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EXISTENCE AND UNIQUENESS OF A FIXED POINT FOR LOCAL CONTRACTIONS

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This paper proves the existence and uniqueness of a fixed point for local contractions without assuming the family of contraction coefficients to be uniformly bounded away from 1. More importantly it shows how this fixed-point result can apply to study the existence and uniqueness of solutions to some recursive equations that arise in economic dynamics.

KEYWORDS: Fixed-point theorem, local contraction, Bellman operator, Koopmans operator, Thompson aggregator, recursive utility.

1. INTRODUCTION

FIXED-POINT RESULTS for local contractions turned out to be useful to solve recursive equations in economic dynamics. Many applications in dynamic programming are presented in Rincón-Zapatero and Rodríguez-Palmero (2003) for the deterministic case and in Matkowski and Nowak (2008) for the stochastic case. Applications to recursive utility problems can be found in Rincón-Zapatero and Rodríguez-Palmero (2007). Previous fixed-point results for local contractions rely on a metric approach.[2] The idea underlying this approach is based on the construction of a metric that makes the local contraction a global contraction in a specific subspace. The construction of an appropriate metric is achieved at the cost of restricting the family of contraction coefficients to be uniformly bounded away from 1. Contrary to the previous literature, we prove a fixed-point result using direct arguments that do not require the application of the Banach contraction theorem for a specific metric. The advantage of following this strategy of proof is that it allows us to deal with a family of contraction coefficients that has a supremum equal to 1. In that respect, the proposed fixed-point result generalizes the fixed-point results for local contractions stated in the literature. An additional benefit is that the stated fixed-

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[2]See Rincón-Zapatero and Rodríguez-Palmero (2003), Matkowski and Nowak (2008), and Rincón-Zapatero and Rodríguez-Palmero (2009).

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point theorem applies to operators that are local contractions with respect to an uncountable family of semidistances.[3]

We exhibit two applications to illustrate that, from an economic perspective, it is important to have a fixed-point result that encompasses local contractions associated with a family of contraction coefficients that are arbitrarily close to 1. The first application deals with the existence and uniqueness of solutions to the Bellman equation in the unbounded case, while the second one addresses the existence and uniqueness of a recursive utility function derived from Thompson aggregators.[4]

The paper is organized as follows. Section 2 defines local contractions and states a fixed-point theorem. Sections 3 and 4 show how the fixed-point result can apply to the issue of existence and uniqueness of solutions to the Bellman and Koopmans equations, respectively. The proof of the fixed-point theorem is postponed to the Appendix, where we also discuss its relation with a fixed-point result established by Hadžić (1979). The proofs of all additional results can be found in Martins-da-Rocha and Vailakis (2008).

2. AN ABSTRACT FIXED-POINT THEOREM

In the spirit of Rincón-Zapatero and Rodríguez-Palmero (2007), we state a fixed-point theorem for operators that are local contractions in an abstract space.[5] Let F be a set and let $\mathcal{D} = (d_j)_{j\in J}$ be a family of semidistances defined on F. We let σ be the weak topology on F defined by the family $\mathcal{D}$. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be σ-Cauchy if it is d_j-Cauchy for each $j \in J$. A subset A of F is said to be sequentially σ-complete if every σ-Cauchy sequence in A converges in A for the σ-topology. A subset $A \subset F$ is said to be σ-bounded if $\operatorname{diam}_j(A) \equiv \sup\{d_j(f, g) : f, g \in A\}$ is finite for every $j \in J$.

DEFINITION 2.1: Let r be a function from J to J. An operator $T : F \to F$ is a *local contraction* with respect to $(\mathcal{D}, r)$ if, for every j, there exists $\beta_j \in [0, 1)$ such that

$$\forall f, g \in F, \quad d_j(Tf, Tg) \le \beta_j d_{r(j)}(f, g).$$

The main technical contribution of this paper is the following existence and uniqueness result of a fixed point for local contractions.

THEOREM 2.1: *Assume that the space F is σ-Hausdorff.*[6] *Consider a function $r : J \to J$ and let $T : F \to F$ be a local contraction with respect to $(\mathcal{D}, r)$. Consider*

[3]In Martins-da-Rocha and Vailakis (2008) two applications are presented to illustrate that, in some circumstances, it is relevant not to restrict the cardinality of the family of semidistances.

[4]Contrary to Blackwell aggregators, Thompson aggregators may not satisfy a uniform contraction property. See Marinacci and Montrucchio (2007) for details.

[5]From now on, we write RZ-RP for Rincón-Zapatero and Rodríguez-Palmero.

[6]That is, for each pair $f, g \in F$, if $f \neq g$, then there exists $j \in J$ such that $d_j(f, g) > 0$.

a nonempty, σ-bounded, sequentially σ-complete, and T-invariant subset $A \subset F$.

(E) *If the condition*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} \operatorname{diam}_{r^{n+1}(j)}(A) = 0 \tag{1}$$

is satisfied, then the operator T admits a fixed point $f^\star$ in A.

(S) *Moreover, if $h \in F$ satisfies*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0, \tag{2}$$

then the sequence $(T^n h)_{n\in\mathbb{N}}$ is σ-convergent to $f^\star$.[7]

The arguments of the proof of Theorem 2.1 are very simple and straightforward. The details are postponed to the Appendix.

