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Fixed point for local contractions: Applications to recursive utility

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The paper shows how fixed-point results for local contractions apply to the study of the existence and uniqueness of recursive utility functions defined on subsets of ℓ^∞_+ and being continuous for a specific topology. Two particular applications are presented that give rise to local contractions associated with an uncountable family of semi-distances.

Key words local contraction, Koopmans operator, recursive utility

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1 Introduction

Many dynamic economic models rely on the assumption that preferences are represented by a functional which is additive over time and discounts future rewards at a constant rate. This basic representation is analytically very tractable and has the following main behavioral feature: it is dynamically consistent and independent of unrealized alternatives. It has, however, some important drawbacks: (i) it implies a constant rate of impatience; (ii) with heterogeneous agents it gives rise to a degenerate long-run distribution: unless all agents have the same discount factor, only the most patient one ends up with a positive consumption level; and (iii) in stochastic environments it is unable to disentangle risk attitudes from the intertemporal elasticity of substitution. The class of recursive utility functions is a generalization of the additive utility family that preserves the nice characteristics of the additive class (time consistency property) and overcomes the aforementioned drawbacks. The idea underlying the construction of recursive preferences is to impose a weak separability between present and future alternatives. This leads to a representation of the utility function in terms of an aggregator function expressing current utility as a function of current choices and future utility derived from future choices.

Two approaches have been followed to construct recursive utility functions. The first one builds on the early work of Koopmans (1960) and is concerned with the axiomatization of preferences

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leading to a recursive representation of utilities. The second approach, due to Lucas and Stokey (1984) and Boyd (1990), treats an aggregator function as the fundamental expression of tastes and then tries to recover the utility function from the assumed properties of the aggregator.[1]

An aggregator function defines a recursion operator over all extended real-valued functions. The aggregator approach explores the connection between a recursive utility function and an aggregator through the study of fixed points of this operator. Therefore, fixed-point results have been very useful to explore thoroughly this link (see Boyd 1990; Durán 2000).

Inspired by the work of Rincón-Zapatero and Rodríguez-Palmero (2003)[2] and Matkowski and Nowak (2011), Martins-da-Rocha and Vailakis (2010) proved a fixed-point result for local contractions without assuming the family of contraction coefficients to be uniformly bounded away from 1. Martins-da-Rocha and Vailakis (2010) exhibit two applications to illustrate that, from an economic perspective, it is important to have a fixed-point result that encompasses local contractions associated with a family of contraction coefficients that are arbitrarily close to 1.

An additional benefit of this fixed-point theorem is that it applies to operators that are local contractions with respect to an uncountable family of semi-distances. This paper aims to illustrate that, in some circumstances, it is relevant not to restrict the cardinality of the family of semi-distances. In particular, we show how the applicability of this fixed-point result for local contractions associated with an uncountable family of semi-distances allows us to address problems of existence and uniqueness of recursive utility functions defined on subsets of ℓ^∞_+. Two applications are presented to illustrate the power of our approach. The first one considers the case of Blackwell aggregators that are unbounded from below and establishes existence and uniqueness of recursive utility functions that are continuous for the product topology. The second application deals with bounded Blackwell aggregators and addresses existence and uniqueness of recursive utility functions that are continuous for the absolute weak topology (and consequently for the Mackey topology).

The paper is organized as follows: Section 2 defines local contractions and states the fixed-point theorem proved in Martins-da-Rocha and Vailakis (2010). Section 3 shows how the fixed-point result can apply to study existence and uniqueness of solutions to the equation of Koopmans (1960). Section 4 concludes.

2 Definitions and the fixed-point result

In this section we recall the fixed-point result stated in Martins-da-Rocha and Vailakis (2010) for operators that are local contractions in an abstract space. Let F be a set and $\mathcal{D} = (d_j)_{j\in J}$ be a family of semi-distances defined on F.[3] We let σ be the weak topology on F defined by the family $\mathcal{D}$. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be σ-Cauchy if it is d_j-Cauchy for each $j \in J$. A subset A of F is said to be sequentially σ-complete if every σ-Cauchy sequence in A converges in A for the σ-topology. A subset $A \subset F$ is said to be σ-bounded if $\mathrm{diam}_j(A) \equiv \sup\{d_j(f,g) : f, g \in A\}$ is finite for every $j \in J$.

