Ann Finance (2012) 8:455–488
DOI 10.1007/s10436-012-0209-y

RESEARCH ARTICLE

On Ponzi schemes in infinite horizon collateralized economies with default penalties

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Received: 11 November 2011 / Accepted: 17 September 2012 / Published online: 17 October 2012

Abstract We show, by means of an example, that in models where default is subject to both collateral repossession and utility punishments, opportunities for doing Ponzi schemes are not always ruled out and (refined) equilibria may fail to exist. This is true even if default penalties are moderate as defined in Páscoa and Seghir (Game Econ Behav 65:270–286, 2009). In our example, asset promises and default penalties are chosen such that, if an equilibrium does exist, agents never default on their promises. At the same time collateral bundles and utility functions are such that the full repayment of debts implies that the asset price should be strictly larger than the cost of collateral requirements. This is sufficient to induce agents to run Ponzi schemes and destroy equilibrium existence.

Keywords Collateral · Default penalties · Ponzi schemes

JEL Classification D52 · D91

1 Introduction

In infinite horizon competitive economies with full commitment, it is well-known that Ponzi schemes must be ruled out in order to guarantee the existence of equilibria. In an environment without commitment, Araujo et al. (2002) showed that Ponzi schemes

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are ruled out (and therefore an equilibrium always exists) if agents are forced to hold collateral when they take debt positions. Páscoa and Seghir (2009) subsequently presented two examples to show that if, in addition to collateral repossession, agents suffer harsh utility penalties when they default, then Ponzi schemes may reappear and equilibria fail to exist. Martins-da Rocha and Vailakis (2012) provided a counterexample to this claim by showing that the economies considered in those examples do have an equilibrium with no trade. This is because the standard equilibrium concept leaves room for spurious inactivity on asset markets when agents have unduly pessimistic expectations about asset deliveries. In particular, in the definition of the competitive equilibrium, the market clearing equation defining the expected delivery rate leaves its value undeterminate when there is no trade.

The finding in Martins-da Rocha and Vailakis (2012) suggests that in models where default is subject to both collateral repossession and utility punishments the equilibrium concept has to be refined to eliminate "undesirable" no-trade equilibria. The paper proposes to adapt the refinement concept of Dubey et al. (2005) to the setting studied by Páscoa and Seghir (2009).

Ferreira and Torres-Martínez (2010) showed that equilibria may fail to exist in the presence of additional (to collateral) enforcement mechanisms, provided that those mechanisms are effective, i.e., they induce payments besides the value of the collateral. When payments exceed the value of the collateral requirements, by non-arbitrage, lenders lend more than the value of those guarantees and, therefore, borrowers may run Ponzi schemes.

Páscoa and Seghir (2009) claimed that moderate default penalties are compatible with equilibrium existence. Our contribution amounts to show that this is not the case. We prove our claim by presenting an example in which, due to moderate default penalties, opportunities for doing Ponzi schemes are not always ruled out and refined equilibria fail to exist.[1] In our example, the default penalties, although moderate, they are severe enough to induce borrowers (at any candidate refined equilibrium) to pay fully their debt at every period.[2]

Choosing the asset promises to be greater than the value of depreciated collateral, we get that lenders always expect to receive more than the value of the depreciated collateral (actually they expect to get fully repaid).[3] For the borrowers, the joint operation of selling the asset and buying the collateral generates utility gains due to the consumption of the collateral good, but also yields negative returns since what they have to repay exceeds the value of depreciated collateral they already posses. We show that it is possible to specify the preference relations and collateral levels in a way that the loss in utility due to the realization of those negative returns is not compensated

[1] In that respect, we provide a counterexample to the existence result claimed in Páscoa and Seghir (2009). If the arguments in the Proof of Theorem 4.1 in Páscoa and Seghir (2009) were correct, then we would get existence of a refined equilibrium when default penalties are moderate.

[2] In other words, we show that moderate penalties can be an effective mechanism (as defined in Ferreira and Torres-Martínez (2010)).

[3] This is not true for equilibria that are not refined since lenders may expect the asset to deliver nothing above the depreciated value of the collateral, despite the fact that default penalties would induce agents to repay fully their debt in case of trade.

by the gains in utility due to the consumption of collateral. Therefore, if a refined equilibrium has to exist asset prices must exceed the price of collateral costs. In this case, as Páscoa and Seghir (2009) already explained, agents are induced to run Ponzi schemes that are not consistent with optimality of plans in a refined equilibrium. Therefore, refined equilibria cannot exist.

Once we know that moderate penalties are effective, a possible route to show that Ponzi schemes reappear consists of looking for restrictions on collateral levels and endowments such that the condition of Theorem 1 in Ferreira and Torres-Martínez (2010) is satisfied. In particular, we would need to assume either that collateral requirements or aggregate endowments vanish in the long run. We opt for a different and direct approach that allows for collateral bundles and aggregate endowments to be uniformly bounded away from zero. A detailed discussion on this issue is provided in Remark 4.4.

The paper is structured as follows. In Sect. 2 we present a simplified version of the model proposed by Páscoa and Seghir (2009) and define the associated equilibrium concept. In Sect. 3 we explain why unduly pessimistic expectations may render the asset market inactive, therefore make it necessary to refine the standard equilibrium concept. A refinement of equilibrium is subsequently proposed in the spirit of Dubey et al. (2005). Section 4 contains our main contribution. It presents an example of an economy with moderate default penalties in which agents are induced to run Ponzi schemes and a refined equilibrium fails to exist. Section 5 concludes in Appendix A we propose a sufficient condition on penalties that rules out Ponzi schemes and we provide a sketch of the proof of the corresponding equilibrium existence result. Appendix B presents necessary and sufficient conditions for the existence of Lagrange multipliers.

2 The model

Páscoa and Seghir (2009) considered an extension of the model developed by Araujo et al. (2002) to allow for the possibility of linear default penalties. Since our aim is to provide an example of non-existence of a refined equilibrium we abstract from a full presentation of their setting and rather consider a specific infinite horizon economy $\mathcal{E}$ without uncertainty and with one short-lived asset. The set $\{0, 1, \ldots, t, \ldots\}$ of time periods is denoted by $\mathcal{T}$.

2.1 Agents and commodities

There exists a finite set L of commodities available for trade at every period. We interpret $x_t \in \mathbb{R}^L_+$ as a claim to consumption at period t. We also write $\mathbf{1}_{\{\ell\}} \in \mathbb{R}^L_+$ for the commodity bundle consisting of one unit of commodity $\ell \in L$ and nothing else. We allow for some commodities to be *non-perishable*, that is, we allow for storable and durable goods. Transformation of commodities is represented by a family $(Y_t)_{t\in\mathcal{T}}$ of linear functionals Y_t from $\mathbb{R}^L_+$ to $\mathbb{R}^L_+$. The bundle $Y_t z_{t-1}$ represents what is obtained at period $t \geqslant 1$ if the bundle $z_{t-1} \in \mathbb{R}^L_+$ is purchased at period $t-1$. At each period there are spot markets for trading every commodity. We let $p = (p_t)_{t\in\mathcal{T}}$ denote the sequence of spot prices where $p_t = (p_t(\ell))_{\ell\in L} \in \mathbb{R}^L_+$ is the price vector at period t.

There is a finite set I of infinitely lived agents. Each agent $i \in I$ is characterized by an endowment sequence $\omega^i = (\omega^i_t)_{t\in\mathcal{T}}$ where $\omega^i_t = (\omega^i_t(\ell))_{\ell\in L} \in \mathbb{R}^L_+$ denotes the endowment bundle available at period t. Each agent chooses a consumption sequence $x = (x_t)_{t\in\mathcal{T}}$ where $x_t \in \mathbb{R}^L_+$. We denote by X the set of consumption sequences. The utility function $U^i : X \longrightarrow [0, +\infty]$ is assumed to be time-additively separable, i.e.,

$$U^i(x) = \sum_{t\in\mathcal{T}} [\beta_i]^t v^i_t(x_t)$$

where $v^i_t : \mathbb{R}^L_+ \longrightarrow [0, \infty)$ represents the instantaneous utility function at period t and $\beta_i \in (0, 1)$ is the discount factor.

2.2 Assets and collateral

There is a single asset which is a short-lived real security available for trade at each period t, paying the dividend $p_{t+1}A_{t+1}$ that corresponds to the value of a bundle $A_{t+1} \in \mathbb{R}^L_+$ under the spot price vector p_{t+1}. We let $q = (q_t)_{t\in\mathcal{T}}$ be the asset price sequence where $q_t \in \mathbb{R}_+$ represents the asset price at period t. For each agent i, we denote by $\theta^i_t \in \mathbb{R}_+$ the purchases and by $\varphi^i_t \in \mathbb{R}_+$ the short-sales of the asset at each period t.

The asset is collateralized in the sense that for every unit of asset sold at a period t, agents should buy a collateral $C_t \in \mathbb{R}^L_+$ that protects lenders in case of default. Implicitly we assume that payments can be enforced through the seizure of the collateral. At a period $t \geqslant 1$, agent i should deliver the promise $V_t(p)\varphi^i_{t-1}$ where $V_t(p) = p_t A_t$. However, agent i may decide to default and choose a delivery d^i_t in units of account. Since the collateral can be seized, this delivery must satisfy $d^i_t \geqslant D_t(p)\varphi^i_{t-1}$ where

$$D_t(p) = \min\{p_t A_t, p_t Y_t C_{t-1}\}.$$

Following Dubey et al. (2005); Páscoa and Seghir (2009) assume that each agent i feels at period t a disutility from defaulting which is represented by a linear function of the extent of default. More precisely, if agent i decides to deliver d^i_t at period t given promises φ^i_{t-1} made at $t-1$, then he suffers at the initial date, the disutility

$$[\beta_i]^t \mu^i_t \frac{\left[V_t(p)\varphi^i_{t-1} - d^i_t\right]^+}{p_t w_t},$$

where $\mu^i_t \in [0, \infty]$ and $p_t w_t$ is the market value of an exogenously given bundle w_t.[4] In that case, agent i may have an incentive to deliver more than the minimum between his debt and the depreciated value of his collateral, i.e., we may have $d^i_t > D_t(p)\varphi^i_{t-1}$. The asset is thought as a pool, i.e., at each period t there is a delivery rate $\kappa_t \in [0, 1]$ that summarizes all different sellers' deliveries. By purchasing one unit of the asset,

[4] The unitary default penalty μ^i_t represents the instantaneous disutility from defaulting in real terms the market value of the bundle w_t.

the lenders correctly anticipate to receive the fraction $V_t(\kappa, p)$ defined by[5]

$$V_t(\kappa, p) = \kappa_t V_t(p) + (1 - \kappa_t) D_t(p).$$

Along the paper we will use repeatedly the following notations. We let A be the space of sequences $a = (a_t)_{t \in \mathcal{T}}$ with[6]

$$a_t = (x_t, \theta_t, \varphi_t, d_t) \in \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+.$$

For each period t, we denote by A^t the set of plans $a \in A$ such that $a_\tau = (0, 0, 0, 0)$ for each $\tau > t$. If a is a plan in A and t is a period, we denote by $a\mathbf{1}_{[0,t]}$ the plan in A^t which coincides with a for every period $\tau \in \{0, \dots, t\}$. We denote by B^t the set of plans a in A^t satisfying $\varphi_t = 0$.

2.3 Budget constraints

In each decision period $t \in \mathcal{T}$, agent i's choice $a^i = (x^i, \theta^i, \varphi^i, d^i) \in A$ must satisfy the following constraints:

(a) solvency constraint:

$$p_t x_t^i + d_t^i + q_t \theta_t^i \leqslant p_t[\omega_t^i + Y_t x_{t-1}^i] + V_t(\kappa, p)\theta_{t-1}^i + q_t \varphi_t^i; \tag{2.1}$$

(b) collateral requirement:

$$C_t \varphi_t^i \leqslant x_t^i; \tag{2.2}$$

(c) minimum delivery:

$$D_t(p)\varphi_{t-1}^i \leqslant d_t^i. \tag{2.3}$$

2.4 The payoff function

Assume that $\pi = (p, q, \kappa)$ is a sequence of prices and delivery rates. Consider that agent i has chosen the plan $a = (x, \theta, \varphi, d) \in A$. He gets the utility $U^i(x) \in [0, \infty]$ defined by

$$U^i(x) = \sum_{t \in \mathcal{T}} [\beta_i]^t v_t^i(x_t)$$

[5] One of the equilibrium conditions will require that lenders' expected return $V_t(\kappa, p)$ coincides with the actual deliveries of the borrowers in the sense that

$$\sum_{i \in I} V_t(\kappa, p)\theta_{t-1}^i = \sum_{i \in I} d_t^i.$$

[6] By convention we let $a_{-1} = (x_{-1}, \theta_{-1}, \varphi_{-1}, d_{-1}) = (0, 0, 0, 0)$.

but he suffers the disutility $W^i(p,a) \in [0,\infty]$ defined by

$$W^i(p,a) = \sum_{t\geqslant 1} [\beta_i]^t \mu_t^i \frac{\left[V_t(p)\varphi_{t-1} - d_t\right]^+}{p_t w_t}.$$

We would like to define the payoff $\Pi^i(p,a)$ of the plan a as $U^i(x) - W^i(p,a)$. Unfortunately, this difference may not be well-defined if $U^i(x)$ and $W^i(p,a)$ are both infinite.[7] We propose to consider the binary relation $\succ_{i,p}$ defined on A by

$$\widetilde{a} \succ_{i,p} a \iff \exists \varepsilon > 0, \quad \exists T \in \mathcal{T}, \quad \forall t \geqslant T, \quad \Pi^{i,t}(p,\widetilde{a}) \geqslant \Pi^{i,t}(p,a) + \varepsilon$$

where

$$\Pi^{i,t}(p,a) = U^{i,t}(x) - W^{i,t}(p,a), \quad U^{i,t}(x) = \sum_{0\leqslant\tau\leqslant t} [\beta_i]^\tau v_\tau^i(x_\tau)$$

and

$$W^{i,t}(p,a) = \sum_{1\leqslant\tau\leqslant t} [\beta_i]^\tau \mu_\tau^i \frac{\left[V_\tau(p)\varphi_{\tau-1} - d_\tau\right]^+}{p_\tau w_\tau}.$$

Observe that if $\Pi^i(p,\widetilde{a})$ and $\Pi^i(p,a)$ exist in $\mathbb{R}$ then $\widetilde{a} \succ_{i,p} a$ if and only if $\Pi^i(p,\widetilde{a}) > \Pi^i(p,a)$. The set $\mathrm{Pref}^i(p,a)$ of plans strictly preferred to plan a by agent i is defined by $\mathrm{Pref}^i(p,a) = \{\widetilde{a} \in A : \widetilde{a} \succ_{i,p} a\}$.

