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Cournot–Nash equilibria in continuum games with non-ordered preferences

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Abstract

In the usual framework of continuum games with externalities, we substantially generalize Cournot–Nash existence results [Balder, A unifying approach to existence of Nash equilibria, Int. J.Game Theory 24 (1995) 79–94; On the existence of Cournot–Nash equilibria in continuum games, J. Math. Econ. 32 (1999) 207–223; A unifying pair of Cournot–Nash equilibrium existence results, J. Econ. Theory 102 (2002) 437–470] to games with possibly non-ordered preferences, providing a continuum analogue of the seminal existence results by Mas-Colell [An equilibrium existence theorem without complete or transitive preferences, J. Math. Econ. 1 (1974) 237–246], Gale and Mas-Colell [An equilibrium existence theorem for a general model without ordered preferences, J. Math. Econ. 2 (1975) 9–15], Shafer and Sonnenschein [Equilibrium in abstract economies without ordered preferences, J. Math. Econ. 2 (1975) 345–348], Borglin and Keiding [Existence of equilibrium actions and of equilibrium: a note on the "new" existence theorems, J. Math. Econ. 3 (1976) 313–316] and Yannelis and Prabhakar [Existence of maximal elements and equilibria in linear topological spaces, J. Math. Econ. 12 (1983) 233–245].

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1. Introduction

For games with finitely many players and payoff/utility functions, Debreu [13] was the first to generalize the non-cooperative equilibrium notion of Nash (and Cournot) by introducing the

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concept of a social system (also called generalized game or abstract economy). The social equilibrium existence result established in Debreu [13] was subsequently used by Arrow and Debreu [1] to prove the existence of Walrasian equilibrium in economies with preference relations represented by utility functions. Following their pioneer work, new existence results have been proposed in the literature based on weaker assumptions than those initially imposed by Arrow and Debreu [1]. Among them, a significant contribution was made by Mas-Colell [30] who provided an equilibrium existence result relaxing the assumption of completeness and transitivity of preference relations. Mas-Colell pointed out, that dropping transitivity or completeness (i.e. having non-ordered preferences), the standard existence arguments through demand correspondences were not anymore applicable. He identified two major difficulties. First, in order to prove that individual demand sets are non-empty, a Weierstrass-type argument is no longer sufficient. [1] Second, when preferences are non-ordered the demand set may not be convex. To circumvent these problems he developed a new approach. In subsequent years, alternative and simpler proofs were also proposed by Gale and Mas-Colell [15], Shafer and Sonnenschein [43] and Borglin and Keiding [10]. Yannelis and Prabhakar [46] generalized these results by allowing for a countably infinite number of agents and an infinite number of commodities.

For continuum games, i.e. games with a non-atomic measure space of agents, Schmeidler [41] proved the existence of pure Nash equilibrium in games where each player's payoff function depends on his own action as well as on the mean of the other players' actions. Mas-Colell [31] presented a reformulated version of Schmeidler's result in distributional terms rather than in function-theoretic terms (see also among others, Khan [17], Khan and Rustichini [20,21], Khan and Sun [22], Balder [3,5] and Rath [37]). Those existence results for games with payoff functions were extended in several directions by Balder [2,4,6], Rath [36], Kim and Yannelis [26] and recently by Balder [8]. A common characteristic of that part of the literature is that, due to the convexifying effect of aggregation, they all dispense with the convexity assumptions on both action sets and preferences.

Following the contributions of Schmeidler [41] and Mas-Colell [30], a new literature emerged [2] attempting to model non-ordered preferences in continuum games/economies with externalities. This literature suffers from many drawbacks. First, despite the convexifying effect on aggregation due to the continuum of agents, those existence results require convex assumptions on both action sets and preferences. The second and more important problem has to do with the incompatibility of the proposed conditions used to establish existence of equilibria. Grodal [3] and Balder [7] showed that many of the required conditions force the *preferred to* correspondence to be empty-valued almost everywhere on the non-atomic part of the measure space. Yet, at the same time, existence results are well known to hold in continuum economies with non-ordered preferences but without externalities (see e.g. Schmeidler [40] and Martins-da-Rocha [27,28]). This is not in contradiction with the inconsistency problem raised by Balder. Indeed the modeling of non-ordered preferences with externalities proposed in the above cited literature does not encompass the traditional modeling of non-ordered preferences without externalities as in Schmeidler [40] and Martins-da-Rocha [27,28].

[1] This difficulty can be solved by using a fixed-point argument.

[2] See e.g. Khan and Vohra [24], Balder and Yannelis [9], Khan and Papageorgiou [18,19], Kim et al. [25], Noguchi [32–34].

[3] Grodal proposed a proof for a finite-dimensional action space. Her proof was adapted to separable Banach spaces by Khan and Papageorgiou [19].

The paper builds on this literature. Its primary aim is to propose an alternative formulation of non-ordered preferences and prove a Cournot–Nash equilibrium existence result that does not suffer from the drawbacks cited in Balder [7]. Our result generalizes in a unified way the previous existence results for: atomic games [4] (or abstract economies) with non-ordered preferences as in Shafer and Sonnenschein [43], Borglin and Keiding [10] and Yannelis and Prabhakar [46], and continuum games with ordered preferences as in Schmeidler [41] and Balder [6,8].