Definition 1 *Let r be a function from J to J. An operator $T : F \to F$ is a local contraction with respect to $(\mathcal{D}, r)$ if for every j there exists $\beta_j \in [0,1)$ such that*

$$\forall f, g \in F, \quad d_j(Tf, Tg) \le \beta_j d_{r(j)}(f,g).$$

1 See Becker and Boyd (1997) for an excellent exposition of these two approaches.

2 See also Rincón-Zapatero and Rodríguez-Palmero (2007) and Rincón-Zapatero and Rodríguez-Palmero (2009).

3 A semi-distance d on a space X is a real-valued function on $X \times X$ that is non-negative, symmetric, satisfies the triangle inequality and in addition $d(x,x) = 0$ for every $x \in X$.

The main technical contribution of Martins-da-Rocha and Vailakis (2010) is the following existence and uniqueness result of a fixed point for local contractions.

Theorem 1 *Assume that the space F is σ-Hausdorff.*[4] *Consider a function $r : J \to J$ and let $T : F \to F$ be a local contraction with respect to $(\mathcal{D}, r)$. Consider a non-empty, σ-bounded, sequentially σ-complete and T-invariant subset $A \subset F$. If the following condition is satisfied*

$$\forall j \in J, \qquad \lim_{n\to\infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} \mathrm{diam}_{r^{n+1}(j)}(A) = 0 \tag{1}$$

then the operator T admits a fixed point f^* in A. Moreover, if $h \in F$ satisfies

$$\forall j \in J, \qquad \lim_{n\to\infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0 \tag{2}$$

then the sequence $(T^n h)_{n\in\mathbb{N}}$ is σ-convergent to f^*.[5]

Consider the particular case where F is sequentially σ-complete. We propose to apply Theorem 1 for a specific set A. Assume that there exists f in F such that the series

$$\sum_{n=0}^{\infty} \beta_j \beta_{r(j)} \ldots \beta_{r^n(j)} d_{r^{n+1}(j)}(f, Tf) \tag{3}$$

is convergent for every $j \in J$. Denote by $\mathcal{O}(f)$ the orbit of f and let A be the σ-closure of $\mathcal{O}(f)$.[6]

As a by-product of Theorem 1 we get a generalization of a fixed-point result proposed in Hadžić (1979).[7]

Corollary 1 *Consider a family $\mathcal{D} = (d_j)_{j\in J}$ of semi-distances defined on a set F such that F is Hausdorff and sequentially complete with respect to the associated topology σ. Let $T : F \to F$ be a local contraction with respect to $(\mathcal{D}, r)$ for some $r : J \to J$. Assume that there exists f in F satisfying (3). Then T admits a unique fixed point in the closure of the orbit of f.*

3 Applications to recursive utility

Consider a model where an agent chooses consumption streams in the space ℓ^∞_+ of non-negative and bounded sequences $\boldsymbol{x} = (x_t)_{t\in\mathbb{N}}$ with $x_t \geq 0$. The space ℓ^∞ is endowed with the sup-norm $\|\boldsymbol{x}\|_\infty \equiv \sup\{|x_t| : t \in \mathbb{N}\}$. We propose to investigate whether it is possible to represent the agent's preference relation on ℓ^∞_+ by a recursive utility function derived from an aggregator

$$W : X \times Y \to Y$$

where $X = \mathbb{R}_+$ and Y is a subset of $[-\infty, \infty)$ containing 0. The answer obviously depends on the assumed properties of the aggregator function W.

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4 That is, for each pair $f, g \in F$, if $f \neq g$ then there exists $j \in J$ such that $d_j(f, g) > 0$.

5 If A is a non-empty subset of F then for each h in F, we let $d_j(h, A) \equiv \inf\{d_j(h, g) : g \in A\}$.

6 The orbit of f is the set $\mathcal{O}(f) \equiv \{T^n f : n \in \mathbb{N}\}$.

7 A detailed comparison of the result in Hadžić (1979) and Theorem 1 is given in Martins-da-Rocha and Vailakis (2010).

Since the seminal contribution of Lucas and Stokey (1984), there has been a wide literature[8] dealing with the issue of existence and uniqueness of a recursive utility function derived from aggregators that satisfy a uniform contraction property (Blackwell aggregators), that is, W is continuous on $X \times Y$, non-decreasing on $X \times Y$ and satisfies a Lipschitz condition with respect to its second argument, in the sense that there exists $\delta \in (0,1)$ such that

$$|W(x,y) - W(x,y')| \leq \delta|y - y'|, \quad \forall x \in X, \quad \forall y, y' \in Y.$$