2.5 The equilibrium concept

We denote by Ξ the set of sequences of prices and delivery rates (p,q,κ) satisfying for all $t \in \mathcal{T}$

$$(p_t, q_t, \kappa_t) \in \mathbb{R}^L_{++} \times \mathbb{R}_+ \times [0,1] \quad \text{and} \quad \sum_{\ell\in L} p_t(\ell) + q_t = 1. \tag{2.4}$$

Given a sequence (p,q,κ) of commodity prices, asset prices and delivery rates, we denote by $B^i(p,q,\kappa)$ the set of plans $a = (x,\theta,\varphi,d) \in A$ satisfying constraints (2.1), (2.2) and (2.3).

Definition 2.1 A *competitive equilibrium* for the economy $\mathcal{E}$ is a family of prices and delivery rates $(p,q,\kappa) \in \Xi$ and an allocation $\boldsymbol{a} = (a^i)_{i\in I}$ with $a^i \in A$ such that

(a) for every agent i, the plan a^i is optimal among the budget feasible plans, i.e.,

$$a^i \in B^i(p,q,\kappa) \quad \text{and} \quad \mathrm{Pref}^i(p,a^i) \cap B^i(p,q,\kappa) = \emptyset; \tag{2.5}$$

[7] This issue is ignored by Páscoa and Seghir (2009).

(b) commodity markets clear at every period, i.e.,[8]

$$\forall t \in \mathcal{T}, \quad \sum_{i \in I} x_t^i = \sum_{i \in I} \left[\omega_t^i + Y_t x_{t-1}^i\right]; \tag{2.6}$$

(c) the asset market clears at every period, i.e.,

$$\forall t \in \mathcal{T}, \quad \sum_{i \in I} \theta_t^i = \sum_{i \in I} \varphi_t^i; \tag{2.7}$$

(d) aggregate borrowers' deliveries match lenders' expectations, i.e.,

$$\forall t \geqslant 1, \quad \sum_{i \in I} V_t(\kappa, p)\theta_{t-1}^i = \sum_{i \in I} d_t^i. \tag{2.8}$$

The set of allocations $\boldsymbol{a} = (a^i)_{i \in I}$ in A satisfying the market clearing conditions (2.6) and (2.7) is denoted by F. Each allocation in F is called physically feasible. A plan $a^i \in A$ is called physically feasible if there exists a physically feasible allocation $\boldsymbol{b} \in$ F such that $a^i = b^i$.

2.6 Assumptions

For each agent i, we denote by $\Omega^i = (\Omega_t^i)_{t \in \mathcal{T}}$ the sequence of accumulated endowments, defined recursively by $\Omega_t^i = Y_t \Omega_{t-1}^i + \omega_t^i$ with $\Omega_0^i = \omega_0^i$. The sequence $\sum_{i \in I} \Omega^i$ of accumulated aggregate endowments is denoted by Ω. This section describes the assumptions imposed on the characteristics of the economy. It should be clear that these assumptions always hold throughout the paper.

Assumption 2.1 (Agents). For every agent i,

(A.1) the sequence of accumulated endowments is strictly positive and uniformly bounded from above, i.e.,

$$\exists \overline{\Omega}^i \in \mathbb{R}_{++}^L, \quad \forall t \in \mathcal{T}, \quad \Omega_t^i \in \mathbb{R}_{++}^L \quad \text{and} \quad \Omega_t^i \leqslant \overline{\Omega}^i;$$

(A.2) for every period t, the utility function v_t^i is concave, continuous and strictly increasing on $\mathbb{R}_+^L$ with $v_t^i(0) = 0$;

(A.3) the infinite sum $U^i(\Omega)$ is finite.[9]

[8] By convention $x_{-1}^i = 0$ and $Y_{-1} = 0$.

[9] This assumption is automatically satisfied if the sequence of functions $(v_t^i)_{t \in \mathcal{T}}$ is uniformly bounded from above by an increasing function $\overline{v}^i$. Indeed, in that case we have

$$U^i(\Omega) \leqslant \overline{v}^i(\overline{\Omega}) \sum_{t \geqslant 0} [\beta_i]^t < \infty$$

where $\overline{\Omega} = \sum_{i \in I} \overline{\Omega}^i$.

Assumption 2.2 (Collateral). For every period t, the collateral bundle C_t is not zero.

Remark 2.1 Let $(\pi, \boldsymbol{a})$ be a competitive equilibrium with $\pi = (p, q, \kappa) \in \Xi$. Since x^i is physically feasible, we have $0 \leqslant x_t^i \leqslant \Omega_t$. Following Assumption (A.2) and (A.3) we have

$$0 \leqslant U^i(x^i) \leqslant U^i(\Omega) < \infty.$$

This implies that $\Pi^i(p, a^i)$ belongs to $[-\infty, \infty)$. Each agent can choose to survive in autarky, i.e., the plan $\mathrm{aut}^i = (\Omega^i, 0, 0, 0)$ belongs to the budget set $B^i(\pi)$. This implies that we cannot have $\mathrm{aut}^i \succ_{i,p} a^i$. Since $W^i(p, \mathrm{aut}^i) = 0$, we must have $W^i(p, a^i) > -\infty$, implying that the payoff $\Pi^i(p, a^i)$ belongs to $\mathbb{R}$.

3 Equilibrium refinement

3.1 Indeterminacy of delivery rates and overpessimistic expectations

Let $(\pi, (a^i)_{i \in I})$ be a competitive equilibrium with $\pi = (p, q, \kappa) \in \Xi$ and $a^i = (x^i, \theta^i, \varphi^i, d^i)$. Fix a period $t \geqslant 1$. Since agent i delivers in period t at least the amount $D_t(p)\varphi_{t-1}^i$, we let $\sigma_t^i \in [0, 1]$ be the individual delivery rate defined by the equation

$$d_t^i = \left[\sigma_t^i V_t(p) + (1 - \sigma_t^i) D_t(p)\right] \varphi_{t-1}^i = \left[\sigma_t^i \{V_t(p) - D_t(p)\} + D_t(p)\right] \varphi_{t-1}^i$$

if agent i has a debt $\varphi_{t-1}^i > 0$, and we pose $\sigma_t^i = 0$ elsewhere. If there is trade in period $t - 1$, i.e., $\varphi_{t-1}^i > 0$ for some agent i, then Eq. (2.8) in the definition of a competitive equilibrium can be restated as follows

$$\kappa_t \sum_{i \in I} \varphi_{t-1}^i = \sum_{i \in I} \sigma_t^i \varphi_{t-1}^i$$

and κ_t can be interpreted as the average delivery rate (per unit of asset sold) above the minimum delivery $D_t(p)$. If there is no trade in period $t - 1$ then the delivery rate κ_t is undeterminate. That is, when the asset is not traded, our equilibrium concept makes no assumption about the expected delivery rate. We claim that pessimistic expectations about delivery (i.e., low values of κ_t) may by itself render the asset market inactive in period $t - 1$ if default penalties are large enough. Our finding shares some similarities with the issue of trivial equilibria pointed out by Dubey et al. (2005). To clarify this link we recall some notations. In Dubey et al. (2005) assets are not collateralized. The repayment rate, denoted by K_t, is defined by the equation

$$K_t V_t(p) \sum_{i \in I} \varphi_{t-1}^i = \sum_{i \in I} d_t^i.$$

As explained in Páscoa and Seghir (2009) (see Remark 3.1), when assets are collateralized agents deliver at least $D_t(p)$ per unit of asset sold. In this case, if $D_t(p)$ and $V_t(p)$ are not zero, rational agents expect K_t to be greater than the ratio $D_t(p)/V_t(p)$, and in particular it must be non-null.[10] This is the reason why in our model we have chosen to parameterize agents' expectations about delivery by the average delivery rate above the minimum delivery, denoted by κ_t. In other words, when there is trade in period $t-1$ we have the relation

$$K_t V_t(p) = \kappa_t\{V_t(p) - D_t(p)\} + D_t(p).$$

In Dubey et al. (2005) it is easy to support equilibria with no trade in the asset on account of absurdly pessimistic expectations about repayment rates. However, in a model with collateral requirements, it is not clear whether such equilibria can be supported.[11]

Martins-da Rocha and Vailakis (2012) presented two examples to argue that the unrefined equilibrium concept introduced by Páscoa and Seghir (2009) is not satisfactory since unreasonable pessimism may render the asset market inactive. The problem comes from Eq. (2.8) that leaves undeterminate agents' expectations about the delivery rate κ_t when there is no trade at equilibrium. What drives the existence of the no trade equilibrium outcome is a simultaneous wedge at autarky between the asset price and the values of the short and long positions. More precisely, in the proposed examples, it is possible to choose the asset price to lie between the value of the long position evaluated using the most pessimistic expectations about deliveries and the value of the short position in the presence of harsh default penalties. The loss of utility when defaulting is so severe that borrowers would fully repay their debts if the asset were traded. This seems to be inconsistent with lenders' pessimistic expectations about deliveries. However, the definition of an (unrefined) equilibrium does not impose any consistency condition on out of equilibrium paths. In particular, under no trade, expectations become indeterminate and the inconsistency is formally absent.

3.2 Removing overpessimistic expectations

It is thus important to refine the equilibrium concept in order to rule out spurious inactivity in asset markets. To address this issue, we follow Dubey et al. (2005) and introduce an equilibrium refinement in which the government intervenes to sell infinitesimal quantities $\varepsilon > 0$ of the asset at each period and fully delivers on its promises. Since the government does not default, it does not need to constitute collateral bundles. However, since it delivers fully $\varepsilon V_t(p)$ but it gets delivered only $\varepsilon V_t(\kappa, p)$, on net the government injects the vector of commodities $\varepsilon b_t(\kappa, p) w_t$ where $b_t(\kappa, p) \geqslant 0$ is defined by the equation

[10] If the promise bundle A_t and the depreciated collateral bundle $Y_t C_{t-1}$ are not zero then $D_t(p)$ (and consequently $V_t(p)$) are not zero since p_t is strictly positive.

[11] See Dubey et al. (2005) and the discussion in Martins-da Rocha and Vailakis (2012).

$$b_t(\kappa, p)p_t w_t = V_t(p) - V_t(\kappa, p).$$

This touch of honesty banishes whimsical pessimism and rules out spurious inactivity on asset markets. We adapt the definition of a competitive equilibrium with the government intervention proposed by Dubey et al. (2005) to our framework.

Definition 3.1 An ε-equilibrium is a family $\pi = (p, q, \kappa) \in \Xi$ of prices and delivery rates and an allocation $(a^i)_{i\in I}$ such that

(i) as in the standard competitive equilibrium concept, for every agent i the plan a^i is optimal among the budget feasible plans and the asset market clears at every period;
(ii) different to the standard competitive equilibrium, commodity markets ε-clear, i.e.,[12]

$$\forall t \in \mathcal{T}, \quad \sum_{i\in I} x_t^i = \sum_{i\in I} [\omega_t^i + Y_t x_{t-1}^i] + \varepsilon b_t(\kappa, p) w_t \tag{3.1}$$

and delivery rates are boosted by the external agent, i.e.,

$$\forall t \geqslant 1, \quad V_t(\kappa, p) \left[\varepsilon + \sum_{i\in I} \theta_{t-1}^i \right] = \varepsilon V_t(p) + \sum_{i\in I} d_t^i. \tag{3.2}$$

Equation (3.2) defining the delivery rate κ_t can be restated as follows

$$\forall t \geqslant 1, \quad \kappa_t \left[\varepsilon + \sum_{i\in I} \theta_{t-1}^i \right] = \varepsilon + \sum_{i\in I} \sigma_t^i \varphi_{t-1}^i$$

where σ_t^i is agent i's individual delivery rate as defined in Sect. 3.1. The delivery rate κ_t is the weighted average of individual rates and is boosted due to the fact that the government delivers fully on its promises. As the government intervention disappears, i.e., ε tends to 0, this boost disappears for periods where the asset is positively traded in the limit.