REMARK 2.1: Theorem 2.1 generalizes an existence result proposed in Hadžić (1979).[8] To be precise, Hadžić (1979) imposed the additional requirement that each semidistance d_j is the restriction of a seminorm defined on a vector space E containing F such that E is a locally convex topological vector space. Under such conditions the existence result cannot be used for the two applications proposed in Sections 3 and 4. Moreover, Hadžić (1979) did not provide any criteria of stability similar to condition (2). A detailed comparison of Theorem 2.1 with the result established in Hadžić (1979) is presented in the Appendix.

REMARK 2.2: If h is a function in A, then condition (2) is automatically satisfied, implying that the fixed point $f^\star$ is unique in A. Actually $f^\star$ is the unique fixed point on the set $B \subset F$ defined by

$$B \equiv \left\{ h \in F : \forall j \in J, \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0 \right\}.$$

REMARK 2.3: If the function r is the identity (i.e., $r(j) = j$), then the operator T is said to be a 0-local contraction and, in that case, conditions (1) and (2) are automatically satisfied. In particular, if a fixed point exists, it is unique on the whole space F.

REMARK 2.4: Assume that the space F is sequentially σ-complete and choose an arbitrary $f \in F$. As in RZ-RP (2007), we can show that the set $F(f)$ defined by

$$F(f) \equiv \{ g \in F : \forall j \in J, d_j(g, f) \leq [1/(1-\beta_j)] d_j(Tf, f) \}$$

[7] If A is a nonempty subset of F, then for each h in F, we let $d_j(h, A) \equiv \inf\{d_j(h, g) : g \in A\}$.

[8] We are grateful to a referee for pointing out this reference.

is nonempty, σ-bounded, σ-closed, and T-invariant. Applying Theorem 2.1 by choosing $A \equiv F(f)$, we obtain the following corollary.

COROLLARY 2.1: *Let $T : F \to F$ be a 0-local contraction with respect to a family $\mathcal{D} = (d_j)_{j\in J}$ of semidistances. Assume that the space F is sequentially σ-complete. Then the operator T admits a unique fixed point $f^\star$ in F. Moreover, for any arbitrary $f \in F$, the sequence $(T^n f)_{n\in\mathbb{N}}$ is σ-convergent to $f^\star$.*

Corollary 2.1 is a generalization of a result first stated in RZ-RP (2003) (see Theorem 1).[9] Unfortunately, the proposed proof in RZ-RP (2003) is not correct. As Matkowski and Nowak (2008) have shown, an intermediate step (Proposition 1b) used in their method of proof is false. RZ-RP (2009) have provided a corrigendum of their fixed-point result, but at the cost of assuming that the family $(\beta_j)_{j\in J}$ of contraction coefficients is uniformly bounded away from 1, that is, $\sup_{j\in J} \beta_j < 1$.[10]

From an economic perspective, the main contribution of this paper is to show that it is important to establish a fixed-point theorem that allows the contraction coefficients to be arbitrarily closed to 1. The economic applications presented in Sections 3 and 4 aim to illustrate this fact.

An additional difference in Theorem 2.1 with respect to the fixed-point results of Matkowski and Nowak (2008) and RZ-RP (2009) is that the family J is not assumed to be countable. Although in many applications it is sufficient to consider a countable family of semidistances, in some circumstances, it may be helpful not to restrict the cardinality of the family of semidistances. The interested reader may refer to Section 5 in Martins-da-Rocha and Vailakis (2008), where two applications are presented.

REMARK 2.5: An interesting observation about Theorem 2.1 is that its proof only requires each β_j to be nonnegative. The requirement that β_j belongs to $[0, 1)$ is used only in the proof of Corollary 2.1.

3. DYNAMIC PROGRAMMING: UNBOUNDED BELOW CASE

We propose to consider the framework of Section 3.3 in RZ-RP (2003). The state space is $X \equiv \mathbb{R}^\ell_+$, there is a technological correspondence $\Gamma : X \to X$, a return function $U : \text{gph}\,\Gamma \to Z \equiv [-\infty, \infty)$, where gph Γ is the graph of Γ, and $\beta \in (0, 1)$ is the discounting factor. Given $x_0 \in X$, we denote by $\Pi(x_0)$ the set of all admissible paths $\widetilde{x} = (x_t)_{t\geq 0}$ defined by

$$\Pi(x_0) \equiv \{\widetilde{x} = (x_t)_{t\geq 0} : \forall t \geq 0, x_{t+1} \in \Gamma(x_t)\}.$$

[9] If the family J is assumed to be countable, then Corollary 2.1 coincides with Theorem 1 in RZ-RP (2007).

[10] Matkowski and Nowak (2008) also proved a similar fixed-point result under this additional assumption.

The dynamic optimization problem consists of solving the maximization problem

$$v^\star(x_0) \equiv \sup\{S(\widetilde{x}) : \widetilde{x} \in \Pi(x_0)\}, \quad \text{where} \quad S(\widetilde{x}) \equiv \sum_{t\geq 0} \beta^t U(x_t, x_{t+1}).$$

We denote by $C(X, Z)$ the space of continuous functions from X to Z, and we let $C^\star(X)$ be the space of functions f in $C(X, Z)$ such that the restriction of f to $X^\star \equiv X \setminus \{0\}$ takes values in $\mathbb{R}$. Among others, we make the following assumptions.