The objective is to find a subspace $\mathbb{X} \subset \ell^\infty_+$ such that W admits a recursive utility function from $\mathbb{X}$ to $\mathbb{R}$. In order to define formally the concept of a recursive utility function we need to introduce some notations. We denote by π the linear functional from ℓ^∞ to $\mathbb{R}$ defined by $\pi\boldsymbol{x} = x_0$ for every $\boldsymbol{x} = (x_t)_{t\in\mathbb{N}}$ in ℓ^∞. We denote by S the operator of ℓ^∞ defined by $S\boldsymbol{x} = (x_{t+1})_{t\in\mathbb{N}}$.

Definition 2 *Let $\mathbb{X}$ be a subset of ℓ^∞ stable under the shift operator S.*[9] *A function $u : \mathbb{X} \to \mathbb{R}$ is a recursive utility function on $\mathbb{X}$ if it verifies the equation of Koopmans (1960):*

$$\forall \boldsymbol{x} \in \mathbb{X}, \quad u(\boldsymbol{x}) = W(\pi\boldsymbol{x}, u(S\boldsymbol{x})).$$

Taking ℓ^∞ as the commodity space is a choice that is made in many intertemporal models.[10] The advantage of ℓ^∞ with respect to other spaces (for instance ℓ^p with $1 \leq p < \infty$) is that it does not impose severe restrictions on the kind of dynamics that can be considered.[11] In addition, the fact that ℓ^∞_+ has a non-empty interior (for the sup-norm) simplifies considerably the application of a separation theorem that underlies the theorems of welfare economics in an intertemporal setting (Lucas and Prescott 1971).

However, the choice of ℓ^∞ as a commodity space introduces some complications on the choice of the appropriate topology. One may consider several topologies on ℓ^∞. There is the topology derived from the sup-norm and the product topology. There are also the weak topology $\sigma(\ell^\infty, \ell^1)$, the Mackey topology $\tau(\ell^\infty, \ell^1)$ defined as the strongest locally convex topology on ℓ^∞ consistent with the duality $\langle \ell^\infty, \ell^1 \rangle$ and the absolute weak topology $|\sigma|(\ell^\infty, \ell^1)$ defined as the smallest locally convex-solid topology on ℓ^∞ consistent with the duality $\langle \ell^\infty, \ell^1 \rangle$.[12] In particular we have[13]

$$\sigma(\ell^\infty, \ell^1) \subset |\sigma|(\ell^\infty, \ell^1) \subset \tau(\ell^\infty, \ell^1).$$

Assuming continuity of preference orderings with respect to one of the aforementioned topologies plays a crucial role in establishing existence of equilibrium in intertemporal models. As shown by Brown and Lewis (1981), assigning to ℓ^∞ one of these topologies is an abstract way of

8 See Epstein and Zin (1989), Boyd (1990), Durán (2000), Durán (2003), Le Van and Vailakis (2005), Rincón-Zapatero and Rodríguez-Palmero (2007), and Marinacci and Montrucchio (2010).

9 That is, for every $\boldsymbol{x} \in \mathbb{X}$ we have $S\boldsymbol{x}$ still belongs to $\mathbb{X}$.

10 See among others Lucas and Prescott (1971), Bewley (1972), Kehoe, Levine, and Romer (1990), Dana and Le Van (1991), Magill and Quinzii (1994), Levine and Zame (1996), and Alipranits, Border, and Burkinshaw (1997). In some models this choice is imposed directly while in some others it is implied by the assumptions made on the production activity.

11 See Chapter 15 in Stockey, Lucas, and Prescott (1989) for a discussion.

12 The Mackey topology is the topology generated by the family of semi-norms $\{\|\cdot\|_{\boldsymbol{q}} : \boldsymbol{q} \in \ell^1\}$ where $\|\boldsymbol{x}\|_{\boldsymbol{q}} = \sup_t |x_t q_t|$ for every $\boldsymbol{x} \in \ell^\infty$. The absolute weak topology is the topology generated by the family of semi-norms $\{\eta_{\boldsymbol{q}} : \boldsymbol{q} \in \ell^1\}$ where $\eta_{\boldsymbol{q}}(\boldsymbol{x}) = \langle |\boldsymbol{x}|, |\boldsymbol{q}| \rangle = \sum_{t\in\mathbb{N}} |x_t q_t|$ for all $\boldsymbol{x} \in \ell^\infty$.