For games with a finite set of players and non-ordered preferences, each *preferred to* correspondence is usually defined on the space of strategy profiles. For each agent $t \in T$, the set of actions strictly preferred to an action $s_t \in S_t$, given the actions $(s_\tau, \tau \neq t) \in \prod_{\tau \neq t} S_\tau$ of the other agents, is represented by a subset $P_t(f_s)$ of agent t's action set S_t, where $f_s : \tau \mapsto f_s(\tau) = s_\tau$ is the action profile associated to the family $s = (s_\tau) \in \prod_{\tau \in T} S_\tau$. In the existing literature [5] it is claimed that the natural extension of this modeling to games with a continuum space of players, is to define in a similar manner each *preferred to* correspondence on the space $\mathcal{S}$ of action profiles which now consists on measurable functions $f : t \ni T \longmapsto f(t) \ni S_t$. For an agent t, the subset $P_t(f) \subset S_t$ represents the set of actions strictly preferred by agent t to the action $f(t)$. In this modeling, the joint evaluation mapping $\mathrm{ev} : (t, f) \mapsto f(t)$ appears to play a central role. As shown by Balder [7], when the set of players is non-atomic, this mapping has non-standard measurability and topological properties which do not seem to be adequate for standard arguments to prove existence of equilibrium by means of measure theory and functional analysis. Observe that when the space of agents is atomic, then the set of action profiles coincides with the product over all agents of action sets, i.e. $\mathcal{S} := \prod_{\tau \in T} S_\tau$. We may then see the *preferred to* correspondence P_t of agent t as a correspondence defined on the space $S_t \times \prod_{\tau \neq t} S_\tau$, or equivalently on the space $S_t \times \mathcal{S}$. Following this observation, we propose, as a natural extension to a measure space of agents, to model preferences of an agent t by a correspondence P_t defined on the product space $S_t \times \mathcal{S}$ and not solely on the space $\mathcal{S}$ of action profiles. The set $P_t(s_t, f) \subset S_t$ then represents the set of actions strictly preferred to s_t by agent t given the actions of the other agents, represented by the action profile $f \in \mathcal{S}$. This modeling of non-ordered preferences (see Remark 2.2) does not suffer from the serious inconsistency pointed out in Balder [7]. Moreover it encompasses both the modeling of payoff/utility functions with externalities as in Schmeidler [41] and Balder [6,8], and the modeling of non-ordered preferences but without externalities as in Schmeidler [40], Cornet et al. [12] and Martins-da-Rocha [27,28].

Our main result is the existence of pure Cournot–Nash equilibrium for continuum games with possibly non-ordered preferences. We prove that this new existence result follows as a simple corollary of well-known existence results established for games with payoff/utility functions. Starting from a game where players have non-ordered preferences, the underlying idea is to define an auxiliary game with payoff functions such that the sets of optimal actions for both games coincide. To get existence of pure Cournot–Nash equilibrium, it is then sufficient to apply to the auxiliary game the existence results in Balder [6,8]. Balder's results are proved by means of Young measure theory. In a companion working paper [29] we propose an independent proof based on a direct application of Kakutani's fixed-point theorem on the space of pure action profiles.

In order to benefit from the convexifying effect of aggregation, we assume that externalities are modeled through finitely many statistics of the aggregate strategy profile. Consequently, if preference relations are transitive (but not necessarily complete), we do not need to assume

[4] That is games with at most countably many agents.

[5] See e.g. Balder [7], Balder and Yannelis [9], Khan and Papageorgiou [18,19], Khan and Vohra [24], Kim et al. [25], Noguchi [32–34] and Yannelis [44,45].

convexity of action sets or preferences. [6] However, when externalities in preferences are modeled by general statistics of the strategy profile, we need [7] to assume convexity assumptions even if preference relations are transitive.

The paper is organized as follows. The model and the main existence result (Theorem 2.1) are presented in Section 2. In Section 3 we provide sufficient conditions on the primitives of a game in order to satisfy the conditions required in Theorem 2.1. Some definitions and technical results are postponed to the Appendix.

2. The model and the main result

We consider the model presented in Balder [6,8]. Let $(T, \mathcal{T}, \mu)$ be an abstract finite measure space. The set T is the set of *players*, which may be a finite set or a continuum such as the unit interval or a mixture of both. For technical reasons we make the following assumption.

Assumption 2.1. The measure space $(T, \mathcal{T}, \mu)$ is complete and separable.

Remark 2.1. This is a *working hypothesis* and we refer to Balder [6, Remark 4.2] and Balder [8, Remarks 2.1.1(iii) and 2.2.1(iv)] for details on arguments and stronger measurability conditions that can be used to remove this additional hypothesis from the final results.

Let S be Hausdorff locally convex topological vector space that is a Suslin space, i.e. S is the continuous image of a Polish space. The space S is the *action space*. Examples of such spaces include separable Banach spaces, equipped with their norm or weak topology, duals of separable Banach spaces, equipped with their weak star topology, separable Fréchet spaces, such as $\mathcal{C}(\mathbb{R})$, equipped with the compact-open topology, or the space of all bounded, signed measures on a completely regular Suslin space. We denote by $S^\star$ the topological dual of S. For each t in T, let $S_t \subset S$ denote the *action set* of player t. We denote by Σ the correspondence from T into S defined by $\Sigma(t) := S_t$.