DP1. The correspondence Γ is continuous with nonempty and compact values.

DP2. The function $U : \operatorname{gph}(\Gamma) \to [-\infty, \infty)$ is continuous on $\operatorname{gph}(\Gamma)$.

DP3. There is a continuous function $q : X^\star \to X^\star$ with $(x, q(x)) \in \operatorname{gph}\Gamma$ and $U(x, q(x)) > -\infty$ for all $x \in X^\star$.

We denote by $\mathcal{B}$ the Bellman operator defined on $C(X, Z)$ as

$$\mathcal{B}f(x) \equiv \sup\{U(x, y) + \beta f(y) : y \in \Gamma(x)\}.$$

Under the previous assumptions, the function $\mathcal{B}f$ belongs to $C(X, Y)$.[11] Moreover, for every $f \in C^\star(X)$, we have $\mathcal{B}f(x) \geq U(x, q(x)) + \beta f(q(x)) > -\infty$ for all $x \in X^\star$. This implies that $\mathcal{B}$ maps $C^\star(X)$ into $C^\star(X)$. Under suitable conditions, the value function $v^\star$ coincides with the fixed point of the Bellman operator $\mathcal{B}$. To establish this relationship, we introduce the following assumptions.[12]

DP4. There exist three functions w_-, w_+, and w in $C^\star(X)$ such that

$$w_- \leq w_+ < w \quad \text{and} \quad \frac{w_- - w}{w_+ - w} = O(1) \quad \text{at} \quad 0$$

together with

(a) $\mathcal{B}w < w$, $\mathcal{B}w_- \geq w_-$, $\mathcal{B}w_+ \leq w_+$,

(b) $(w_+ - w)/(\mathcal{B}w - w) = O(1)$ at 0,

(c) for any $x_0 \in X^\star$, the set $\Pi^0(x_0)$ is nonempty[13] and for each admissible path $(x_t)_{t\geq 0}$ in $\Pi^0(x_0)$, it follows that

$$\lim_{t\to\infty} \beta^t w_-(x_t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \beta^t w_+(x_t) = 0.$$

[11] We cannot apply the classical Berge maximum theorem since the range of the function U includes $-\infty$. We use Lemma 2 in Alvarez and Stokey (1998).

[12] Given two functions f and g in $C^\star(X)$ with $g(x) \neq 0$ in a neighborhood of 0, we say that $f/g = O(1)$ at 0 if there exists a neighborhood V of 0 in X such that f/g is bounded in $V \setminus \{0\}$.

[13] $\Pi^0(x_0)$ is the subset of $\Pi(x_0)$ of all admissible paths $\widetilde{x}$ in $\Pi(x_0)$ such that $S(\widetilde{x})$ exists and satisfies $S(\widetilde{x}) > -\infty$.

DP5. There exists a countable increasing family $(K_j)_{j\in\mathbb{N}}$ of nonempty and compact subsets of X such that for any compact subset K of X, there exists j with $K \subset K_j$ and such that $\Gamma(K_j) \subset K_j$ for all $j \in \mathbb{N}$.

We denote by $[w_-, w_+]$ the order interval in $C^\star(X)$, that is, the space of all functions $f \in C^\star(X)$ satisfying $w_- \leq f \leq w_+$. The following theorem is analogous to the main result in Section 3.3 (see Theorem 6) of RZ-RP (2003).[14]

THEOREM 3.1: *Assume* DP1–DP5. *Then the following statements hold*:

(a) *The Bellman equation has a unique solution f in* $[w_-, w_+] \subset C^\star(X)$.

(b) *The value function $v^\star$ is continuous in $X^\star$ and coincides with the fixed point f.*

(c) *For any function g in* $[w_-, w_+]$, *the sequence* $(\mathcal{B}^n g)_{n\in\mathbb{N}}$ *converges to* $v^\star$ *for the topology associated with the family* $(d_j)_{j\in\mathbb{N}}$ *of semidistances defined on* $[w_-, w_+]$ *by*

$$d_j(f, g) \equiv \sup_{x\in K_j^\star}\left|\ln\left(\frac{f - w}{w_+ - w}(x)\right) - \ln\left(\frac{g - w}{w_+ - w}(x)\right)\right|,$$

where $K_j^\star = K_j \setminus \{0\}$.