Definition 3.2 A competitive equilibrium $(\pi, (a^i)_{i\in I})$ is called a *refined equilibrium* if for every $\varepsilon > 0$ small enough there exists an ε-equilibrium $(\pi(\varepsilon), (a^i(\varepsilon))_{i\in I})$ such that

$$\lim_{\varepsilon\to 0} (\pi(\varepsilon), (a^i(\varepsilon))_{i\in I}) = (\pi, (a^i)_{i\in I}).$$

4 Ponzi schemes and non-existence of refined equilibrium

It is now natural to investigate under which conditions a refined equilibrium exists. To answer this question we should find first conditions under which an ε-equilibrium

[12] By convention we let $a_{-1} = (x_{-1}, \theta_{-1}, \varphi_{-1}, d_{-1}) = (0, 0, 0, 0)$ and $b_0(\kappa, p) = 0$.

exists. It is straightforward to adapt the arguments in Araujo et al. (2002); Dubey et al. (2005); Páscoa and Seghir (2009) to get existence (under standard assumptions) of an ε-equilibrium for economies with a finite horizon. One may then consider a sequence of finite-horizon ε-equilibria $(\pi^T, (a^{i,T})_{i\in I})_{T\in\mathcal{T}}$ where the horizon T tends to infinite. Following the arguments in Páscoa and Seghir (2009) it is straightforward to prove that there exists a subsequence converging to some $(\pi, (a^i)_{i\in I})$. In order to prove that the limit $(\pi, (a^i)_{i\in I})$ is an ε-equilibrium, the only difficulty is to show that the plan a^i is optimal in the budget set defined by the price sequence π.

Páscoa and Seghir (2009) claimed that a sufficient condition for optimality of the limiting plan a^i (and consequently for existence of an equilibrium) is to assume that default penalties are *moderate*, in the sense that, for infinitely many periods, the penalty associated with the maximal default for a physically feasible plan is less than the utility from consuming the current endowment.[13] This statement appears to be very intuitive: if default penalties are moderate then default does not hurt much and nothing prevent agents to fully default in the long run. Ponzi schemes should be avoided, since, after a while, the joint operation of short-selling an asset and purchasing the collateral bundle should not allow to transfer wealth between periods. In what follows we show, by means of a specific example, that although intuitive, the condition proposed by Páscoa and Seghir (2009) is not sufficient to ensure existence.[14] In the proposed example default penalties are moderate but severe enough to induce all agents (at an ε-equilibrium) to never default on their promises. At the same time collateral bundles and utility functions are such that the full repayment of debts implies that the asset price should be strictly larger than the cost of collateral requirements. This is sufficient to induce agents to run Ponzi schemes which are not compatible with the existence of an ε-equilibrium.

4.1 Moderate default penalties

Fix some $\varepsilon > 0$. We start by introducing some notation and making additional assumptions. We assume that there exists $b \geqslant 0$ such that for every period $t \geqslant 1$ we have $A_t \leqslant bw_t + Y_t C_{t-1}$. This implies that the maximal default $\overline{b}_t$ per unit of asset is bounded from above by b, i.e.,[15]

[13] Páscoa and Seghir (2009) assumed that default penalties are moderate and claimed in Theorem 4.1 that an equilibrium exists. Actually, the only difficult step (which is also the only step where the assumption of moderate penalties is used) of their proof consists of proving that if a sequence of finite horizon equilibria converges then for every agent, the limiting plan is optimal for the infinite horizon budget set. If their arguments were correct we would also get existence of an ε-equilibrium when default penalties are moderate since optimality of a plan among budget feasible plans is independent of whether we consider equilibria or ε-equilibria (individual demand sets coincide for both concepts).

[14] The mistake in the intuitive argument we provide (and in the proof proposed by Páscoa and Seghir (2009)) is that when contemplating an alternative budget feasible plan, an agent does not restrict his choices to be physically feasible. In particular, depending on the sequence of prices, a budget feasible plan may have a sequence of asset short-sales that is inconsistent with the scarcity of goods (recall that when short-selling an agent should constitute collateral in terms of goods).

[15] The set $\Delta(L)$ is the simplex in $\mathbb{R}^L_+$, i.e., $\Delta(L) = \{p \in \mathbb{R}^L_+ : \sum_{\ell\in L} p(\ell) = 1\}$.

$$\overline{b}_t \equiv \sup_{p \in \Delta(L)} \frac{\left[pA_t - pY_t C_{t-1}\right]^+}{pw_t} \leqslant b.$$

We denote by $W = (W_t)_{t \geqslant 1}$ the sequence defined recursively by $W_t = w_t + Y_t W_{t-1}$ where $W_1 = w_1$. We assume that there exists an upper bound $\overline{W}$ of the sequence W. For each period t, we denote by M_t the real number

$$\min_{\ell \in L} \frac{\Omega_t^\varepsilon(\ell)}{C_t(\ell)}$$

where Ω_t^ε is an upper bound on aggregate resources at period t defined recursively by[16]

$$\Omega_t^\varepsilon = \omega_t + \varepsilon \overline{b}_t w_t + Y_t \Omega_{t-1}^\varepsilon \quad \text{and} \quad \Omega_0^\varepsilon = \omega_0$$

where $\omega_t = \sum_{i \in I} \omega_t^i$. We let $\overline{\Omega}^\varepsilon$ be the uniform upper bound on aggregate resources defined by

$$\overline{\Omega}^\varepsilon \equiv \overline{\Omega} + \varepsilon b \overline{W}.$$

Observe that under Assumption 2.2, we have $M_t < \infty$. Finally, for every period $t \geqslant 1$ we let

$$H_t = M_{t-1} \overline{b}_t.$$

The quantity H_t is an upper bound of the amount in real terms that an agent may default if his plan is feasible in the ε-economy. The proof of the following proposition is straightforward and omitted.

Proposition 4.1 *If a in A is a plan physically feasible in the ε-economy and (p, q, κ) in Ξ is a sequence of prices and delivery rates, then for each period t, we have*

$$\varphi_t \leqslant M_t \quad \textit{and} \quad \frac{\left[V_t(p)\varphi_{t-1} - d_t\right]^+}{p_t w_t} \leqslant H_t.$$

Páscoa and Seghir (2009) introduced the concept of α-*moderate* default penalties that is adapted here to ε-economies. Fix a sequence $\alpha = (\alpha_t)_{t \in \mathcal{T}}$ with $\alpha_t \in (1, \infty)$.

Definition 4.1 Default penalties are said to be α-*moderate* with respect to utility functions, if for every agent i there exists an infinite subset Mod^i of $\mathcal{T}$ such that

$$\forall t \in \mathrm{Mod}^i, \quad \alpha_t \mu_t^i H_t \leqslant v_t^i(\omega_t^i). \tag{4.1}$$

Default penalties are said *moderate* with respect to utility functions, if they are α-moderate for some $\alpha \in (1, \infty)^{\mathcal{T}}$.

[16] Observe that the term $b_t(\kappa, p)$ appearing in the market clearing condition (3.1) satisfies $b_t(\kappa, p) \leqslant \overline{b}_t$.

In other words, when default penalties are α-moderate, then for infinitely many periods the penalty associated with the maximal default for a feasible plan, is less than the utility from consuming the current endowment.

4.2 Lagrange multipliers when default penalties are moderate

Throughout this subsection we fix an economy $\mathcal{E}$ with moderate default penalties and provide necessary conditions (in particular first order conditions) for the existence of an ε-equilibrium. Our conditions rely on a technical existence result of Lagrange multipliers that is presented in Appendix B. While standard, the arguments are delicate. For instance, the fact that default penalties are moderate plays a crucial role.[17] It is not clear to us whether the first order conditions that we provide are still valid if default penalties are not moderate.

Assume that $(\pi, \boldsymbol{a})$ is an ε-equilibrium where $\pi \in \Xi$ is a sequence $\pi = (\pi_t)_{t\in\mathcal{T}}$ of prices and delivery rates with $\pi_t = (p_t, q_t, \kappa_t)$ and $\boldsymbol{a} = (a^i)_{i\in I}$ is an allocation of plans $a^i = (a^i_t)_{t\in\mathcal{T}}$ in A with $a^i_t = (x^i_t, \theta^i_t, \varphi^i_t, d^i_t)$. Applying the theorem of Appendix B,[18] we can prove that for each agent i there exist,[19]

(i) a sequence of non-negative Lagrange multipliers $(\gamma^i_t)_{t\in\mathcal{T}}$ corresponding to the sequence of budget constraints (2.1);
(ii) for each commodity ℓ, a sequence $(\chi^i_t(\ell))_{t\in\mathcal{T}}$ of non-negative Lagrange multipliers corresponding to the sequence of collateral requirements (2.2);
(iii) a sequence of non-negative Lagrange multipliers $(\rho^i_t)_{t\in\mathcal{T}}$ corresponding to the sequence of minimum delivery constraints (2.3);[20]
(iv) two sequences of non-negative Lagrange multipliers $(\alpha^i_{\theta,t})_{t\in\mathcal{T}}$ and $(\alpha^i_{\varphi,t})_{t\in\mathcal{T}}$ corresponding to the non-negative constraints on portfolio purchases and sales

such that for any period $\tau \geqslant 1$ and each finite sequence $a = (a_t)_{t\in\mathcal{T}} \in \widetilde{A}^\tau$,

$$\sum_{0\leqslant t\leqslant \tau+1} \mathcal{L}^i_t(a_t, a_{t-1}) \leq \sum_{t\in\mathcal{T}} \Pi^i_t(a^i_t, a^i_{t-1}) \tag{4.2}$$

where $\widetilde{A}$ is the set of sequences $(a_t)_{t\in\mathcal{T}}$ with $a_t = (x_t, \theta_t, \varphi_t, d_t)$ satisfying[21]

$$x_t \in \mathbb{R}^L, \quad \theta_t \in \mathbb{R}, \quad \varphi_t \in \mathbb{R} \quad \text{and} \quad d_t \in \mathbb{R}$$

[17] The fact that default penalties are moderate is used in Claim 4.1 which is essential in order to get condition (d) in the Theorem of Appendix B.

[18] We thank Juan Pablo Torres–Martínez for pointing out that this issue is delicate and deserves some attention.

[19] One should apply the theorem in Appendix B by choosing $L_t = L \cup \{1, 2, 3\}$ or equivalently $\mathbb{R}^{L_t} = \mathbb{R}^L \times \mathbb{R}^3$. Condition (L.3) in Appendix B follows from Assumptions (A.1) and (A.2). Condition (b) in the theorem of Appendix B follows from Remark 2.1. For more details, we refer to Appendix B.

[20] We let $\rho^i_0 = 0$ since there is no delivery at the initial period $t = 0$.

[21] By convention, we let $a_{-1} = (0, 0, 0, 0)$.

and $\widetilde{A}^\tau$ the set of sequences $(a_t)_{t\in\mathcal{T}} \in \widetilde{A}$ with horizon τ, i.e., $a_t = 0$ for each period $t > \tau$. The Lagrangian $\mathcal{L}_t^i(a_t, a_{t-1})$ is defined by

$$\mathcal{L}_t^i(a_t, a_{t-1}) = \Pi_t^i(a_t, a_{t-1}) + \sum_{\ell\in L} \chi_t^i(\ell)\{x_t(\ell) - C_t(\ell)\varphi_t\}$$
$$+\gamma_t^i g_t^i(a_t, a_{t-1}) + \rho_t^i h_t^i(a_t, a_{t-1}) + \alpha_{\theta,t}^i \theta_t + \alpha_{\varphi,t}^i \varphi_t$$

where

$$\frac{\Pi_t^i(a_t, a_{t-1})}{[\beta_i]^t} = v_t^i(x_t) - \mu_t^i \frac{\left[V_t(p)\varphi_{t-1} - d_t\right]^+}{p_t w_t},$$

with

$$g_t^i(a_t, a_{t-1}) = p_t\{\omega_t^i + Y_t x_{t-1}\} - p_t x_t + q_t[\varphi_t - \theta_t] + V_t(\kappa, p)\theta_{t-1} - d_t$$

and

$$h_t^i(a_t, a_{t-1}) = d_t - D_t(p)\varphi_{t-1}.$$

Remark 4.1 Because of the minimum delivery constraint we do not need to restrict the delivery to be non-negative, and because of the collateral requirement constraint we do not need to restrict the consumption plan to be non-negative. This is the reason why there are no Lagrange multipliers corresponding to the non-negative constraints on consumption bundles and deliveries.

Since default penalties are moderate, the ε-equilibrium $(\pi, \boldsymbol{a})$ should satisfy the following property.

