have $\langle \psi, x^i\rangle = \langle \psi, e^i\rangle$.

[22] The first welfare theorem implies that endowments are efficient here. The equilibrium is then unique.

[23] To see this rigorously, note that by Itô's formula, $\psi(t) = 1 + \int_0^t \psi(s) 1_{\{W_s>0\}}\, dW(s) + \int_0^t \psi(s)\, dL(s) + \frac{1}{2}\int_0^t \psi(s) 1_{\{W_s>0\}}\, ds$. The stochastic integral is a martingale, and the last integrand is absolutely continuous. The Stieltjes integral $\int_0^t \psi(s)\, dL(s)$ is non-decreasing, but not absolutely continuous.

As usual in the general equilibrium literature, we consider the following list of *standard* assumptions.

Assumption 1 *(C).* For each agent i,

(C.1) the initial endowment e^i belongs to E_+ and is not zero, i.e., $e^i > 0$,
(C.2) the utility function V^i is concave,
(C.3) the utility function V^i is norm-continuous.[24]

We recall a well-known property of optimality for allocations.

Definition 2. An attainable allocation $\boldsymbol{x} \in \mathcal{A}$ is said to be an *Edgeworth* equilibrium if there is no $0 \neq \boldsymbol{\lambda} \in (\mathbb{Q} \cap [0,1])^I$ and some allocation $\boldsymbol{y}$ such that $V^i(y^i) > V^i(x^i)$ for each i with $\lambda^i > 0$ and satisfying $\sum_{i \in I} \lambda^i y^i = \sum_{i \in I} \lambda^i e^i$.

The reader should observe that this concept is "price free" in the sense that it is an intrinsic property of the commodity space. It is proved in Martins-da-Rocha and Riedel [38] that every economy satisfying Assumption C admits an Edgeworth equilibrium. It is straightforward to check that every Arrow–Debreu equilibrium is an Edgeworth equilibrium. The main difficulty consists in proving the converse.

3.2. Properness of preferences and existence

We propose to follow the classical literature[25] dealing with infinite-dimensional commodity (and price) spaces by introducing the concept of proper economies. It is a well-known fact that without some properness hypotheses on preferences, equilibrium existence may fail when the positive cone of the commodity space has empty interior.

Definition 3 *(τ-properness).* Let τ be a Hausdorff locally convex linear topology on E. An economy $(\boldsymbol{V}, \boldsymbol{e})$ is τ-proper if for every Edgeworth equilibrium $\boldsymbol{x}$, for each i, there is a set $\widehat{P}^i(x^i)$ such that[26]

(i) the vector $x^i + e$ is a τ-interior point of $\widehat{P}^i(x^i)$,
(ii) the set $\widehat{P}^i(x^i)$ is convex and satisfies the following additional convexity property

$$\forall z \in \widehat{P}^i\big(x^i\big) \cap E_+,\ \forall t \in (0,1), \quad tz + (1-t)x^i \in \widehat{P}^i\big(x^i\big) \cap E_+,$$

---

[24] Actually, it is sufficient to assume that V^i is upper semi-continuous on the order interval $[0,e]$. That is, if $(x_n)_{n \in \mathbb{N}}$ is a sequence in $[0,e]$ which norm-converges to x in $[0,e]$, then

$$\limsup_{n \to \infty} V^i(x_n) \leqslant V^i(x).$$

[25] We refer, among others, to Mas-Colell [39], Richard and Zame [47], Yannelis and Zame [52], Aliprantis, Brown and Burkinshaw [3,4], Zame [53], Richard [46], Araujo and Monteiro [13], Mas-Colell and Richard [40], Mas-Colell and Zame [41], Podczeck [43], Anderson and Zame [11,12], Tourky [50], Deghdak and Florenzano [21], Tourky [51], Aliprantis, Tourky and Yannelis [8], Shannon and Zame [49], Florenzano [30], Aliprantis, Monteiro and Tourky [9], and Aliprantis, Florenzano and Tourky [5,6].

[26] Recall that $\boldsymbol{e}$ is the family $(e^i)_{i \in I}$ and e is aggregate endowment $e = \sum_{i \in I} e^i$.