13 See Aliprantis and Border (1999), p. 292.

formalizing the idea that agents are impatient. In particular, continuity of preference orders with respect to the Mackey topology permits equilibria of finite horizon economies to approximate the equilibria of infinite horizon economies since it implies that consumption in the very distant future is unimportant.

In what follows we show how our fixed-point result can apply to prove existence of recursive utility functions, defined on subsets of ℓ_+^∞ endowed with a specific topology, in two particular frameworks.

3.1 Aggregators unbounded from below

In this subsection we allow for Blackwell aggregators that are unbounded from below. More precisely, we assume that $W(x,y) \in \mathbb{R}$ for every $x \neq 0$ and $y \in \mathbb{R}$ but we allow for $W(0,y) = -\infty$. In particular we have $Y = [-\infty, \infty)$.

Let $\mathcal{A}$ be the space of sequences $\boldsymbol{a} \in \ell_+^\infty$ such that

$$\sum_{t\in\mathbb{N}} \delta^t |W(a_t, 0)| < \infty.$$

Let also $\mathcal{K}$ be the set of all order intervals $\mathbb{K} \equiv [\boldsymbol{a}, b\mathbf{1}]$ where $\boldsymbol{a} \in \mathcal{A}$ and $b > \|\boldsymbol{a}\|_\infty$.[14] Observe that if $\mathbb{K}$ belongs to $\mathcal{K}$ then $S\mathbb{K} = \{S\boldsymbol{x}: \boldsymbol{x} \in \mathbb{K}\}$ also belongs to $\mathcal{K}$. We take $\mathbb{X}$ to be the union of all intervals $\mathbb{K}$. Obviously, $\mathbb{X}$ is a subset of ℓ_+^∞ stable under S.

Let F be the space of functions $V : \mathbb{X} \to \mathbb{R}$ which are bounded and continuous for the product topology on every $\mathbb{K} \in \mathcal{K}$. For every set $\mathbb{K} \in \mathcal{K}$ we let $d_\mathbb{K}$ be the semi-distance on F defined by

$$d_\mathbb{K}(U,V) \equiv \sup\{|U(\boldsymbol{x}) - V(\boldsymbol{x})| : \boldsymbol{x} \in \mathbb{K}\} = \|U - V\|_\mathbb{K}.$$

The space F is sequentially complete with respect to the topology defined by the family $\mathcal{D} \equiv (d_\mathbb{K})_{\mathbb{K}\in\mathcal{K}}$. We let $r : \mathcal{K} \to \mathcal{K}$ be the mapping defined by $r(\mathbb{K}) = S\mathbb{K}$. Given $U \in F$ we let $TU : \mathbb{X} \to \mathbb{R}$ be the function defined by $[TU](\boldsymbol{x}) = W(\pi\boldsymbol{x}, U(S\boldsymbol{x}))$. Since W is continuous and non-decreasing, the mapping T is an operator on F, that is, it maps F into F. We have the following result.

Proposition 1 *There exists a recursive utility function $U : \mathbb{X} \to \mathbb{R}$ bounded and continuous for the product topology on every order interval $\mathbb{K}$ in $\mathcal{K}$. Moreover, U is the unique fixed point of T on the set of all functions $V : \mathbb{X} \to \mathbb{R}$ bounded and continuous for the product topology on every order interval $\mathbb{K}$ in $\mathcal{K}$ satisfying*

$$\forall \mathbb{K} \in \mathcal{K}, \quad \lim_{t\to\infty} \delta^t \sup_{\boldsymbol{x}\in\mathbb{K}} |V(S^t\boldsymbol{x})| = 0. \tag{4}$$

The proof of Proposition 1 is based on an application of Corollary 1 with an uncountable family of semi-distances.

PROOF: Since W satisfies a Lipschitz contraction property, we get that T is a local contraction with respect to $(\mathcal{D}, r)$. More precisely, we have

$$d_\mathbb{K}(U,V) \le \delta d_{r(\mathbb{K})}(U,V).$$

[14] We denote by $\mathbf{1}$ the sequence $\boldsymbol{x} = (x_t)_{t\in\mathbb{N}}$ in ℓ^∞ defined by $x_t = 1$ for every t. The order interval $[\boldsymbol{a}, b\mathbf{1}]$ is the set $\{\boldsymbol{x} \in \ell^\infty : a_t \le x_t \le b, \forall t \in \mathbb{N}\}$.