Claim 4.1 For every agent i and any period $t \geqslant 1$, there exist $\tau \geqslant t$ and a budget feasible $(\tau + 1)$-period sequence $\check{a}^i$ in $B^i(p, q, \kappa) \cap B^{\tau+1}$ such that

$$\Pi^i(p, \check{a}^i) \geqslant \Pi^{i,\tau}(p, a^i) \quad \text{and} \quad \check{a}^i \mathbf{1}_{[0,\tau]} = a^i \mathbf{1}_{[0,\tau]}. \tag{4.3}$$

Proof Fix an agent i and a period $t \geqslant 1$. Since Mod^i is infinite, there exists $T \in \mathrm{Mod}^i$ satisfying $T > t$. We pose $\tau = T - 1$. It is straightforward to check that we can choose $\check{a}^i$ defined as follows

$$\forall s \in \mathcal{T}, \quad \check{a}_s^i = \begin{cases} a_s^i & \text{if } s \leqslant \tau \\ (\omega_{\tau+1}^i, 0, 0, d_{\tau+1}) & \text{if } s = \tau + 1 \\ (0, 0, 0, 0) & \text{if } s > \tau + 1 \end{cases}$$

where $d_{\tau+1} = D_{\tau+1}(p)\varphi_\tau^i$. In other words, when choosing the plan $\check{a}^i$, agent i decides to fully default on his debt at period $T = \tau + 1$ and consume his initial endowment. Because $\tau + 1$ belongs to Mod^i, the utility from consuming his endowment compensates the disutility from defaulting and we get Eq. (4.3). □

Now, we can apply Claim 4.1 to show that condition (d) in the theorem of Appendix B is satisfied. Therefore, for any period $t \in \mathcal{T}$ and every commodity ℓ, we have

$$\gamma_t^i g_t^i(a_t^i, a_{t-1}^i) = 0, \quad \chi_t^i(\ell)\{x_t^i(\ell) - C_t(\ell)\varphi_t^i\} = 0$$

together with

$$\rho_t^i h_t^i(a_t^i, a_{t-1}^i) = 0, \quad \alpha_{\theta,t}^i \theta_t^i = 0 \quad \text{and} \quad \alpha_{\varphi,t}^i \varphi_t^i = 0.$$

It follows that

$$\sum_{0 \leqslant t \leqslant \tau+1} \mathcal{L}_t^i(a_t, a_{t-1}) \leqslant \sum_{t \in \mathcal{T}} \Pi_t^i(a_t^i, a_{t-1}^i) = \sum_{t \in \mathcal{T}} \mathcal{L}_t^i(a_t^i, a_{t-1}^i). \tag{4.4}$$

In particular, we can deduce that there exist for each agent i,[22]

(i) a family of super-gradients $(\nabla v_t^i)_{t \in \mathcal{T}}$ where ∇v_t^i belongs to $\partial v_t^i(x_t^i)$ the super-differential of v_t^i at x_t^i;
(ii) a family of super-gradients $(\delta_t^i)_{t \geqslant 1}$ where δ_t^i is a super-gradient of $\Delta \mapsto [\Delta]^+$ at $\Delta_t^i = V_t(p)\varphi_{t-1}^i - d_t^i$,

such that

(a) first order condition for consumption:

$$\forall t \in \mathcal{T}, \quad [\beta_i]^t \nabla v_t^i + \gamma_{t+1}^i p_{t+1} Y_{t+1} + \chi_t^i = \gamma_t^i p_t; \tag{4.5}$$

(b) first order condition for asset purchases:

$$\forall t \in \mathcal{T}, \quad \gamma_t^i q_t = \alpha_{\theta,t}^i + \gamma_{t+1}^i V_{t+1}(\kappa, p); \tag{4.6}$$

(c) first order condition for deliveries:

$$\forall t \geqslant 1, \quad [\beta_i]^t \mu_t^i \frac{\delta_t^i}{p_t w_t} + \rho_t^i = \gamma_t^i; \tag{4.7}$$

(d) first order condition for asset sales:

$$\forall t \geqslant 1, \quad \gamma_t^i q_t + \alpha_{\varphi,t}^i = \rho_{t+1}^i D_{t+1}(p) + \chi_t^i C_t + [\beta_i]^{t+1} \mu_{t+1}^i \frac{\delta_{t+1}^i}{p_{t+1} w_{t+1}} V_{t+1}(p). \tag{4.8}$$

Based on the above necessary conditions, we will prove two results. First, we will show that, for any sequence of collateral bundles and utility functions (among a certain class), we can choose asset promises and moderate default penalties such that default

22 See Statement 2 in the theorem of Appendix B.

is incompatible with equilibrium. Second, given the asset promises and moderate default penalties suitably chosen, we will show that we can choose a specific sequence of collateral bundles and utility functions such that the full repayment of debt in a non-trivial equilibrium implies that the asset price q_t is strictly larger than the cost $p_t C_t$ of constituting the collateral requirement. Páscoa and Seghir (2009) already explained that this condition induce agents to run Ponzi schemes which are not not consistent with optimality of plans in an ε-equilibrium.

In order to understand why moderate default penalties may induce agents to fully repay their debt, we need to present the following important intermediary step.

Proposition 4.2 *For every agent i and every period $\tau \in Mod^i$, the "discounted" market value $\gamma^i_{\tau+1} p_{\tau+1} \omega^i_{\tau+1}$ of the initial endowment is bounded by the maximum discounted continuation value $\sum_{t \geqslant \tau} [\beta_i]^t v^i_t(\overline{\Omega}^{\varepsilon}) < \infty$.*

Proof Fix an agent i and a period $\tau \in \mathrm{Mod}^i$. Let a in $\widetilde{A}$ be defined by

$$\forall t \geqslant 0, \quad a_t = \begin{cases} a^i_t & \text{if } t < \tau \\ (\omega^i_\tau, 0, 0, d_\tau) & \text{if } t = \tau \\ (0, 0, 0, 0) & \text{if } t > \tau \end{cases}$$

where $d_\tau = D_\tau(p)\varphi^i_{\tau-1}$. In other words, choosing the plan a, agent i is following the optimal plan a^i up to period τ, defaulting fully on his debt at period τ and consuming his initial endowment. Since the plan a belongs to $\widetilde{A}^\tau$, we can apply Eq. (4.4) to get[23]

$$\mathcal{L}^i_\tau(a_\tau, a_{\tau-1}) + \mathcal{L}^i_{\tau+1}(a_{\tau+1}, a_\tau) \leqslant \sum_{t \geqslant \tau} \Pi^i_t(a^i_t, a^i_{t-1}).$$

Since

$$\mathcal{L}^i_\tau(a_\tau, a_{\tau-1}) \geqslant \Pi^i_\tau(a_\tau, a_{\tau-1}) \geqslant (\beta_i)^\tau \left[v^i(\omega^i_\tau) - \mu^i_\tau H_\tau \right]$$

and

$$\mathcal{L}^i_{\tau+1}(a_{\tau+1}, a_\tau) = \gamma^i_{\tau+1} g^i_{\tau+1}(a_{\tau+1}, a_\tau) \geqslant \gamma^i_{\tau+1} p_{\tau+1} \omega^i_{\tau+1}$$

we get

$$[\beta_i]^\tau v^i_\tau(\omega^i_\tau) - [\beta_i]^\tau \mu^i_\tau H_\tau + \gamma^i_{\tau+1} p_{\tau+1} \omega^i_{\tau+1} \leqslant \sum_{t \geqslant \tau} [\beta_i]^t v^i_t(x^i_t).$$

[23] Observe that for any period $t < \tau$, we have $\mathcal{L}^i_t(a_t, a_{t-1}) = \mathcal{L}^i_t(a^i_t, a^i_{t-1}) = \Pi^i_t(a^i_t, a^i_{t-1})$; and for any period $t > \tau + 1$ we have $\mathcal{L}^i_t(a_t, a_{t-1}) = \mathcal{L}^i_t(0, 0) \geqslant 0$.

Since default penalties are moderate at period τ, i.e., $\tau \in \mathrm{Mod}^i$, we have $\mu^i_\tau H_\tau \leqslant v^i_\tau(\omega^i_\tau)$ implying the desired result:

$$\gamma^i_{\tau+1} p_{\tau+1} \omega^i_{\tau+1} \leqslant \sum_{t \geqslant \tau} [\beta_i]^t v^i_t(x^i_t) \leqslant \sum_{t \geqslant \tau} [\beta_i]^t v^i_t(\overline{\Omega}^\varepsilon) < \infty.$$

□

Remark 4.2 We obtain a bound on the marginal utility of wealth at period $\tau + 1$ when τ belongs to Mod^i, i.e., when default penalties are moderate. If default penalties are not moderate we do not know if it is possible to exhibit a similar bound.

4.3 Moderate default penalties precluding default

We are now ready to show that moderate default penalties may induce agents to optimally decide to make full payments. In order to clarify the presentation, we assume from now on that the economy satisfies the following list of additional assumptions.

Assumption 4.1 There exist a function $v^i : \mathbb{R}^L_+ \to \mathbb{R}_+$, a strictly positive bundle $\underline{\omega} \in \mathbb{R}^L_{++}$ and a uniformly bounded sequence $(b_t)_{t \geqslant 1}$ with $b_t > 0$ such that for every agent i and every period t,

(A.4) the endowment bundle ω^i_t is bounded from below by $\underline{\omega}$;
(A.5) the "normalization" bundle w_t coincides with $\underline{\omega}$;
(A.6) the promise A_t of the asset satisfies $A_t = b_t \underline{\omega} + Y_t C_{t-1}$;
(A.7) the utility function v^i_t coincides with v^i.

Observe that Assumption (A.2) implies that v^i is concave, continuous, strictly increasing with $v^i(0) = 0$. Moreover, under Assumptions (A.5) and (A.6) we have $\overline{b}_t = b_t$ and the maximum amount H_t in real terms that an agent may default if his plan is feasible, satisfies $H_t = M_{t-1} b_t$. Since $(b_t)_{t \geqslant 1}$ is uniformly bounded, we can let b be the least upper bound $\sup_{t \geqslant 1} b_t$.

We claim that we can choose default penalties such that they are moderate but at the same time severe enough to preclude default at equilibrium. To find such default penalties we make use of the bound obtained in Proposition 4.2.[24] Our aim is to show that restricting default penalties to be moderate does not prevent us to choose them severe enough to preclude default at equilibrium. The intuition is very simple. Default penalties are moderate if the disutility $\mu^i_t H_t$ felt at period t for defaulting the amount H_t is compensated by the utility $v^i_t(\omega^i_t)$ of consuming the initial endowment. However, the maximum amount H_t that an agent may default if his plan is feasible can be made as small as desired independently of the utility $v^i_t(\omega^i_t)$ and the unitary default penalty μ^i_t.

[24] We borrowed from Páscoa and Seghir (2009) the idea that we can choose exogenously the default penalty such that, endogenously at equilibrium, no agent will decide to default. This is possible due to the bound on marginal wealth obtained in Proposition 4.2. We only succeeded to find such a bound when default penalties are moderate. In particular, we do not know for the examples proposed by Páscoa and Seghir (2009) whether Ponzi schemes reappear when unduly pessimistic expectations on asset deliveries are ruled out.

Under the simplifying assumptions made above, we have $H_t = M_{t-1} b_t$. We can always choose the promise A_t close enough to the depreciated collateral $Y_t C_{t-1}$ (by choosing b_t close enough to 0) such that H_t is as small as desired. This implies that given any sequence of default penalties, we can always choose the sequence of promises such that penalties are moderate. Then, as shown below, we can use Proposition 4.2 to find default penalties μ_t^i severe enough to preclude default.

Proposition 4.3 *Fix $\alpha > 1$ and choose default penalties as follows*

$$\forall t \geqslant 1, \quad \mu_t^i = \mu^i \quad \textit{where} \quad \mu^i = \alpha \frac{v^i(\overline{\Omega}^\varepsilon)}{\beta_i(1-\beta_i)}$$

and the promises' coefficients b_t as follows

$$\forall t \geqslant 1, \quad b_t = \frac{1}{\alpha} \min_{i \in I} \frac{v^i(\omega_t^i)}{\mu^i} \times \frac{\max_{\ell \in L} C_{t-1}(\ell)}{\max_{\ell \in L} \overline{\Omega}^\varepsilon(\ell)}.$$

Default penalties are α-moderate, more precisely, for every agent i we have $\mathrm{Mod}^i = \mathcal{T}$. Moreover, if there is a competitive equilibrium for $\mathcal{E}$ then every agent pays his debt at any period $t \geqslant 1$.