(iii) we can extend preferences in the following way

$$\widehat{P}^i(x^i) \cap E_+ \cap A_{x^i} \subset P^i(x^i) \subset \widehat{P}^i(x^i) \cap E_+$$

where $A_{x^i} \subset E$ is a radial set at x^i.[27]

We say that an economy is *strongly* τ-proper if condition (iii) in Definition 3 is replaced by the following condition (iii′):

(iii′) we can extend preferences in the following way

$$\widehat{P}^i(x^i) \cap E_+ = P^i(x^i).$$

Strong τ-properness was introduced by Tourky [51] and is used, among others, by Aliprantis, Tourky and Yannelis [8], and Aliprantis, Florenzano and Tourky [5,6]. We refer to Aliprantis, Tourky and Yannelis [7] for a comparison of the different notions of properness used in the literature. Observe that if $\mathcal{E} = (\boldsymbol{V}, \boldsymbol{e})$ is an economy (satisfying the following monotonicity Assumption M) such that for each i, it is possible to extend V^i to a τ-continuous and concave function $\widehat{V}^i : E \to \mathbb{R}$, then the economy is τ-proper.[28] In other words, τ-properness can be seen as a strengthening of τ-continuity. Moreover, τ-properness is slightly weaker than strong τ-properness. However this slight difference is crucial in order to compare properness with the existence of smooth sub-gradients.[29] We borrow the following definition of smooth sub-gradients from Bank and Riedel [17] (see also Martins-da-Rocha and Riedel [38]). Recall that K is the space of processes in F that represent norm-continuous linear functionals on E.

Definition 4. An economy $(\boldsymbol{V}, \boldsymbol{e})$ has smooth sub-gradients in K if for each i, for every $x \in E_+$, there exists a non-negative optional process $\nabla V^i(x) \in K_+ = K \cap F_+$ with

(U.1) for each $j \in I$, we have $\langle \nabla V^i(x), e^j \rangle > 0$,
(U.2) the vector $\nabla V^i(x)$ satisfies the subgradient property

$$\forall y \in E_+, \quad V^i(y) - V^i(x) \leqslant \langle \nabla V^i(x), y - x \rangle,$$

(U.3) this subgradient is continuous in the sense that,

$$\forall y \in E_+, \quad \lim_{\varepsilon \downarrow 0} \langle \nabla V^i(\varepsilon y + (1-\varepsilon)x), y - x \rangle = \langle \nabla V^i(x), y - x \rangle.$$

The economy $(\boldsymbol{V}, \boldsymbol{e})$ is said to satisfy Property U if (U.1), (U.2) and (U.3) are satisfied.

Remark 3. Let $\mathcal{E} = (\boldsymbol{V}, \boldsymbol{e})$ be an economy. Preferences of agent i are said increasing if $V^i(x+y) \geqslant V^i(x)$ for every x, y in E_+; strictly increasing if $V^i(x+y) > V^i(x)$ for every x, y in E_+ with $y \neq 0$. Note that if $\mathcal{E}$ satisfies Property U, then preferences of agent i are increasing; they are strictly increasing if and only if $\nabla V^i(x)$ is strictly positive for every $x \in E_+$.

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[27] A subset A of E is radial at $x \in A$ if for each $y \in E$, there exists $\bar{\alpha} \in (0, 1]$ such that $(1-\alpha)x + \alpha y$ belongs to A for every $\alpha \in [0, \bar{\alpha}]$.

[28] Take $\widehat{P}^i(x) := \{y \in E\colon \widehat{V}^i(y) > \widehat{V}^i(x)\}$.

[29] See also Assumption A.7 in Shannon and Zame [49].

Remark 4. Let $(\boldsymbol{V}, \boldsymbol{e})$ be an economy satisfying Property U, then for each i, j in I, the initial endowment e^j is strongly desirable for agent i in the sense that

$$\forall x \in E_+,\ \forall t > 0, \quad V^i\big(x + te^j\big) > V^i(x).$$

Remark 5. Assume that Assumption (U.2) is satisfied,

(a) if preferences of agent i are strictly increasing and $e^j > 0$ for each $j \in J$, then Assumption (U.1) is satisfied,
(b) if preferences of agent i are increasing and for each $j \in I$, there exists a strictly positive integrable adapted process ξ^j such that $de^j(t) = \xi^j(t)\,dt$, then Assumption (U.1) is satisfied.

In order to compare norm-properness and the existence of smooth sub-gradients in K, we consider the following monotonicity assumption.

Assumption 2 *(M).* For every Edgeworth equilibrium $\boldsymbol{x}$, for each agent i, the following property is satisfied:

$$\forall j \in I,\ \forall t > 0, \quad x^i + te^j + E_+ \subset P^i\big(x^i\big).$$

Remark 6. From Remarks 3–4, Assumption C and conditions (U.1) and (U.2) imply Assumption M.

It is proved in Martins-da-Rocha and Riedel [38] that under Assumption (C.2), the existence of smooth sub-gradients in K implies that the economy is weakly proper,[30] in particular it is norm-proper.

It was left as open question in Hindy and Huang [31] whether a norm-proper economy admits a continuous equilibrium price. The existence results available in the literature are not general enough to be applied directly to our framework. The topology derived from the intertemporal norm does not give rise to the mathematical properties known to be sufficient for the existence of an Arrow–Debreu equilibrium.[31] The main contribution of this section is to prove that if an economy is proper with respect to the intertemporal norm it admits a compatible equilibrium, i.e., a price functional that is continuous on the positive cone E_+.