We let $\overline{T}$ in $\mathcal{T}$ be some fixed measurable subset of players that contains the purely atomic part of $(T, \mathcal{T}, \mu)$. The set $\overline{T}$ is the set of players that will satisfy additional convexity assumptions. We let $\widehat{T}$ denote the set $T \setminus \overline{T}$ and we let $\overline{\mathcal{T}}$ (resp. $\widehat{\mathcal{T}}$) be the trace σ-algebra of $\mathcal{T}$ on $\overline{T}$ (resp. $\widehat{T}$). We suppose that the following holds.

Assumption 2.2. For every $t \in T$, the set S_t is non-empty and compact, and the graph

$$\operatorname{gph}\Sigma := \{(t, s) \in T \times S : s \in S_t\}$$

of the correspondence Σ belongs [8] to $\mathcal{T} \otimes \mathcal{B}(S)$. Moreover, for every $t \in \overline{T}$, the set S_t is convex.

[6] We need conditions on primitives to ensure that each preference relation has a maximal element on any non-empty compact set. Either we assume that the preference relation is transitive and we apply a Weierstrass-type argument, or we assume that the preference relation is convex and we apply a fixed-point argument.

[7] See Rath et al. [38] for a counterexample. We also refer to Khan and Sun [23] where the convexifying effect of aggregation is restored when the space of players is modeled by a hyperfinite Loeb measure space. In a very recent preprint Podczeck [35] has shown that no recourse to non-standard measure theory is needed for purification as practiced in [23].

[8] The symbol $\mathcal{B}(S)$ refers to the Borel σ-algebra on S.

The trace σ-algebra of $\mathcal{T} \otimes \mathcal{B}(S)$ on $\operatorname{gph}\Sigma$ is denoted by $\mathcal{D}$. An action profile is a function $f : T \to S$ that is measurable with respect to $\mathcal{T}$ and $\mathcal{B}(S)$ or, equivalently [9] that is scalarly measurable, i.e. for all $s^\star \in S^\star$, the scalar functions $t \mapsto \langle f(t), s^\star \rangle$ is $\mathcal{T}$-measurable. Let $\mathcal{S}$ denote the set of all action profiles. An action profile f is feasible if $f(t)$ belongs to S_t for a.e. $t \in T$. The set of all feasible action profiles is denoted by $\mathcal{S}_\Sigma$. Also, let $\overline{\mathcal{S}}_\Sigma$ be the set of all restrictions to $\overline{T}$ of functions in $\mathcal{S}_\Sigma$; it is only this set that needs to be topologized. We endow $\overline{\mathcal{S}}_\Sigma$ with the feeble topology introduced by Balder [6,8] and defined as the coarsest topology for which all functionals

$$J_g : f \mapsto \int_{\overline{T}} g(t, f(t))\mu(dt), \quad g \in \overline{\mathcal{G}}_{LC,\Sigma}$$

are continuous. Here $\overline{\mathcal{G}}_{LC,\Sigma}$ is the collection of all $\overline{\mathcal{T}} \otimes \mathcal{B}(S)$-measurable functions $g : \overline{T} \times S \to \mathbb{R}$ for which $g(t, .)$ is linear and continuous on S (i.e. belongs to $S^\star$) for every $t \in \overline{T}$ and for which there is an integrable function ϕ_g in $\mathcal{L}^1_{\mathbb{R}}(\overline{T}, \overline{\mathcal{T}}, \mu)$ with $\sup_{s \in S_t} |g(t, s)| \leqslant \phi_g(t)$ for all $t \in \overline{T}$. The feeble topology can simultaneously subsume the two customary topologies that have been used in the literature on games with a measure space of players (we refer to Balder [6,8] for precisions and examples). Following Balder [8, Remark 4.3.1] the feeble topology on $\overline{\mathcal{S}}_\Sigma$ is semimetrizable.

Let us now define as the externality of each player $t \in T$ the mapping $d := (\bar{d}, \hat{d}) : \mathcal{S}_\Sigma \to \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, which is defined by

$$\bar{d}(f) := f|_{\overline{T}} \quad \text{and} \quad \hat{d}(f) := \left(\int_{\widehat{T}} g_i(t, f(t))\mu(dt), \ i \in \{1, \dots, m\} \right).$$

Here $f|_{\overline{T}} \in \overline{\mathcal{S}}_\Sigma$ stands for the restriction to $\overline{T}$ of f. Also, for each $i \in \{1, \dots, m\}$, the function $g_i : \operatorname{gph}\Sigma \cap (\widehat{T} \times S) \to \mathbb{R}$ is given and satisfies the following condition.

Assumption 2.3. For each $i \in \{1, \dots, m\}$, the function g_i belongs to $\widehat{\mathcal{G}}_{C,\Sigma}$.

Here $\widehat{\mathcal{G}}_{C,\Sigma}$ is the collection of all $\widehat{\mathcal{T}} \otimes \mathcal{B}(S)$-measurable functions $g : \widehat{T} \times S \to \mathbb{R}$ for which $g(t, .)$ is continuous on S for every $t \in \widehat{T}$ and for which there is an integrable function ϕ_g in $\mathcal{L}^1_{\mathbb{R}}(\widehat{T}, \widehat{\mathcal{T}}, \mu)$ with $\sup_{s \in S_t} |g(t, s)| \leqslant \phi_g(t)$ for all $t \in \widehat{T}$. The externality $d(f)$ depends on the action $f(t)$ of almost every player t in $\overline{T}$ and depends only on the aggregate $\hat{d}(f)$ over all $\widehat{T}$.