Using the convexity property of the Bellman operator, RZ-RP (2003, p. 1553) proved that the operator $\mathcal{B}$ is a 0-local contraction with respect to the family $(d_j)_{j\in\mathbb{N}}$ where the contraction coefficient β_j is defined by

$$\beta_j \equiv 1 - \exp\{-\mu_j\} \quad \text{with} \quad \mu_j \equiv \sup\{d_j(f, \mathcal{B}w) : f \in [w_-, w_+]\}.$$

Observe that for each j and each pair of functions f, g in $[w_-, w_+]$ we have

$$d_j(f, g) = \sup_{x\in K_j^\star}\left|\ln\left(\frac{f - w}{g - w}(x)\right)\right|,$$

implying that

$$\mu_j = \max\bigl\{\|\ln\theta_+\|_{K_j^\star}, \|\ln\theta_-\|_{K_j^\star}\bigr\},$$

where $\theta_+ \equiv (w - w_+)/(w - \mathcal{B}w)$ and $\theta_- \equiv (w - w_-)/(w - \mathcal{B}w)$.[15] Since the family $(K_j)_{j\in\mathbb{N}}$ covers the space X, we get

$$\sup_{j\in J}\mu_j = \max\{\|\ln\theta_+\|_{X^\star}, \|\ln\theta_-\|_{X^\star}\}.$$

[14] Our set of assumptions is slightly different from the one used by RZ-RP (2003). In particular, condition DP4(b) is not imposed in RZ-RP (2003). We make this assumption to ensure that the distance $d_j(f, \mathcal{B}w)$ is well defined. See Appendix C in Martins-da-Rocha and Vailakis (2008) for details.

[15] If f is a function in $C(X, Y)$ and K is a subset of X, we let $\|f\|_K \equiv \sup\{|f(x)| : x \in K\}$.

If either the function $\ln\theta_+$ or the function $\ln\theta_-$ is unbounded, then the supremum $\sup_{j\in J}\beta_j$ of the contraction coefficients is 1. In this case, the fixed-point results of Matkowski and Nowak (2008) and RZ-RP (2009) cannot apply to prove Theorem 3.1. In contrast, Theorem 2.1 makes it possible to provide a straightforward proof of Theorem 3.1. We can find in RZ-RP (2003) two examples that give rise to an unbounded sequence $(\mu_j)_{j\in J}$. In both examples the production technology has decreasing returns, while the return function is logarithmic in the first example (Example 10 in RZ-RP (2003)) and homogeneous in the second (Example 11 in RZ-RP (2003)).[16]

4. RECURSIVE PREFERENCES FOR THOMPSON AGGREGATORS

Consider a model where an agent chooses consumption streams in the space ℓ^∞_+ of nonnegative and bounded sequences $\mathbf{x} = (x_t)_{t\in\mathbb{N}}$ with $x_t \geq 0$. The space ℓ^∞ is endowed with the sup-norm $\|\mathbf{x}\|_\infty \equiv \sup\{|x_t| : t \in \mathbb{N}\}$. We propose to investigate whether it is possible to represent the agent's preference relation on ℓ^∞_+ by a recursive utility function derived from an aggregator

$$W : X \times Y \to Y,$$

where $X = \mathbb{R}_+$ and $Y = \mathbb{R}_+$. The answer obviously depends on the assumed properties of the aggregator function W.[17]

After the seminal contribution of Lucas and Stokey (1984), a wide literature has dealt with the issue of the existence and uniqueness of a recursive utility function derived from aggregators that satisfy a uniform contraction property (Blackwell aggregators). We refer to Becker and Boyd (1997) for an excellent exposition of this literature.[18] In what follows, we explore whether a unique recursive utility function can be derived from Thompson aggregators.

Throughout this section, we assume that W satisfies the following conditions:

ASSUMPTION 4.1: *W is a Thompson aggregator as defined by Marinacci and Montrucchio* (2007), *that is*, *the following conditions are satisfied*:

W1. *The function W is continuous*, *nonnegative*, *nondecreasing*, *and satisfies* $W(0, 0) = 0$.

W2. *There exists a continuous function $f : X \to Y$ such that* $W(x, f(x)) \leq f(x)$.[19]

[16] Refer to Sections 3.1 and 3.2 in Martins-da-Rocha and Vailakis (2008) for details.

[17] Throughout this section, some arguments are omitted. We refer to Appendix D in Martins-da-Rocha and Vailakis (2008) for details.

[18] See also Epstein and Zin (1989), Boyd (1990), Duran (2000, 2003), Le Van and Vailakis (2005), and Rincón-Zapatero and Rodríguez-Palmero (2007).

[19] Marinacci and Montrucchio (2007) assumed that there is a sequence $(x^n, y^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2_+$ with $(x^n)_{n\in\mathbb{N}}$ increasing to infinity and $W(x^n, y^n) \leq y^n$ for each n. This assumption, together with the others, implies that for each $x \in X$, there exists $y_x \in Y$ such that $W(x, y_x) \leq y_x$. We require that $x \mapsto y_x$ can be chosen to be continuous.

W3. *The function W is concave in the second variable at* 0.[20]
W4. *For every $x > 0$ we have $W(x, 0) > 0$.*

REMARK 4.1: Marinacci and Montrucchio (2007) proposed a list of examples of Thompson aggregators that do not satisfy a uniform contraction property. For instance, consider $W(x, y) = (x^\eta + \beta y^\sigma)^{1/\rho}$, where η, σ, ρ, $\beta > 0$ together with the following conditions: $\sigma < 1$ and either $\sigma < \rho$ or $\sigma = \rho$ and $\beta < 1$. Another example is the aggregator introduced by Koopmans, Diamond, and Williamson (1964): $W(x, y) = (1/\theta)\ln(1 + \eta x^\delta + \beta y)$ with θ, β, δ, $\eta > 0$. This aggregator is always Thompson, but it is Blackwell only if $\beta < \theta$.