Proof It is straightforward to check that $\mathrm{Mod}^i = \mathcal{T}$ for every agent i. Indeed, under Assumptions (A.5) and (A.6) we have

$$H_t = M_{t-1} b_t \quad \text{and} \quad M_{t-1} \leqslant \frac{\max_{\ell \in L} \overline{\Omega}^\varepsilon(\ell)}{\max_{\ell \in L} C_{t-1}(\ell)}.$$

It follows that

$$\forall i \in I, \quad \forall t \geqslant 1, \quad \alpha \mu_t^i H_t = \alpha \mu^i M_{t-1} b_t \leqslant v_t^i(\omega_t^i). \tag{4.9}$$

We propose now to prove that if $(\pi, \boldsymbol{a})$ is a competitive equilibrium then every agent pays his debt at any period. Fix $t \geqslant 1$ and assume by way of contradiction that agent i is not paying his debt at date t. The super-gradient δ_t^i associated to the default penalty must then satisfy $\delta_t^i = 1$. From the first order condition for deliveries (4.7) we get

$$[\beta_i]^t \mu_t^i \leqslant \gamma_t^i p_t \underline{\omega} \leqslant \gamma_t^i p_t \omega_t^i. \tag{4.10}$$

Combining Proposition 4.2 and Eq. (4.10) we get the following contradiction

$$\mu_t^i = \mu^i \leqslant \frac{v^i(\overline{\Omega}^\varepsilon)}{\beta_i(1-\beta_i)}.$$

□

4.4 Moderate default penalties leading to Ponzi schemes

Given the choices of promises and default penalties made in Proposition 4.3, if $(\pi, \boldsymbol{a})$ is an ε-equilibrium then every agent fully repays his debt at every period. Since the government always pays his debt, we get that $\kappa_t = 1$ for every period t.[25] We would like to compare the asset price q_t and the cost $p_t C_t$ of constituting the collateral bundle. When a lender wants to transfer wealth from period t to period $t+1$, two strategies are available. The first one consists of purchasing the asset in period t: the lender accepts to pay q_t in exchange of the future wealth $p_{t+1} A_{t+1}$. As a second strategy, the lender may prefer to purchase the collateral C_t, paying $p_t C_t$, in exchange of the future wealth $p_{t+1} Y_{t+1} C_t$. Following the second strategy, the lender will also enjoy utility from consuming the collateral bundle in period t.

Our aim is to show that we can choose the collateral bundle C_t and the utility function v^i such that the marginal utility from consuming the collateral bundle in period t does not compensate the marginal consumption corresponding to the difference $p_{t+1} b_{t+1} = p_{t+1}[A_{t+1} - Y_{t+1} C_t]$ of the payoffs associated to the above strategies. As a consequence we will have $q_t > p_t C_t$ for every period t which is inconsistent with optimality of individual plans.[26]

Theorem 4.1 *Choose default penalties and promises coefficients as in Proposition 4.3 (implying that default penalties are moderate). Moreover, assume that*

(i) there are two goods, $L = \{\ell, g\}$*;*
(ii) for every t*, the collateral bundle* C_t *is only in terms of good* g*, more precisely,*

$$\exists c > 0, \quad \forall t \geqslant 0, \quad C_t = c\mathbf{1}_{\{g\}};$$

(iii) the utility function v^i *is separable in goods, i.e., there exist two functions* v^i_ℓ *and* v^i_g *defined on* $[0, \infty)$*, concave, differentiable, strictly increasing with* $v^i_\ell(0) = v^i_g(0) = 0$ *such that*

$$\forall x = (x(\ell), x(g)) \in \mathbb{R}^L_+, \quad v^i(x) = v^i_\ell(x(\ell)) + v^i_g(x(g)).$$

Let $\widehat{\underline{\omega}} = (1/\#I)\underline{\omega}$ *and choose the functions* v^i_ℓ *and* v^i_g *satisfying*[27]

$$c\nabla v^i_g(\widehat{\underline{\omega}}(g)) < \beta_i \underline{b} \nabla v^i(\overline{\Omega})\underline{\omega}, \tag{4.11}$$

[25] If we do not consider an ε-equilibrium, then one may have $\kappa_t < 1$ and no trade in period t. In that case, our argument does not apply.

[26] In that respect, we show that the arguments in (Páscoa and Seghir, 2009, Theorem 4.1) are not correct.

[27] As usual $\nabla v^i(x) = (\nabla v^i_\ell(x(\ell)), \nabla v^i_g(x(g)))$ is the gradient of v^i at x where ∇v^i_ℓ and ∇v^i_g are the differential of v^i_ℓ and v^i_g respectively.

where[28]

$$\underline{b} = \aleph \min_{i \in I} \frac{\beta^i(1-\beta^i)v^i(\underline{\omega})}{v^i(\overline{\Omega}^\varepsilon)} \quad \textit{and} \quad \aleph = \frac{1}{\alpha^2} \times \frac{c}{\max\{\overline{\Omega}^\varepsilon(\ell), \overline{\Omega}^\varepsilon(g)\}}.$$

Then Ponzi schemes are not ruled out, i.e., an ε-equilibrium cannot exist, despite default penalties being moderate.

Remark 4.3 It is straightforward to provide examples of utility functions satisfying Eq. (4.11). Indeed, fix any pair of functions $f^i, v^i_\ell : [0,\infty) \to [0,\infty)$ concave, differentiable, strictly increasing satisfying $f^i(0) = v^i_\ell(0) = 0$ and for $\xi > 0$, pose $v^{i,\xi}_g = \xi f^i$, or equivalently

$$v^{i,\xi}(x) = \xi f^i(x(g)) + v^i_\ell(x(\ell)).$$

Observe that

$$c\nabla v^{i,\xi}_g(\widehat{\underline{\omega}}(g)) = \xi c \nabla f^i(\widehat{\underline{\omega}}(g))$$

and

$$\nabla v^{i,\xi}(\overline{\Omega})\underline{\omega} = \underline{\omega}(\ell)\nabla v^i_\ell(\overline{\Omega}(\ell)) + \xi \underline{\omega}(g)\nabla f^i(\overline{\Omega}(g)).$$

Passing to the limit, we get

$$\lim_{\xi \to 0} c\nabla v^{i,\xi}_g(\widehat{\underline{\omega}}(g)) = 0 \quad \text{and} \quad \lim_{\xi \to 0} \nabla v^{i,\xi}(\overline{\Omega})\underline{\omega} = \underline{\omega}(\ell)\nabla v^i_\ell(\overline{\Omega}(\ell)) > 0.$$

It follows that there exists $\xi > 0$ small enough such that

$$c\nabla v^{i,\xi}_g(\widehat{\underline{\omega}}(g)) < \beta_i \underline{b} \nabla v^{i,\xi}(\overline{\Omega})\underline{\omega}.$$

We have thus exhibited a non-empty class of utility functions satisfying Eq. (4.11).

Proof of Theorem 4.1 Assume by way of contradiction that there exists an ε-equilibrium $(\pi, \boldsymbol{a})$ with $\pi = (p, q, \kappa)$. Following Proposition 4.3, every agent i pays his debt at every period $t \geqslant 1$. Since the government always pays his debt, this implies that $\kappa_t = 1$ for every period t. It follows that $(p, q, \boldsymbol{a})$ is also a competitive equilibrium of the economy $\mathcal{E}'$ with perfect commitment (the government plays no role in $\mathcal{E}'$ since it does not need to inject goods, i.e., $b_t(\kappa, p) = 0$.) in the sense that each agent i maximizes the utility $U^i(x)$ among the actions (x, θ, φ) satisfying, for each period $t \geqslant 0$, the following budget constraint

$$p_t x_t + q_t\theta_t + V_t(p)\varphi_{t-1} \leqslant q_t\varphi_t + V_t(p)\theta_{t-1} + p_t\{\omega^i_t + Y_t x_{t-1}\} \tag{4.12}$$

[28] Observe that if we replace ω_t by $\underline{\omega}$ in the definition of b_t given in Proposition 4.3 then we get $\underline{b}$. In particular we have $b_t \geqslant \underline{b}$ for every $t \geqslant 1$.

together with the collateral requirement constraint

$$C_t \varphi_t \leqslant x_t. \tag{4.13}$$

We propose to prove that for every period t we have $q_t > p_t C_t$. Assume by way of contradiction that there exists $t \in \mathcal{T}$ such that $q_t \leqslant p_t C_t$. Since markets clear, we have

$$\sum_{i \in I} x_t^i = \sum_{i \in I} \Omega_t^i$$

implying that there exists at least one agent i such that

$$x_t^i(g) \geqslant (1/\#I)\Omega_t(g) \geqslant (1/\#I)\underline{\omega}(g) = \widehat{\underline{\omega}}(g) > 0.$$

Given $\nu > 0$, we consider the alternative choice $(\widetilde{x}^\nu, \widetilde{\theta}^\nu, \widetilde{\varphi}^\nu)$ which coincides with the action $(x^i, \theta^i, \varphi^i)$ except for consumption vectors at periods t and $t+1$ and asset purchases at period t. More precisely, we pose

$$\widetilde{x}_t^\nu = x_t^i - \nu c \mathbf{1}_{\{g\}}, \quad \widetilde{\theta}_t^\nu = \theta_t^i + \nu \quad \text{and} \quad \widetilde{x}_{t+1}^\nu = x_{t+1}^i + \nu b_{t+1} \underline{\omega}$$

where $\nu > 0$ is chosen small enough to ensure $\widetilde{x}_t^\nu \geqslant 0$. The alternative action $(\widetilde{x}^\nu, \widetilde{\theta}^\nu, \widetilde{\varphi}^\nu)$ consists on reducing the consumption of good g by νc units. The gain in purchasing power is $\nu p_t C_t$. Since the price q_t of the asset is lower than $p_t C_t$, agent i can purchase ν units of the asset, implying that the budget restriction (4.12) is satisfied at date t. Consuming $\widetilde{x}_t^\nu$ at period t instead of x_t^i implies a loss of $\nu p_{t+1} Y_{t+1} C_t$ units of account in period $t+1$. This loss is more than compensated by the extra return

$$\nu p_{t+1} A_{t+1} = \nu b_{t+1} p_{t+1} \underline{\omega} + \nu p_{t+1} Y_{t+1} C_t$$

associated to the alternative portfolio $\widetilde{\theta}^\nu$. The extra wealth $\nu b_{t+1} p_{t+1} \underline{\omega}$ can be used to purchase at period $t+1$ the νb_{t+1} additional units of the bundle $\underline{\omega}$. We have thus proved that the budget restriction (4.12) is also satisfied at date t. Therefore the alternative action $(\widetilde{x}^\nu, \widetilde{\theta}^\nu, \widetilde{\varphi}^\nu)$ belongs to the budget set of the perfect commitment economy $\mathcal{E}'$. In particular, we must have $U^i(x^i) \geqslant U^i(\widetilde{x}^\nu)$. However, posing $\Delta U^i \equiv U^i(x^i) - U^i(\widetilde{x}^\nu)$, we have

$$\frac{\Delta U^i}{\nu[\beta_i]^t} = \frac{v_g^i(x_t^i(g)) - v_g^i(x_t^i(g) - \nu c)}{\nu} + \beta_i \frac{v^i(x_{t+1}^i) - v^i(x_{t+1}^i + \nu b_{t+1}\underline{\omega})}{\nu}.$$

Since

$$C_t \nabla v^i(x_t^i) = c \nabla v_g^i(x_t^i(g)) \leqslant c \nabla v_g^i(\widehat{\underline{\omega}}(g))$$

and[29]

$$b_{t+1}\nabla v^i(x^i_{t+1})\underline{\omega} \geqslant \underline{b}\nabla v^i(\overline{\Omega})\underline{\omega}$$

we get

$$\begin{aligned}\lim_{\nu\to 0}\frac{U^i(x^i)-U^i(\widetilde{x}^\nu)}{\nu[\beta_i]^t} &= C_t\nabla v^i(x^i_t)-\beta_i b_{t+1}\nabla v^i(x^i_{t+1})\underline{\omega}\\ &\leqslant c\nabla v^i_g(\widehat{\underline{\omega}}(g))-\beta_i\underline{b}\nabla v^i(\overline{\Omega})\underline{\omega}.\end{aligned}$$

It follows from Eq. (4.11) that there exists $\nu > 0$ small enough such that we obtain the contradiction $U^i(x^i) - U^i(\widetilde{x}^\nu) < 0$. □

Remark 4.4 In what follows we explain why Theorem 4.1 cannot follow as a simple corollary of Theorem 1 in Ferreira and Torres-Martínez (2010). In our example, the collateral bundle C_t is time independent and contains only units of good g, i.e., $C_t = C = c\mathbf{1}_{\{g\}}$ with $c > 0$. Translating to our framework the condition (recalling that the enforcement coefficient λ is equal to 1 in our refined equilibrium) on collateral levels imposed in Ferreira and Torres-Martínez (2010), we get

$$c < \Psi_t \equiv \frac{1}{\overline{\pi}_t}\underline{\pi}_{t+1}[b_{t+1}\underline{\omega}+Y_{t+1}C]$$

where

$$\overline{\pi}_t = \frac{U^i(\Omega^\epsilon)}{\min\{\omega^i_t(\ell),\omega^i_t(g)\}}$$

and for every good k,

$$\underline{\pi}_{t+1}(k) = \frac{1}{2\Omega^\epsilon_{t+1}(k)}\min_{0\leqslant x\leqslant\Omega^\epsilon_{t+1}}\beta^t[v^i(x+2\Omega^\epsilon_{t+1}(k)\mathbf{1}_{\{k\}})-v^i(x)].$$

Under our assumptions (endowments are uniformly bounded from below and above) we have that $\overline{\pi}_t > \overline{\pi} > 0$ for all $t \geqslant 0$ while $\underline{\pi}_{t+1} \to 0$, implying that $c = 0$: a contradiction.