For each $t \geq 1$ we have

$$\|T0\|_{r^t(\mathbb{K})} = \sup_{\boldsymbol{x} \in \mathbb{K}} |W(x_t, 0)|.$$

Since $\mathbb{K}$ belongs to $\mathcal{K}$, it follows that the series

$$\sum_{t=0}^{\infty} \delta^t \|T0\|_{r^t(\mathbb{K})}$$

is convergent. We can then apply Corollary 1 to get the existence of a fixed point U of the operator T which is unique in A, the closure of the orbit $\mathcal{O}(0)$ of 0.

Now, fix a function $V : \mathbb{X} \to \mathbb{R}$ continuous for the product topology on every order interval $\mathbb{K}$ in $\mathcal{K}$ satisfying (4). We have to prove that for every $\mathbb{K} \in \mathcal{K}$,

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}\in\mathbb{K}} |U(\boldsymbol{x}) - W(x_0, W(x_1, \ldots, W(x_t, V(S^{t+1}\boldsymbol{x}))\ldots))| = 0. \tag{5}$$

In other words, we should prove that

$$\forall \mathbb{K} \in \mathcal{K}, \quad \lim_{t\to\infty} d_{\mathbb{K}}(T^t V, U) = 0.$$

According to Theorem 1, it is sufficient to prove that

$$\forall \mathbb{K} \in \mathcal{K}, \quad \lim_{t\to\infty} \delta^t d_{r^t(\mathbb{K})}(V, A) = 0.$$

Since 0 belongs to A, we have

$$d_{r^t(\mathbb{K})}(V, A) \leq d_{r^t(\mathbb{K})}(V, 0) = \|V\|_{S^t\mathbb{K}}$$

and the desired result follows from (4). □

3.2 Weak absolute continuous utility function

In this subsection we restrict our attention to aggregators that are bounded from below. More precisely, we assume that $Y = [0, \infty)$ and for simplicity we impose $W(0, 0) = 0$. We will also assume that for any $y \in Y$, the function $x \mapsto W(x, y)$ is concave. This class of aggregators has been studied by Koopmans, Diamond, and Williamson (1964). We show that under our assumptions, there exists a recursive utility function defined on ℓ_+^∞ that is continuous for the Mackey topology.[15] More precisely, we have the following result.

Proposition 2 *There exists a recursive utility function $U : \ell_+^\infty \to \mathbb{R}_+$ which is continuous for the absolute weak topology. Moreover, the function U is the unique recursive utility function among all functions $V : \ell_+^\infty \to \mathbb{R}_+$ continuous for the absolute weak topology and satisfying*

$$\lim_{t\to\infty} \delta^t \sup_{\boldsymbol{x}\in\mathbb{K}} \{V(S^t\boldsymbol{x})\} = 0 \tag{6}$$

for every non-empty set $\mathbb{K} \subset \ell_+^\infty$ compact for the absolute weak topology.

[15] Stroyan (1983) also proves existence and uniqueness of a Mackey continuous recursive utility function. However, the arguments of his proof rely on non-standard analysis.

The proof of Proposition 2 is based on an application of Corollary 1 with an uncountable family of semi-distances. It proceeds in several steps. Claims 1 and 2 are devoted to the construction of an uncountable family of compact (for the absolute weak topology) sets that is stable under the shift operator and covers ℓ^∞_+. Claim 3 identifies a subset of real-valued functions that are continuous (for the absolute weak topology) on ℓ^∞_+. This space of functions constitutes the space F in Corollary 1. Given a compact set, we subsequently define a semi-distance on F by taking the uniform metric on the fixed compact set. The family of all aforementioned semi-distances makes F sequentially complete. As in the previous proposition, Corollary 1 gives existence of a fixed point U of the operator T which is unique in A, the closure of the orbit $\mathcal{O}(0)$ of the null function. The last step involves to establish uniqueness on the whole space of continuous (for the absolute weak topology) functions defined on ℓ^∞_+ and satisfying (6).