To define formally the concept of a recursive utility function, we need to introduce some notations. We denote by π the linear functional from ℓ^∞ to $\mathbb{R}$ defined by $\pi \mathbf{x} = x_0$ for every $\mathbf{x} = (x_t)_{t\in\mathbb{N}}$ in ℓ^∞. We denote by σ the operator of ℓ^∞ defined by $\sigma\mathbf{x} = (x_{t+1})_{t\in\mathbb{N}}$.

DEFINITION 4.1: Let $\mathbb{X}$ be a subset of ℓ^∞ that is stable under the shift operator σ.[21] A function $u : \mathbb{X} \to \mathbb{R}$ is a *recursive utility function* on $\mathbb{X}$ if

$$\forall \mathbf{x} \in \mathbb{X}, \quad u(\mathbf{x}) = W(\pi\mathbf{x}, u(\sigma\mathbf{x})).$$

We propose to show that we can use the Thompson metric introduced by Thompson (1963) to prove the existence of a continuous recursive utility function when the space $\mathbb{X}$ is the subset of all sequences in ℓ^∞_+ which are uniformly bounded away from 0, that is, $\mathbb{X} \equiv \{\mathbf{x} \in \ell^\infty : \inf_{t\in\mathbb{N}} x_t > 0\}$.[22] The topology on $\mathbb{X}$ derived from the sup-norm is denoted by τ. This space of feasible consumption patterns also appears in Boyd (1990).

4.1. *The Operator*

In the spirit of Marinacci and Montrucchio (2007) we introduce the following operator. First, denote by $\mathcal{V}$ the space of sequences $V = (v_t)_{t\in\mathbb{N}}$, where v_t is a τ-continuous function from $\mathbb{X}$ to $\mathbb{R}_+$. The real number $v_t(\mathbf{x})$ is interpreted as the utility at time t derived from the consumption stream $\mathbf{x} \in \mathbb{X}$. For each sequence of functions $V = (v_s)_{s\in\mathbb{N}}$ and each period t, we denote by $[TV]_t$ the function from $\mathbb{X}$ to $\mathbb{R}_+$ defined by

$$\forall \mathbf{x} \in \mathbb{X}, \quad [TV]_t(\mathbf{x}) \equiv W(x_t, v_{t+1}(\mathbf{x})).$$

Since W and v_{t+1} are continuous, the function $[TV]_t$ is continuous. In particular, the mapping T is an operator on $\mathcal{V}$, that is, $T(\mathcal{V}) \subset \mathcal{V}$.

[20] In the sense that $W(x, \alpha y) \geq \alpha W(x, y) + (1 - \alpha)W(x, 0)$ for each $\alpha \in [0, 1]$ and each x, $y \in \mathbb{R}_+$.
[21] That is, for every $\mathbf{x} \in \mathbb{X}$ we have that $\sigma\mathbf{x}$ still belongs to $\mathbb{X}$.
[22] See also Montrucchio (1998) for another reference where the Thompson metric is used.

We denote by $\mathcal{K}$ the family of all sets $\mathbb{K} = [a\mathbf{1}, b\mathbf{1}]$ with $0 < a < b < \infty$.[23] We consider the subspace F of $\mathcal{V}$ composed of all sequences V such that on every set $\mathbb{K}$, the family $V = (v_t)_{t\in\mathbb{N}}$ is uniformly bounded from above and away from 0, that is, $V = (v_t)_{t\in\mathbb{N}}$ belongs to F if for every $0 < a < b < \infty$ there exist $\underline{v}$ and $\overline{v}$ such that

$$\forall t \in \mathbb{N}, \forall \mathbf{x} \in [a\mathbf{1}, b\mathbf{1}], \quad 0 < \underline{v} \leq v_t(\mathbf{x}) \leq \overline{v} < \infty.$$

Observe that T maps F into F since W is monotone with respect to both variables.[24] The objective is to show that T admits a unique fixed point $V^\star$ in F. The reason is that if $V^\star = (v_t^\star)_{t\in\mathbb{N}}$ is a fixed point of T, then the function $v_0^\star$ is a recursive utility function. Indeed, we will show that for each consumption stream $\mathbf{x} \in \mathbb{X}$ and every time t, we have $\lim_{n\to\infty}[T^n 0]_t(\mathbf{x}) = v_t^\star(\mathbf{x})$. Since $[T^n 0]_t(\sigma\mathbf{x}) = [T^n 0]_{t+1}(\mathbf{x})$ and

$$[T^n 0]_t(\mathbf{x}) = W\big(x_t, W(x_{t+1}, \dots, W(x_{t+n}, 0)\dots)\big),$$

passing to the limit we get that $v_t^\star(\sigma\mathbf{x}) = v_{t+1}^\star(\mathbf{x})$. This property is crucial to proving that $v_0^\star$ is a recursive utility on $\mathbb{X}$. Indeed, we have $v_0^\star(\mathbf{x}) = [TV^\star]_0(\mathbf{x}) = W(x_0, v_1^\star(\mathbf{x})) = W(x_0, v_0^\star(\sigma\mathbf{x}))$.[25]