Each player $t \in T$ must choose her actions in accordance with the other players as follows: given the action profile $f \in \mathcal{S}_\Sigma$, player t's socially feasible actions constitute a given subset $A_t(d(f)) \subset S_t$.

Now we depart from the model presented in Balder [6,8] by considering a more general framework to model the preference of each player. [10] Every player $t \in T$ has a *preferred to* correspondence

$$P_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to 2^{S_t}.$$

Given the action profile $\bar{f} \in \overline{\mathcal{S}}_\Sigma$, the externality vector $y \in \mathbb{R}^m$ and an action $s \in S_t$, the set $P_t(s, \bar{f}, y)$ represents the set of actions $s' \in S_t$ that agent t strictly prefers to action s.

[9] The equivalence is not an easy matter and we refer to Schwartz [42] for details.

[10] Actually the model with non-ordered preferences is more general only in appearance since we will prove that it can be reduced to the standard one with ordered preferences.

Example 2.1. In Balder [6,8], each agent t is endowed with a payoff function $U_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$. In that case, the correspondence P_t is defined by

$$P_t(s, \bar{f}, y) := \{s' \in S_t : U_t(s', \bar{f}, y) > U_t(s, \bar{f}, y)\}.$$

Remark 2.2. Observe that given an agent $t \in T$, the *preferred to* correspondence P_t is defined on $S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$. In the literature (see Balder and Yannelis [9], Khan and Papageorgiou [18,19], Khan and Vohra [24], Kim et al. [25], Noguchi [32–34] and Yannelis [44,45]), the *preferred to* correspondence is defined only on $\overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$. Moreover it is claimed in Balder [7] that for a game with payoff functions $U_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$, the canonical *preferred to* correspondence for an agent $t \in \overline{T}$ is defined by

$$P_t(\bar{f}, y) := \{s \in S_t : U_t(s, \bar{f}, y) > U_t(\bar{f}(t), \bar{f}, y)\}.$$

We claim that this modeling of preferences is not relevant. First, because it is proved in Balder [7] that the usual conditions used in the literature for this model force the *preferred to* correspondences to be empty valued almost everywhere in the non-atomic part of the measure space of agents. Second, because this model does not encompass the literature dealing with games or abstract economies with a measure space of agents but without externalities: in Schmeidler [40], Cornet et al. [12] and Martins-da-Rocha [27,28] the *preferred to* correspondence P_t is defined on S_t. We propose an alternative modeling of non-ordered preferences with externalities by considering a *preferred to* correspondence P_t defined on the space $S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$. We refer to Balder [8, Section 2.4] for a discussion about the consistency question regarding to our modeling of preferences.

Definition 2.1. For each $t \in T$, $\bar{f} \in \overline{\mathcal{S}}_\Sigma$ and $y \in \mathbb{R}^m$, we denote by $M_t(\bar{f}, y)$ the set of optimal actions in the socially feasible set $A_t(\bar{f}, y)$, i.e.

$$M_t(\bar{f}, y) = \{s \in A_t(\bar{f}, y) : P_t(s, \bar{f}, y) \cap A_t(\bar{f}, y) = \emptyset\}.$$

Example 2.2. If agent t is endowed with a payoff function $U_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$ such that the correspondence P_t is defined as in Example 2.1, then $M_t(\bar{f}, y)$ coincides with the set $\operatorname{argmax}_{s \in A_t(\bar{f}, y)} U_t(s, \bar{f}, y)$.

We present hereafter the list of assumptions the optimal actions correspondence will be required to satisfy.

Assumption 2.4 (*convexity*). For each player $t \in \overline{T}$, the correspondence M_t has convex values.

Assumption 2.5 (*continuity*). For each player $t \in T$, the correspondence $(\bar{f}, y) \mapsto M_t(\bar{f}, y)$ is upper-semicontinuous with non-empty closed values.

Assumption 2.6 (*measurability*). For every $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, the correspondence $t \mapsto M_t(\bar{f}, y)$ has a measurable graph, i.e.

$$\{(t, s) \in \operatorname{gph} \Sigma : s \in M_t(\bar{f}, y)\} \in \mathcal{D}.$$

Remark 2.3. Assumption 2.4 is satisfied if for every player $t \in \overline{T}$, for every $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ and every action $s \in S_t$, the sets $A_t(\bar{f}, y)$ and $\{s' \in S_t : s \notin P_t(s', \bar{f}, y)\}$ are convex. In particular, if the *preferred to* correspondence is defined by a payoff function $U_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$ then the last property is satisfied if $s \mapsto U_t(s, \bar{f}, y)$ is quasi-concave.

We provide in Section 3 conditions on the primitives A and P of a game such that Assumptions 2.5 and 2.6 are satisfied.

Theorem 2.1. *Under Assumptions* 2.1–2.6, *the game* $\Gamma := (T, \Sigma, A, P)$ *has a pure Cournot–Nash equilibrium, that is, there exists an action profile* $f_\star \in \mathcal{S}_\Sigma$ *such that for almost every player* $t \in T$,

$$f_\star(t) \in A_t(d(f_\star)) \quad \textit{and} \quad P_t(f_\star(t), d(f_\star)) \cap A_t(d(f_\star)) = \emptyset.$$

The outline of the proof is as follows. We consider a game Γ satisfying Assumptions 2.1–2.6. We define an auxiliary game Γ' with payoff functions such that optimal action profiles for Γ' and optimal action profiles for Γ coincide. We check that the auxiliary game Γ' satisfies the set of assumptions needed to apply Theorem 2.2.1 in Balder [8]. We then get the existence of a Cournot–Nash equilibrium for Γ' which is also a Cournot–Nash equilibrium for the initial game Γ. Following this approach, our existence result for non-ordered preferences appears to be a simple by-product of the existence results for ordered preferences in Balder [6,8].