Proposition 1. *Under Assumptions* C *and* M*, if an economy is norm-proper then it admits a compatible price.*

Proof. Consider an economy satisfying Assumptions C and M and assume that it is norm-proper. From the norm-properness of utility functions, there exists a family $(\psi^i)_{i\in I}$ where ψ^i belongs to K and supports agent i's preferences. In order to apply Proposition 2 and Theorem 2 in Martins-da-Rocha and Riedel [38], it is sufficient to prove that the maximum of two processes in K is a process in H. Actually this is a consequence of the fact that H is stable by taking the max. Indeed, let ϕ and ψ be two processes in H, i.e., ϕ and ψ are non-negative, right-continuous with left-limits, bounded in L^q, and the projection of a raw continuous process bounded in L^1.

30 I.e., proper for the weak topology $\sigma(E, K)$.
31 The topological dual space $(E, \|\cdot\|)'$ endowed with dual order defined by the cone $E^\star_+$ is not a vector lattice.

We denote by θ the process defined by $\theta_t = \max\{\phi_t, \psi_t\}$. We have to show that θ belongs to H_+. It is non-negative, right-continuous with left-limits and bounded in L^q. It remains to show that θ is the optional projection of a raw continuous process in L^1. For this we can again check the conditions of the main result in Bismut [19]. To this end we have to show that θ is of class (D) and continuous in expectations. As θ is bounded in L^q, it is of class (D). Continuity in expectation is preserved by taking the max, and the proof is done. □

Remark 7. Actually, Theorem 1 is still valid if the norm-properness of each utility function V^i is replaced by the τ-properness for any linear topology τ on E such that any linear functional τ-continuous on E is represented by a vector in H.

Remark 8. Observe that contrary to Bank and Riedel [17] and Martins-da-Rocha and Riedel [38], we do not need to assume that the filtration $\mathbb{F}$ is quasi-left-continuous. This is an assumption on the way new information is revealed to the agents. Economically, an information flow corresponds to a quasi-left-continuous filtration[32] if information surprises (in the sense of Hindy and Huang [31]) occur only at times which cannot be predicted. The announcement of a policy change of the Federal reserve is an example for an information surprise which occurs at a time known in advance.

3.3. All compatible prices are possible in equilibrium

We know already that we cannot expect to find prices in the Hindy–Huang dual (Example 1) and that we have compatible equilibrium prices in general (Proposition 1). We show now that

- for all utility functions of the Hindy–Huang–Kreps type and every compatible price $\psi \in H$, one can construct an economy with these preferences such that ψ is an equilibrium price. In this sense, there is no hope to improve on our price space;
- one can have equilibrium prices that are not semi-martingales even in a representative agent world if we allow for state-dependent utility functions.

Let us start with *Hindy–Huang–Kreps preferences*, i.e., preferences given by utility functionals of the form

$$V^i(x) = \mathbb{E} \int_{[0,T]} u^i\big(t, Y(x)(t)\big)\kappa(dt)$$

where $u^i : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ denotes a felicity function for agent i, and the quantity

$$Y(x)(t) = \int_{[0,t]} \beta e^{-\beta(t-s)}\, dx(s)$$

describes the investor's level of satisfaction obtained from his consumption up to time $t \in [0, T]$. The constant $\beta > 0$ measures how fast satisfaction decays.

[32] See Hindy and Huang [31] for a precise definition. An information flow generated by a Brownian motion or a Poisson process is quasi-left-continuous.

We consider the linear mapping $\phi : E \to E$ defined by

$$\forall t \in [0,T], \quad \phi(x)(t) = \int\limits_{[0,t]} \exp\{\beta s\}\, dx(s).$$

For each $x \in E$, the vector $\phi(x)$ is defined by the optional random measure

$$d\big[\phi(x)\big](t) = \exp\{\beta t\} dx(t).$$

The linear mapping ϕ is bijective and the inverse mapping ϕ^{-1} is given by

$$\forall t \in [0,T], \quad \phi^{-1}(x)(t) = \int\limits_{[0,t]} \exp\{-\beta s\}\, dx(s).$$

We introduce on E the following norm ρ:

$$\forall x \in E, \quad \rho(x) := \big\|\phi(x)\big\| = \mathbb{E} \int\limits_{[0,T]} \big|\phi(x)(t)\big| \kappa(dt).$$