Proof: We now denote by $\mathcal{K}$ the set of all subsets $\mathbb{K}$ of ℓ^∞_+ such that

$$\sum_{t\in\mathbb{N}} \delta^t \sup_{\boldsymbol{x}\in\mathbb{K}}\{W(x_t,0)\} < \infty.$$

Let $\boldsymbol{x}$ be any element in ℓ^∞_+. Observe that $0 \le x_t \le \|\boldsymbol{x}\|_\infty$ for all $t \in \mathbb{N}$. Since W is non-decreasing we get $0 \le W(x_t,0) \le W(\|\boldsymbol{x}\|_\infty, 0)$ for all $t\in\mathbb{N}$, implying that

$$\sum_{t\in\mathbb{N}} \delta^t W(x_t,0) < \infty.$$

In particular, for every $\boldsymbol{x}\in\ell^\infty_+$, the set $\{\boldsymbol{x}\}$ belongs to $\mathcal{K}$.

Choose $\eta > 0$ such that

$$\sum_{t\in\mathbb{N}} \delta^t W(\eta,0) < 1. \tag{7}$$

We denote by $\mathcal{K}(\eta)$ the family of all non-empty sets $\mathbb{K}\subset\ell^\infty_+$ such that there exists $\boldsymbol{x}\in\mathbb{K}$ satisfying

$$\sup_{\boldsymbol{z}\in\mathbb{K}}\sum_{t\in\mathbb{N}} \delta^t W(x_t,0)|z_t - x_t| < \infty \quad \text{and} \quad \sup_{\boldsymbol{z}\in\mathbb{K}}\sum_{t\in\mathbb{N}} \delta^t W(\eta,0)|z_t - x_t| < \infty.$$

Claim 1 The family $\mathcal{K}(\eta)$ is stable under S, contains all non-empty subsets of ℓ^∞_+ that are compact for the absolute weak topology, and covers ℓ^∞_+.

Proof of Claim 1 Let $\boldsymbol{x}$ be any consumption stream in ℓ^∞_+. The set $\{\boldsymbol{x}\}$ belongs to $\mathcal{K}(\eta)$. This implies that $\mathcal{K}(\eta)$ is non-empty and covers ℓ^∞_+. The stability of $\mathcal{K}(\eta)$ is obvious. Now, let $\mathbb{K}$ be a non-empty set of ℓ^∞_+ that is compact for the absolute weak topology. Since $\mathbb{K}$ is non-empty, we let $\boldsymbol{x}$ be any element of $\mathbb{K}$. We have already proved that

$$\sum_{t\in\mathbb{N}} \delta^t W(x_t,0) < \infty.$$

Therefore, the sequence $\boldsymbol{q}$ belongs to ℓ^1_+ where $q_t = \delta^t W(x_t,0)$ for every $t\in\mathbb{N}$. Observe that the sequence $\boldsymbol{r} = (r_t)_{t\in\mathbb{N}}$ defined by $r_t = \delta^t W(\eta,0)$ also belongs to ℓ^1_+. Since $\mathbb{K} - \{\boldsymbol{x}\}$ is compact for $|\sigma|(\ell^\infty,\ell^1)$, there exists $M > 0$ such that

$$\sup_{z\in\mathbb{K}}\sum_{t\in\mathbb{N}}\delta^t W(x_t,0)|z_t-x_t| = \sup_{z\in\mathbb{K}}\langle|\boldsymbol{z}-\boldsymbol{x}|,\boldsymbol{q}\rangle < M$$

and

$$\sup_{z\in\mathbb{K}}\sum_{t\in\mathbb{N}}\delta^t W(\eta,0)|z_t-x_t| = \sup_{z\in\mathbb{K}}\langle|\boldsymbol{z}-\boldsymbol{x}|,\boldsymbol{q}\rangle < M.$$

This implies that $\mathbb{K}$ belongs to $\mathcal{K}(\eta)$. □

Claim 2 The family $\mathcal{K}(\eta)$ is a subset of $\mathcal{K}$.

Proof of Claim 2 Let $\mathbb{K}$ be a set in $\mathcal{K}(\eta)$ and let $\boldsymbol{x}$ be any element of $\mathbb{K}$ and $M>0$ such that

$$\sup_{z\in\mathbb{K}}\sum_{t\in\mathbb{N}}\delta^t W(x_t,0)|z_t-x_t| < M \quad \text{and} \quad \sup_{z\in\mathbb{K}}\sum_{t\in\mathbb{N}}\delta^t W(\eta,0)|z_t-x_t| < M.$$