Proof of Theorem 2.1. Let $\Gamma := (T, \Sigma, A, P)$ be a game satisfying Assumptions 2.1–2.6. Since S is Suslin there exists a metric d on S which is weaker than the original topology. Hence on compact subsets of S the original topology and the d-topology coincide. Observe that each Borel σ-algebra corresponding to the d-topology coincides with $\mathcal{B}(S)$, because S is Suslin (Corollary 2 of Theorem II.10 of Schwartz [42]).

For each player $t \in T$, consider the payoff function $V_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$, defined by

$$\forall (s, \bar{f}, y) \in S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m, \quad V_t(s, \bar{f}, y) := -d(s, M_t(\bar{f}, y)).$$ [11]

Claim 1. *For every* $t \in T$, *the function* V_t *is upper-semicontinuous.*

Proof of Claim 1. Let $t \in T$ and $c \in \mathbb{R}$, we have to prove that

$$L := \{(s, \bar{f}, y) \in S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \colon d(s, M_t(\bar{f}, y)) \leqslant c\}$$

is closed. Let $(s_n, \bar{f}_n, y_n)$ be a sequence in L which converges to $(s, \bar{f}, y) \in S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$. For each $n > 0$, there exists $\tilde{s}_n$ in $M_t(\bar{f}_n, y_n)$ such that $d(s_n, \tilde{s}_n) \leqslant c + 1/n$. Since S_t is compact, passing to a subsequence if necessary, we can suppose that $(\tilde{s}_n)$ is convergent to some $\tilde{s}$ in S_t. From Assumption 2.5, the graph of the correspondence M_t is closed,[12] which implies that $\tilde{s}$ belongs to $M_t(\bar{f}, y)$. Hence we have

$$d(s, M_t(\bar{f}, y)) \leqslant d(s, \tilde{s}) = \lim_n d(s_n, \tilde{s}_n) \leqslant c. \qquad \square$$

[11] If A is a subset of S and $s \in S$ then $d(s, A) := \inf\{d(s, a) : a \in A\}$.

[12] From Assumptions 2.2 and 2.5, the correspondence M_t is upper-semicontinuous with non-empty closed values in the compact set S_t. It is well known (see Florenzano [14, Appendix A]) that this condition is sufficient and necessary for the graph of the correspondence M_t to be closed.

Claim 2. *For each* $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, *the restriction of the function* $(t, s) \mapsto V_t(s, \bar{f}, y)$ *to* gph Σ *is* $\mathcal{D}$*-measurable.*

Proof of Claim 2. Let $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, we let $F : T \to 2^S$ be the correspondence defined by $F(t) := M_t(\bar{f}, y)$. From Assumption 2.6 the correspondence F has a measurable graph. From Castaing–Valadier [11, Theorem III.22] (see also Sainte Beuve [39]), there exists a sequence (σ_n) of measurable functions $\sigma_n : T \to S$ such that $(\sigma_n(t))$ is dense in $F(t)$ for every $t \in T$. It follows that for each $s \in S$,

$$d(s, F(t)) = \inf\{d(s, \sigma_n(t)) : n \in \mathbb{N}\}.$$

Hence for each $s \in S$ the function $t \mapsto d(s, F(t))$ is measurable, and for each $t \in T$, the function $s \mapsto d(s, F(t))$ is continuous. Applying Lemma III.14 in Castaing–Valadier [11], we get that $(t, s) \mapsto d(s, F(t))$ is $\mathcal{T} \otimes \mathcal{B}(S)$-measurable. Since $F(t)$ is a subset of S_t, it follows that the restriction of $(t, s) \mapsto d(s, F(t))$ to gph Σ is $\mathcal{D}$-measurable. □

Claim 3. *For every* $t \in T$, *for every* $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$,

$$\operatorname{argmax}_{s \in S_t} V_t(s, \bar{f}, y) = M_t(\bar{f}, y).$$

Proof of Claim 3. Let $t \in T$ and $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ be fixed. From Assumption 2.5, the set $M_t(\bar{f}, y)$ is non-empty, hence there exists $\sigma \in S_t$ such that $V_t(\sigma, \bar{f}, y) = 0$. Now since $V_t(s, \bar{f}, y) \leqslant 0$ for each $s \in S_t$, we have that

$$\operatorname{argmax}_{s \in S_t} V_t(s, \bar{f}, y) = \{s \in S_t\colon d(s, M_t(\bar{f}, y)) = 0\}.$$

From Assumption 2.5, the set $M_t(\bar{f}, y)$ is closed. Hence the claim follows. □

We consider now the truly non-cooperative game $\Gamma' := (T, \Sigma, A', P')$ defined by

$$\forall (\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m, \quad A'_t(\bar{f}, y) := S_t$$

and

$$\forall s \in S_t, \quad P'_t(s, \bar{f}, y) := \{s' \in S_t\colon V_t(s', \bar{f}, y) > V_t(s, \bar{f}, y)\}.$$