It is proved in Martins-da-Rocha and Riedel [38, Lemma 2] that the norm-topology and the ρ-topology coincide on E_+, and that the ρ-topological dual $(E,\rho)'$ coincides with the norm-topological dual $(E, \|.\|)'$. In order to apply Proposition 1 it is sufficient to prove that V^i is ρ-proper. From Martins-da-Rocha and Riedel [38, Theorem] this is a consequence of the following conditions: for each $i \in I$,

(V.1) for each $t \in [0,T]$, the function $u^i(t,.) : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, concave, and differentiable on $(0,\infty)$,[33]
(V.2) for each $c \in \mathbb{R}_+$, the function $u^i(.,c) : [0,T] \to \mathbb{R}$ is $\mathcal{B}$-measurable and the function $u^i(.,0)$ belongs to $L^1(\mathcal{B},\kappa)$,
(V.3) for each $t \in [0,T]$ the right-derivative $u^i_c(t,0+)$ exists and the function $u^i_c(.,0+)$ belongs to $L^\infty_+(\mathcal{B},\kappa)$.

So we know that Hindy–Huang–Kreps preferences have compatible equilibrium prices. Let us now ask the converse question: is it possible that *any* compatible price can be an equilibrium price in some economy with Hindy–Huang–Kreps preferences? Here is the answer.

Theorem 2. *For Hindy–Huang–Kreps utility functions $\boldsymbol{V} = (V^i)_{i \in I}$ and any compatible price $\psi \in H$ that is strictly positive, there exist endowments $\boldsymbol{e} = (e^i)_{i\in I}$ such that $(\psi, \boldsymbol{e})$ is an Arrow–Debreu equilibrium of the economy $(\boldsymbol{V}, \boldsymbol{e})$.*

Proof. A compatible price $\psi \in H$ is of class (D) because it is uniformly bounded by an integrable random variable and it is continuous in expectation because the associated price functional is continuous with respect to the intertemporal topology on the consumption set; in particular, $\mathbb{E}\psi(\tau_n) \to \mathbb{E}\psi(\tau)$ for stopping times $\tau_n \to \tau$. According to Theorem 3 in Bank and El Karoui [16] (compare also Section 2.1 in [16]), there exist optional processes e^i such that

[33] The derivative of $c \mapsto u^i(t,c)$ at $c_0 > 0$ is denoted by $u^i_c(t,c_0)$.

$$\nabla V^i\big(e^i\big)(t) = \mathbb{E}\left[\int_t^T u_c^i\big(t, Y^i\big(e^i\big)(s)\big)e^{-\beta^i(t-s)\,ds}\,\middle|\,\mathcal{F}_t\right] = \psi(t).$$

If we take these $\boldsymbol{e} = (e^i)_{i\in I}$ as endowments, the first-order condition of utility maximization is satisfied (compare Bank and Riedel [18]); hence, ψ is an equilibrium price in this economy. □

Our next example considers a Hindy–Huang–Kreps utility functional where the period felicity is affected by some unobservable shock ξ.

Example 2. Let

$$V(x) = \mathbb{E}\int_{[0,T]} u\big(t, Y(x)(t), \xi(t)\big)\kappa(dt)$$

for some (not necessarily adapted) stochastic process ξ. The gradient[34] of this utility function is given by

$$\nabla V(x)(t) = \mathbb{E}\left[\int_t^T u_c\big(t, Y(x)(s), \xi(s)\big)e^{-\beta(t-s)\,ds}\,\middle|\,\mathcal{F}_t\right].$$

This gradient is the prototype of an equilibrium state price in a one agent world. If we take x to be deterministic, e.g., and ξ to be, say, a fractional Brownian motion, then this gradient is not a semi-martingale. Hence, we cannot get semi-martingale prices in general.

4. Proof of Theorem 1

We give the proof of Theorem 1 for the case $p = 1$. Later, we indicate how to obtain the result for $p > 1$.

4.1. *Sufficiency*

First, we show that every element that satisfies the conditions of the theorem induces a compatible price.

Lemma 1. *Let ψ be a cadlag process that satisfies the assumptions in Theorem* 1. *Then the mapping $\pi = \langle\psi, \cdot\rangle$ is a compatible price.*

Proof. Let $M \geqslant 0$ be an upper bound for ψ, i.e., $M \geqslant \sup_t \psi_t$ almost surely. As processes $z \in E$ have integrable variation, $\langle\psi, \cdot\rangle$ is well defined on E:

$$\forall z \in E, \quad \left|\mathbb{E}\int \psi\, dz\right| \leqslant M\|z\|_{\mathrm{s}} < \infty.$$

Since $\langle\psi, \cdot\rangle$ is obviously linear and non-negative, it belongs to $E_+^\star$.

[34] See Duffie and Skiadas [27] for the computation which readily translates to our slightly more general utility function.