As it was claimed in the introduction one could modify our economy in such a way that it would be possible to apply Theorem 1 in Ferreira and Torres-Martínez (2010). The authors exhibit two examples in which their condition is verified. The first involves an economy in which the process of collateral constraints converges to zero while the second involves an economy in which aggregate endowments converge to zero. We argue in favor of following our route for two reasons: (1) our Theorem 4.1 is rather simple making the paper self-contained; (2) it shows that effectively persistent

[29] Recall that $b_{t+1}/\underline{b} = v^i(\omega^i_t)/v^i(\underline{\omega}) \geqslant 1$.

mechanisms (moderate penalties) may induce Ponzi schemes even if collateral levels are constant (or aggregate endowments are uniformly bounded away from 0).[30]

5 Concluding remarks and related literature

We show that when unduly pessimistic expectations are ruled out, equilibria fail to exist even if default penalties are moderate. More precisely, we provide a specific example showing that moderate default penalties can be severe enough to induce agents to pay fully their debt at every period. This fact can induce agents to run Ponzi schemes and destroys equilibrium existence.

It will be interesting to study whether there is a (non-trivial) condition relating default penalties to primitives that precludes agents to run Ponzi schemes, therefore ensuring that a refined equilibrium always exists. Providing a general condition ensuring existence goes beyond the scope of this paper. However, in Appendix A, we present such a condition that works for a specific class of models. In particular, we show that when utility is separable in commodities, Ponzi schemes are ruled out provided that the marginal utility from consuming the collateral becomes eventually larger than the marginal default penalty. In a recent paper, Páscoa and Seghir (2011) provide an existence result that applies to models where there are no collateral utility gains, i.e., collateral is a productive asset as in Kubler and Schmedders (2003). In such settings it is possible to find an upper bound on penalty coefficients that makes the collateral cost never fall below the promise price.

Acknowledgments We would like to thank Luis H. B. Braido, Martine Quinzii and Juan Pablo Torres-Martínez for useful comments and suggestions. This paper has benefited from comments by seminar participants and discussants at the University of Warwick (CRETA seminar), the University of Illinois at Urbana-Champaign (Conference in honor of Wayne Shafer), the University of Copenhagen (Workshop on Economic Theory), the XVII European Workshop on General Equilibrium Theory, the 2009 SAET Conference on current trends in Economics, and the IV Vigo workshop on Frontiers in Economic Theory and Applications. Financial support from CNPq and the GIP ANR of the Risk Foundation (Groupama Chair) is gratefully acknowledged by V. Filipe Martins-da-Rocha. Yiannis Vailakis acknowledges the financial support of an ERC starting grant (FP7, Ideas specific program, Project 240983 DCFM).

Appendix A: Sufficient conditions on primitives to rule out Ponzi schemes

In this section we exhibit a condition relating the marginal utility of consuming the collateral bundle and the marginal penalty of default which ensures that the limits of equilibria of truncated economies are competitive equilibria of the infinite horizon economy.[31] Our objective is neither to provide a general existence result nor to give a rigorous proof.[32] We only intend to give a sketch of the proof to illustrate the intuition behind this condition.

[30] This in particular shows that the sufficient condition proposed by Ferreira and Torres-Martínez (2010) is not necessary.

[31] The arguments can easily be adapted to prove existence of an ε-equilibrium and then of a refined equilibrium.

[32] The extension to a model with uncertainty and incomplete markets is only a matter of notation.

Fix an horizon $T > 1$ and consider a competitive equilibrium $(\pi, \boldsymbol{a})$ of the economy truncated at period T where $\pi = (p, q, \kappa)$ and $a^i = (x^i, \theta^i, \varphi^i, d^i)$. Fix an agent i and a period $t < T$. Assume that the following inequality is satisfied

$$\exists i \in I, \quad dv_t^i(x_t^i; C_t) \geqslant \beta_i \mu_{t+1}^i \overline{b}_{t+1} \tag{A.1}$$

where $dv_t^i(x_t^i; C_t)$ is the derivative of v_t^i at x_t^i in the direction of C_t,[33] and $\overline{b}_{t+1}$ is the maximum default in real terms at date $t+1$ defined by

$$\overline{b}_{t+1} = \sup_{p \in \Delta(L)} \frac{[pA_{t+1} - pY_{t+1}C_t]^+}{pw_{t+1}}.$$

We claim that we must have $q_t \leqslant p_t C_t$. Indeed, assume by contradiction that there exists $\alpha > 0$ such $q_t = p_t C_t + \alpha p_t \mathbf{1}_L$.[34] Suppose agent i considers deviating from φ_t^i by "shorting" ε more security, using the receipt $q_t \varepsilon$ to buy ε of the collateral and $\alpha\varepsilon$ of the vector $\mathbf{1}_L$ (i.e., $\alpha\varepsilon$ units of each good). This strategy is budget feasible at date t, but also at date $t+1$. Indeed, agent i may decide to pay back $D_{t+1}(p)\varepsilon$ which is smaller than the value of depreciated collateral $Y_{t+1}C_t\varepsilon$ constituted at period t. The decrease in utility at date t due to default penalties at date $t+1$ is

$$\varepsilon \beta_i \mu_{t+1}^i \frac{[p_{t+1}A_{t+1} - p_{t+1}Y_{t+1}C_t]^+}{p_{t+1}w_{t+1}} \leqslant \varepsilon \beta_i \mu_{t+1}^i \overline{b}_{t+1}.$$

Inequality (A.1) implies that the deviation strictly increases agent i's utility,[35] contradicting that a^i is optimal.

Inequality (A.1) depends on the consumption allocation $(x_t^i)_{i \in I}$ which is an endogenous variable. It is possible to exhibit a stronger condition only in terms of primitives. Indeed, consider the following property:

$$\forall z_t \in \mathrm{F}_t, \quad \exists i \in I, \quad dv_t^i(z_t^i; C_t) \geqslant \beta_i \mu_{t+1}^i \overline{b}_{t+1} \tag{A.2}$$

33 If $f : \mathbb{R}^n \to [-\infty, \infty)$ is a concave function with $f(x) > -\infty$, then the derivative of f at x in the direction of $y \in \mathbb{R}^n$ is

$$df(x; y) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda y) - f(x)}{\lambda}.$$

34 We must have $p_t \mathbf{1}_L > 0$ since v_t^i is strictly increasing.

35 Observe that

$$\lim_{\varepsilon \downarrow 0} \frac{v_t^i(x_t^i + \varepsilon(C_t + \alpha \mathbf{1}_L)) - v_t^i(x_t^i)}{\varepsilon} = dv_t^i(x_t^i; C_t + \alpha \mathbf{1}_L) > dv_t^i(x_t^i; C_t).$$

where F_t is the set of consumption bundles $\boldsymbol{z}_t = (z_t^i)_{i\in I}$ physically feasible at date t, i.e.,

$$\sum_{i\in I} z_t^i = \Omega_t.$$

Under Eq. (A.2) condition (A.1) is automatically satisfied since x_t^i is physically feasible for any competitive equilibrium.[36]

Consider now that there is an infinite set Barr $\subset \mathcal{T}$ such that Eq. (A.2) is satisfied for every date $t \in$ Barr. Let $(\pi^T, \boldsymbol{a}^T)$ be a competitive equilibrium for the truncated economy $\mathcal{E}^T$ with finite-horizon T. Observe that for every $t \in$ Barr with $t < T$, we must have $q_t^T \leqslant p_t^T C_t$. Following the (standard) arguments in Páscoa and Seghir (2009), passing to a subsequence if necessary, we can assume that the sequence $(\pi^T, \boldsymbol{a}^T)$ converges to some $(\pi, \boldsymbol{a})$ such that all markets clear and each action a^i is budget feasible and optimal among finite-horizon actions in the budget set $B^i(\pi)$. The difficulty is to show that a^i is optimal among budget feasible infinite horizon actions. Observe that for every $t \in$ Barr one must have $q_t \leqslant p_t C_t$. If the utility function $U^i(x)$ is bounded for every consumption sequence $x \in X$, then it is easy to show that $\boldsymbol{a}$ is optimal among all feasible plans. We can then conclude that $(\pi, \boldsymbol{a})$ is a competitive equilibrium.

Appendix B: Necessary and sufficient conditions for the existence of Langrange multipliers

In this appendix we consider an abstract infinite dimensional maximization problem and we present some necessary and sufficient conditions for optimality. Necessary conditions for optimality by means of Lagrange multipliers are first presented (see Sect. B.1). We then provide sufficient conditions in Sect. B.2. In the last section we restate the previous results for the specific economic model of the associated paper.

For each period $t \in \mathcal{T}$, we fix a finite set L_t of "types of action".[37] We denote by $C(L)$ the space of all sequencees $c = (c_t)_{t\in\mathcal{T}}$ where c_t is a vector in $\mathbb{R}^{L_t}$. By

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36 Condition (A.2) is satisfied if

$$\forall i \in I, \quad \inf_{y\in[0,\Omega_t]} dv_t^i(y; C_t) \geqslant \beta_i \mu_{t+1}^i \overline{b}_{t+1}.$$

Observe that by concavity we have $dv_t^i(y; C_t) \geqslant v_t^i(y + C_t) - v_t^i(y)$. Since $[0, \Omega_t]$ is a compact set and v_t^i is continuous, there exists $y_t^i \in [0, \Omega_t]$ such that

$$\inf_{y\in[0,\Omega_t]} dv_t^i(y; C_t) \geqslant v_t^i(y_t^i + C_t) - v_t^i(y_t^i).$$

Since the function v_t^i is strictly increasing, we have $v_t^i(y_t^i + C_t) > v_t^i(y_t^i)$ implying that the infimum is not 0. It follows that Condition (A.2) can be satisfied for strictly positive default penalties.

37 In the economic model of this paper, an action a_t is a vector $(x_t, \theta_t, \varphi_t, d_t)$ called a plan where $x_t \in \mathbb{R}^L$, $\theta_t \in \mathbb{R}$, $\varphi_t \in \mathbb{R}$, and $d_t \in \mathbb{R}$. For this case, we have $L_t = L \cup \{1, 2, 3\}$.

convention, we pose $L_{-1} = \{1\}$ and $c_{-1} = 0$ for any sequence $c \in C(L)$. For each period $T \geqslant 1$, we let $C^T(L)$ be the subset of $C(L)$ defined by

$$C^T(L) = \{c \in C(L) \,:\, \forall t \in \mathcal{T}, \quad t > T \Rightarrow c_t = 0\}.$$

Fixing a finite set K_t of "constraints" on actions[38] we can define in a similar way the sets $C(K)$ and $C^T(K)$ by replacing L_t by K_t. For period t, we fix an objective function $f_t : \mathbb{R}^{L_t} \times \mathbb{R}^{L_{t-1}} \longrightarrow \mathbb{R} \cup \{-\infty\}$ and a constraint function $g_t : \mathbb{R}^{L_t} \times \mathbb{R}^{L_{t-1}} \longrightarrow \mathbb{R}^{K_t}$. For each period $T \geqslant 1$ and each sequence $c \in C(L)$, we let

$$f^T(c) = \sum_{0 \leqslant t \leqslant T} f_t(c_t, c_{t-1}).$$

When the limit exists, we denote by $f(c)$ the following sum

$$f(c) = \lim_{T \to \infty} f^T(c).$$

Given $c \in C(L)$, we denote by $g(c)$ the sequence in $C(K)$ defined by

$$\forall t \in \mathcal{T}, \quad [g(c)]_t = g_t(c_t, c_{t-1}) \quad \text{or equivalently} \quad g(c) = (g_t(c_t, c_{t-1}))_{t \in \mathcal{T}}.$$

The vector $g_t(c_t, c_{t-1})$ in $\mathbb{R}^{K_t}$ is denoted by $(g_{t,k}(c_t, c_{t-1}))_{k \in K_t}$ where $g_{t,k}$ is interpreted as the kth constraint function from $\mathbb{R}^{L_t} \times \mathbb{R}^{L_{t-1}}$ to $\mathbb{R}$.

We assume that

(L.1) for each period t, the functions f_t and g_t are concave with $f_t(0,0) = 0$ and $g_t(0,0) \geqslant 0$;

(L.2) for each period t, the function g_t is continuous[39] and the function f_t, when restricted to its domain $\mathrm{dom}(f_t)$, is continuous;[40]

(L.3) for each period $T \geqslant 1$, there exists a sequence $\widehat{c} \in C^T(L)$ such that

$$f(\widehat{c}) \geqslant 0, \quad g_{T+1}(0, \widehat{c}_T) \geqslant 0 \quad \text{and} \quad \forall t \in \{0, \ldots, T\}, \quad g_t(\widehat{c}_t, \widehat{c}_{t-1}) \in \mathbb{R}^{K_t}_{++}.$$

B.1 Necessary conditions

Applying sequentially a finite dimensional convex separation argument, we can prove the following result.