We claim that the game Γ' satisfies Assumptions 2.2.5–2.2.7 in Balder [8]. Indeed, the game Γ' is truly non-cooperative, in the sense that for each $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, for every $t \in T$, one has $A'_t(\bar{f}, y) = S_t$. Then Assumption 2.2.5 in [8] is trivially satisfied. From Claims 1 and 2, Assumption 2.2.6 in [8] is satisfied. Observe that for every $t \in T$, the function

$$(\bar{f}, y) \longmapsto \sup\{V_t(s, \bar{f}, y)\colon s \in S_t\}$$

is identical to zero, so Assumption 2.2.7(i) in [8] is trivially satisfied. Assumption 2.2.7(ii) in [8] follows from Claim 3 and Assumption 2.4.

Now we can apply Theorem 2.2.1 in Balder [8] to get the existence of a Cournot–Nash equilibrium $f_\star \in \mathcal{S}_\Sigma$ for the game Γ', i.e. for almost every $t \in T$,

$$f_\star(t) \in \operatorname{argmax}_{s \in S_t} V_t(s, d(f_\star)).$$

From Claim 3, one has $f_\star(t) \in M_t(d(f_\star))$ for a.e. $t \in T$. This means that $f_\star$ is a Cournot–Nash equilibrium for the game Γ. $\square$

Remark 2.4. Following the arguments in Balder [8, Section 4.3] we may prove, as a corollary of our pure Cournot–Nash equilibrium existence result (Theorem 2.1), a generalization of the mixed Cournot–Nash equilibrium existence result by Balder [8, Theorem 2.1] to games with non-ordered preferences.

Remark 2.5. The proof of Theorem 2.1 proposed in this paper relies on Theorem 2.2.1 in Balder [8] which is proved by means of Young measure theory. An independent proof purely by means of the feeble topology is possible (see e.g. Martins-da-Rocha and Topuzu [29]). It was already announced in Balder [6, Section 5] and Balder [8, Remark 4.3.1] that such a proof was possible for ordered preferences. However, the proof we propose in [29] deals not only with ordered preferences but also with non-ordered preferences.

3. Assumptions on primitives

Let $\Gamma = (T, \Sigma, A, P)$ be a game. We provide in this section conditions on the primitives A and P under which Assumptions 2.5 and 2.6 are satisfied. We first consider a list of assumptions on the correspondence A of socially feasible actions.

Assumption 3.1. For every $(t, \bar{f}, y) \in T \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$,

(i) the set $A_t(\bar{f}, y)$ is a non-empty and closed subset of S_t;
(ii) the correspondence $A_t : \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to 2^{S_t}$ is upper-semicontinuous;
(iii) the set $\{(t, s) \in \operatorname{gph}\Sigma : s \in A_t(\bar{f}, y)\}$ belongs to $\mathcal{D}$.

Remark 3.1. Assumption 3.1 coincides with Assumption 2.2.5 in Balder [8].

We consider now a list of assumptions on the *preferred to* correspondence P.

Assumption 3.2. For every $(t, \bar{f}, y) \in T \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$,

(i) for every $s \in S_t$, one has $s \notin P_t(s, \bar{f}, y)$ and one of the two following conditions is satisfied:
  a. the correspondence $s \mapsto P_t(s, \bar{f}, y)$ is transitive, [13]
  b. $A_t(\bar{f}, y)$ is convex and $s \notin \operatorname{co}P_t(s, \bar{f}, y)$ for each $s \in S_t$;
(ii) the correspondence $s \mapsto P_t(s, \bar{f}, y)$ has open lower-sections;
(iii) the set $\{(s, \bar{g}, z) \in S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m : P_t(s, \bar{g}, z) \cap A_t(\bar{g}, z) \neq \emptyset\}$ is open;
(iv) the graph of the correspondence $(\tau, s) \mapsto P_\tau(s, \bar{f}, y)$ belongs to $\mathcal{T} \otimes \mathcal{B}(S) \otimes \mathcal{B}(S)$;
(v) the correspondence $s \mapsto P_t(s, \bar{f}, y)$ has open upper-sections.

Remark 3.2. Following Proposition A.1 in Appendix A, condition (iii) in Assumption 3.2 may be replaced by one of the three following conditions:

(iii.1) the lower-sections and upper-sections of the correspondence P_t are open, and $A_t(\bar{f}, y)$ is the closure of $B_t(\bar{f}, y)$ where $B_t : \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to 2^{S_t}$ has open lower-sections;

[13] The correspondence $s \mapsto P_t(s, \bar{f}, y)$ is transitive if for each (s, s', s'') in S_t^3, $[s' \in P_t(s, \bar{f}, y)$ and $s'' \in P_t(s', \bar{f}, y)]$ implies $s'' \in P_t(s, \bar{f}, y)$.

(iii.2) the graph of the correspondence $P_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to 2^{S_t}$ is open and the correspondence A_t is lower-semicontinuous;

(iii.3) the correspondence P_t is lower-semicontinuous and $A_t(\bar{f}, y) = S_t$.

Conditions similar to (iv) and (v) in Assumption 3.2 also appear in Hildenbrand [16], Cornet et al. [12] and Martins-da-Rocha [27,28].