Let ξ be a continuous process which optional projection ${}^o\xi$ coincides with ψ. We first establish continuity of the functional $\langle \psi, \cdot \rangle$ on the space

$$E_+^k := \{x \in E_+ \colon x_T \leqslant k \text{ a.s.}\}$$

for arbitrary $k > 0$. Let $(x^n) \subset E_+^k$ be a sequence converging to some $x \in E_+^k$ for the intertemporal norm. As we have

$$\mathbb{E} \int \psi \, dz = \mathbb{E} \int \xi \, dz$$

for all $z \in E$, it is enough to prove that

$$\lim_n \mathbb{E} \int \xi \, dx^n = \mathbb{E} \int \xi \, dx.$$

Suppose this is not true. Then there is a subsequence (y^n) of (x^n) such that

$$\lim_n d^n := \lim_n \mathbb{E} \int \xi \, dy^n = d \neq c := \mathbb{E} \int \xi \, dx.$$

Without loss of generality, we can assume that on a set of probability 1, the sequence (y^n) converges weakly in the sense of measures on the time axis to x (see Lemma 1 in Martins-da-Rocha and Riedel [38] or Hindy, Huang and Kreps [33, Proposition 5]). Then we have $\lim_n \int \xi \, dy^n = \int \xi \, dx$ almost surely because ξ is continuous. From

$$\left| \int \xi \, dy_n \right| \leqslant k \sup_{t \in [0,T]} |\xi_t|$$

and the assumption on ξ, we get by dominated convergence that $\lim_n d^n = c$: a contradiction.

Now let (x^n) be an arbitrary sequence in E_+ that converges to x. For each $k \in \mathbb{N}$, we let x_k^n and x_k be the optional random measures defined by

$$dx_k^n = dx^n \wedge [\delta_0 k] \quad \text{and} \quad dx_k = dx \wedge [\delta_0 k].$$

Observe that for every $t \in [0, T]$ we have $x_k^n(t) = \min\{x^n(t), k\}$ and $x_k(t) = \min\{x(t), k\}$. It follows immediately from dominated convergence that

$$\lim_k \|x_k - x\|_{\mathrm{s}} = 0 \tag{1}$$

and

$$\forall k \in \mathbb{N}, \quad \lim_n \left\| x_k^n - x_k \right\| = 0. \tag{2}$$

Observe that for every (k, n) we have

$$\left| \left\langle \psi, x_k^n - x^n \right\rangle \right| \leqslant M \left\| x_k^n - x^n \right\|_{\mathrm{s}} = M \mathbb{E}\left[x^n - k\right]^+ \leqslant M \left\| x^n - x \right\| + M \mathbb{E}[x_k - k]^+. \tag{3}$$

For $\epsilon > 0$, relation (1) allows us to find k_0 such that

$$\left| \langle \psi, x - x_k \rangle \right| \leqslant M \|x - x_k\|_{\mathrm{s}} < \epsilon \quad \text{and} \quad M \mathbb{E}[x_k - k]^+ \leqslant \varepsilon. \tag{4}$$

Now fix $k = k_0$, it follows from (3) and (4) that for all $n \in \mathbb{N}$

$$\left| \left\langle \psi, x - x^n \right\rangle \right| \leqslant \left| \langle \psi, x - x_k \rangle \right| + \left| \left\langle \psi, x_k - x_k^n \right\rangle \right| + \left| \left\langle \psi, x_k^n - x^n \right\rangle \right| \tag{5}$$

$$\leqslant 2\epsilon + \left| \left\langle \psi, x_k - x_k^n \right\rangle \right| + M \left\| x^n - x \right\|. \tag{6}$$

By the fact that π is continuous on E_+^k and (2), we can choose n_0 such that for all $n \geqslant n_0$

$$\left|\langle \psi, x_k - x_k^n \rangle\right| < \varepsilon \quad \text{and} \quad M\left\|x^n - x\right\| \leqslant \varepsilon$$

and we finally obtain

$$\left|\langle \psi, x - x^n \rangle\right| < 4\epsilon$$

for $n \geqslant n_0$. This shows that π is continuous on E_+. □

4.2. Necessity

The converse is the much more demanding part. Given a compatible price $\pi \in H_+$, we have to find a *density* ψ that represents π. We will frequently use the following continuity lemma that yields suitable upper bounds.