4.2. *The Thompson Metric*

Fix a set $\mathbb{K}$ in $\mathcal{K}$. We propose to introduce the semidistance $d_{\mathbb{K}}$ on F defined as

$$d_{\mathbb{K}}(V, V') \equiv \max\{\ln M_{\mathbb{K}}(V|V'), \ln M_{\mathbb{K}}(V'|V)\},$$

where

$$M_{\mathbb{K}}(V|V') \equiv \inf\{\alpha > 0 : \forall \mathbf{x} \in \mathbb{K}, \forall t \in \mathbb{N}, v_t(\mathbf{x}) \leq \alpha v_t'(\mathbf{x})\}.$$

Let $V^\infty \in \mathcal{V}$ be the sequence of functions $(v_t^\infty)_{t\in\mathbb{N}}$ defined by $v_t^\infty(\mathbf{x}) \equiv f(\|\mathbf{x}\|_\infty)$. Observe that $[TV^\infty]_t(\mathbf{x}) \leq v_t^\infty(\mathbf{x})$ for every $t \in \mathbb{N}$ and every $\mathbf{x}$ in $\mathbb{X}$. We denote by V^0 the sequence of functions $T0 = ([T0]_t)_{t\in\mathbb{N}}$, that is, $V^0 = (v_t^0)_{t\in\mathbb{N}}$ with $v_t^0(\mathbf{x}) = W(x_t, 0)$. The monotonicity of T then implies that T maps the order interval $[V^0, V^\infty]$ into $[V^0, V^\infty]$. Moreover, both V^0 and V^∞ belong to F. We can then adapt the arguments of Theorem 9 in Marinacci and Montrucchio

[23] We denote by $\mathbf{1}$ the sequence $\mathbf{x} = (x_t)_{t\in\mathbb{N}}$ in ℓ^∞ defined by $x_t = 1$ for every t. The order interval $[a\mathbf{1}, b\mathbf{1}]$ is the set $\{\mathbf{x} \in \ell_+^\infty : a \leq x_t \leq b, \forall t \in \mathbb{N}\}$.

[24] We can easily check that for every $V = (v_t)_{t\in\mathbb{N}}$ in F and for every $\mathbb{K} \equiv [a\mathbf{1}, b\mathbf{1}]$, we have $W(a, \underline{v}) \leq [TV]_t(\mathbf{x}) \leq W(b, \overline{v})$.

[25] Observe that the time t utility $v_t^\star(\mathbf{x})$ of the consumption stream $\mathbf{x}$ does not depend on the past consumption since $v_t^\star(\mathbf{x}) = v_{t-1}^\star(\sigma\mathbf{x}) = \cdots = v_0^\star(\sigma^t\mathbf{x})$.

(2007, Appendix B) to show that T is a 0-local contraction on $[V^0, V^\infty]$ with respect to the family $\mathcal{D} = (d_{\mathbb{K}})_{\mathbb{K}\in\mathcal{K}}$. More precisely, we can prove that

$$d_{\mathbb{K}}(TV, TV') \le \beta_{\mathbb{K}} d_{\mathbb{K}}(V, V'),$$

where $\beta_{\mathbb{K}} \equiv 1 - [\mu_{\mathbb{K}}]^{-1}$ and $\mu_{\mathbb{K}} \equiv M_{\mathbb{K}}(V^\infty|V^0)$. Recall that

$$M_{\mathbb{K}}(V^\infty|V^0) \equiv \inf\bigl\{\alpha > 0 : \forall \mathbf{x} \in \mathbb{K}, \forall t \in \mathbb{N}, f(\|\mathbf{x}\|_\infty) \le \alpha W(x_t, 0)\bigr\},$$

implying that

$$\mu_{\mathbb{K}} = \sup_{\mathbf{x}\in\mathbb{K}} \sup_{t\in\mathbb{N}} \frac{f(\|\mathbf{x}\|_\infty)}{W(x_t, 0)} = \sup_{\mathbf{x}\in\mathbb{K}} \frac{f(\|\mathbf{x}\|_\infty)}{\inf_{t\in\mathbb{N}} W(x_t, 0)} = \frac{f(b)}{W(a, 0)}.$$

The set $[V^0, V^\infty]$ is sequentially complete with respect to the family $\mathcal{D}$. Therefore, we can apply Corollary 2.1 to get the existence of a unique fixed point $V^\star = (v_t^\star)_{t\in\mathbb{N}}$ of T in $[V^0, V^\infty]$.[26] The function $u^\star \equiv v_0^\star : \mathbb{X} \to \mathbb{R}_+$ is then a recursive utility function associated with the aggregator W and continuous for the sup-norm topology.[27] We have thus provided a sketch of the proof of the following result.[28]

THEOREM 4.1: *Given a Thompson aggregator W, there exists a recursive utility function $u^\star : \mathbb{X} \to \mathbb{R}$ which is continuous on $\mathbb{X}$ for the sup-norm. Moreover, this function is unique among all continuous functions which are bounded on every order interval of $\mathcal{K}$.*

REMARK 4.2: In the spirit of Kreps and Porteus (1978), Epstein and Zin (1989), Ma (1998), Marinacci and Montrucchio (2007), and Klibanoff, Marinacci, and Mukerji (2009), we can adapt the arguments above so as to deal with uncertainty.