We provide a corollary of Theorem 2.1 for games with possibly non-ordered preferences satisfying a list of conditions on primitives.

Corollary 3.1. *Let* $\Gamma = (T, \Sigma, A, P)$ *be a game satisfying Assumptions* 2.1–2.4, 3.1 *and* 3.2. *Then the game* Γ *has a pure Cournot–Nash equilibrium.*

Proof. We can check that under Assumptions 2.1 and 2.2, Assumptions 3.1 and 3.2 imply Assumptions 2.5 and 2.6.

First, we prove that Assumption 3.1(iii) and Assumptions 3.2(iv–v) imply Assumption 2.6. Indeed, let $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$ be fixed. For simplicity, the sets $A_t(\bar{f}, y)$, $P_t(s, \bar{f}, y)$ and $M_t(\bar{f}, y)$ are denoted by A_t, $P_t(s)$ and M_t. From Castaing–Valadier [11, Theorem III.22], there exists a sequence (g_n) of measurable functions $g_n : T \to S$ such that $(g_n(t))$ is dense in A_t for every $t \in T$. From Assumption 3.2(v) the set $P_t(s)$ is open in S_t for each $(t, s) \in \operatorname{gph} \Sigma$. This implies that the set

$$G := \left\{(t, s) \in \operatorname{gph} \Sigma \colon\ P_t(s) \bigcap A_t \neq \emptyset\right\}$$

coincides with the set

$$\bigcup_{n \in \mathbb{N}} \{(t, s) \in \operatorname{gph} \Sigma \colon\ g_n(t) \in P_t(s)\}.$$

As a consequence of Assumption 3.2(iv), the set G belongs to $\mathcal{D}$. Observing that $\operatorname{gph} A \setminus \operatorname{gph} M = (\operatorname{gph} A) \cap G$, it follows from Assumption 3.1(iii) that $\operatorname{gph} M$ belongs to $\mathcal{D}$.

Second, following standard arguments (see e.g. Yannelis and Prabhakar [46]), Assumption 3.1(i) and Assumption 3.2(i–ii) imply that for each $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, the set $M_t(\bar{f}, y)$ is a non-empty closed subset of S_t.

Finally, Assumption 3.1(ii) and Assumption 3.2(iii) imply that the correspondence M_t is upper-semicontinuous. □

Remark 3.3. Corollary 3.1 generalizes the equilibrium existence results for games with countably many agents and non-ordered preferences provided in Yannelis and Prabhakar [46, Theorem 6.1]. In [46] it is only assumed that each action set S_t is a non-empty compact convex metrizable subset of a locally convex topological vector space S. Observe that an obvious extension, inspired by the place where the Suslin property of the action set is used in the proof of Theorem 2.2.1 in Balder [8], is as follows. One could introduce two separate action universes, viz. S^{na} (for players in the non-atomic part of $(T, \mathcal{T}, \mu)$) and S^{pa} (for players in the pure atomic part of $(T, \mathcal{T}, \mu)$). In such a setup, only S^{na} would have to satisfy the Suslin property, and S^{pa} could be any locally convex topological vector space S such that each action set S_t, for each $t \in T^{pa}$, is a non-empty compact convex metrizable subset of S.

Remark 3.4. In the proof of Corollary 3.1, Assumption 3.2(v) was only used to prove the validity of Assumption 2.6. Actually when each agent is endowed with a payoff function, Assumption 3.2(v) is superfluous. [14]

We provide now a corollary of Corollary 3.1 for games with payoff functions.

Assumption 3.3. For each $t \in T$, agent t is endowed with a payoff function $U_t : S_t \times \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m \to [-\infty, +\infty]$ such that

(i) the function U_t is continuous;
(ii) for every $(\bar{f}, y) \in \overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$, the function $(\tau, s) \mapsto U_\tau(s, \bar{f}, y)$ is $\mathcal{D}$-measurable.

Corollary 3.2. *Let* $\Gamma = (T, \Sigma, A, P)$ *be a game satisfying Assumptions* 2.1–2.4, 3.1 *and* 3.3. *Then the game* Γ *has a pure Cournot–Nash equilibrium.*

Proof. We can check that under Assumptions 2.1–2.4 and Assumption 3.1, Assumption 3.3 implies Assumption 3.2. Indeed, since the *preferred to* correspondences are defined by payoff functions, Assumption 3.2(i) is automatically satisfied. Assumptions 3.2(ii) and 3.2(v) are a direct consequence of Assumption 3.3(i). Assumption 3.2(iv) follows from Assumption 3.3(ii). Assumption 3.2(iii) follows from Assumptions 3.3(i) and 3.1.

Remark 3.5. Corollary 3.2 coincides with Theorem 2.1 in Balder [6]. In particular, Corollary 3.1 generalizes Theorem 2.1 in Balder [6] to games with possibly non-ordered preferences.

Remark 3.6. Following Remark 3.4 we can replace condition (i) of Assumption 3.3 by the following weaker condition

(i′) the function U_t is upper-semicontinuous and the function

$$(\bar{f}, y) \mapsto \sup_{s \in A_t(\bar{f}, y)} U_t(s, \bar{f}, y)$$

is lower-semicontinuous on $\overline{\mathcal{S}}_\Sigma \times \mathbb{R}^m$.

Then we also obtain Theorem 2.2.1 in Balder [8] as a corollary of Theorem 2.1.