Lemma 2. *Compatible prices are continuous with respect to the strong topology.*

Proof. As the strong topology is stronger than the intertemporal topology, a compatible price π is $\|\cdot\|_s$-continuous on E_+. But the space E is a topological vector lattice with respect to the strong topology. Hence, the lattice operations are continuous with respect to the strong topology. It follows that π is $\|\cdot\|_s$-continuous on the whole space E. □

As the space $(E, \|\cdot\|_s)$ is a Banach space, the preceding lemma yields a constant $K > 0$ such that

$$\forall z \in E, \quad \left|\pi(z)\right| \leqslant K\|z\|_s. \tag{7}$$

Denote by $\mathcal{T}$ the set of all stopping times $\tau \leqslant T$. A $\mathcal{T}$-system is a family $(z^\tau)_{\tau\in\mathcal{T}}$ of random variables that satisfy (see Dellacherie and Lenglart [22])

1. consistency: for σ, τ stopping times $z^\tau = z^\sigma$ on the set $\{\tau = \sigma\}$,
2. measurability: every random variable z^τ is $\mathcal{F}_\tau$-measurable.

Fix a random variable $\tau \in \mathcal{T}$. Define a linear mapping Q^τ from $L^1(\mathcal{F}_\tau, \mathbb{P})$ into $\mathbb{R}$ by setting

$$\forall Z \in L^1(\mathcal{F}_\tau, \mathbb{P}), \quad Q^\tau(Z) = \pi(\delta_\tau Z).$$

Being a continuous linear mapping, it can be represented by a random variable $z^\tau \in L^\infty(\mathcal{F}_\tau, \mathbb{P})$ such that

$$\forall Z \in L^1(\mathcal{F}_\tau, \mathbb{P}), \quad Q^\tau(Z) = \mathbb{E}\big(Zz^\tau\big).$$

As π is non-negative we actually have $z^\tau \geqslant 0$ a.s.

Claim 1. *The family $(z^\tau)_{\tau\in\mathcal{T}}$ forms a $\mathcal{T}$-system.*

Proof. It is sufficient to show that $z^\sigma \mathbf{1}_{\{\sigma=\tau\}} = z^\tau \mathbf{1}_{\{\sigma=\tau\}}$ almost surely. As both $z^\sigma \mathbf{1}_{\{\sigma=\tau\}}$ and $z^\tau \mathbf{1}_{\{\sigma=\tau\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-measurable, it is enough to show

$$\forall Z \in L^1(\mathcal{F}_{\sigma\wedge\tau}, \mathbb{P}), \quad \mathbb{E}z^\sigma \mathbf{1}_{\{\sigma=\tau\}} Z = \mathbb{E}z^\tau \mathbf{1}_{\{\sigma=\tau\}} Z.$$

Take such a Z in $L^1(\mathcal{F}_{\sigma\wedge\tau},\mathbb{P})$. Then

$$\begin{aligned}\mathbb{E}z^\sigma\mathbf{1}_{\{\sigma=\tau\}}Z &= Q^\sigma(\mathbf{1}_{\{\sigma=\tau\}}Z)\\ &= \pi(\delta_\sigma\mathbf{1}_{\{\sigma=\tau\}}Z)\\ &= \pi(\delta_\tau\mathbf{1}_{\{\sigma=\tau\}}Z)\\ &= Q^\tau(\mathbf{1}_{\{\sigma=\tau\}}Z)\\ &= \mathbb{E}z^\tau\mathbf{1}_{\{\sigma=\tau\}}Z.\end{aligned}$$

This concludes the proof. □

The question is: can we find a process $(\psi_t)_{t\in[0,T]}$ such that $\psi_\tau = z^\tau$ almost surely for all stopping times $\tau\in\mathcal{T}$? Such a question is called a *problem of aggregation* in the théorie générale of stochastic processes. In general, aggregation is not possible without some continuity requirement (see Dellacherie and Lenglart [22]). Therefore, we establish the following claim.

Claim 2. *The $\mathcal{T}$-system $(z^\tau)_{\tau\in\mathcal{T}}$ is continuous in expectation in the sense that*

$$\lim_n \mathbb{E}z^{\tau_n} = \mathbb{E}z^\tau$$

for all sequences of stopping times (τ_n) with $\lim_n\tau_n=\tau$.

Proof. Let (τ_n) be a sequence of stopping times satisfying $\lim\tau_n=\tau$. Then, the sequence of optional random measures (δ_{τ_n}) converges to δ_τ in the intertemporal topology. Continuity of the price functional π implies

$$\lim_n \mathbb{E}z^{\tau_n} = \lim_n \pi(\delta_{\tau_n}) = \pi(\delta_\tau) = \mathbb{E}z^\tau. \qquad \Box$$

Claim 3. *There exists a non-negative, adapted and cadlag process $\psi\in L^\infty(\mathbb{P},\mathcal{B})$ that aggregates (or recollects) $(z^\tau)_{\tau\in\mathcal{T}}$ in the sense that $\psi_\tau=z^\tau$ almost surely for every stopping time $\tau\in\mathcal{T}$.*