We denote by $\mathbb{N}_\eta$ the subset of all $t\in\mathbb{N}$ such that $x_t\le\eta$. Let $t\in\mathbb{N}_\eta$. If $z_t\ge\eta$ then by concavity of $W(\cdot,0)$ we have

$$|W(z_t,0)-W(x_t,0)| \le \frac{W(\eta,0)}{\eta}|z_t-x_t|.$$

If $z_t<\eta$ then

$$|W(z_t,0)-W(x_t,0)| \le 2W(\eta,0).$$

It follows that for every $\boldsymbol{z}\in\mathbb{K}$ we have

$$\begin{aligned}\sum_{t\in\mathbb{N}_\eta}\delta^t|W(z_t,0)-W(x_t,0)| &\le \sum_{t\in\mathbb{N}_\eta}\delta^t\left[\frac{W(\eta,0)}{\eta}|z_t-x_t| + 2W(\eta,0)\right]\\ &\le M/\eta + 2.\end{aligned}$$

Now if $t\notin\mathbb{N}_\eta$ then $x_t>\eta>0$ and by concavity of $W(\cdot,0)$ we have for every $\boldsymbol{z}\in\mathbb{K}$

$$|W(z_t,0)-W(x_t,0)| \le \frac{W(x_t,0)}{x_t}|z_t-x_t| \le \frac{W(x_t,0)}{\eta}|z_t-x_t|.$$

This implies that for every $\boldsymbol{z}\in\mathbb{K}$

$$\begin{aligned}\sum_{t\in\mathbb{N}}\delta^t|W(z_t,0)-W(x_t,0)| &\le \sum_{t\in\mathbb{N}_\eta}\delta^t|W(z_t,0)-W(x_t,0)|\\ &\quad+\sum_{t\notin\mathbb{N}_\eta}\delta^t|W(z_t,0)-W(x_t,0)|\\ &\le \sum_{t\in\mathbb{N}_\eta}\delta^t|W(z_t,0)-W(x_t,0)|\\ &\quad+\sum_{t\notin\mathbb{N}_\eta}\delta^t\frac{W(x_t,0)}{\eta}|z_t-x_t|\\ &\le (M/\eta+2)+M/\eta = 2(M/\eta+1).\end{aligned}$$

We have shown that

$$\sum_{t\in\mathbb{N}} \delta^t \sup_{z\in\mathbb{K}}\{W(z_t,0)\} \le \sum_{t\in\mathbb{N}} \delta^t W(x_t,0) + \sup_{z\in\mathbb{K}} \sum_{t\in\mathbb{N}} \delta^t |W(z_t,0) - W(x_t,0)| \le \sum_{t\in\mathbb{N}} \delta^t W(x_t,0) + 2(M/\eta+1) < \infty.$$

This implies that the set $\mathbb{K}$ belongs to $\mathcal{K}$. □

We let H be the space of functions $U : \ell^\infty_+ \to \mathbb{R}$ which are continuous on ℓ^∞_+ for the absolute weak topology and we let F be the space of functions $U : \ell^\infty_+ \to \mathbb{R}$ which are bounded and continuous for the product topology on every set $\mathbb{K}$ of $\mathcal{K}(\eta)$.

Claim 3. Any function in F is also continuous on ℓ^∞_+ for the absolute weak topology, that is, F is a subset of H.

Proof of Claim 3 Let $V : \mathbb{X} \to \mathbb{R}$ be function in F. Let $(\boldsymbol{x}^\alpha)_{\alpha\in A}$ be a net in ℓ^∞_+ converging to $\boldsymbol{x}$ in ℓ^∞_+ for the absolute weak topology. Recall that we have

$$\sum_{t\in\mathbb{N}} \delta^t W(x_t,0) < \infty$$

implying that the sequences $\boldsymbol{q}$ and $\boldsymbol{r}$ defined in the proof of Claim 1 also belong to ℓ^1_+. The convergence of $(\boldsymbol{x}^\alpha)_{\alpha\in A}$ to $\boldsymbol{x}$ for the absolute weak topology implies that

$$\lim_{\alpha\in A}\langle \boldsymbol{q}, |\boldsymbol{x}^\alpha - \boldsymbol{x}|\rangle = 0 \quad \text{and} \quad \lim_{\alpha\in A}\langle \boldsymbol{r}, |\boldsymbol{x}^\alpha - \boldsymbol{x}|\rangle = 0.$$

Therefore, there exists $\alpha_0 \in A$ such that for all $\alpha \ge \alpha_0$ we have

$$\sum_{t\in\mathbb{N}} \delta^t W(x_t,0)|x^\alpha_t - x_t| \le 1 \quad \text{and} \quad \sum_{t\in\mathbb{N}} \delta^t W(\eta,0)|x^\alpha_t - x_t| \le 1.$$