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[14] For simplicity, we omit to specify the parameter $(\bar{f}, y)$. Assume that there exists a function $U : \operatorname{gph} \Sigma \to [-\infty, +\infty]$ such that $P_t(s) = \{s' \in S_t : U(t, s') > U(t, s)\}$. Then $M_t = \{s \in A_t : U(t, s) \geqslant V(t)\}$ where $V(t) = \sup\{U(t, s') : s' \in A_t\}$. Assumption 3.2(iv) implies that the function U is $\mathcal{D}$-measurable. Applying Lemma III.39 in Castaing–Valadier [11] the function V is $\mathcal{A}$-measurable (the completeness of $\mathcal{A}$ plays a crucial role for this result). It then follows that $\operatorname{gph} M \in \mathcal{D}$.

Appendix A. Continuous correspondences

Let X and Y two semimetrizable topological vector spaces and F a correspondence from X to Y.

Definition A.1. The correspondence F is said: to be *upper-semicontinuous* if for each open set $V \subset Y$ the set $\{x \in X \colon F(x) \subset V\}$ is open; to have a *closed graph* if the set $\{(x, y) \in X \times Y \colon y \in F(x)\}$ is closed.

Remark A.1. We recall the following well-known equivalence result (see e.g. Florenzano [14, Appendix A]). If there exists a compact set $K \subset Y$ such that for each $x \in X$, $F(x) \subset K$ then the correspondence F is upper-semicontinuous with closed values if and only if it has a closed graph.

Definition A.2. The correspondence F is said: to have an *open graph* if $\{(x, y) \in X \times Y \colon\ y \in F(x)\}$ is open; to have *open lower-sections* if for each $y \in Y$, the set $\{x \in X \colon F(x) \ni y\}$ is open; to have *open upper-sections* if for each $x \in Y$, the set $F(x)$ is open; and to be *lower-semicontinuous* if for each non-empty open set $V \subset Y$ the set $\{x \in X \colon\ F(x) \cap V \neq \emptyset\}$ is open.

Proposition A.1. *Let S and Δ be Hausdorff topological spaces and consider a correspondence P from $S \times \Delta$ to S and A from Δ to S. Consider the following conditions*:

(iii.1) *the correspondence P has open lower-sections and open upper-sections and for each $\delta \in \Delta$, $A(\delta)$ is the closure of $B(\delta)$ where the correspondence $B : \Delta \to S$ has open lower-sections*;
(iii.2) *the graph of the correspondence $P : S \times \Delta \to S$ is open and the correspondence A is lower-semicontinuous*;
(iii.3) *the correspondence P is lower-semicontinuous and $A(\delta) = S$ for each $\delta \in \Delta$.*

Then each one of the above conditions imply the following one:

(iii) *the set $\{(s, \delta) \in S \times \Delta :\ P(s, \delta) \cap A(\delta) \neq \emptyset\}$ is open.*

The proof of Proposition A.1 is mostly inspired by Yannelis and Prabhakar [46].

Proof of: (iii.1) implies (iii). For each $(s, \delta) \in S \times \Delta$, $P(s, \delta)$ is open hence $P(s, \delta) \cap A(\delta) \neq \emptyset$ if and only if $P(s, \delta) \cap B(\delta) \neq \emptyset$. Let $(s, \delta) \in S \times \Delta$ such that $P(s, \delta) \cap B(\delta) \neq \emptyset$ and let $\sigma \in S$ be such that $\sigma \in P(s, \delta) \cap B(\delta)$. Since P has open lower-sections there exist open sets U and V in S and Δ such that $(s, \delta) \in U \times V$ and for each $(s', \delta') \in U \times V$, $P(s', \delta') \ni \sigma$. Since B has open lower-sections there exists an open set W in Δ such that $\delta \in W$ and for each $\delta' \in W$, $B(\delta') \ni \sigma$. It then follows that $(s, \delta) \in U \times (V \cap W)$ and for each $(s', \delta') \in U \times (V \cap W)$, $P(s', \delta') \cap B(\delta') \neq \emptyset$. $\square$

Proof of: (iii.2) implies (iii). Let $(s, \delta) \in S \times \Delta$ such that $P(s, \delta) \cap A(\delta) \neq \emptyset$. Let $\sigma \in S$ be such that $\sigma \in P(s, \delta) \cap A(\delta)$. Since P has an open graph, there exists two open sets U, V in S and an open set W in Δ such that $(s, \delta, \sigma) \in U \times W \times V$ and for each $(s', \delta', \sigma') \in U \times W \times V$, $P(s', \delta') \ni \sigma'$. The correspondence A is lower semi-continuous hence the set $\{\delta' \in \Delta : A(\delta') \cap V \neq \emptyset\}$ is open. Therefore there exists an open set W' in Δ such that $\delta \in W'$ and for each $\delta' \in W'$, $A(\delta') \cap V \neq \emptyset$. It then follows that $(s, \delta) \in U \times (W \cap W')$ and for each $(s', \delta') \in U \times (W \cap W')$, $P(s', \delta') \cap A(\delta') \neq \emptyset$. $\square$

Proof of: (iii.3) implies (iii). For each $(s,\delta) \in S \times \Delta$, $P(s,\delta) \cap A(\delta) \neq \emptyset$ if and only if $P(s,\delta) \cap S \neq \emptyset$. The set S is open, hence (iii) follows from the lower semi-continuity of the correspondence P. $\square$

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