REMARK 4.3: Consider the Koopmans–Diamond–Williamson (KDW) aggregator

$$W(x, y) = (1/\theta)\ln(1 + \eta x^\delta + \beta y)$$

for any $\theta, \beta, \delta, \eta > 0$. Applying Theorem 4.1, we get the existence of a recursive utility function defined on $\mathbb{X}$ and continuous for the sup-norm. When

[26] Observe that the family of contraction coefficients is such that $\sup_{\mathbb{K}\in\mathcal{K}} \beta_{\mathbb{K}} = 1$. Actually, uniqueness is obtained on the whole set F. See Appendix D in Martins-da-Rocha and Vailakis (2008) for details.

[27] Since $u^\star(\mathbf{x}) = \lim_{n\to\infty} W(x_0, W(x_1, \dots, W(x_n, 0)\dots))$ we can deduce that $u^\star$ is nondecreasing.

[28] See Appendix D in Martins-da-Rocha and Vailakis (2008) for details.

$\beta < \theta$, the aggregator W is Blackwell and the existence of a continuous recursive utility function can be established by applying the continuous existence theorem in Boyd (1990) or Becker and Boyd (1997). We propose to show that the case $\beta \geq \theta$ is not covered by the continuous existence theorem. Observe first that the lowest $\alpha > 0$ that satisfies the uniform Lipschitz condition

$$|W(x, y) - W(x, y')| \leq \alpha |y - y'|$$

for all $x > 0$ and $y, y' \geq 0$ is $\alpha = \beta/\theta$. Assume by way of contradiction that the conditions of the continuous existence theorem are met. Then there exists a positive continuous function $\varphi : \mathbb{X} \to (0, \infty)$ such that

$$M \equiv \sup_{\mathbf{x} \in \mathbb{X}} \frac{W(\pi \mathbf{x}, 0)}{\varphi(\mathbf{x})} < \infty \quad \text{and} \quad \chi \equiv \sup_{\mathbf{x} \in \mathbb{X}} \alpha \frac{\varphi(\sigma \mathbf{x})}{\varphi(\mathbf{x})} < 1.$$

For every $\mathbf{x} \in \mathbb{X}$ and every $n \geq 1$, we obtain

$$\begin{aligned} \alpha^n W(x_n, 0) &\leq M \alpha^n \varphi(\sigma^n \mathbf{x}) \\ &\leq M \left[\alpha \frac{\varphi(\sigma^n \mathbf{x})}{\varphi(\sigma^{n-1} \mathbf{x})} \times \cdots \times \alpha \frac{\varphi(\sigma \mathbf{x})}{\varphi(\mathbf{x})} \right] \varphi(\mathbf{x}) \\ &\leq M \chi^n \varphi(\mathbf{x}). \end{aligned}$$

Choosing $\mathbf{x} = a\mathbf{1}$ for any $a > 0$, we get

$$\forall n \geq 1, \quad \alpha^n W(a, 0) \leq M \chi^n \varphi(a\mathbf{1}).$$

Since $\alpha \geq 1$ and $\chi < 1$, it follows that $W(a, 0) = 0$ for every $a > 0$—a contradiction.

APPENDIX

PROOF OF THEOREM 2.1: Consider a set F and a family $\mathcal{D} = (d_j)_{j \in J}$ of semi-distances on F such that F is σ-Hausdorff, where we recall that σ is the weak topology defined by the family $\mathcal{D}$. Fix $r : J \to J$ and let $T : F \to F$ be a local Lipschitz function with respect to $(\mathcal{D}, r)$ in the sense that for every j, there exists $\beta_j \geq 0$ such that[29]

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g).$$

Consider a nonempty, σ-bounded, sequentially σ-complete, and T-invariant subset $A \subset F$. We recall the two results presented in Theorem 2.1:

[29] If $\beta_j \in [0, 1)$ for each j, then F is a local contraction. The concept of a local Lipschitz function was first introduced by Hadžić (1979) in a more specific framework.

(E) If the condition

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} \operatorname{diam}_{r^{n+1}(j)}(A) = 0 \tag{2.1}$$

is satisfied, then the operator T admits a fixed point $f^\star$ in A.

(S) Moreover, if $h \in F$ satisfies

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{n+1}(j)} d_{r^{n+1}(j)}(h, A) = 0, \tag{2.2}$$

then the sequence $(T^n h)_{n\in\mathbb{N}}$ is σ-convergent to $f^\star$.

We first prove the existence (E) of the fixed point.