It follows that the set

$$\mathbb{K} \equiv \{\boldsymbol{x}\} \cup \{\boldsymbol{x}^\alpha : \alpha \ge \alpha_0\}$$

belongs to $\mathcal{K}(\eta)$.[16] Since $(\boldsymbol{x}^\alpha)_{\alpha\ge\alpha_0}$ converges for the absolute weak topology, it also converges for the product topology.[17] Since the restriction of V to $\mathbb{K}$ is continuous for the product topology, we get that

$$\lim_{\alpha\ge\alpha_0} V(\boldsymbol{x}^\alpha) = V(\boldsymbol{x}).$$

□

For each $\mathbb{K} \in \mathcal{K}(\eta)$ we let $d_\mathbb{K}$ be the semi-distance on F defined by

$$d_\mathbb{K}(U,V) \equiv \sup_{\boldsymbol{x}\in\mathbb{K}} |U(\boldsymbol{x}) - V(\boldsymbol{x})|.$$

[16] The family $\mathcal{K}(\eta)$ was introduced because we do not know if the set $\{\boldsymbol{x}\} \cup \{\boldsymbol{x}^\alpha : \alpha \ge \alpha_0\}$ is compact for the absolute weak topology.

[17] Fix any $s \in \mathbb{N}$ and let $\boldsymbol{q}$ be defined by $q_t = 0$ if $t \ne s$ and $q_s = 1$. The sequence $\boldsymbol{q}$ belongs to ℓ^1_+.

The space F is sequentially complete for the topology defined by the family $\mathcal{D} \equiv (d_{\mathbb{K}})_{\mathbb{K}\in\mathcal{K}(\eta)}$. For any function U in F, we let TU be the function defined on ℓ_+^∞ by $[TU](\boldsymbol{x}) = W(\pi\boldsymbol{x}, U(S\boldsymbol{x}))$. We can show that T maps F into F and is a local contraction with respect to $(\mathcal{D}, r)$ where $r(\mathbb{K}) = S\mathbb{K}$. Let A be the closure of the orbit $\mathcal{O}(0)$ of the null function. As in the proof of Proposition 1 (this is because of Claim 1) the series

$$\sum_{t=0}^{\infty} \delta^t \|T0\|_{r^t(\mathbb{K})}$$

is convergent. We can then apply Corollary 1 to get the existence of a fixed point U of the operator T which is unique in A. Claim 3 implies that U is continuous on ℓ_+^∞ for the absolute weak topology.

Denote by $C(|\sigma|)$ the set of all non-empty subset of ℓ_+^∞ which are compact for the absolute weak topology. We already proved (see Claim 2) that $C(|\sigma|)$ is a subset of $\mathcal{K}(\eta)$. If $\mathbb{K}$ belongs to $C(|\sigma|)$ then we can extend the definition of $d_{\mathbb{K}}$ to the larger space H. Indeed, every function in H is continuous for the absolute weak topology and therefore must be bounded on $\mathbb{K}$. Moreover, the mapping T can be extended to H and satisfies $T(H) \subset H$.

Now fix a function $V : \ell_+^\infty \to \mathbb{R}$ continuous for the absolute weak topology, that is, $V \in H$ and satisfying

$$\lim_{t\to\infty} \delta^t \sup_{\boldsymbol{x}\in\mathbb{K}} |V(S^t\boldsymbol{x})| = 0$$

for every non-empty set $\mathbb{K} \in C(|\sigma|)$. To show that V must coincide with U it suffices that

$$\forall \mathbb{K} \in C(|\sigma|), \quad \lim_{t\to\infty} \delta^t d_{r^t(\mathbb{K})}(V, A) = 0.$$

The argument to prove this result is the same as in Proposition 1. □

4 Conclusion

The paper exploits the applicability of a fixed-point result for local contractions associated with an uncountable family of semi-distances to provide new insights on the existence and uniqueness of recursive utility functions derived from aggregator functions. Two applications are presented to illustrate the power of this approach: the first one considers the case of unbounded Blackwell aggregators while the second one tackles the case of aggregators bounded from below. Existence and uniqueness of recursive utility functions is established among all functions defined on subsets of ℓ^∞ that are continuous for the product topology (the unbounded form below case) and for the Mackey topology (the bounded from below case).

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