Theorem *Assume that there exists $c^\star \in C(L)$ such that*

[38] In the economic model of this paper, the constraints are the solvency constraint, the collateral requirement, the minimum delivery constraint and non-negativity constraints. For this case, we have $K_t = \{1, 2, 3, 4\} \cup L$.

[39] Since the domain $\mathrm{dom}(g_t)$ of the function g_t is the whole space $\mathbb{R}^{L_t} \times \mathbb{R}^{L_{t-1}}$, concavity already implies that g_t is continuous.

[40] We denote by $\mathrm{dom}(f_t)$ the set of all points $(c_1, c_2) \in \mathbb{R}^{L_t} \times \mathbb{R}^{L_{t-1}}$ such that $f_t(c_1, c_2) \in \mathbb{R}$. Then, continuity is in the sense that $\widehat{f_t} : \mathrm{dom}(f_t) \to \mathbb{R}$ defined by $\widehat{f_t}(c_1, c_2) = f_t(c_1, c_2)$ is continuous on $\mathrm{dom}(f_t)$.

(a) *the sequence* $c^\star$ *satisfies the constraints* $g(c^\star) \geqslant 0$*;*
(b) *the sum* $f(c^\star)$ *is well defined;*
(c) *the sequence* $c^\star$ *is optimal among finite-horizon sequences, i.e., for any period* $\tau \geqslant 1$*, for every finite-horizon sequence* $c \in C^\tau(L)$ *satisfying the constraints* $g(c) \geqslant 0$*, we have* $f(c^\star) \geqslant f(c)$*.*

Then the following properties hold.

1. *There exists* $\Psi \in C(K)$ *with* $\Psi_t \in \mathbb{R}_+^{K_t}$ *such that for any period* $\tau \geqslant 1$ *and any finite-horizon sequence* $c \in C^\tau(L)$*,*

$$\sum_{0 \leqslant t \leqslant \tau+1} \mathcal{L}_t(c_t, c_{t-1}) \leqslant f(c^\star) \tag{B.1}$$

where $\mathcal{L}_t(c_t, c_{t-1}) = f_t(c_t, c_{t-1}) + \Psi_t \cdot g_t(c_t, c_{t-1})$*.*

2. *If moreover, we have*
   (d) for any period $t \geqslant 1$*, there exist* $\tau \geqslant t$ *and a finite-horizon sequence* $\check{c} \in C^{\tau+1}(L)$ *satisfying* $g(\check{c}) \geqslant 0$*,* $f(\check{c}) \geqslant f^\tau(c^\star)$ *and* $c^\star \mathbf{1}_{[0,\tau]} = \check{c}\mathbf{1}_{[0,\tau]}$
   then[41]

$$\forall t \in \mathcal{T}, \quad \Psi_t \cdot g_t(c_t^\star, c_{t-1}^\star) = 0. \tag{B.2}$$

In particular we obtain the following variational property: for every sequence $\widetilde{c}$ *in* $C(L)$ *and every period* $T \geqslant 1$ *we have*

$$\sum_{t=0}^{T} \mathcal{L}_t(\widetilde{c}_t, \widetilde{c}_{t-1}) + \mathcal{L}_{T+1}(c_{T+1}^\star, \widetilde{c}_T) \leqslant \sum_{t=0}^{T+1} \mathcal{L}_t(c_t^\star, c_{t-1}^\star). \tag{B.3}$$

Therefore, for every period $T \geqslant 1$*, there exist a family of super-gradients*[42]

$$(\nabla \mathcal{L}_0^\star, \ldots, \nabla \mathcal{L}_T^\star) \quad \textit{where} \quad \nabla \mathcal{L}_t^\star = (\nabla_1 \mathcal{L}_t^\star, \nabla_2 \mathcal{L}_t^\star) \in \partial \mathcal{L}_t(c_t^\star, c_{t-1}^\star)$$

[41] Under (a)–(d) we obtain for every finite-horizon sequence $c \in C^\tau(L)$,

$$\sum_{0 \leqslant t \leqslant \tau+1} \mathcal{L}_t(c_t, c_{t-1}) \leqslant \sum_{t \in \mathcal{T}} \mathcal{L}_t(c_t^\star, c_{t-1}^\star) = \sum_{t \in \mathcal{T}} f_t(c_t^\star, c_{t-1}^\star).$$

[42] If $\nabla \mathcal{L}_t^\star$ is a super-gradient of $\mathcal{L}_t$ at $(c_t^\star, c_{t-1}^\star)$ there exist two vectors $\nabla_1 \mathcal{L}_t^\star \in \mathbb{R}^{L_t}$ and $\nabla_2 \mathcal{L}_t^\star \in \mathbb{R}^{L_{t-1}}$ such that

$$\mathcal{L}_t(\widetilde{c}_t, \widetilde{c}_{t-1}) - \mathcal{L}_t(c_t^\star, c_{t-1}^\star) \leqslant \nabla_1 \mathcal{L}_t^\star \times (\widetilde{c}_t - c_t^\star) + \nabla_2 \mathcal{L}_t^\star \times (\widetilde{c}_{t-1} - c_{t-1}^\star)$$

for every pair $(\widetilde{c}_t, \widetilde{c}_{t-1})$ in $\mathbb{R}^{L_t} \times \mathbb{R}^{L_{t-1}}$. The super-gradient $\nabla \mathcal{L}_t^\star$ is then assimilated with the pair $(\nabla_1 \mathcal{L}_t^\star, \nabla_2 \mathcal{L}_t^\star)$. Observe that $\nabla_1 \mathcal{L}_t^\star$ belongs to the super-differential of the function $x \mapsto \mathcal{L}_t(x, c_{t-1}^\star)$ at $c_t^\star$ and $\nabla_2 \mathcal{L}_t^\star$ belongs to the super-differential of the function $x \mapsto \mathcal{L}_t(c_t^\star, x)$ at $c_{t-1}^\star$.

and ξ_{T+1} a super-gradient of the function $x \mapsto \mathcal{L}_{T+1}(c^\star_{T+1}, x)$ such that

$$\forall t \in \{0, \dots, T-1\}, \quad \nabla_1 \mathcal{L}^\star_t + \nabla_2 \mathcal{L}^\star_{t+1} = 0 \quad \textit{and} \quad \nabla_1 \mathcal{L}^\star_T + \xi_{T+1} = 0 \tag{B.4}$$

Proof of the theorem Fix a period $T \geqslant 1$. We let A be the subset of $\mathbb{R} \times C^{T+1}(K)$ defined by

$$A = \Big\{(\alpha, d) \in \mathbb{R} \times C^{T+1}(K) : \exists c \in C^T(L), \quad \alpha \leqslant f(c) - f(c^\star) \quad \text{and} \quad d \leqslant g(c)\Big\}$$

and we let $B = (0, \infty) \times C^{T+1}_+(K)$ where

$$C^{T+1}_+(K) = \prod_{0 \leqslant t \leq T+1} \mathbb{R}^{K_t}_+.$$

Following Assumptions (L.1)–(L.3) and conditions (a)–(c), the sets A and B are disjoint non-empty convex subsets of $\mathbb{R} \times C^{T+1}(K)$. It follows from the finite dimensional separating hyperplane theorem that there exists a non-zero pair $(\mu^T, \Psi^T) \in \mathbb{R}_+ \times C^{T+1}_+(K)$ such that

$$\forall c \in C^T(L), \quad \mu^T f(c) + \sum_{0 \leq t \leq T+1} \Psi^T_t \cdot g_t(c_t, c_{t-1}) \leq \mu^T f(c^\star). \tag{B.5}$$

Following Assumption (L.3), we can take $\mu^T = 1$ without any loss of generality.

Fix a period $\tau \in \mathcal{T}$. The objective is to prove that the sequence $(\Psi^T_\tau)_{T \geqslant 1}$ converges in $\mathbb{R}_+$. Following Assumption (L.3), there exists a process $\widehat{c} \in C^\tau(L)$ such that

$$f(\widehat{c}) \geqslant 0, \quad g_{\tau+1}(0, \widehat{c}_\tau) \geqslant 0 \quad \text{and} \quad \forall t \in \{0, \dots, \tau\}, \quad \varepsilon_t \equiv g_t(\widehat{c}_t, \widehat{c}_{t-1}) \in \mathbb{R}^{K_t}_{++}.$$

Observe that for any $t > \tau + 1$ we have $g_t(\widehat{c}_t, \widehat{c}_{t-1}) = g_t(0, 0) \geqslant 0$. It follows from Eq. (B.5) that for all $k \in K_\tau$,

$$\Psi^T_\tau(k) \leqslant \frac{f(c^\star)}{\varepsilon_\tau(k)}.$$

Using a diagonal procedure and passing to a subsequence if necessary, we can prove that there exists $\Psi \in C_+(K)$ such that

$$\forall \tau \in \mathcal{T}, \quad \Psi_\tau = \lim_{T \to \infty} \Psi^T_\tau.$$

Now we fix a period $\tau \geqslant 1$ and a finite-horizon sequence $c \in C^{\tau}(L)$. For each $T > \tau$, it follows from Eq. (B.5) and Assumption (L.1) that[43]

$$f(c) + \sum_{0 \leqslant t \leqslant \tau+1} \Psi_t^T \cdot g_t(c_t, c_{t-1}) \leq f(c) + \sum_{0 \leqslant t \leqslant T+1} \Psi_t^T \cdot g_t(c_t, c_{t-1}) \leqslant f(c^\star).$$

Passing to the limit when T goes to infinite, we get the desired result (B.1):

$$\sum_{0 \leqslant t \leqslant \tau+1} f_t(c_t, c_{t-1}) + \Psi_t \cdot g_t(c_t, c_{t-1}) \leqslant f(c^\star).$$

Now assume that (d) is satisfied. Fix a period $t \in \mathcal{T}$, there exist $\tau \geqslant t$ and a finite-horizon sequence $\check{c} \in C^{\tau+1}(L)$ satisfying $g(\check{c}) \geqslant 0$, $f(\check{c}) \geqslant f^{\tau}(c^\star)$ and $\check{c}\mathbf{1}_{[0,\tau]} = c^\star \mathbf{1}_{[0,\tau]}$. Observe that $f^{\tau+2}(\check{c}) = f(\check{c}) \geqslant f^{\tau}(c^\star)$. Choosing $c = \check{c}$ in Eq. (B.1), it follows that

$$f^{\tau}(c^\star) + \Psi_t \cdot g_t(c_t^\star, c_{t-1}^\star) \leqslant f^{\tau+2}(\check{c}) + \sum_{0 \leqslant s \leqslant \tau+2} \Psi_s \cdot g_s(\check{c}_s, \check{c}_{s-1}) \leqslant f(c^\star).$$

Since

$$\lim_{\tau \to \infty} f^{\tau}(c^\star) = f(c^\star)$$

we get the desired result (B.2).

Now fix a sequence $\widetilde{c}$ in $C(L)$ and a period $T \geqslant 1$. For every $\tau > T$ we let c be the sequence in $C^{\tau}(L)$ defined by

$$c_t = \begin{cases} \widetilde{c}_t & \text{if } t \leqslant T \\ c_t^\star & \text{if } T+1 \leqslant t \leqslant \tau \\ 0 & \text{if } \tau+1 \leqslant t \end{cases}$$

It follows from Eqs. (B.1) and (B.2) that

$$\sum_{0 \leqslant t \leqslant \tau+1} \mathcal{L}_t(c_t, c_{t-1}) \leqslant \sum_{t \in \mathcal{T}} \mathcal{L}_t(c_t^\star, c_{t-1}^\star).$$

[43] Observe that the horizon of the sequence c is τ. Since $T > \tau$, it follows that c also belongs to $C^T(L)$. From Eq. (B.5) we get

$$f(c) + \sum_{0 \leqslant t \leqslant T+1} \Psi_t^T \cdot g_t(c_t, c_{t-1}) \leqslant f(c^\star).$$

From Assumption (L.1) we know that $g_t(c_t, c_{t-1}) = g_t(0, 0) \geqslant 0$ for every $t > \tau + 1$. Therefore we get

$$f(c) + \sum_{0 \leqslant t \leqslant \tau+1} \Psi_t^T \cdot g_t(c_t, c_{t-1}) \leqslant f(c) + \sum_{0 \leqslant t \leqslant T+1} \Psi_t^T \cdot g_t(c_t, c_{t-1}) \leqslant f(c^\star).$$

Given the construction of the sequence c, we have

$$\sum_{t=0}^{T}\{\mathcal{L}_t(\widetilde{c}_t,\widetilde{c}_{t-1})-\mathcal{L}_t(c^\star_t,c^\star_{t-1})\}+\{\mathcal{L}_{T+1}(c^\star_{T+1},\widetilde{c}_T)-\mathcal{L}_{T+1}(c^\star_{T+1},c^\star_T)\}$$
$$\leqslant \sum_{s\geqslant\tau+2} f_s(c^\star_s,c^\star_{s-1}).$$