Proof. By Theorem 6 in Dellacherie and Lenglart [22], every non-negative $\mathcal{T}$-system can be aggregated by an optional process ψ. The process ψ is non-negative because so are every z^τ. From Claim 2 the process ψ is continuous in expectation. If we prove that ψ is uniformly integrable, then we can apply Dellacherie and Meyer [23, Theorem 48][35] to conclude that ψ is cadlag. We first prove that ψ is bounded in L^1. Let τ be a stopping time and observe that

$$0\leqslant \mathbb{E}\psi_\tau = \mathbb{E}z^\tau = \pi(\delta_\tau).$$

The process δ_τ belongs to E and (7) yields $\pi(\delta_\tau)\leqslant K$ implying that $\mathbb{E}\psi_\tau<\infty$. To establish uniform integrability, we have to show that for all $\varepsilon>0$ there exists $\delta>0$ such that for all sets $A\in\mathcal{F}$ with $\mathbb{P}(A)\leqslant\delta$ one has $\mathbb{E}\mathbf{1}_A\psi_\tau\leqslant\varepsilon$ for every stopping time τ. For $A\in\mathcal{F}$ and a stopping time τ, let $c=\delta_\tau\mathbb{E}[\mathbf{1}_A|\mathcal{F}_\tau]$. Since the process c belongs to E we have

$$\mathbb{E}\mathbf{1}_A\psi_\tau = \mathbb{E}\big(\mathbb{E}[\mathbf{1}_A|\mathcal{F}_\tau]\psi_\tau\big) = \pi(c)\leqslant K\|c\|_{\mathrm{tot}} = K\mathbb{P}(A).$$

Setting $\delta=\varepsilon/K$, we obtain uniform integrability. □

[35] The theorem is formulated for bounded processes only. The comment 50(f) in Dellacherie and Meyer [23] shows that it is enough to have uniform integrability.

Claim 4. *The process ψ is bounded.*

Proof. Fix $\alpha > 0$ and let τ be the stopping time

$$\tau = \begin{cases} \inf\{t \geqslant 0\colon \psi_t \geqslant n\} & \text{if } \sup_{0\leqslant t\leqslant T} \psi_t \geqslant \alpha, \\ \infty & \text{elsewhere.} \end{cases}$$

Recall that the random variable ψ_τ is given by $\psi_\tau(\omega) = \psi_{\tau(\omega)}(\omega)\mathbf{1}_{\{\tau<\infty\}}$. Consider the optional random measure $c = \delta_\tau \mathbf{1}_{\{\tau<\infty\}}$. We have

$$\pi(c) = \mathbb{E}z^\tau \mathbf{1}_{\{\tau<\infty\}} = \mathbb{E}\psi_\tau \mathbf{1}_{\{\tau<\infty\}}.$$

Since ψ is cadlag, we get $\pi(c) = \mathbb{E}\psi_\tau \mathbf{1}_{\{\tau<\infty\}} \geqslant \alpha\mathbb{P}\{\tau < \infty\}$. On the other hand, (7) yields

$$\pi(c) \leqslant K\|c\|_{\mathrm{s}} = K\mathbb{E}\|\delta_\tau \mathbf{1}_{\{\tau<\infty\}}\|_{\mathrm{tot}} = K\mathbb{P}\{\tau < \infty\}.$$

Choosing $\alpha > K$ then shows that $\mathbb{P}\{\tau < \infty\} = 0$. Hence the process ψ is bounded, i.e., $\psi \in L^\infty(\mathbb{P}, B(T))$.

In general, ψ is not going to be continuous. However, we have the following result that goes back to Bismut [19] and Emery [29].

Claim 5. *There exists a* not necessarily adapted *continuous process ξ with*

$$\mathbb{E} \sup_{t\in[0,T]} |\xi_t| < \infty$$

whose optional projection is ψ, that is

$$\psi_\tau = \mathbb{E}\big[\xi_\tau \big| \mathcal{F}_\tau\big]$$

for all stopping times $\tau \in \mathcal{T}$.

Proof. This is the main theorem in Bismut [19] and Emery [29]. According to the notations in Bismut [19] and Emery [29], we have to check the conditions that ψ is regular and of class (D). As ψ is bounded, it is of class (D). A process is regular if and only if the predictable projection of ψ is equal to ψ_-. This is equivalent to continuity in expectation from below (see Dellacherie and Meyer [23, 50(d)]). As ψ is even continuous in expectation, it is regular. □

Claim 6. *For every bounded consumption plan $x \in E_+$ with $x_T \in L^\infty(\mathbb{P})$ we have $\pi(x) = \langle\psi, x\rangle$, i.e.,*

$$\pi(x) = \mathbb{E}\int \psi\, dx.$$

Proof. By construction, we have for every stopping time τ and $\mathcal{F}_\tau$-measurable random variable h

$$\pi(\delta_\tau h) = \mathbb{E}z^\tau h = \mathbb{E}\psi_\tau h = \mathbb{E}\xi_\tau h.$$