(E) Fix an element g in A. Since T is a local contraction, for every pair of integers $q > n > 0$, we have

$$\begin{aligned} d_j(T^q g, T^n g) &\le \beta_j d_{r(j)}(T^{q-1} g, T^{n-1} g) \le \cdots \\ &\le \beta_j \beta_{r(j)} \cdots \beta_{r^{n-1}(j)} d_{r^n(j)}(T^{q-n} g, g). \end{aligned}$$

Since A is T-invariant, $T^{q-n} g$ belongs to A and we get

$$d_j(T^q g, T^n g) \le \beta_j \beta_{r(j)} \cdots \beta_{r^{n-1}(j)} \operatorname{diam}_{r^n(j)}(A).$$

It follows from condition (2.1) that the sequence $(T^n g)_{n\in\mathbb{N}}$ is d_j-Cauchy for each j. Since A is assumed to be sequentially σ-complete, there exists $f^\star$ in A such that $(T^n g)_{n\in\mathbb{N}}$ is σ-convergent to $f^\star$. Since the sequence $(T^n g)_{n\in\mathbb{N}}$ converges for the topology σ to $f^\star$, we have

$$\forall j \in J, \quad d_j(Tf^\star, f^\star) = \lim_{n\to\infty} d_j(Tf^\star, T^{n+1} g).$$

Recall that the operator T is a local contraction with respect to $(\mathcal{D}, r)$. This implies that

$$\forall j \in J, \quad d_j(Tf^\star, f^\star) \le \beta_j \lim_{n\to\infty} d_{r(j)}(f^\star, T^n g).$$

Since convergence for the σ-topology implies convergence for the semidistance $d_{r(j)}$, we get that $d_j(Tf^\star, f^\star) = 0$ for every $j \in J$. This in turn implies that $Tf^\star = f^\star$ since σ is Hausdorff. Hence, (E) is proved.

Now we prove the stability (S) criterion.

(S) Fix an arbitrary $h \in F$. For each $j \in J$ and every $n \ge 1$, we have

$$\begin{aligned} d_j(T^{n+1} h, T^{n+1} f^\star) &\le \beta_j d_{r(j)}(T^n h, T^n f^\star) \\ &\le \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, f^\star) \\ &\le \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} \big[d_{r^{n+1}(j)}(h, A) + \operatorname{diam}_{r^{n+1}(j)}(A) \big]. \end{aligned}$$

Since $Tf^\star = f^\star$, it follows from conditions (2.1) and (2.2) that $(T^n h)_{n\in\mathbb{N}}$ is d_j-convergent to $f^\star$. Since this is true for every j, we have thus proved that $(T^n h)_{n\in\mathbb{N}}$ is σ-convergent to $f^\star$. *Q.E.D.*

Assume now that F is sequentially σ-complete. We propose to apply Theorem 2.1 for a specific set A. Assume that there exists f in F such that the series

$$\sum_{n=0}^{\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(f, Tf) \tag{A.1}$$

is convergent for every $j \in J$. Denote by $\mathcal{O}(f)$ the orbit of f and let A be the σ-closure of $\mathcal{O}(f)$.[30]

CLAIM A.1: *The set A is T-invariant and sequentially σ-complete.*

PROOF: We first prove that A is σ-bounded. Fix $j \in J$ and observe that

$$\begin{aligned} \operatorname{diam}_j(A) &\equiv \sup\{d_j(f, g) : f, g \in A\} \\ &= \operatorname{diam}_j(\mathcal{O}(f)) \leq 2 \sup_{n\in\mathbb{N}} d_j(T^{n+1} f, f). \end{aligned}$$

Since T is a local Lipschitz function with respect to $(\mathcal{D}, r)$, we get that for every $n \geq 1$,

$$\begin{aligned} d_j(T^{n+1} f, f) &\leq d_j(Tf, f) + \beta_j d_{r(j)}(Tf, f) + \cdots \\ &\quad + \beta_j \beta_{r(j)} \cdots \beta_{r^{n-1}(j)} d_{r^n(j)}(Tf, f). \end{aligned}$$

This implies that

$$\operatorname{diam}_j(A) \leq 2\left[d_j(f, Tf) + \sum_{n=0}^{\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(f, Tf) \right] < \infty \tag{A.2}$$

and the set A is σ-bounded. From (A.2) we have that for each $n \geq 1$,

$$\beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} \operatorname{diam}_{r^{n+1}(j)}(A) \leq 2 \sum_{k=0}^{\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^{k+n}(j)} d_{r^{k+n+1}(j)}(f, Tf)$$

implying that (1) follows from (A.1). *Q.E.D.*

We can thus apply Theorem 2.1 to get the following corollary which generalizes Lemma 2 in Hadžić (1979).[31]

[30]The orbit of f is the set $\mathcal{O}(f) \equiv \{T^n f : n \in \mathbb{N}\}$.

[31]Hadžić (1979) allowed the operator T to be multivalued. The arguments of the proof of Theorem 2.1 can easily be adapted to deal with multivalued operators.

COROLLARY A.1: *Consider a family $\mathcal{D} = (d_j)_{j\in J}$ of semidistances defined on a set F such that F is Hausdorff and sequentially complete with respect to the associated topology σ. Let $T : F \to F$ be a locally Lipschitz operator with respect to $(\mathcal{D}, r)$ for some $r : J \to J$. Assume that there exists f in F satisfying* (A.1). *Then T admits a unique fixed point in the closure of the orbit of f.*

REMARK A.1: Hadžić (1979) assumed that each semidistance d_j is the restriction of a semi-norm defined on a vector space E containing F such that E is a locally convex topological vector space. We have proved that this assumption is superfluous. Moreover, Hadžić (1979) did not provide any criteria of stability similar to condition (2.2).

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