Passing to the limit when $\tau\to\infty$ and using the fact that the infinite sum $f(c^\star)$ is well-defined, we get the desired result

$$\sum_{t=0}^{T}\mathcal{L}_t(\widetilde{c}_t,\widetilde{c}_{t-1})+\mathcal{L}_{T+1}(c^\star_{T+1},\widetilde{c}_T)\leqslant\sum_{t=0}^{T}\mathcal{L}(c^\star_t,c^\star_{t-1})+\mathcal{L}_{T+1}(c^\star_{T+1},c^\star_T).$$

We let H be the function defined on $\mathbb{R}^{L_0}\times\cdots\times\mathbb{R}^{L_T}$ by

$$H(c_0,c_1,\ldots,c_T)=\sum_{t=0}^{T}\mathcal{L}_t(c_t,c_{t-1})+\mathcal{L}_{T+1}(c^\star_{T+1},c_T).$$

This is a concave function having a global maximum at $(c^\star_0,\ldots,c^\star_T)$. It follows that the super-differential of H at $(c^\star_0,\ldots,c^\star_T)$ is non-empty and contains 0. Observe that

$$H(c_0,\ldots,c_T)=\sum_{t=0}^{T}H_t(c_0,\ldots,c_T)+h_{T+1}(c_0,\ldots,c_T)$$

where

$$H_t(c_0,\ldots,c_T)=\mathcal{L}_t(c_t,c_{t-1})\quad\text{and}\quad h_{T+1}(c_0,\ldots,c_T)=\mathcal{L}_{T+1}(c^\star_{T+1},c_T).$$

It follows that

$$0\in\sum_{t=0}^{T}\partial H_t(c^\star_0,\ldots,c^\star_T)+\partial h_{T+1}(c^\star_0,\ldots,c^\star_T). \tag{B.6}$$

Observe that a super-gradient in $\partial H_t(c^\star_0,\ldots,c^\star_T)$ is a vector in $\mathbb{R}^{L_0}\times\cdots\times\mathbb{R}^{L_T}$ of the following form

$$(0,\ldots,0,\underbrace{\nabla_2\mathcal{L}^\star_t}_{t-1},\underbrace{\nabla_1\mathcal{L}^\star_t}_{t},0,\ldots,0) \tag{B.7}$$

where $(\nabla_1\mathcal{L}^\star_t,\nabla_2\mathcal{L}^\star_t)$ is a super-gradient in $\partial\mathcal{L}_t(c^\star_t,c^\star_{t-1})$. Observe moreover that a super-gradient in $\partial h_{T+1}(c^\star_0,\ldots,c^\star_T)$ takes the following form

$$(0,\ldots,0,\xi_{T+1}) \tag{B.8}$$

where $\xi_{T+1} \in \mathbb{R}^{L_t}$ is a super-gradient of the function $x \mapsto \mathcal{L}_{T+1}(c^\star_{T+1}, x)$ at $c^\star_T$. Combining Eqs. (B.6), (B.7) and (B.8) we get the desired result (B.4). □

B.2 Sufficient conditions

In this subsection, we investigate under which conditions the first-order conditions (B.4) in the theorem are sufficient to obtain optimality. Let $c^\star \in C(L)$ satisfying the conditions (a) and (b) of the theorem, i.e., the sequence $c^\star$ satisfies the constraint $g(c^\star) \geqslant 0$ and the sum $f(c^\star)$ is well defined. Let $\Psi \in C(K)$ be a sequence of Lagrange multipliers $\Psi_t = (\Psi_{t,k})_{k \in K_t} \in \mathbb{R}^{K_t}_+$ such that the first-order conditions (B.4) and the binding conditions (B.2) are satisfied. Actually we will assume a stronger property: there exists a sequence of super-gradients

$$(\nabla \mathcal{L}^\star_t)_{t \in \mathcal{T}} \quad \text{where} \quad \nabla \mathcal{L}^\star_t = (\nabla_1 \mathcal{L}^\star_t, \nabla_2 \mathcal{L}^\star_t) \in \partial \mathcal{L}_t(c^\star_t, c^\star_{t-1})$$

such that

$$\forall t \in \mathcal{T}, \quad \nabla_1 \mathcal{L}^\star_t + \nabla_2 \mathcal{L}^\star_{t+1} = 0 \tag{B.9}$$

where we recall that $\mathcal{L}_t = f_t + \Psi_t \cdot g_t$.

Let c be a sequence in $C(L)$ satisfying the constraints $g(c) \geqslant 0$. Given $T \geqslant 1$, we denote by $\mathcal{L}^T(c)$ the Lagrangian up to period T defined by

$$\mathcal{L}^T(c) = \sum_{t=0}^{T} \mathcal{L}_t(c_t, c_{t-1}).$$

By concavity and definition of the super-gradients, we have

$$\mathcal{L}^T(c) - \mathcal{L}^T(c^\star) \leqslant \sum_{t=0}^{T} \left[\nabla_1 \mathcal{L}^\star_t \cdot (c_t - c^\star_t) + \nabla_2 \mathcal{L}^\star_t \cdot (c_{t-1} - c^\star_{t-1}) \right].$$

Rearranging the previous sum we get

$$\mathcal{L}^T(c) - \mathcal{L}^T(c^\star) \leqslant \nabla_1 \mathcal{L}^\star_T \cdot (c_T - c^\star_T) + \sum_{t=0}^{T-1} \left[\nabla_1 \mathcal{L}^\star_t + \nabla_2 \mathcal{L}^\star_{t+1} \right] (c_t - c^\star_t).$$

Using the Euler Eq. (B.9) we get

$$\mathcal{L}^T(c) - \mathcal{L}^T(c^\star) \leqslant \nabla_1 \mathcal{L}^\star_T \cdot (c_T - c^\star_T).$$

Since $\mathcal{L}_t = f_t + \Psi \cdot g_t$ it follows that there exist a super-gradient $\nabla f^\star_t$ of f_t at $(c^\star_t, c^\star_{t-1})$ and for each $k \in K_t$ a super-gradient $\nabla g^\star_{t,k}$ of $g_{t,k}$ at $(c^\star_t, c^\star_{t-1})$ such that

$$\nabla \mathcal{L}^\star_t = \nabla f^\star_t + \sum_{k \in K_t} \Psi_t \nabla g^\star_{t,k}.$$

We can decompose the above super-gradients as follows

$$\nabla f_t^\star = (\nabla_1 f_t^\star, \nabla_2 f_t^\star) \quad \text{and} \quad \nabla g_{t,k}^\star = (\nabla_1 g_{t,k}^\star, \nabla_2 g_{t,k}^\star)$$

and $\nabla_1 \mathcal{L}_T^\star$ can be written as follows[44]

$$\nabla_1 \mathcal{L}_T^\star = \nabla_1 f_T^\star + \Psi_T \star \nabla_1 g_T^\star.$$

We then get

$$\mathcal{L}^T(c) - \mathcal{L}^T(c^\star) \leqslant \left[\nabla_1 f_T^\star + \Psi_T \star \nabla_1 g_T^\star\right] \cdot (c_T - c_T^\star).$$

Since for every period t we have

$$\Psi_t \cdot g_t(c_t, c_{t-1}) \geqslant 0 \quad \text{and} \quad \Psi_t \cdot g_t(c_t^\star, c_{t-1}^\star) = 0$$

it follows that

$$f^T(c) - f^T(c^\star) \leqslant \mathcal{L}^T(c) - \mathcal{L}^T(c^\star) \leqslant \left[\nabla_1 f_T^\star + \Psi_T \star \nabla_1 g_T^\star\right] \cdot (c_T - c_T^\star).$$

We obtain immediately the following properties:

Claim If the sequence $c^\star$ satisfies the following transversality condition

$$\liminf_{T\to\infty} \left[\nabla_1 f_T^\star + \Psi_T \star \nabla_1 g_T^\star\right] \cdot (-c_T^\star) \leqslant 0 \tag{B.10}$$

then $c^\star$ is optimal among finite-horizon sequences, i.e., if $c \in C^\tau(L)$ for some $\tau \in \mathcal{T}$ and satisfies $g(c) \geqslant 0$ then we have $f(c) \leqslant f(c^\star)$.

Now if c is a (possibly infinite) sequence with $g(c) \geqslant 0$ and satisfying the following transversality condition

$$\liminf_{T\to\infty} \left[\nabla_1 f_T^\star + \Psi_T \star \nabla_1 g_T^\star\right] \cdot (c_T - c_T^\star) \leqslant 0 \tag{B.11}$$

then we have[45]

$$\liminf_{T\to\infty} f^T(c) \leqslant f(c^\star).$$

In particular, if $f(c)$ is well-defined then we get $f(c) \leqslant f(c^\star)$.

[44] For simplicity, the sum

$$\sum_{k\in K_t} \Psi_{T,k} \nabla_1 g_{T,k}^\star$$

is denoted by $\Psi_T \star \nabla_1 g_T^\star$.

[45] Replacing "lim inf" by "lim sup" in Eq. (B.11) we get $\limsup_{T\to\infty} f^T(c) \leqslant f(c^\star)$.

B.3 The economic model of the paper

In the infinite horizon economy of this paper, every agent solves a maximization problem that is a particular case of the abstract problem presented above. We fix a price sequence $\pi = (p_t, q_t, \kappa_t)_{t\in\mathcal{T}}$ and consider a generic agent without specifying the index i.

Let $L_t = L \cup \{1, 2, 3\}$ and identify the choice c_t with the plan $(x_t, \theta_t, \varphi_t, d_t)$. We then have

$$f_t(c_t, c_{t-1}) = \beta^t \widehat{v}_t(x_t) - \beta^t \mu_t \frac{[V_t(p)\varphi_{t-1} - d_t]^+}{p_t w_t}$$

where $\widehat{v}_t$ coincides with v_t on $\mathbb{R}^L_+$ and is identical to $-\infty$ elsewhere. From now on we abuse notations and identify v_t with $\widehat{v}_t$.

Let $K_t = \{1, 2, 3, 4\} \cup L$ and choose

$$g_{t,1}(c_t, c_{t-1}) = p_t\{\omega_t + Y_t x_{t-1}\} - p_t x_t + q_t[\varphi_t - \theta_t] + V_t(\kappa, p)\theta_{t-1} - d_t$$

together with

$$g_{t,2}(c_t, c_{t-1}) = d_t - D_t(p)\varphi_{t-1}, \quad g_{t,3}(c_t, c_{t-1}) = \theta_t, \quad g_{t,4}(c_t, c_{t-1}) = \varphi_t$$

and

$$g_{t,\ell}(c_t, c_{t-1}) = x_t(\ell) - C_t(\ell)\varphi_t.$$

Observe that under the assumptions of our model, Assumptions (L.1)–(L.3) are automatically satisfied. A sequence $(\Psi_t)_{t\in\mathcal{T}}$ with $\Psi_t \in \mathbb{R}^{K_t} = \mathbb{R}^4 \times \mathbb{R}^L$ is denoted by

$$\Psi_t = (\gamma_t, \rho_t, \alpha_{\theta,t}, \alpha_{\varphi,t}, \chi_t) \quad \text{where} \quad \chi_t = (\chi_t(\ell))_{\ell\in L} \in \mathbb{R}^L.$$

The first order conditions (B.4) translate into the following form:

(a) first order condition for consumption:

$$\forall t \in \mathcal{T}, \quad \beta^t \nabla v_t + \gamma_{t+1} p_{t+1} Y_{t+1} + \chi_t = \gamma_t p_t; \tag{B.12}$$

(b) first order condition for asset purchases:

$$\forall t \in \mathcal{T}, \quad \gamma_t q_t = \alpha_{\theta,t} + \gamma_{t+1} V_{t+1}(\kappa, p); \tag{B.13}$$

(c) first order condition for deliveries:

$$\forall t \geqslant 1, \quad \beta^t \mu_t \frac{\delta_t}{p_t w_t} + \rho_t = \gamma_t; \tag{B.14}$$

(d) first order conditions for asset sales:

$$\forall t \geqslant 1, \quad \gamma_t q_t + \alpha_{\varphi,t} = \rho_{t+1} D_{t+1}(p) + \chi_t C_t + \beta^{t+1} \mu_{t+1} \frac{\delta_{t+1}}{p_{t+1} w_{t+1}} V_{t+1}(p); \quad \text{(B.15)}$$

where ∇v_t is a super-gradient of v_t at x_t and $-\delta_t$ is a super-gradient of $\Delta \mapsto -[\Delta]^+$ at $\Delta_t = V_t(p)\varphi_{t-1} - d_t$. The binding conditions (B.2) translate in the following form[46]

$$\begin{aligned} p_t x_t + q_t \theta_t + d_t &= p_t\{\omega_t + Y_t x_{t-1}\} + q_t \varphi_t + V_t(\kappa, p)\theta_{t-1} \\ \rho_t\{d_t - D_t(p)\varphi_{t-1}\} = 0 \quad &\text{and} \quad \alpha_{\theta,t}\theta_t = \alpha_{\varphi,t}\varphi_t = 0, \end{aligned}$$

and for every $\ell \in L$,

$$\chi_t(\ell)\{x_t^i(\ell) - C_t(\ell)\varphi_t\} = 0.$$

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[46] Observe that from Eq. (B.12) we have $\gamma_t p_t \geqslant \beta^t \nabla v_t \gg 0$ implying that $\gamma_t > 0$.