Via linearity, we obtain $\pi(z) = \langle\psi, z\rangle$ for every simple random measure z. As simple random measures are dense with respect to the intertemporal norm in E_+ and ξ is continuous, we get the result for optional random measures with bounded variation in $L^\infty(\mathbb{P})$ (observe that Bismut [19] obtains this in his proof). □

Since ψ belongs to $L^\infty(\mathbb{P}, B(T))$, the random variable $\psi^\star$ defined by

$$\psi^\star = \sup_{t\in[0,T]} \psi_t$$

belongs to $L^\infty(\mathbb{P})$. It follows that for every consumption plan $x \in E_+$ the quantity $\langle\psi, x\rangle$ is well defined as

$$\langle\psi, x\rangle = \mathbb{E}\int \psi\,dx \leqslant \mathbb{E}\psi^* x_T < \infty.$$

We can now prove that $\pi(x) = \langle\psi, x\rangle$ for every $x \in E_+$. From Claim 6, we know that $\pi(x) = \langle\psi, x\rangle$ for all bounded x in $L^\infty(\mathbb{P}, B(T))$. Now let $x \in E_+$ be given. Set $dx_n = dx \wedge n\delta_0$, i.e., $x_n(t) = \min\{x_t, n\}$ for every $t \in [0, T]$. For each n the optional random measure x_n is bounded and the sequence $(x_n(\omega))$ converges for the total variation norm $\|\cdot\|_{\mathrm{tot}}$ to $x(\omega)$ from below for all ω. Consequently, for all non-negative measurable functions f we have

$$\lim_n \int f\,dx_n = \int f\,dx.$$

In particular, we have $\lim_n \int \psi\,dx_n = \int \psi\,dx$ almost surely. By monotone convergence, we obtain

$$\lim_n \pi(x_n) = \lim_n \mathbb{E}\int \psi\,dx_n = \mathbb{E}\int \psi\,dx = \langle\psi, x\rangle$$

and as π is continuous with respect to the strong topology, $\pi(x) = \langle\psi, x\rangle$ follows. This concludes the proof of the theorem.

4.3. The proof for $p > 1$

For $p > 1$, the proof follows almost verbatim the above proof for $p = 1$. However, one cannot use the argument given above that establishes boundedness of the process ψ. Instead, one has to use a different argument to prove that the supremum of ψ is in L^q. This argument is given next.

Claim 7. *The supremum*

$$\psi^* := \sup_{t\in[0,T]} \psi_t$$

satisfies

$$\forall H \in L^p(\mathbb{P}), \quad \mathbb{E}\psi^*|H| \leqslant (K + 1)\|H\|_{L^p}.$$

In particular the random variable ψ^* belongs to L^q.

Proof. Let S be a random time (not necessarily a stopping time) and $h \in L^p_+(\mathbb{P})$. Denote by $z = \delta_S h$ and by $x = (z)^o$ its optional dual projection. Then x is an optional random measure and

$$\mathbb{E}\psi_S h = \mathbb{E}\int \psi\,dz = \mathbb{E}\int \psi\,dx = \pi(x) \leqslant K\|x\|_{\mathrm{s}}.$$

The process x is non-decreasing and $\mathcal{F} = \mathcal{F}_T$, hence

$$\|x\|_{\mathrm{s}} = \mathbb{E}\|x\|_{\mathrm{tot}} = \mathbb{E}x_T = \mathbb{E}h \leqslant \|h\|_{L^p}.$$

Let S be a cross-section of the set

$$\{(\omega, t)\colon\ \psi_t(\omega) \geqslant \psi^\star(\omega) - 1\}.$$

Then we have

$$\mathbb{E}\psi^* h \leqslant \mathbb{E}(\psi_S + 1)h \leqslant (K+1)\|h\|_{L^p}. \qquad \square$$

5. Conclusion

We show how the economically sensible intertemporal topology introduced by Hindy and Huang [31] allows to derive general structural results about equilibrium state prices. Using the *théorie générale* of stochastic processes, we show that price functionals that are continuous on the consumption set can be represented by state prices with right-continuous sample paths that admit left-limits. Moreover, the state price is the optional projection of a process with continuous sample paths that is not necessarily adapted. Our results fit also in the finance literature working with the no-arbitrage principle. In a recent paper, Jouini, Napp and Schachermayer [34] study general spaces in which a version of the Kreps–Yan theorem holds true. Their results show that our commodity and price spaces are consistent with no-arbitrage. This result would also follow from the existence of equilibrium established in this paper as there can be no arbitrage in equilibrium, of course.

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